Variance-Constrained Filtering for Uncertain Stochastic Systems With Missing Measurements

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***Abstract*—In this note, we consider a new filtering problem for linear uncertain discrete-time stochastic systems with missing measurements. The parameter uncertainties are allowed to be norm-bounded and enter into the state matrix. The system measurements may be unavailable (i.e., missing data) at any sample time, and the probability of the occurrence of missing data is assumed to be known. The purpose of this problem is to design a linear filter such that, for all admissible parameter uncertainties and all possible incomplete observations, the error state of the filtering process is mean square bounded, and the steady-state variance of the estimation error of each state is not more than the individual prescribed upper bound. It is shown that, the addressed filtering problem can effectively be solved in terms of the solutions of a couple of algebraic Riccati-like inequalities or linear matrix inequalities. The explicit expression of the desired robust filters is parameterized, and an illustrative numerical example is provided to demonstrate the usefulness and flexibility of the proposed design approach.**

***Index Terms*—Incomplete observation, Kalman filtering, linear matrix inequality, missing signal, robust filtering.**

I. Introduction

The well-known Kalman filtering is one of the most successful H_2 filtering approaches widely used in various fields of signal processing and control. However, it has now been recognized that the standard Kalman filtering algorithm will generally not guarantee satisfactory performance when there exist parameter uncertainties in the system model; see, e.g., [2]. To improve the robustness, in recent years, many alternative design methods have been developed, among them, we just mention the H_∞ filtering and robust filtering approaches; see, for example, [9], [15], [20], and the references therein. Generally speaking, the H_∞ filtering approach aims at minimizing the H_∞ norm of the transfer function from noises to the estimation error, while the robust filtering approach guarantees an upper bound to the quadratic cost in spite of various parameter uncertainties, and then minimizes this upper bound locally.

In practical engineering, however, it is often the case that, for a class of filtering problems such as the tracking of a maneuvering target, the performance objectives are naturally described as the upper bounds on the error variances of estimation; see, e.g., [22] and [27]. Unfortunately, it is usually difficult to utilize traditional methods to deal with this class of *constrained variance* filtering problems. For instance, the theory of weighted least-squares estimation minimizes a weighted scalar sum of the error variances of the state estimation, but minimizing a scalar sum does not ensure that the multiple variance requirements will be satisfied [23]. Motivated by this fact, a novel filtering method, namely, error covariance assignment (ECA) theory (see, e.g., [14], [27], and [28]), was developed to provide a closed form solution for directly assigning the specified steady-state estimation error covariance. Subsequently, the idea of ECA theory has been applied in investigating the so-called variance-constrained filtering problems for parameter uncertain systems [25] and sampled-data systems [26], where a prespecified upper bound is placed onto the steady-state estimation error variance. It is worth mentioning that the specified variance constraints may not be minimal, but should meet given engineering requirements. Therefore, after assigning to the error dynamic system a specified variance upper bound, there remains *much* freedom which can be used to attempt to *directly* achieve other desired performance requirements, but the traditional optimal (robust) Kalman filtering methods may not have such an advantage.

So far in the previously mentioned literature, it is assumed that the measurements always contain the signal. However, in practical applications such as target tracking, there may be a nonzero probability that any observation consists of noise alone if the target is absent, i.e., the measurements are not consecutive but contain missing observations. The missing observations are caused by a variety of reasons, e.g., the high maneuverability of the tracked target, a certain failure in the measurement, intermittent sensor failures, accidental loss of some collected data, or some of the data may be jammed or coming from a high noise environment, etc. [16].

Basically, the standard definition of covariance in the data statistical analysis does not directly apply if some of the measurements are unavailable. Thus, the popular robust and/or H_∞ filtering approaches, which are dependent on the system output covariance, do not suit the case when there are missing measurements. In [5] and [16], the system identification problem was studied for data with missing observations. As for filtering problem, only a very limited number of filter design methods for system output signals with missing measurements have been developed. In [11], the effect of missing data on the steady-state performance of a tracking filter was shown to be crucial. In [6], Chen proposed a suboptimal Kalman filtering method to cope with the case of measurement data missing. In [13], a measurement model with a binary multiplicative noise was introduced to account for the influence from the missing data. The similar model was employed in [14] to study the filter design problem with error covariance assignment, where the parameter uncertainties were not taken into account. Some more relevant references can also be found in [7] and [14]. Recently, in [18] and [19], the robust filtering problem with missing data was investigated by using a jump Riccati equation approach, where a notion of incompleteness matrix function was proposed to quantify the missing data. Up to now, to the best of the authors' knowledge, the issue of *variance-constrained* filtering on *parameter uncertain* systems with *missing measurements* has not been fully investigated and remains to be important and challenging.

