New Control Variates For Pricing Basket and Asian Options

by

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DECLARATION

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Kam Jipreze	Date
Prof. Paresh Date (Supervisor)	Date

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To my family

"What Did it Cost? .... Everything"

-Gamorra & Thanos

Abstract

In this thesis, we investigate new control variates for simulation-based pricing of options where the option price is a function of the sum of (or integral of) lognormal random variables. We use two different approaches: one is the use of Hermite polynomial approximation of the relevant function and another is the use of upper and lower bounds on the option prices obtained using the properties of Brownian motion. We provide detailed numerical experiments to illustrate the use of these approaches for accurate and low variance pricing basket and Asian options. First order Hermite polynomial approximation also gives a reasonable direct approximation to the basket or Asian option price for at the money and in-the-money options.

List of Publications

- Jipreze, K., & Date, P. (2024). New control variates for pricing basket options. *IMA Journal of Management Mathematics*, dpae023.

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Symbol	Description
$\phi(.)$	Pdf of a standard Gaussian random variable
$\phi_W(.)$	Pdf of a standard Brownian motion $W(t)$
$\phi_{M,W}(.,.)$	Joint density of a Brownian motion and its running maximum
$\phi_M(.)$	Joint density of the running maximum of a Brownian motion
$\phi_m(.)$	Joint density of the running minimum of a Brownian motion
$\phi_{(M,n)}(.)$	Pdf of the maximum of n independent Brownian motions
$\phi_{(m,n)}(.)$	Pdf of the minimum of n independent Brownian motions
$\phi_{(M^*,n)}(.)$	Pdf of the maximum of the running maximums of n independent Brownian motions
$\phi_{(m^8,n)}(.)$	Pdf of the minimum of the running minimums of n independent Brownian motions
$\varrho_M(.,.)$	Joint pdf of the minimum of the absolute value of Brownian motion and its running maximum
$f_{\beta,\tilde{\beta}}(.,.)$	Pdf of the Brownian bridge β and its running maximum $\tilde{\beta}$
$g_{\beta,\tilde{\beta}}(.,.)$	Pdf of the Brownian bridge β and its running minimum $\tilde{\beta}$
$\mathcal{N}(0,1)$	Standard normal distribution
$\mathbb{1}_A(.)$	indicator function on set A
$\Phi(.)$	Cdf of a standard normal random variable
$\|.\|$	Absolute value function

Chapter 1

Introduction

1.1 Overview

Basket and Asian options are exotic options whose payoff depend on the sum of prices of underlying asset/s. If the prices are assumed to be lognormal, weakly path dependent options on a single asset generally admit a closed-form representation, however, pricing basket and Asian options is analytically intractable even when the underlying asset price follows a lognormal distribution. The most accurate way of pricing basket and Asian options is by the use of Monte Carlo methods. However, Monte Carlo is computationally very intensive and, without the use of specialized variance reduction methods, leads to estimates with very high variance. Most of the analytical work in this area has to do with price and bounds estimations (as in the works of Milevsky and Posner in [28], Gentle in [13], Curran in [8]), while most numerical methods involve the use of control variates (Korn in [22], Dingeç in [11], Shiraya in [36]), in an effort to achieve low variance estimates of the option prices. In the remaining chapter, we look at the general theory on basket and Asian options and review existing literature on them.

1.2 Basket Options

A basket option is a type of financial derivative where the underlying asset is a group of assets, commodities, securities or currencies. Some examples of basket options include index options and currency baskets etc. Basket options have a similar payoff structure to standard options, since they depend on the final value of the asset or groups of assets. Given a basket of assets, whose final value is $S(T)$ at a maturity time T, its payoff is given by

$$\Big(S(T) - K\Big)^{+}, \tag{1.1}$$

for a nonnegative strike K.

Basket options are typically traded over-the-counter and are customized depending on the buyer's requirements. Besides being able to efficiently and simultaneously hedge risk on several assets at the same time, basket options are also relatively low-priced compared to buying options on each of the individual assets in the basket. However, the downside of holding such a financial derivative is liquidity. If an investor holding a call/put basket option wanted to get rid of his position, he/she would have to purchase a put/call option.

Some of the benefits of basket options are

1 Basket options are more cost-effective than purchasing individual options on each asset in the basket.

2 Basket options hedges exposure more efficiently than options on the individual assets in the basket.

3 They are customized to suit investors needs/requirements.

4 Low transaction costs: Transactions costs are lower since a basket option is a single transaction compared to purchasing inidividual options on the assets in the basket.

Some of the drawbacks of basket options include:

1 Liquidity problems: Given that a call/put basket option is customized based on investors preferences, they would need to purchase a put/call option to get rid of this position.

2 Basket options are difficult to price because there is no closed-form formula to price them.

The most popular type of basket options is the currency option which is typically used by multinational corporations to mitigate numerous currency exposures.
Despite the payoff of a basket option being similar to that of a standard option, the behaviour of the basket differs from that assets constituting the basket. As a result of this, pricing a basket option is quite different from pricing standard options and in fact, basket options have no known closed-form solution. This is because the basket of assets have no known distribution and thus density. Under the assumption that the individual assets in the basket follow a geometric Brownian motion (GBM) model and using no arbitrage arguments, the

price of a basket option at an earlier time 0 with maturity T is given by

$$e^{-rT}\mathbb{E}\left[\left(\sum_{i=1}^{n}\omega_i S_i(T)-K\right)^+\right] \tag{1.2}$$

where n is the number of assets in the basket, $S_i(T)$ is the price of the ith asset in the basket at time T, K is the nonnegative strike. Each asset $S_i(T)$ is lognormally distributed and $S(T)$ is the sum of n lognormally distributed random variables, which is in fact not lognormal or known to follow any distribution.

Most of the analytical work on pricing basket options in existence are approximations, since the distribution of the basket of assets is unknown. Analytic approximations are obtained by using a random variable whose dynamics closely match that of the basket. Kemna [20] and Gentle [13] obtained closed-form approximation for the price of a basket option using the geometric mean of the assets in the basket. This was done using the fact geometric mean of a sequence are a lower bound on the arithmetic mean of the sequence. Moreover, the geometric mean of the assets are lognormally distributed and can give closed-form estimates. This approach was found to generally under value basket option prices. Milevsky and Posner [28] obtained closed-form estimates for the basket option price, by observing that the distribution of the sum of infinite lognormal random variables can be approximated by the reciprocal gamma distribution. The only drawback is that it was found to under-price out-of-the-money call options when compared to Monte Carlo prices. Ju [19] used Taylor series expansion to price basket and Asian options. Another form of closed-form approximation for pricing basket options is methods of moment-matching. This involves finding a suitable lognormal random variable with first and second moments similar to those of the basket of assets, thus obtaining suitable parameters for which the basket option can be priced under the assumption that it is lognormal. Methods of moment-matching were used by Brigo [5] and Henriksen [15] to approximate the price of a basket option. Paletta [?] also used exact moment-matching and Hermite expansion to price and hedge basket options with shifted jump-diffusions. Curran [8] used conditional moment-matching by conditioning on the geometric mean to obtain closed-form estimates for the basket option price. Alternatively, bounds estimation was used in the absence of closed-form analytic solutions and these were lower and upper bounds on the price of basket options. Rogers-Shi [35] found a lower bound on the payoff of the sum of lognormal randomm variables conditioned on a suitable random variable. More recently, Xu and Zheng [39] obtained good estimates for the basket option price using a weighted sum of the lower bound and conditional second moments using local volatility jump-diffusion models. Despite several analytical approximations to pricing basket option, Monte Carlo simulations remain the most accurate way of valuing such options. The major drawback of this approach is the high variance of its estimates. De Luigi [9] used adaptive numerical techniques to price low-dimensional basket options and found that this technique served as good control variates in the pricing of high dimensional ones. Dingeç [11] used control variates and conditional Monte Carlo to price basket options. They found that using the geometric average as a control variate, together with conditional Monte Carlo, yielded more efficiency than solely using the geometric price as a control variate. Korn [22] proposed the use of a limiting geometric mean as an approximation to arithmetic mean to obtain closed-form estimates for the basket option price. Also, the use of this limiting geometric mean as a control variate provided efficient Monte Carlo estimates. Xu and Zheng [39] used control variates to price basket option under jump diffusion models.

This was extended to stochastic volatility models with jumps using asymptotic expansion as control variates by Shiraya [36]. This method of asymptotic expansion was also previously used by Xu and Zheng[39], with Forward Partial Integral Differential Equation (PIDE) to approximate the basket option price. Dingeç [10] proposed new control variate models using time-changed Brownian motions for pricing and sensitivity analysis of Basket options. Shiraya [37] proposed a class of control variates for pricing basket options driven by Lévy processes with the use of subordinated Wiener processes and was extended by Zhang [40] to exponential subordinated Wiener processes.

1.3 Asian Options

An Asian option also called an average option refers to an option whose payoff is based on an average value of the asset during some time period of the options lifetime. It is an option whose payoff depends on the time average of the underlying asset price over the lifetime of the option. Asian options are exotic or path-dependent options. Typically, this average is either sampled on a discrete or a continuous basis. The payoff of a European Asian call option is given by

$$\left(\frac{1}{T}\int_0^T S(u)du - K\right)^+, \tag{1.3}$$

in the continuous case and

$$\left(\frac{1}{M+1}\sum_{i=1}^{M} S(t_i) - K\right)^+, \tag{1.4}$$

for the discretely monitored case and $t_i \in [0, T]$ such that $t_i = i\frac{T}{m}$ $\forall i = 1, ..., M$.