In this note, we are concerned with the variance-constrained filtering problem for uncertain discrete-time stochastic systems with probabilistic missing measurements. We aim at designing a linear filter such that, for all admissible parameter uncertainties and all possible incomplete observations: 1) the error state of the filtering process is mean square bounded and 2) the steady-state variance of the estimation error of each state is not more than the individual prescribed upper bound. It is shown that, the solution to the addressed filtering problem is related to a couple of algebraic Riccati-like inequalities or linear matrix inequalities. The explicit expression of the desired robust filters is derived, and a numerical example is offered to illustrate the usefulness of the proposed design approach.

The rest of this note is arranged as follows. Section II formulates the robust variance-constrained filter design problem for uncertain discrete-time systems with missing measurements. The solution of this

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problem is given in Section III. We demonstrate the use of the proposed theory in Section IV by means of a numerical example. In Section V, some concluding remarks end the note by pointing out possible extensions and future research directions.

Notation: The notations in this note are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I is the identity matrix with compatible dimension. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathcal{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Prob$\{\cdot\}$ means the occurrence probability of the event "$\cdot$." The shorthand $\mathrm{diag}(M_1, M_2, \ldots, M_N)$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \ldots, M_N$. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation and Assumptions

Consider the following linear uncertain discrete-time stochastic system:

$$x(k+1) = (A + \Delta A)x(k) + w(k) \tag{1}$$

and the measurement equation

$$y(k) = \gamma(k)Cx(k) + v(k) \tag{2}$$

where $x \in \mathbb{R}^n$ is a state vector, $y \in \mathbb{R}^p$ is a measured output vector, and A and C are known constant matrices. $w(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^p$ are mutually uncorrelated zero mean Gaussian white noise sequences with respective covariances $W > 0$ and $V > 0$. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$, and is uncorrelated with both $w(k)$ and $v(k)$. ΔA is a real-valued perturbation matrix that represents parametric uncertainty being of the following form:

$$\Delta A = MFN \quad FF^T \leq I \tag{3}$$

and M and N are known constant matrices of appropriate dimensions which specify how the elements of the nominal matrix A are affected by the uncertain parameters in F. The uncertainties in ΔA are said to be admissible if (3) holds. The stochastic variable $\gamma(k) \in \mathbb{R}$ is a Bernoulli distributed white sequence taking values on 0 and 1 with

$$\mathrm{Prob}\{\gamma(k) = 1\} = \mathcal{E}\{\gamma(k)\} := \bar{\gamma} \tag{4}$$

where $\bar{\gamma}$ is a known positive constant, and $\gamma(k) \in \mathbb{R}$ is assumed to be independent of $w(k)$, $v(k)$, and $x(0)$. Therefore, we have

$$\mathrm{Prob}\{\gamma(k) = 0\} = 1 - \bar{\gamma} \tag{5}$$

$$\sigma_\gamma^2 := \mathcal{E}\left\{(\gamma(k) - \bar{\gamma})^2\right\} = (1 - \bar{\gamma})\bar{\gamma}. \tag{6}$$

Remark 1: The parameter uncertainty structure as in (3) has been frequently used in the problems of robust filtering and control of uncertain systems (see, e.g., [4], [9], [15], [20], [21], [24], and [25]). Many practical systems possess parameter uncertainties which can be either exactly modeled or overbounded by (3). Observe that the unknown matrix F in (3) can even be allowed to be time varying and state dependent, i.e., $F = F(t, x(t))$, as long as (3) is satisfied. On the other hand, the system measurement mode (2) was first introduced in [13], and has subsequently been used in many papers (see, e.g., [14]) to account for the probabilistic measurement missing. The corresponding probability $\bar{\gamma}$ could be estimated through statistical tests.

Throughout this note, we will need the following assumption.

Assumption 1: The matrix A is nonsingular and Schur stable (i.e., all eigenvalues of A are nonzero and located within the unit circle in the complex plane).

Introducing now a new stochastic sequence

$$\tilde{\gamma}(k) := \gamma(k) - \bar{\gamma} \tag{7}$$

we can see that $\tilde{\gamma}(k)$ is a scalar zero mean stochastic sequence with variance

$$\sigma_{\tilde{\gamma}}^2 = (1 - \bar{\gamma})\bar{\gamma}. \tag{8}$$

The linear full-order filter considered in this note is of the following structure:

$$\hat{x}(k+1) = G\hat{x}(k) + K\left(y(k) - \bar{\gamma}C\hat{x}(k)\right) \tag{9}$$

where $\hat{x}(k)$ stands for the state estimate and G and K are the filter parameters to be scheduled.

The steady-state estimation error covariance is defined by

$$P := \lim_{k\to\infty} P(k) := \lim_{k\to\infty} \mathcal{E}\left[e(k)e^T(k)\right] \quad e(k) = x(k) - \hat{x}(k). \tag{10}$$

From (1), (2), (7), and (9), we have

$$\begin{aligned} y(k) - \bar{\gamma}C\hat{x}(k) &= \gamma(k)Cx(k) + v(k) - \bar{\gamma}C\hat{x}(k) \\ &= \tilde{\gamma}(k)Cx(k) + \bar{\gamma}Ce(k) + v(k) \end{aligned}$$

and, subsequently

$$\begin{aligned} e(k+1) = &\left(A + \Delta A - G - \tilde{\gamma}(k)KC\right)x(k) \\ &+ (G - \bar{\gamma}KC)e(k) + w(k) - Kv(k). \end{aligned} \tag{11}$$

Define

$$x_f(k) := \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$$

$$A_f := \begin{bmatrix} A & 0 \\ A - G - \tilde{\gamma}(k)KC & G - \bar{\gamma}KC \end{bmatrix} \tag{12}$$

$$A_n := \begin{bmatrix} A & 0 \\ A - G & G - \bar{\gamma}KC \end{bmatrix} \quad J := \begin{bmatrix} 0 & 0 \\ \sigma_{\tilde{\gamma}}KC & 0 \end{bmatrix} \tag{13}$$

$$M_f := \begin{bmatrix} M \\ M \end{bmatrix} \quad N_f := [N \quad 0] \quad \Delta A_f = M_f F N_f \tag{14}$$

$$W_f := B_f B_f^T := \begin{bmatrix} W & W \\ W & W + KVK^T \end{bmatrix} \tag{15}$$

$$X(k) := \mathcal{E}\left[x_f(k)x_f^T(k)\right] := \begin{bmatrix} X_{xx}(k) & X_{xe}(k) \\ X_{xe}^T(k) & X_{ee}(k) \end{bmatrix}. \tag{16}$$

Considering (1) and (11), we obtain the following augmented system:

$$x_f(k+1) = (A_f + \Delta A_f)x_f(k) + B_f w_f(k) \tag{17}$$

where $w_f(k)$ denotes a zero mean Gaussian white noise sequence with unity intensity $I > 0$.

Remark 2: It is mentionable that there is a stochastic variable $\tilde{\gamma}(k)$ involved in A_f, which reflects the characteristic of the missing measurement for the addressed filtering problem, and the augmented system (17) is, therefore, essentially a stochastic parameter system. Note that a more general stochastic parameter system is the so-called Markovian jumping system, where the stochastic variable $\tilde{\gamma}(k)$ may be an ergodic finite state Markov chain. Markovian jumping systems have received much research attention in the past decade; see [3], [4], and the references therein. The robust Kalman filtering problem has recently been studied in [21] for linear jumping systems by solving two sets of *coupled* algebraic Riccati equations. Nevertheless, focusing on the particular data missing problem, we let the stochastic variable $\tilde{\gamma}(k)$ be a Bernoulli sequence and, therefore, we are able to obtain more practical solutions. For example, the algorithm developed in this note will not involve solving *coupled* matrix equations/inequalities. Another motivation is that, instead of the minimum variance filtering, we consider the constrained variance filtering problem here, which should give more design freedom. This will be demonstrated later in Remark 6.