This averaging feature of Asian options makes them less likely to be significantly affected by manipulation of the underlying asset price. They are very similar to standard options because they have similar payoff structures and differ mainly in the sense that standard options are lognormal and Asian options have no known distribution or closed-form solution. This makes them difficult to price analytically and the most accurate way of pricing Asian options is by Monte Carlo simulations. However, this approach to pricing Asian option is very computationally intensive.

Asian options are difficult to price since they do not follow any known distribution, despite the underlying asset's distribution being lognormal. This is because the sum/average of lognormal random variables is not lognormal. Most of the research into Asian option can be divided into three main categories. The first one can be categorized as focusing mainly on closed-form estimates/approximations of Asian options. This typically involves finding a distribution that closely resembles that of the random variable as seen in the work of Geman-Yor [12], Kemna [20], Milevsky [28], Curran [8] to mention but a few.

This also includes methods of moment-matching which involves approximating the average value of the asset with a lognormal random variable. This is done by matching the first two moments of a lognormal random variable with those of the average value of the asset. This provided a basis for Asian options to be priced under the assumption that they are approximately lognormal. Some of the research on this was carried out by Brigo [5], Levy [26], Henriksen [15] and Tunaru [25] etc. The second is the method of bounds estimation. Given that the average value of an asset has no closed-form solution, finding suitable bounds on the actual price is important for an investor. One of the most popular bounds on Asian was discovered by Rogers-Shi [35].

The third method is a numerical approach and this is done by using appropriate numerical methods such as control variates to obtain efficient Monte Carlo estimates of the price of an Asian option. Most of the recent work done (Lai [24], Shiraya [36], Zhang [40] etc.) in the pricing of Asian options is in developing new control variates, to obtain accurate, efficient Asian option prices.

More recently, Fourier and Laplace transform based models have gained wide acceptance in their use for pricing exotic options (particularly basket and Asian options). This is due to their efficiency in dealing with complicated payoff structures by using the characteristic functions of the underlying asset(s) to reduce the pricing problem to that requiring an inversion of a transformed density. This approach was first explored in the seminal works of Geman-Yor [12] to price continuously-monitored Asian options. Some of the more recent excursions of this approach in pricing basket and Asian option can be seen in the works of Bayer [1], Bayer [2] and Zhang [42].

1.4 Statement of the Problem

Despite basket and Asian options being different in the sense one depends on the value of a group of assets and the other depending on its average over a time period, they are quite similar in terms of the pricing structure. Under the geometric Brownian motion model (GBM), asset prices are lognormal but the distribution of the basket of assets or the average value of an asset is not lognormal and this is due to the fact that the sum of lognormal random variables is not lognormal. Without loss of generality, the associated pricing problem can be stated as: given n normal random variables $x_1, ..., x_n$ with a non-negative number K such that

$$\int_A (e^{x_1} + ... + e^{x_n} - K)\, \mathbb{Q}(dx_1...dx_n), \tag{1.5}$$

where $A = \{(x_1, ...x_n) : \sum_{i=1}^{n} e^{x_i} > K\}$.

The density $\mathbb{Q}(dx_1...dx_n)$ is unknown and is generally not lognormal. As a result of this, options of this functional form have no known closed-form solution even though the work of Milevsky in [28], suggested that the sum of infinite lognormals follows an inverse gamma distribution. Monte Carlo methods remains probably one of the most accurate way till date of pricing basket and Asian options. However, its implementation is usually computationally intensive and the estimates are usually inefficient due to their high variance. Despite the plethora of literature and work available on the pricing of options on a single underlying asset, it is a well-known fact there is no closed-form solution for the price of a basket option, and accurate basket option prices are obtained using Monte Carlo methods. This is because the value of the underlying basket of assets is not lognormal and in fact has no known distribution. The possible existence of an analytical solution for a call option on a basket of assets was pointed out by Hobson [16], but further stated that such a solution would be difficult to obtain. Many of the available methods for valuing basket options in closed-form are either analytical or numerical approximations.

In this thesis, we model the underlying asset(s) using a Geometric Brownian Motion (GBM) model. Despite the existence of more sophisticated models such as the Stochastic Volatility Models (SVM) and jump-diffusion models which can capture volatility skew. In practice asset price volatilities are quoted in terms of Black-Scholes implied volatility and the GBM model remains the basis on which practitioner theories are built. Furthermore, the GBM model is analytically tractable and does not suffer from over-fitting of the parameters. This analytical tractability allows us to keep our modelling approach at the center stage without having to make unrealistic simplifications/assumptions. As a result of this, we provide good analytic approximations for the price of basket and Asian options, using first order Hermite polynomials. In the basket option case, this approach leads to a reformulation of the representation of the value of the basket, resulting in a Black-Scholes type solution. While in the Asian option case, our approximations for the average value of the underlying is an evaluation of a function of Gaussian random variables. We also provide good bounds on the option prices using the distributional properties of the underlying Brownian motion(s) and the convexity of the option payoff. Finally, we apply use these price estimates and bounds as control variates to obtain fast, accurate, low variance Monte Carlo prices of basket and Asian options. These result are benchmarked against standard control variates such as the (modified) Geometric lower bound which is famous Kemna and Vorst [20] approximation in the Asian option case, and the Gentle's [13] approximation for the Basket option.

1.5 Organisation of Thesis

This thesis is organised into seven chapters. This chapter introduces the general theory and literature on pricing basket and Asian options and a formal statement of the associated pricing problem. Chapter 2 incorporates basic results in probability and martingale theory, and thereafter derive some results which we shall use in later chapters. Some of these results include convergence of maximum and minimum of independent Brownian motions, joint distributions of a Brownian motion and its running maximum (or minimum), joint distribution of a Brownian bridge and its running maximum (or minimum), joint density of running maximums or minimum of a Brownian motion. These results are later used in Chapter's 3 and 4, to derive analytic bounds on the price of basket and Asian options. Chapter 3 presents the dynamics of basket option under the assumption that the individual assets follow a Geometric Brownian Motion framework and its time-changed representation. We derive closed-form approximations for basket option in a Gaussian and a lognormal framework. In the Gaussian approach, we approximate the value of the basket by approximating the martingale component of the assets for small time periods, which then leads to closed-form basket option prices. In the lognormal approach, we estimate the value of the basket using first order Hermite polynomials and its parameters are estimated using third order Taylor series expansions, and the lognormal estimate is then used to price basket options. However, this approach is limited to pricing low-dimensional portfolios, as a result we suggest an analytical adaptation which allows us to price reasonable large portfolios. We also derive bounds on the basket option price using convexity arguments of the payoff function of the basket option. Further analytic bounds are obtained on the basket option price are obtained by replacing the underlying Brownian motions with there joint maximum or minimum. These bounds are generally not analytically tractable and requires integrability conditions or conditioning arguments, which we provide to obtain explicit closed-form results. Chapter 4 presents analogous methods used to estimate the price of basket options and its bounds to Asian options. We estimate the price of an Asian option by estimating the martingale component of the underlying assets using first order Hermite polynomial, then using this approximation to work out the average value of the asset and its option price. We also derive the bounds on the average value of the underlying by replacing the underlying Brownian motion with its running maximum or minimum, and consequently the bounds on the Asian option price using several different approaches. Chapter 5 covers the general methodology of control variates. We provide algorithms for the control variates (for both basket and Asian options), which we use for our numerical experiments. These control variates are estimates and bounds on basket and Asian option price derived in chapters 3 and 4. We also include another control variate for pricing Asian options which we term "future-valued basket" (FVB), which we refer to in Chapter 6. In Chapter 6, we present and compare the numerical results from Monte Carlo simulation for basket and Asian call options for a variety of positions of moneyness and maturity times. For basket options, assets' volatilities are obtained from real world market indices and we simulate the option prices for two and five-asset baskets, for different strikes and maturities. Simulations are also carried out to obtain Asian option prices for different maturities and strikes. Chapter 7 summarises our results and findings. We also suggest possible future research in this area.

Chapter 2

Control Variate Methodology & Mathematical Premliminaries

2.1 Overview

The control variate methodology is a variance reduction method used obtain efficient Monte Carlo estimates. A lot of work of work in the pricing of basket and Asian options employ this approach to achieve option prices as in the works of [36], [37] and [41]. In this chapter, we show the importance of correlation in the choice of a suitable control variate and on variance reduction. We also introduce the concept of martingales and Brownian motions. We provide a few known results of martingales which are essential to our arbitrage free pricing framework. Brownian motions are stochastic processes with almost surely continuous sample paths, which are used to model asset prices. Understanding the distribution of such a stochastic process is key to effectively pricing and hedging such assets/derivatives.Later in this chapter we derive distributional properties of Brownian motions, which we will employ in chapter 3 and 4 in the pricing of path-dependent options, as well as in finding suitable bounds on them.

2.2 Control Variate Methodology

The control variate method is a technique used to reduce the variance of Monte Carlo estimates and provide an efficient estimate. The main idea behind this approach is approximating a random variable with a similar known distribution, and using the knowledge of the known distribution to reduce the error of the Monte Carlo estimates.