Using the statistics of the noises $w(k)$, $v(k)$ and, in particular, $\tilde{\gamma}(k)$, the state covariance $X(k)$ defined in (16) is found to satisfy

$$X(k+1) = (A_n + \Delta A_f)X(k)(A_n + \Delta A_f)^T + JX(k)J^T + W_f. \tag{18}$$

We know from [1] and [8] that, if the state of (17) is mean square bounded, the steady-state covariance X of (17) defined by

$$X := \lim_{k\to\infty} X(k) \tag{19}$$

exists and satisfies the following discrete-time modified Lyapunov equation:

$$X = (A_n + \Delta A_f)X(A_n + \Delta A_f)^T + JXJ^T + W_f. \tag{20}$$

Remark 3: It follows from [1] and [8] that, there exists a unique symmetric positive–semidefinite solution to (20) iff

$$\rho\left\{(A_n + \Delta A_f) \otimes (A_n + \Delta A_f) + J \otimes J\right\} < 1 \tag{21}$$

where ρ is the spectral radius and $\otimes$ is the Kronecker product. Furthermore, we also know from [1] and [8] that the condition (21) is equivalent to the mean square boundedness of the state of (17). Hence, we conclude here that, if there exists a positive definite solution to the (20), then (21) holds, and the convergence of $X(k)$ in (16) will be guaranteed to a constant value X.

The purpose of this note is to design the filter parameters, G and K, such that for all admissible perturbations ΔA: 1) the state of the augmented system (17) is mean square bounded, i.e., (21) holds; and 2) the steady-state error covariance X_{ee} satisfies

$$[X_{ee}]_{ii} \le \alpha_i^2, \qquad i = 1, 2, \ldots, n. \tag{22}$$

where $[X_{ee}]_{ii}$ means the steady-state variance of the ith error state, and $\alpha_i^2 (i = 1, 2, \ldots, n)$ denotes the prespecified steady-state error estimation variance constraint on the ith state.

Remark 4: In (22), individual upper bound constraint, which can be obtained according to the engineering requirement, is imposed on individual steady-state estimation error variance. This idea has been applied in dealing with the pointing problem for NASA's ACES structure, such as the Hubble Space Telescope [29]. Note that conventional minimum variance filtering method aims to minimize a weighted scalar sum of the estimation error variances, and is therefore not able to ensure that the multiple variance requirements will be satisfied [23].

In the next section, a solution to the problem addressed above will be given. A two-step approach will be developed. Specifically, we will first characterize an upper bound on the steady-state error covariance X satisfying (20) in terms of some free parameters, and let this upper bound meet the prespecified variance constraints, and then we will parameterize all desired filter gains with which the resulting steady-state error covariance is not more than the obtained upper bound.

III. Main Results and Proofs

The following lemmas will be needed in establishing our main results.

Lemma 1: [24] Let a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q_f > 0$ be such that $N_f Q_f N_f^T < \varepsilon I$, and $\Delta A_f = M_f F N_f$ with $FF^T \le I$. Then

$$\begin{aligned}(A_n + \Delta A_f)&Q_f(A_n + \Delta A_f)^T \\ &\le A_n\left(Q_f^{-1} - \varepsilon^{-1}N_f^T N_f\right)^{-1} A_n^T + \varepsilon M_f M_f^T \end{aligned} \tag{23}$$

holds for all admissible perturbations ΔA.

Lemma 2: [25] For a given negative definite matrix $\Pi < 0$ ($\Pi \in \mathbb{R}^{n\times n}$), there always exists a matrix $L \in \mathbb{R}^{n\times p}$ $(p \le n)$ such that $\Pi + LL^T < 0$.

Lemma 3: (Schur complement) Given constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0$$

or

$$\begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

For presentation convenience, we denote

$$\begin{aligned}\Phi :=& (A-G)\left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1}(A-G)^T \\ &+ \varepsilon MM^T + W \end{aligned} \tag{24}$$

$$R := \left(\bar{\gamma}^2 + \sigma_{\tilde{\gamma}}^2\right) CP_2C^T + V \tag{25}$$

$$\Pi := \Phi + GP_2G^T - P_2 - \bar{\gamma}^2 GP_2C^T R^{-1} CP_2G^T \tag{26}$$

where $\bar{\gamma}$ and $\sigma_{\tilde{\gamma}}$ are defined in (4) and (8), respectively.

The following theorem reveals that the solution to the problem of variance-constrained filtering with missing measurements is related to two quadratic matrix inequalities.