Consider a random variable Y, which is a function $h(X)$ whose distribution is not known, but the distribution of X is known. We can estimate the value of Y, by using a random variable $Z = h^*(X)$ whose distribution is known by using the random variable Ψ which is an estimator of Y such that

$$\Psi = Y - \lambda\left(Z - \mathbb{E}(Z)\right), \tag{2.1}$$

where λ is a constant and Z is referred to as a control variate of Y (see Glasserman [14]). By generating samples for Y_i and Z_i for a considerable sample size n, it can be observed that for large n that,

$$\frac{1}{n}\sum_{i=1}^{n}\Psi_i = \frac{1}{n}\sum_{i=1}^{n}Y_i - \lambda\left(\frac{1}{n}\sum_{i=1}^{n}Z_i - \mathbb{E}(Z)\right) \to \mathbb{E}(Y). \tag{2.2}$$

Alternatively, this can be written as

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}\Psi_i - \mathbb{E}(Y)\right| > \epsilon\right) = 0, \tag{2.3}$$

for any $\epsilon > 0$.

This shows that Ψ is an estimator of Y. To see how Ψ is an efficient estimator, we observed the variance of Ψ , which is given by

$$Var(\Psi) = Var(Y) + \lambda^2 Var(Z) - 2\lambda Cov(Y, Z). \tag{2.4}$$

To minimize the variance of Ψ, we take its partial derivatives with respect to λ and set it to zero to obtain

$$\lambda^* = \frac{Cov(Y, Z)}{Var(Z)}. \tag{2.5}$$

Substituting λ^* back into 2.4 to obtain a reduction in the variance given by

$$Var(\Psi) = Var(Y) - \frac{Cov(Y, Z)^2}{Var(Z)}. \tag{2.6}$$

Thus making the expression of Ψ given below

$$\Psi = Y - \lambda^* \left(Z - \mathbb{E}(Z)\right), \tag{2.7}$$

to be an efficient estimator of Y.

To achieve significant variance reduction in the (Monte Carlo) estimates, there must be strong control correlation between the variable we wish to price Y and the chosen control variate X. To see the role of correlation in variance reduction, we have provided an analytic working showing this relationship. To prove this, we observe the variance function 2.6 and see that

$$Var(\Psi) = Var(Y) - \frac{Cov(Y,Z)^2}{Var(Z)}, \tag{2.8}$$

$$= Var(Y)\left[1 - \frac{Cov(Y,Z)^2}{Var(Y)Var(Z)}\right] \tag{2.9}$$

$$= Var(Y)\left[1 - \left(\frac{Cov(Y,Z)}{Var(Y)^{1/2}Var(Z)^{1/2}}\right)^2\right] \tag{2.10}$$

where the object given by $\frac{Cov(Y,Z)}{Var(Y)^{1/2}Var(Z)^{1/2}}$ is the correlation $\rho(Y,Z)$ between the variables Y and Z. This shows that the stronger the correlation between the variables Y and Z, the greater the variance reduction. This relationship can also viewed from a regression standpoint, i.e. given a regression model of the form:

$$Y = \alpha + \beta Z + \epsilon, \tag{2.11}$$

where ϵ is the random error process, α is the intercept, and β is the coefficient of the linear regression. It turns out that the optimal β in 2.11 estimated by minimising the sum of squared errors is given by

$$\beta^* = \frac{\mathrm{Cov}(Y,Z)}{var(Z)}, \tag{2.12}$$

which is identical to the optimal λ in 2.5. The performance of the regression model is evaluated by its R^2, which measures the proportion of variation in the dependent variable Y that can be attributed to the independent variable Z. Also, the R^2 is also given by the square of the correlation between Y and Z, i.e. $R^2 = \rho^2(Y,Z)$. Hence, the higher the R^2, the higher the correlation and the greater the variance reduction achieved. An alternative representation for the R^2 is given by

$$R^2 = \lambda^2 \frac{1}{var(Z)/var(Y)} \tag{2.13}$$

$$= \lambda^2 \frac{1}{\tilde{\sigma}_{ZY}^2}, \tag{2.14}$$

where $\tilde{\sigma}_{ZY}^2$ represents the normalised variance of Z with respect to that of Y.

In chapter 5, we provide different control variates used to obtain efficient Monte Carlo prices of basket and Asian options, as well as their numerical implementation schemes. We also provide the optimal values for the parameter λ for our control variates in chapter 6.

2.3 Mathematical Preliminaries

In this section, we develop the theory of the random variables whose densities will be used in the analytic pricing and bounds estimation of both basket and Asian options.

2.3.1 Martingales

A martingale $\{X(t)\}_{0\leq t\leq T}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following properties

1 $X(t)$ is integrable i.e. $\mathbb{E}[|X(t)|] < \infty$.

2 $\mathbb{E}[|X(t)||\mathcal{F}(s)] = X(s)\ \forall s \leq t$.

3 If $\mathbb{E}[|X(t)||\mathcal{F}(s)] \geq X(s)$ then $\{X(t)\}\ \forall s \leq t$ is a submartingale.

4 If $\mathbb{E}[|X(t)||\mathcal{F}(s)] \leq X(s)\ \forall s \leq t$ then $\{X(t)\}$ is a supermartingale.

Theorem 2.3.1. *Martingale convergence theorem: Let $X(t)$ be a martingale bounded in $\mathcal{L}^1$, then $\lim_{t\to\infty} X(t) = X(\infty)$ $\mathbb{P}-a.e$ and in $\mathcal{L}^1$.*

Theorem 2.3.2. *Martingale representation theorem: Let $g \in \mathcal{L}^2(\mathcal{F}(t))$ and $X(t)$ be an $\mathcal{F}(t)$-martingale, then $X(t)$ admits a representation of the form*

$$X(t) = X(0) + \int_0^t g(s)dW(s). \tag{2.15}$$

Theorem 2.3.3. *Ito's Lemma: Let $f(t, W(t)) \in \mathcal{C}^{1,2}$, then f has the following stochastic dynamics given by*

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial W}dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial dW^2}\left(dW(t)\right)^2, \tag{2.16}$$

such that $dtdW(t) = (dt)^2 = 0$ and $(dW(t))^2 = dt$

2.3.2 Derived densities and distributions of Brownian motions

In this section, we shall derive some key results of (functions of) Brownian motions which we will employ in chapters 3 and 4, in pricing and obtaining analytic bounds on basket and Asian option price. Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a fixed probability space, equipped with a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$, generated by n independent Brownian motions. For a fixed time $t > 0$, we define $M_n(t)$ and $m_n(t)$ as a sequence indexed on n, of the maximum and minimum respectively of n independent Brownian motions.

Proposition 2.3.1. *The densities $\phi_{(M,n)}$ of the maximum $M_n(t)$ and $\phi_{(m,n)}$ of the minimum $m_n(t)$ of n independent Brownian motions $W_1(t), ..., W_n(t)$ are respectively given by*

$$\phi_{(M,n)}(y) = \frac{n}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right)\left(\Phi\left(\frac{y}{\sqrt{t}}\right)\right)^{n-1},$$

and

$$\phi_{(m,n)}(y) = \frac{n}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right)\left(\Phi\left(-\frac{y}{\sqrt{t}}\right)\right)^{n-1},$$

for any $y \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proposition 2.3.2. *For a fixed time $t > 0$, the sequence $\{M_n(t)\}_{n\geq 2}$ is a non-decreasing sequence and $\mathcal{F}_n(t)$-submartingale in n.*

Proof. For any fixed time $0 \leq t \leq T$, $\{M_n(t)\}_{n\geq 2}$ is continuous and $\mathcal{F}_n(t)$-adapted, since the underlying Wiener processes are continuous and measurable with respect to $\mathcal{F}(t)$.

To show integrability, for a fixed t and $n \geq 2$, we use the following inequality to obtain,

$$\mathbb{E}\left[|M_n(t)|\right] \leq \mathbb{E}\left(\sum_{j=1}^{n}|W_j(t)|\right) < \infty. \tag{2.17}$$

To show the submartingale property in n, we define $\mathcal{F}_n(t)$ as the filtration up to time t, observing n independent Wiener processes. Also, we simplify $M_n(t)$ by the following

$$M_n(t) = \max\left(M_{n-1}(t), W_n(t)\right), \tag{2.18}$$

$$= M_{n-1}(t) + \left(W_n(t) - M_{n-1}(t)\right)^{+}. \tag{2.19}$$

Therefore,

$$\mathbb{E}\left[M_n(t) \mid \mathcal{F}_{n-1}(t)\right] = \mathbb{E}\left[M_{n-1}(t) + (W_n(t) - M_{n-1}(t))^{+}|\mathcal{F}_{n-1}(t)\right], \tag{2.20}$$

$$= M_{n-1}(t) + \mathbb{E}\left[(W_n(t) - M_{n-1}(t))^{+}|\mathcal{F}_{n-1}(t)\right] \tag{2.21}$$

Since, $(W_n(t) - M_{n-1}(t))^{+} \geq 0$, we obtain that

$$\mathbb{E}\left[M_n(t) \mid \mathcal{F}_{n-1}(t)\right] \geq M_{n-1}(t). \tag{2.22}$$

$\square$

Proposition 2.3.3. *For a fixed time $t > 0$, the sequence $\{m_n(t)\}_{n\geq 2}$ is a non-increasing sequence and $\mathcal{F}_n(t)$-supermartingale in n.*

Proof. For a fixed time $t \geq 0$, $\{M_n(t)\}_{n\geq 2}$ is continuous, adapted and is $\mathcal{F}_n(t)$-measurable, since the underlying Wiener processes are measurable with respect to $\mathcal{F}(t)$.

We show integrability for $n \geq 2$ and for a fixed time t, by the following inequality to obtain

$$\mathbb{E}\left(|m_n(t)|\right) < \mathbb{E}\left(\sum_{j=1}^{n}|W_j(t)|\right) < \infty. \tag{2.23}$$

To show the supermartingale property in n, we define $\mathcal{F}_n(t)$ as the filtration up to time t, observing n Wiener processes. Also, we simplify $m_n(t)$ by the following

$$m_n(t) = \min\left(m_{n-1}(t), W_n(t)\right), \tag{2.24}$$

$$= m_{n-1}(t) - \left(m_{n-1}(t) - W_n(t)\right)^+. \tag{2.25}$$

Therefore,

$$\mathbb{E}\left(m_n(t) \mid \mathcal{F}_{n-1}(t)\right) = \mathbb{E}\left[m_{n-1}(t) - \left(m_{n-1}(t) - W_n(t)\right)^+ |\mathcal{F}_{n-1}(t)\right], \tag{2.26}$$

$$= m_{n-1}(t) - \mathbb{E}\left[\left(m_{n-1}(t) - W_n(t)\right)^+ |\mathcal{F}_{n-1}(t)\right], \tag{2.27}$$

$$\leq m_n(t). \tag{2.28}$$

$\square$

Proposition 2.3.4. *The maximum $M_n(t)$ of n Brownian motions converges $\mathbb{P}$-a.e. and in $\mathcal{L}^1$ to $M_\infty(t)$ as $n \to \infty$.*

Proof. To prove this, it sufficient to show that $\mathbb{E}\left[|M_\infty(t)|\right] < \infty$ since $M_n(t)$ is an increasing sequence in n.

We prove this using proof by induction. By integration by parts, we find that $\mathbb{E}\left[|M_n(t)|\right] < \infty$ is true for $n = 2$ and 3. Assuming this is true for $n = k$, this implies that

$$\mathbb{E}\left[|M_k(t)|\right] = \int_0^\infty yk\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi^{k-1}\left(\frac{y}{\sqrt{t}}\right)dy, \tag{2.29}$$

where $\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right) = \frac{1}{\Delta}\phi\left(\frac{y}{\sqrt{t}}\right)$ and $\Delta = 1 - \frac{1}{2^n}$ is the normalisation factor over the positive half-line.

For $n = k+1$,

$$\begin{aligned}
\mathbb{E}\left[|M_{k+1}(t)|\right] &= \int_0^\infty y(k+1)\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi^{k}\left(\frac{y}{\sqrt{t}}\right)dy, && (2.30)\\
&= \int_0^\infty y(k+1)\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi^{k}\left(\frac{y}{\sqrt{t}}\right)dy, && (2.31)\\
&= \int_0^\infty yk\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi^{k}\left(\frac{y}{\sqrt{t}}\right)dy + \int_0^\infty y\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi^{k}\left(\frac{y}{\sqrt{t}}\right)dy, && (2.32)\\
&\leq \int_0^\infty yk\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi^{k-1}\left(\frac{y}{\sqrt{t}}\right)dy + \int_0^\infty y\tilde{\phi}\left(\frac{y}{\sqrt{t}}\right)\Phi\left(\frac{y}{\sqrt{t}}\right)dy, && (2.33)\\
&= \mathbb{E}\left[|M_k(t)|\right] + \frac{1}{\Delta}\int_0^\infty y\phi\left(\frac{y}{\sqrt{t}}\right)\Phi\left(\frac{y}{\sqrt{t}}\right)dy, && (2.34)\\
&= \mathbb{E}\left[|M_k(t)|\right] + \frac{\sqrt{2}+1}{4\sqrt{\pi}} < \infty. && (2.35)
\end{aligned}$$

This suggests that $M_k(t)$ is bounded for all k and $\sup_{k\in\mathbb{N}}\mathbb{E}\left[|M_k(t)|\right] < \infty$. Thus, $M_k(t)$ is bounded in $\mathcal{L}^1$ and by monotone convergence theorem increases to $M_\infty(t)$. □

Proposition 2.3.5. *The minimum $m_n(t)$ of n Brownian motions converges $\mathbb{P}$-a.e. and in $\mathcal{L}^1$ to $m_\infty(t)$ respectively as $n \to \infty$.*

Proof. The proof of this is quite similar to the case of maximum of Brownian motions. □

Proposition 2.3.6. *Given the running maximums and minimums of n independent Brownian motions, the density of their joint maximum $M_n^*(t)$ of the running maximums and the joint minimum of the running minimums $m_n^*(t)$ are respectively given by*

$$\phi_{(M^*,n)}(y) = \frac{2n}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right)\left(2\Phi\left(\frac{y}{\sqrt{t}}\right) - 1\right)^{n-1}, \ y \geq 0$$

and

$$\phi_{(m^*,n)}(y) = \frac{2n}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right)\left(1 - 2\Phi\left(\frac{y}{\sqrt{t}}\right)\right)^{n-1}, \ y < 0,$$

for any $y \in \mathbb{R}$.

Proof. To prove the above, we make use of their distributional properties. The joint distribution of the running maximum of n independent Brownian motions is given by

$$\mathbb{Q}\left(M_n^*(t) \leq y\right) = \left(\mathbb{Q}\left(M_i^*(t) \leq y\right)\right)^n \tag{2.36}$$

$$= \Phi_{(M^*,n}(y) \tag{2.37}$$

We then take the derivatives of the above distribution with respect to y to obtain,

$$\phi_{(M^*,n)}(y) = \frac{2n}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right)\left(2\Phi\left(\frac{y}{\sqrt{t}}\right) - 1\right)^{n-1}, \ y \geq 0. \tag{2.38}$$

For the running minimum of n independent Brownian motions,

$$1 - \mathbb{Q}\left(m_n^*(t) \geq y\right) = 1 - \left(\mathbb{Q}\left(m^*(t) \geq y\right)\right)^n, \tag{2.39}$$

$$= \Phi_{(m^*,n)}(y). \tag{2.40}$$

Taking derivatives of the distribution $\Phi_{(m^*,n)}$ of the joint minimum with respect to y to obtain the density $\phi_{(m^*,n)}$, we get that

$$\phi_{(m^*,n)}(y) = \frac{2n}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right)\left(1 - 2\Phi\left(\frac{y}{\sqrt{t}}\right)\right)^{n-1}, \ y < 0. \tag{2.41}$$

□

Proposition 2.3.7. *The maximum of the running maximums of* n *independent Brownian motions is a non-decreasing sequence in* n *and* t*, for* $n \geq 2$ *and* $t \geq 0$ *and is an* $\{\mathcal{F}(t)\}$*-submartingale.*

Proof. It is fairly obvious that the maximum of the running maximums is a continuous, adapted and non-decreasing sequence, since its underlying running maximums are non-decreasing, adapted and continuous as well.

We can show integrability of this joint maximum of running maximums as follows

$$\mathbb{E}\Big(|M_n^*(t)|\Big) \leq \mathbb{E}\Big(\sum_{j=1}^{n}|M^*(t)|\Big) < \infty. \tag{2.42}$$

To show the submartingale property, for any $0 \leq s \leq t$ we have

$$\mathbb{E}\Big(M_n^*(t)\Big|\mathcal{F}(s)\Big) = \mathbb{E}\Big(M_n^*(t) - M_n^*(s) + M_n^*(s)\Big|\mathcal{F}(s)\Big), \tag{2.43}$$

$$= M_n^*(s) + \mathbb{E}\Big(M_n^*(t) - M_n^*(s)\Big|\mathcal{F}(s)\Big), \tag{2.44}$$

$$\geq M_n^*(s). \tag{2.45}$$

□

Proposition 2.3.8. *The minimum of the running minimums of* n *independent Brownian motions is a non-increasing sequence in* n *and* t*, for* $n \geq 2$ *and* $t \geq 0$*.*

Proof. It is fairly obvious that the maximum of the running maximums is a continuous, adapted and non-decreasing sequence, since its underlying running maximums are non-decreasing, adapted and continuous as well.

We can show integrability of this joint minimum of running minimums by

$$\mathbb{E}\Big(|m_n^*(t)|\Big) \leq \mathbb{E}\Big(\sum_{j=1}^{n}|m_n^*(t)|\Big) < \infty. \tag{2.46}$$

To show the supermartingale property, for any $0 \leq s \leq t$ we have,

$$\mathbb{E}\Big(m_n^*(t)\Big|\mathcal{F}(s)\Big) = \mathbb{E}\Big(m_n^*(t) - m_n^*(s) + m_n^*(s)\Big|\mathcal{F}(s)\Big), \tag{2.47}$$

$$= m_n^*(s) + \mathbb{E}\Big(m_n^*(t) - m_n^*(s)\Big|\mathcal{F}(s)\Big), \tag{2.48}$$

$$\leq m_n^*(s). \tag{2.49}$$

□

In general for any time $t \geq 0$, $W(t)$ satisfies the following inequality

$$W(t) \leq |W(t)| \wedge M(t) \leq |W(t)| \vee M(t), \tag{2.50}$$

where W(t) and M(t) is an $\mathcal{F}(t)-$Brownian motion and its running maximum respectively.