Theorem 1: Assume that there exists a positive scalar $\varepsilon > 0$ such that the following two quadratic matrix inequalities:

$$\begin{aligned}AP_1A^T - P_1 + AP_1N^T(\varepsilon I - NP_1N^T)^{-1} \times& \\ NP_1A^T + \varepsilon MM^T + W &< 0 \end{aligned} \tag{27}$$

$$\Pi = \Phi + GP_2G^T - P_2 - \bar{\gamma}^2 GP_2C^T R^{-1} CP_2G^T < 0 \tag{28}$$

respectively have positive–definite solutions $P_1 > 0$ $(NP_1N^T \le \varepsilon I)$ and $P_2 > 0$, where

$$G = A + (\varepsilon MM^T + W)(A^{-1})^T\left(P_1^{-1} - \varepsilon^{-1}N^T N\right). \tag{29}$$

Moreover, let $L \in \mathbb{R}^{n\times p}$ $(p \le n)$ be an arbitrary matrix satisfying $\Pi + LL^T < 0$ (see Lemma 2), and $U \in \mathbb{R}^{p\times p}$ be an arbitrary orthogonal matrix (i.e., $UU^T = I$). Then, the filter (9) with the parameters determined by (29) and

$$K = \bar{\gamma}GP_2C^T R^{-1} + LUR^{-\frac{1}{2}} \tag{30}$$

will be such that, for all admissible perturbations ΔA: 1) the state of the augmented system (17) is mean square bounded; and 2) the steady-state error covariance X_{ee} meets $X_{ee} < P_2$.

Proof: Define $P_f := \mathrm{diag}(P_1, P_2)$. Then, it follows directly from Lemma 1 and the definitions (24)–(26) that

$$\begin{aligned}(A_n + \Delta A_f)P_f(A_n + \Delta A_f)^T -& \\ P_f + JP_fJ^T + W_f \le& \\ A_n\left(P_f^{-1} - \varepsilon^{-1}N_f^T N_f\right)^{-1} A_n^T +& \\ \varepsilon M_f M_f^T - P_f + JP_fJ^T + W_f := \Psi :=& \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & \Psi_{22}\end{bmatrix} \end{aligned} \tag{31}$$

where

$$\Psi_{11} = A\left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1}A^T - P_1 + \varepsilon MM^T + W \tag{32}$$

$$\Psi_{12} = A\left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1}(A-G)^T + \varepsilon MM^T + W \tag{33}$$

$$\begin{aligned}\Psi_{22} =& (A-G)\left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1}(A-G)^T \\ &+ (G - \bar{\gamma}KC)P_2(G - \bar{\gamma}KC)^T + \varepsilon MM^T \\ &- P_2 + \sigma_{\tilde{\gamma}}^2 KCP_2C^TK^T + W + KVK^T. \end{aligned} \tag{34}$$

It follows immediately from the matrix inverse lemma that

$$\left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1} = P_1 + P_1N^T(\varepsilon I - NP_1N^T)^{-1}NP_1$$

and, therefore, (27) implies that $\Psi_{11} < 0$. Moreover, substituting the expression of G in (29) into (33) leads to $\Psi_{12} = 0$ easily.

Next, we will consider Ψ_{22}. By using the definitions (24)–(26), we can rearrange (34) as follows:

$$\begin{aligned}\Psi_{22} &= \Phi + (G - \bar{\gamma}KC)P_2(G - \bar{\gamma}KC)^T - P_2 \\ &\quad + \sigma_{\bar{\gamma}}^2 KCP_2C^TK^T + KVK^T \\ &= \Phi + GP_2G^T - P_2 + K\Big[\big(\bar{\gamma}^2 + \sigma_{\bar{\gamma}}^2\big)CP_2C^T + V\Big]K^T \\ &\quad - \bar{\gamma}GP_2C^TK^T - \bar{\gamma}KCP_2G^T \\ &= \Phi + GP_2G^T - P_2 - \bar{\gamma}^2 GP_2C^TR^{-1}CP_2G^T \\ &\quad + \Big(KR^{\frac{1}{2}} - \bar{\gamma}GP_2C^TR^{-\frac{1}{2}}\Big) \\ &\quad \times \Big(KR^{\frac{1}{2}} - \bar{\gamma}GP_2C^TR^{-\frac{1}{2}}\Big)^T \\ &= \Pi + \Big(KR^{\frac{1}{2}} - \bar{\gamma}GP_2C^TR^{-\frac{1}{2}}\Big) \\ &\quad \times \Big(KR^{\frac{1}{2}} - \bar{\gamma}GP_2C^TR^{-\frac{1}{2}}\Big)^T. \end{aligned} \tag{35}$$