Proposition 2.3.9. *The density $\varrho_M(y)$ of the minimum of the absolute value of a Brownian motion and its running maximum is given by*

$$\varrho_M(y) = \mathbb{1}\{y \geq 0\}\frac{1}{\sqrt{t}}\left[\phi\left(\frac{3y}{\sqrt{t}}\right) - \phi\left(\frac{y}{\sqrt{t}}\right)\right],$$

for any $y \geq 0$.

Proof. We define $R(t) = min(|W(t)|, M(t))$. The distribution of $R(t)$ is given by

$$\mathbb{Q}(R(t) \leq y) = 1 - \mathbb{Q}(R(t) \geq y), \tag{2.51}$$

$$= 1 - \mathbb{Q}(M(t) \geq y, |W(t)| \geq y). \tag{2.52}$$

Also,

$$\mathbb{Q}(M(t) \geq y, |W(t)| \geq y) = \int_0^y \int_y^\infty \phi_{(M,W)}(m, w)dwdm + \int_0^y \int_{-\infty}^{-y} \phi_{(M,W)}(m, w)dwdm, \tag{2.53}$$

where $\phi_{(M,W)}(m,w)$ is the joint pdf of a Brownian motion and its running maximum.

We set $I_1 = \int_0^y \int_y^\infty \phi_{(M,W)}(m,w)dwdm$ and $I_2 = \int_0^y \int_{-\infty}^{-y} \phi_{(M,W)}(m,w)dwdm$.

Also, we set $\varphi(m,y) = \int_y^\infty \phi_{(M,W)}(m,w)dw$.

We take partial derivatives of I_1 with respect to y so that,

$$\frac{\partial}{\partial y}\left(\int_0^y \int_y^\infty \phi_{(M,W)}(m,w)dwdm\right) = \frac{\partial}{\partial y}\int_y^\infty \phi_{(M,W)}(m,w)dw, \tag{2.54}$$

$$= \varphi(y,y) + \int_0^y \frac{\partial}{\partial y}\varphi(m,y)dm, \tag{2.55}$$

where

$$\varphi(y,y) = \int_y^\infty \phi_{(M,W)}(y,w)dw, \tag{2.56}$$

$$= \int_0^\infty \frac{2(2y-w)}{t\sqrt{t}}\phi\left(\frac{2y-w}{\sqrt{t}}\right)dw, \tag{2.57}$$

$$= -\frac{2}{\sqrt{t}}\int_{-\infty}^y \left(\frac{z}{t}\right)\phi\left(\frac{z}{\sqrt{t}}\right)dz, \quad (\textit{where } z = 2y-w) \tag{2.58}$$

$$= -\frac{2}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right). \tag{2.59}$$

Also,

$$\int_0^y \frac{\partial}{\partial y}\varphi(m,y)dm = -\int_0^y \phi_{(M,W)}(m,y)dm, \tag{2.60}$$

$$= -\int_0^\infty \frac{2(2m-y)}{t\sqrt{t}}\phi\left(\frac{2m-y}{\sqrt{t}}\right)dm, \tag{2.61}$$

$$= \frac{1}{\sqrt{t}}\left(\phi\left(-\frac{y}{\sqrt{t}}\right) - \phi\left(-\frac{y}{\sqrt{t}}\right)\right), \tag{2.62}$$

$$= 0. \tag{2.63}$$

We take the partial derivatives of I_2 with respect to y,

$$\frac{\partial}{\partial y}\left(\int_0^y \int_{=\infty}^y \phi_{(M,W)}(m,w)dwdm\right) = \varphi(y,y) + \int_0^y \frac{\partial}{\partial y}\varphi(m,y)dm, \tag{2.64}$$

$$= \int_0^\infty \frac{2(2y-w)}{t\sqrt{t}}\phi\left(\frac{2y-w}{\sqrt{t}}\right)dw + \int_0^y \frac{\partial}{\partial y}\varphi(m,y)dm,$$

$$= \frac{2}{\sqrt{t}}\phi\left(\frac{y}{2\sqrt{t}}\right) + \int_0^y \frac{\partial}{\partial y}\varphi(m,y)dm, \tag{2.65}$$

$$= \frac{2}{\sqrt{t}}\phi\left(\frac{y}{\sqrt{t}}\right) - \int_0^y \frac{2(2m+y)}{t\sqrt{t}}\phi\left(\frac{2m+y}{\sqrt{t}}\right)dw,$$

$$= \frac{2}{\sqrt{t}}\phi\left(\frac{y}{2\sqrt{t}}\right) - \frac{1}{\sqrt{t}}\left(\phi\left(\frac{3y}{\sqrt{t}}\right) - \phi\left(\frac{y}{\sqrt{t}}\right)\right). \tag{2.66}$$

Combining the partial derivatives of I_1 and I_2 and substituting back into 2.52 to obtain,

$$\mathbb{Q}\left(R(t) \in dy\right) = \mathbb{1}\{y \geq 0\}\frac{1}{\sqrt{t}}\left(\phi\left(\frac{3y}{\sqrt{t}}\right) - \phi\left(\frac{y}{\sqrt{t}}\right)\right)dy. \tag{2.67}$$

Furthermore, the density $\mathbb{Q}\left(\min(R_1(t), ..., R_n(t)) \in dy\right)$ of the joint maximum of the minimum of running maximum and the absolute value of a Brownian motion for n independent Brownian motions is given by

$$\mathbb{Q}\left(\min(R_1(t), ..., R_n(t)) \in dy\right) = n\left(\left(\phi\left(\frac{3y}{\sqrt{t}}\right) - \phi\left(\frac{y}{\sqrt{t}}\right)\right)dy\right)^{n-1}\varrho_M(y)dy,$$

$$= n\left(\Phi\left(\frac{3y}{\sqrt{t}}\right) - \Phi\left(\frac{y}{\sqrt{t}}\right)\right)^{n-1}\varrho_M(y)dy. \tag{2.68}$$

□

Proposition 2.3.10. *The density $\varrho_m(y)$ of the maximum of the negative path of a Brownian motion reflected below the x-axis and the running minimum of a Brownian motion is given by*

$$\varrho_m\left(y\right) = \frac{1}{\sqrt{t}}\left[\phi\left(\frac{y}{\sqrt{t}}\right) - \phi\left(\frac{3y}{\sqrt{t}}\right)\right],$$

where $y \leq 0$.

Proof. We can use the the fact $-|W(t)| \vee m(t) = -R(t)$ and is the reflection of $R(t)$ below the x-axis and we obtain above density. $\square$

Proposition 2.3.11. *The joint density* f_{β,β^*} *of a Brownian bridge and its running maximum have the following form*

$$f_{\beta,\beta^*}(m,w) = \exp\left(-\frac{z}{T}w - \frac{z^2}{T}\right)\frac{(2m-w)}{\pi t\sqrt{tT}}\exp\left(-\frac{1}{2t}(2m-w)^2\right),$$

for $w \leq m$, $m \geq 0$ and $z \in \mathbb{R}$.

Proof. We define a Brownian bridge at a time t over the interval $[0,T]$ by

$$\beta(t,T) = W(t) - \frac{t}{T}W(T).$$

If we fix $W(T) = z$, we see that the conditional distribution of $\beta(t,T)$ given $W(T) = z$ is a drifted Brownian motion and conditionally normally distributed as $\mathcal{N}\left(-\frac{t}{T}z, t\right)$.

Furthermore, we define an equivalent measure $\tilde{\mathbb{Q}}$, which is absolutely continuous with respect to $\mathbb{Q}$, such that $W(t)$ is a Brownian motion with no drift.

We define a change of measure variable $\theta(t)$ such that

$$\theta(t) = \exp\left(\frac{z}{T}W(t) - \frac{1}{2}\left(\frac{z}{T}\right)^2 t\right), \tag{2.69}$$

$$= \exp\left(\frac{z}{T}\tilde{W}(t) + \frac{1}{2}\left(\frac{z}{T}\right)^2 t\right), \tag{2.70}$$

and

$$\tilde{\mathbb{Q}}(A) = \int_A \theta(T)d\mathbb{Q} \;\forall\; A \in \mathcal{F}. \tag{2.71}$$

The joint pdf $\tilde{g}_{\beta,\beta^*}$ of $\beta(t,T)$ given that $W(T)=z$ and its running maximum $\beta^*(t,T)$ under $\tilde{\mathbb{Q}}$ is given by

$$\tilde{g}_{\beta,\beta^*}(m,w) = \frac{2(2m-w)}{t\sqrt{t}}\phi\left(\frac{2m-w}{\sqrt{t}}\right), \tag{2.72}$$

where $w \leq m$ and $m \geq 0$, which is in fact just the joint density of a Brownian $W(t)$ and its running maximum $M(t)$.

The conditional distribution under $\mathbb{Q}$ is given by

$$\mathbb{Q}\left(\beta(t,T) \leq w, \beta^*(t,T) \leq m | W(T) = z\right) = \mathbb{E}\left[\mathbb{1}_{\{\beta(t,T)\leq w, \beta^*(t,T)\leq m\}}\right], \tag{2.73}$$

$$= \tilde{\mathbb{E}}\left[\frac{1}{\theta(T)}\mathbb{1}_{\{\beta(tT)\leq w, \beta^*(tT)\leq m\}}\right]. \tag{2.74}$$

We simplify this further to obtain that,

$$\mathbb{Q}\left(\beta(t,T) \leq w, \beta^*(t,T) \leq m | W(T) = z\right) = \int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T}\frac{2(2x-y)}{t\sqrt{t}}\phi\left(\frac{2x-y}{\sqrt{t}}\right)dxdy, \tag{2.75}$$

where $\alpha = -\dfrac{z}{T}$.

To obtain the conditional density g_{β,β^*} of a Brownian bridge $\beta(t,T)$ and its running maximum $\beta^*(t,T)$ in the $\mathbb{Q}-$measure we take derivatives to w and m, which yields,

$$g_{\beta,\beta^*}(m,w) = e^{\alpha y - \frac{1}{2}\alpha^2 T}\frac{2(2m-w)}{t\sqrt{t}}\phi\left(\frac{1}{\sqrt{t}}(2m-w)\right), \tag{2.76}$$

and thus its unconditional density f_{β,β^*} is given by

$$f_{\beta,\beta^*}(m,w) = \exp\left(-\frac{z}{T}w - \frac{z^2}{2T}\right)\frac{2(2m-w)}{t\sqrt{tT}}\phi\left(\frac{1}{\sqrt{t}}(2m-w)\right)\phi\left(\frac{z}{\sqrt{T}}\right), \tag{2.77}$$

$$= \exp\left(\frac{z}{T}w - \frac{z^2}{2T}\right)\frac{2(2m-w)}{t\sqrt{tT}}\phi\left(\frac{1}{\sqrt{t}}(2m-w)\right)\phi\left(\frac{z}{\sqrt{T}}\right), \tag{2.78}$$

$$= \exp\left(-\frac{z}{T}w - \frac{z^2}{T}\right)\frac{(2m-w)}{\pi t\sqrt{tT}}\exp\left(\frac{1}{2t}(2m-w)^2\right). \tag{2.79}$$

which is the required results. $\square$

Proposition 2.3.12. *The joint density g_{β,β^*} of a Brownian bridge $\beta(t,T)$ and its running minimum $\tilde{\beta}(t,T)$ have the following the form*

$$g_{\beta,\tilde{\beta}}(m,w) = \exp\left(-\frac{z}{T}w - \frac{z^2}{T}\right)\frac{(w-2m)}{\pi t\sqrt{tT}}\exp\left(-\frac{1}{2t}(w-2m)^2\right),$$

for $w \geq m$ and $m \leq 0$.

Proof. The proof of this is very similar to the density of a Brownian bridge and its maximum. We use the fact that for any process $W(t)$,

$$\max_{0\leq s\leq t} W(s) = \min_{0\leq s\leq t} -W(s). \tag{2.80}$$

Substituting this identity into 2.79, we get the required results. $\square$

Proposition 2.3.13. *The running maximum of a Brownian motion satisfies the following stochastic dynamics*

$$dM(t) = \mathbb{1}\{W(t) > M(t-)\}dW(t) + \frac{1}{2}dL^M(t),$$

where $L^M(t)$ is the local time of the Brownian motion $W(t)$ at its running maximum $M(t)$ at a time t.

Proposition 2.3.14. *The running minimum of a Brownian motion satisfies the following stochastic dynamics*

$$dm(t) = \mathbb{1}\{W(t) < m(t-)\}dW(t) + \frac{1}{2}dL^m(t),$$

where $L^m(t)$ is the local time of the Brownian motion $W(t)$ at its running minimum $m(t)$ at a time t.

Proposition 2.3.15. *The density of* $\sum_{i=1}^{n} c_i M(t_i)$ *is given by*

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k M(t_k) \in dy\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipy} \int_{z_{n-1}}^{\infty} \dots \int_{0}^{\infty} e^{ip(c_1 z_1 + c_2 z_2 + \dots + c_n z_n)}$$
$$\times \prod_{k=1}^{n} \frac{1}{c_k^2} q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,$$

where $i = \sqrt{-1}$*,* $q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) = \sqrt{\frac{2}{\pi(t_k - t_{k-1})}} \exp\left[-\frac{(z_k - z_{k-1})^2}{2(t_k - t_{k-1})}\right]$ *and* $t_k = k\frac{T}{n}$ $\forall k \leq n$*.*

Proof. We consider a partition of time T such that $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{n-1} \leq t_n = T$ and the distribution of $\sum_{k=1}^{n} c_k M(t_k)$ is given by

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k M(t_k) \leq y\right) = \mathbb{E}\left[\mathbb{1}\left\{\sum_{k=1}^{n} c_k M(t_k) \leq y\right\}\right], \tag{2.81}$$

and we take its derivative with respect to y, to obtain its density. To do this, we first work out the joint density of $M(t_1), \dots, M(t_n)$.

$$\mathbb{Q}\left(M(t_1) \in dx_1, \dots, M(t_{n-1}, t_n) \in dx_n\right) = \mathbb{Q}\left(M(t_1) \in dx_1, \dots, M(t_n) - M(t_{n-1}) \in dx_n\right),$$
$$= \prod_{k=0}^{n-1} \left(\frac{2}{\pi(t_{k+1} - t_k)}\right)^{\frac{1}{2}} \exp\left[-\frac{x_{k+1}^2}{2(t_{k+1} - t_k)}\right] dx_{k+1}.$$

For simplicity, we set

$$q(t_k - t_{k-1}, x_k) = \left(\frac{2}{\pi(t_k - t_{k-1})}\right)^{\frac{1}{2}} \exp\left[-\frac{x_k^2}{2(t_k - t_{k-1})}\right].$$

The distribution of $\sum_k c_k M(t_k)$ is given by,

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k M(t_k) \leq y\right) = \int_0^\infty \int_0^\infty ... \int_0^\infty \mathbb{1}\{c_1 M(t_1) + ... + c_n M(t_n) \leq y\}$$

$$\times \mathbb{Q}\left(M(t_1) \in dx_1, M(t_1, t_2) \in dx_2, ..., M(t_{n-1}, t_n) \in dx_n\right),$$

$$= \int_0^\infty \int_0^\infty ... \int_0^\infty \mathbb{1}\{c_1 x_1 + c_2(x_1 + x_2) + ... + c_n(x_1 + ... + x_n) \leq y\}$$

$$\times q(t_1, x_1) q(t_2 - t_1, x_2) ... q(t_n - t_{n-1}, x_n) dx_1 dx_2 ... dx_n \tag{2.82}$$

Setting $c_1 x_1 = z_1$, $c_2(x_1 + x_2) = z_2$,..., and $c_n(x_1 + x_2 + ... + x_n) = z_n$ and transforming the RHS of 2.82, using Jacobian transforms so that

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k M(t_k) \leq y\right) = \int_0^\infty \int_0^\infty ... \int_0^\infty \mathbb{1}\{z_1 + z_2 + ... + z_n \leq y\} \prod_{k=1}^{n} \frac{1}{c_k^2}$$

$$\times q(t_1, z_1) q(t_2 - t_1, z_2 - z_1) ... q(t_n - t_{n-1}, z_n - z_{n-1}) dz_1 dz_2 ... dz_n,$$

and its corresponding density is given by

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k M(t_k) \in dy\right) = \int_{z_{n-1}}^\infty \int_{z_{n-2}}^\infty ... \int_0^\infty \frac{d}{dy} \mathbb{1}\{z_1 + z_2 + ... + z_n \leq y\} \prod_{k=1}^{n} \frac{1}{c_k^2}$$

$$\times q(t_1, z_1) q(t_2 - t_1, z_2) ... q(t_n - t_{n-1}, z_n) dz_1 dz_2 ... dz_n dy,$$

$$= \int_{z_{n-1}}^\infty \int_{z_{n-2}}^\infty ... \int_0^\infty \delta\left(z_1 + z_2 + ... + z_n - y\right) \prod_{k=1}^{n} \frac{1}{c_k^2}$$

$$\times q(t_1, z_1) q(t_2 - t_1, z_2 - z_1) ... q(t_n - t_{n-1}, z_n - z_{n-1}) dz_1 dz_2 ... dz_n dy. \tag{2.83}$$

Using the Fourier representation of a delta function which is given by

$$\delta(x - \alpha) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ip(x - \alpha)} dp, \tag{2.84}$$

for any $\alpha \in \mathbb{R}$.

We can simplify 2.83 to become

$$\begin{aligned}
\mathbb{Q}\left(\sum_{k=1}^{n} c_k M(t_k) \in dy\right) &= \int_{-\infty}^{\infty} e^{ipy} \int_{z_{n-1}}^{\infty} \int_{z_{n-2}}^{\infty} \dots \int_{0}^{\infty} e^{ip(z_1+z_2+\dots+z_n)} \prod_{k=1}^{n} \frac{1}{c_k^2} \\
&\quad \times q(t_1, z_1) q(t_2 - t_1, z_2 - z_1) \dots q(t_n - t_{n-1}, z_n - z_{n-1}) dz_1 dz_2 \dots dz_n dp dy \\
&= \frac{dy}{2\pi} \int_{-\infty}^{\infty} e^{ipy} \int_{z_{n-1}}^{\infty} \int_{z_{n-2}}^{\infty} \dots \int_{0}^{\infty} e^{ip(z_1+z_2+\dots+z_n)} \prod_{k=1}^{n} \frac{1}{c_k^2} \\
&\quad \times \prod_{k=1}^{n} \left(\frac{2}{\pi(t_{k+1} - t_k)}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{z_{k+1} - z_k}{\sqrt{t_{k+1} - t_k}}\right)^2\right] dz_1 dz_2 \dots dz_n dp
\end{aligned} \tag{2.85}$$

Thus, we are able to show the required results. $\square$

Proposition 2.3.16. *The density of* $\sum_{k=1}^{n} c_k m(t_k)$ *is given by*

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k m(t_k) \in dy\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipy} \int_{\infty}^{z_{n-1}} \int_{\infty}^{z_{n-2}} \dots \int_{-\infty}^{0} e^{ip(z_1+z_2+\dots+z_n)}$$
$$\times \prod_{k=1}^{n} \frac{1}{c_k^2} q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,$$

where $q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right) = \sqrt{\frac{2}{\pi(t_k - t_{k-1})}} \exp\left[-\frac{(z_k - z_{k-1})^2}{2(t_k - t_{k-1})}\right]$, $i = \sqrt{-1}$ *and* $t_k = k\frac{T}{n}$ $\forall k \leq n$.