Noticing the expression of $K = \bar{\gamma}GP_2C^TR^{-1} + LUR^{-1/2}$ in (30) and the fact that $UU^T = I$, we have

$$\Big(KR^{\frac{1}{2}} - \bar{\gamma}GP_2C^TR^{-\frac{1}{2}}\Big)\Big(KR^{\frac{1}{2}} - \bar{\gamma}GP_2C^TR^{-\frac{1}{2}}\Big)^T = LL^T.$$

Thus, it follows from (35), the definition of the matrix $L(L \in \mathbb{R}^{n\times p})$ and (28) that $\Psi_{22} = \Pi + LL^T < 0$.

To this end, we can conclude that $\Psi < 0$. Therefore, it follows from (31) that

$$(A_n + \Delta A_f)P_f(A_n + \Delta A_f)^T - P_f + JP_fJ^T \le -W_f + \Psi < 0 \tag{36}$$

which leads to (21). As discussed earlier in Section II (see Remark 3), we know that the state of the augmented system (17) is mean square bounded, and there exists a symmetric positive–semidefinite solution to (20). The first claim of this theorem is then proved.

Furthermore, subtract (20) from (36) to give

$$\begin{aligned}(A_n + \Delta A_f)(P_f - X)(A_n + \Delta A_f)^T \\ -(P_f - X) + J(P_f - X)J^T \le \Psi < 0\end{aligned} \tag{37}$$

which indicates again from Remark 3 that $P_f - X \ge 0$ and, therefore

$$X_{ee} = [X]_{22} \le [P_f]_{22} = P_2$$

This completes the proof of this theorem. ■

Remark 5: It is clear from Theorem 1 that, if the quadratic matrix inequalities (27) and (28), respectively, have positive–definite solutions $P_1 > 0$, $P_2 > 0$, and $P_2 > 0$ satisfies

$$[P_2]_{ii} \le \alpha_i^2, \qquad i = 1, 2, \ldots, n \tag{38}$$

then the filter (9) determined by (29) and (30) will be such that: 1) the state of the augmented system (17) is mean square bounded; and 2) $[X_{ee}]_{ii} < [P_2]_{ii} \le \alpha_i^2, i = 1, 2, \ldots, n$. Hence, the design objective of variance-constrained robust filter with missing measurements will be accomplished. Note that the existence of a positive–definite solution to (27) implies the asymptotical Schur stability of system matrix A, and the nonsingularity of A is required in (29). This means that Assumption 1 should hold.

We now briefly discuss the solvability of the quadratic matrix inequalities (27) and (28), which play a key role in designing the expected filters. We can observe that, fortunately, the parameter P_2 of (28) is not included in (27). Therefore, we could first solve (27) for $\varepsilon > 0$ and $P_1 > 0$, and then solve the standard Riccati-like matrix inequality (28) for $P_2 > 0$.

By using the Schur Lemma (Lemma 3), we can transform (27) into the following linear matrix inequality (LMI):

$$\begin{bmatrix} AP_1A^T - P_1 + \varepsilon MM^T + W & AP_1N^T \\ NP_1A^T & -\varepsilon I + NP_1N^T \end{bmatrix} < 0 \tag{39}$$

The inequality (39), together with the inequality constraint

$$-\varepsilon I + NP_1N^T < 0 \tag{40}$$

are both linear on $\varepsilon > 0$ and $P_1 > 0$. Therefore, we can employ the standard LMI techniques in [10] to check the solvability of the original matrix inequality (27). After P_1 is obtained, (28) becomes a standard Riccati-like matrix inequality, and some related solving algorithms can be found in [17], etc. It is mentionable that, in the past decade, LMIs have gained much attention for their computational tractability and usefulness in signal processing and control engineering, and the number of control problems that can be formulated as LMI problems is large and continues to grow. The LMIs can now be solved efficiently by the powerful Matlab LMI Toolbox [10].