Proof. We consider a partition of time T such that $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{n-1} \leq t_n = T$ and the distribution of $\sum_{k=1}^{n} c_k m(t_k)$ is given by

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k m(t_k) \leq y\right) = \mathbb{E}\left[\mathbb{1}\left\{\sum_{k=1}^{n} c_k m(t_k) \leq y\right\}\right], \tag{2.86}$$

and we take its derivative with respect to y, to obtain its density. To do this, we first work out the joint density of $m(t_1), ..., m(t_n)$.

$$\mathbb{Q}\left(m(t_1) \in dx_1, ..., m(t_n) - m(t_{n-1}) \in dx_n\right) = \mathbb{Q}\left(m(t_1) \in dx_1, ..., m(t_{n-1}, t_n) \in dx_n\right),$$

$$= \prod_{k=0}^{n-1}\left(\frac{2}{\pi(t_{k+1} - t_k)}\right)^{\frac{1}{2}} \exp\left[-\frac{x_{k+1}^2}{2(t_{k+1} - t_k)}\right] dx_{k+1}.$$

For simplicity, we set

$$q'(t_k - t_{k-1}, x_k) = \left(\frac{2}{\pi(t_k - t_{k-1})}\right)^{\frac{1}{2}} \exp\left[-\frac{x_k^2}{2(t_k - t_{k-1})}\right].$$

The distribution of $\sum_k m(t_k)$ is given by,

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k m(t_k) \leq y\right) = \int_{-\infty}^{0}\int_{-\infty}^{0}...\int_{-\infty}^{0} \mathbb{1}\{c_1 m(t_1) + ... + c_n m(t_n) \leq y\}$$

$$\times\, \mathbb{Q}\left(m(t_1) \in dx_1, m(t_1, t_2) \in dx_2, ..., m(t_{n-1}, t_n) \in dx_n\right),$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{0}...\int_{-\infty}^{0} \mathbb{1}\{c_1 x_1 + c_2(x_1 + x_2) + ... + c_n(x_1 + ... + x_n) \leq y\}$$

$$\times\, q(t_1, x_1) q(t_2 - t_1, x_2) ... q(t_n - t_{n-1}, x_n) dx_1 dx_2 ... dx_n. \tag{2.87}$$

Setting $c_1 x_1 = z_1$, $c_2(x_1 + x_2) = z_2$,..., and $c_n(x_1 + x_2 + ... + x_n) = z_n$ and transforming the RHS of 2.87 so that

$$\mathbb{Q}\left(\sum_{k=1}^{n} c_k m(t_k) \leq y\right) = \int_{-\infty}^{0}\int_{-\infty}^{0}...\int_{-\infty}^{0} \mathbb{1}\{z_1 + z_2 + ... + z_n \leq y\} \prod_{k=1}^{n} \frac{1}{c_k^2}$$

$$\times\, q'(t_1, z_1) q'(t_2 - t_1, z_2 - z_1) ... q'(t_n - t_{n-1}, z_n - z_{n-1}) dz_1 dz_2 ... dz_n,$$

and its corresponding density is given by

$$
\begin{aligned}
\mathbb{Q}\left(\sum_{k=1}^{n} c_k m(t_k) \in dy\right) &= \int_{z_{n-1}}^{\infty}\int_{z_{n-2}}^{\infty}...\int_{0}^{\infty} \frac{d}{dy}\mathbb{1}\{z_1+z_2+...+z_n \leq y\}\prod_{k=1}^{n}\frac{1}{c_k^2} \\
&\times q'(t_1, z_1)q'(t_2-t_1, z_2)...q'(t_n-t_{n-1}, z_n)dz_1dz_2...dz_ndy, \\
&= \int_{z_{n-1}}^{\infty}\int_{z_{n-2}}^{\infty}...\int_{-\infty}^{0} \delta\left(z_1+z_2+...+z_n-y\right)\prod_{k=1}^{n}\frac{1}{c_k^2} \\
&\times q'(t_1, z_1)q'(t_2-t_1, z_2-z_1)...q'(t_n-t_{n-1}, z_n-z_{n-1})dz_1dz_2...dz_ndy, \\
&= \int_{-\infty}^{\infty} e^{ipy}\int_{z_{n-1}}^{\infty}\int_{-\infty}^{z_{n-2}}...\int_{-\infty}^{0} e^{ip(z_1+z_2+...+z_n)}\prod_{k=1}^{n}\frac{1}{c_k^2} \\
&\times q'(t_1, z_1)q'(t_2-t_1, z_2-z_1)...q'(t_n-t_{n-1}, z_n-z_{n-1})dz_1dz_2...dz_ndpdy, \\
&= \frac{dy}{2\pi}\int_{-\infty}^{\infty} e^{ipy}\int_{-\infty}^{z_{n-1}}\int_{-\infty}^{z_{n-2}}...\int_{-\infty}^{0} e^{ip(z_1+z_2+...+z_n)}\prod_{k=1}^{n}\frac{1}{c_k^2} \\
&\times \prod_{k=1}^{n}\left(\frac{2}{\pi(t_{k+1}-t_k)}\right)^{\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\frac{z_{k+1}-z_k}{\sqrt{t_{k+1}-t_k}}\right)^2\right]dz_1dz_2...dz_ndp.
\end{aligned}
\tag{2.88}
$$

Thus, we are able to show the required results. □

2.4 Summary

In this chapter, we have observed the general framework and methodology of control variates. We have also shown the role of played by correlation in the reduction of the variance of estimates when using control variates. In regressional analysis, this can be viewed in terms of the R^2 of the model, which we have been able to express in terms of the optimal λ and the normalised variance of the control variate with respect to that of our variable of interest. We have also derived the dynamics of (functions of) Brownian motions and their densities which we use later on to obtain analytic estimates for the price of basket and Asian options, as well as their bounds.

Chapter 3

Basket Options

3.1 Overview

Under the Black-Scholes framework, assets follow a Geometric Brownian motion model i.e. they are lognormally distributed. However, the same cannot be said about a basket of assets. In this chapter, we study the behaviour of a basket of assets in the Black-Scholes framework and obtain suitable approximations for the price of a basket option. In section 3.2, we observe the dynamical features of the basket of assets such as it stochastic dynamics, moments and its time change representation. In section 3.3, we obtain a lognormal approximation for the basket, allowing for closed-form approximation of a basket option in the aforementioned setting. We also make a generalisation for a large number of assets in the basket, specifically for use as a control variate for simulation purposes. In section 3.4, we demonstrate the Gaussian behaviour of the basket in small time intervals, leading to closed-form solutions for short maturity basket options. Despite basket options not having an analytical formula, much work has been done in obtaining closed-form bounds (Kemna [20], Gentle [13], Xu and Zheng [39]). We derive closed-form bounds on the price of a basket option using the distributional properties of a Brownian motion, which is covered in the rest of the chapter.

3.2 Basket Option Dynamics

We assume the existence of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), \mathbb{Q})$ and the filtration $\{\mathcal{F}(t)\}$ is generated by d independent Brownian motions. The basket comprises n assets whose price processes at any time t are given by $S_1(t), S_2(t), ..., S_n(t)$ and satisfy the Black-Scholes dynamics given by

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t), \tag{3.1}$$

for each i and the value $S(t)$ of the basket of n assets at any time t is given by

$$S(t) = \sum_{i=1}^{n} \omega_i S_i(t), \tag{3.2}$$

where ω_i are non-negative portfolio weights of the assets in the basket such that $\sum_{i=1}^{n} \omega_i = 1$, and r is a constant, risk-free interest rate. The value of the basket $S(t)$ satisfies the following SDE

$$\frac{dS(t)}{S(t)} = rdt + \Lambda dW(t), \tag{3.3}$$

where $W(t)$ is an $\{\mathcal{F}(t)\}$-Brownian motion such that

$$W(t) = \int_0^t \frac{1}{\Lambda} \sum_{i=1}^{n} \sum_{k=1}^{d} \omega_i \sigma_{ik} \frac{S_i(u)}{S(u)} dW_k(u), \tag{3.4}$$

and Λ is the volatility of the value process $\{S(t)\}$ of the basket satisfying the relation

$$\Lambda^2 = \frac{1}{S^2(t)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{d} \omega_i \omega_j S_i(t) S_j(t) \sigma_{ik} \sigma_{jk}. \tag{3.5}$$

The distribution of the basket satisfying the above dynamics can be estimated by studying the conditional/unconditional moments of the value of the basket.