Remark 6: A typical feature of the present parameterization design approach is that, there exists much explicit freedom, such as the choices of the free parameters $L(L \in \mathbb{R}^{n\times p}$ satisfies $\Pi + LL^T < 0)$, the orthogonal matrix $U \in \mathbb{R}^{p\times p}$, etc. This makes it possible that more performance constraints (e.g., the transient requirement and reliability behavior on the filtering process) could be taken into account within the same framework, which gives us one of the future research topics. We also point out that, the results of this note can also be extended to the case when there are some deterministic parameter uncertainties on the system outputs. The reason why we discuss the case when the parameter uncertainties only enter into the state matrix is just to make our theory more understandable and to avoid unnecessarily complicated notations.

As a summary, we give our main results as follows.

Corollary 1: If there exist a positive scalar $\varepsilon > 0$ and two positive–definite matrices $P_1 > 0$, $P_2 > 0$ such that the LMIs (39) (40) and the matrix Riccati inequality (28) hold, and $P_2 > 0$ satisfies $[P_2]_{ii} \le \alpha_i^2$ $(i = 1, 2, \ldots, n.)$, then the filter (9) determined by (29) and (30) will achieve the desired robust filtering performance for uncertain systems with missing measurements.

IV. Numerical Example

In this section, we demonstrate the theory developed in this note by means of a target tracking example.

Assume that the maneuvering target is accelerating with random bursts of gas from its reaction control system (RCS) thrusters and, hence, the state vector could consist of the position and velocity. When tracking a maneuvering target through a radar system, we wish that the target would be kept inside a "window" as frequent as possible. The size of such a window depends on the acceptable filtering error variance. The target is said to be missing if it is outside the window. Our task is therefore to design a filter such that, in the presence of missing measurement and modeling error, the probability for the target to stay within a given window would be kept at a given level. Such a requirement can then be expressed in terms of the variance constraints on the individual error state; see [12] and the references therein. Motivated by this background, we consider a linear uncertain discrete-time stochastic system (1) and (2) with parameters given by

$$A = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & -0.5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 0.1 & 0.05 \\ -0.02 & 0.8 \end{bmatrix}$$
$$N = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad W = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad V = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

and the probability for complete observation is assumed to be 0.9, i.e., the missing probability is 0.1.

The purpose of this example is to design the filter parameters, G and K, such that for all admissible perturbations ΔA, the augmented system (17) is mean square bounded, and the steady-state error covariance X_{ee} satisfies

$$[X_{ee}]_{11} \le 0.8 \quad [X_{ee}]_{22} \le 4.$$

Solving LMIs (39) and (40) for ε, P_1, and then the Riccati-like matrix inequality (28) for P_2, we obtain

$$\varepsilon = 1.8286$$
$$P_1 = \begin{bmatrix} 5.8346 & 0.0064 \\ 0.0064 & 3.6628 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0.7765 & 0.0052 \\ 0.0052 & 3.6983 \end{bmatrix}.$$

One of the filter parameters, G, is calculated from (29) as follows:

$$G = \begin{bmatrix} 0.5437 & 0.0768 \\ 0.2040 & -1.1470 \end{bmatrix}.$$

To obtain another parameter, K, we choose $L = 0.5I_2$ such that $\Pi + LL^T < 0$ and select the orthogonal matrix U as I_2. Then, it follows from (30) that

$$K = \begin{bmatrix} 0.8040 & 0.0725 \\ 0.1246 & -0.8165 \end{bmatrix}.$$

Alternatively, to show the design flexibility, we choose U as $-I_2$, and subsequently have

$$K = \begin{bmatrix} -0.1346 & 0.0732 \\ 0.1253 & -1.3490 \end{bmatrix}.$$

It is not difficult to verify that the specified mean square boundedness as well as the steady-state error variance constraint are achieved.

V. Conclusion

In this note, the linear filtering problem has been considered for parameter uncertain discrete-time stochastic systems where there is a nonzero probability of signal being absent in the measurement. This problem has been approached by assigning an upper bound to the steady-state error covariance, and by parameterizing the set of all filter gains that could achieve such an upper bound. It has been shown that, the problem is solvable if several LMIs or Riccati-like matrix inequalities have positive definite solutions. In particular, the characterization of the desired filter gains has been given in terms of some "free" parameters, and much design flexibility have been offered, which could be utilized to achieve more expected performance requirements. A numerical example has been provided to illustrate the effectiveness of the proposed design approach.

Possible future research directions include real-time applications of the proposed filtering theory in telecommunications, and further extensions of the present results to more complex systems, such as sampled-data systems, bilinear systems, and a class of nonlinear systems.

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