Proposition 3.2.1. *The first and second conditional moments of the basket in 3.3, evolves respectively according to the following ODEs*

$$\partial_t \mathbb{E}_s[S(t)] = r\mathbb{E}_s[(t)],$$

$$\partial_t \mathbb{E}_s[S^2(t)] = \left(r + \Lambda^2\right) \mathbb{E}_s[S^2(t)].$$

with initial conditions $\mathbb{E}_s[S(s)] = S(s)$ *and* $\mathbb{E}_s[S^2(t)] = S^2(s)e^{(r+\Lambda^2)(t-s)}$*, whose respective solutions are given by*

$$\mathbb{E}_s[S(t)] = S(s)e^{r(t-s)},$$

$$\mathbb{E}_s[S^2(t)] = S^2(s)e^{(r+\Lambda^2)(t-s)}, \quad \forall 0 \le s \le t.$$

where $\mathbb{E}_s[-]$ *is the conditional expectation with respect to* $\mathcal{F}(s)$ *and* $\partial_t(.)$ *is the partial derivative with respect to time.*

Corollary 3.2.1. *The conditional variance* $V(s,t)$ *of the basket is given by*

$$V(s,t) = S^2(s)e^{r(t-s)}\left[e^{\Lambda^2} - e^{r(t-s)}\right].$$

Given the representation of the basket in 3.3, we can see that Λ has the functional form $\Lambda = \Lambda(t, S(t))$, due to its dependence on the price level of the basket at time t. This suggests that basket options would be suitably priced using local volatility models as done by Xu and

Zheng in [39]. Despite this above representation, we in fact see that GBM models are quite adequate and useful for pricing options on a group of assets. To demonstrate this, we begin by setting $Y(t) = \int_0^t \Lambda dW(s)$, which is simply the integral of the the second factor on the RHS of 3.3. This process $Y(t)$ is an $\mathcal{F}(t)-$continuous local martingale. The continuity of $Y(t)$ is fairly obvious, since the underlying randomness is generated by independent Brownian motions which are continuous $\mathbb{P}-$a.e. The quadratic variation $\langle Y \rangle_t$ of such a process $Y(t)$ is given by

$$\langle Y \rangle_t = \int_0^t \Lambda^2 ds. \tag{3.6}$$

By a direct application of the Dubin-Dambis-Schwarz theorem, we can rewrite the process as

$$Y(t) = \hat{W}(\tau_t), \tag{3.7}$$

Proposition 3.2.2. *The stochastic differential equation satisfied by the basket at any time t satisfies the following dynamical equation given by*

$$\frac{dS(t)}{S(t)} = rdt + d\hat{W}(\tau_t),$$

where $\hat{W}(\tau_t)$ is an $\mathcal{F}(\tau_t)$-Brownian motion such that $\tau_t = \inf\{t : \int_0^t \Lambda^2 du > t\}$.

This above representation of the stochastic dynamics of the basket in fact suggests that, we can rewrite its dynamics in terms of a Geometric Brownian Motion whose volatility is unity. Given these features of the basket given above, we attempt to provide a suitable lognormal approximation for the price of a basket option.

The price of the basket call option at an earlier time 0 with maturity date T denoted $C_B(0,T,K)$ is given by

$$C_B(0,T,K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S(T)-K)^+\right], \tag{3.8}$$

where K is the nonnegative strike price of the basket option. The price of the basket call option $C_B(t,T,K)$ at any earlier time t prior to the maturity date T is given by

$$C_B(t,T,K) = e^{-r(T-t)}\mathbb{E}^{\mathcal{Q}}[(S(T)-K)^+|\mathcal{F}(t)]. \tag{3.9}$$

To obtain the expectation in 3.8 and 3.9, we require the distribution of the basket which is generally unknown.

3.3 Lognormal Approach to Basket Option Pricing

Given assets which follow a Geometric Brownian motion model, being able to consistently price the basket of assets in a lognormal framework is extremely desirable. This was the main motivation behind methods of moment-matching (Brigo [5], Henriksen [15], Paletta [?]). In this section, we estimate the price of a basket option using first order Hermite polynomials in attempt to fit suitable parameters, which allow our basket option to be price in the lognormal framework.

3.3.1 First Order Hermite Polynomial Approximation

We know for a fact that $S(t)$ is lognormal for $n = 1$ and $C_B(0,T)$ is available in closed-form in terms of Black-Scholes formula. If we can approximate the summation $\sum_{i=1}^{n} \omega_i S_i(t)$

by a lognormal random variable with a known finite variance in closed-form, we can obtain a Black-Scholes type solution of the price of the basket call option. Instead of trying to approximate $S(t)$ directly *e.g.* using moments, we construct a linear approximation of $y(t) := \ln(S(t))$ in terms of $\ln(S_i(t))$, using Hermite polynomial basis. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), \mathbb{Q})$, $\{\mathcal{F}(t)\}$ represents the filtration generated by d independent Brownian motions $W_1(t), ..., W_d(t)$ and $\mathbb{Q}$ is the risk neutral measure. We assume the asset prices $S_i(t)$ are $\mathcal{F}(t)$-measurable and follow a Geometric Brownian Motion (GBM) model given by

$$S_i(t) = S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right) t + \sum_{j=1}^{d} \sigma_{ij} W_j(t)\right], \quad \forall i = 1, ..., n, \tag{3.10}$$

and satisfy the SDE in 3.1. The price $C(0, T, K)$ of a basket call option at a time 0, with a nonnegative strike K and maturing at time T is

$$C(0, T, K) = e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[(S(T) - K)^+\right]. \tag{3.11}$$

Given the function ψ, which is the log of the terminal value of the basket, we can re-write ψ as a function of standard normal variables such that

$$\begin{aligned}
\psi(u) &= \log\left(S(T)\right), && (3.12)\\
&= \log\left(\sum_{i=1}^{n} \omega_i S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma_i^2\right) T + \sum_{j=1}^{d} \sigma_{ij} W_j(T)\right)\right), && (3.13)\\
&= \log\left(\sum_{i=1}^{n} \omega_i S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma_i^2\right) T + \sqrt{T} \sum_{j=1}^{d} \sigma_{ij} u_j\right)\right). && (3.14)
\end{aligned}$$

where $u_j \sim \mathcal{N}(0, 1) \ \forall j$.

We can transform nonlinear functions of Gaussian random variables into a linear combination of standard normal random variables using Hermite polynomials. Suppose ψ belongs to a class of functions $\mathcal{Y}$ such that

$$\mathcal{Y} = \left\{ \psi(u) : \int_{-\infty}^{\infty} \phi(u; 0, I)\psi^2(u)du < \infty, \forall j \right\}, \tag{3.15}$$

where $\phi(u; 0, I)$ is the density of a standard normal vector $u = (u_1, ..., u_d)$ with covariance matrix I. We define the first order Hermite polynomials $\{h_j^{(1)}(u)\}_{j=1}^{d}$ as

$$h_j^{(1)}(u) = (-1)\frac{\partial \phi(u; 0, I)}{\partial u_j}\phi^{-1}(u; , 0, I), \tag{3.16}$$

$$= u_j. \tag{3.17}$$

where $h_j^{(0)}(u) = 1 \ \forall j$, which satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} h_j^{(1)}(u)h_k^{(1)}(u)\phi(u; 0, I)du = \delta_{jk}, \tag{3.18}$$

where $\delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$

We can express $\psi(u)$ as

$$\psi(u) \approx \sum_{j=0}^{d} b_j h_j^{(1)}(u),$$

$$= b_0 + \sum_{j=1}^{d} b_j u_j. \tag{3.19}$$

To obtain b_0, multiply 3.19 by $\phi(u)$ and integrate over u

$$\int_{-\infty}^{\infty} \psi(u)\phi(u)du = b_0 \int_{-\infty}^{\infty} \phi(u)du + \sum_{j=1}^{d} b_j \int_{-\infty}^{\infty} u_j\phi(u)du, \tag{3.20}$$

$$= b_0. \tag{3.21}$$

Thus,

$$b_0 = \int_{-\infty}^{\infty} \psi(u)\phi(u)du. \tag{3.22}$$

To obtain b_j, we multiply 3.19 by $u_k f(u)$ and integrate over u to get

$$\int_{-\infty}^{\infty} u_k \psi(u)\phi(u)du = b_0 \int_{-\infty}^{\infty} u_k \phi(u)du + \sum_{j=1}^{d} b_j \int_{-\infty}^{\infty} u_j u_k \phi(u)du, \tag{3.23}$$

$$= \sum_{j=1}^{d} b_j \delta_{jk}, \tag{3.24}$$

yielding,

$$b_j = \int_{-\infty}^{\infty} u_j \psi(u)\phi(u)du, \quad 1 \leq j \leq d. \tag{3.25}$$

Proposition 3.3.1. *The closed-form estimate of the basket option price at a time* 0 *is given by*

$$C(0,T,K) \approx e^{-rT} \left[\exp\left(\frac{1}{2y} \left[(b_0 + y)^2 - b_0^2 \right] \right) \Phi\left(\frac{b_0 + y - \ln K}{\sqrt{y}} \right) - K\Phi\left(\frac{b_0 - \ln K}{\sqrt{y}} \right) \right]. \tag{3.26}$$

Proof. We can then approximate the price at time 0 of a basket option maturing at time T, whose assets satisfy 3.2 by

$$C(0,T,K) \approx e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[\left(e^{\psi(u)} - K \right)^+ \right]. \tag{3.27}$$

The approximate terminal payoff is given by

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(e^{\psi(u)}-K)^{+}\right] &= \int_{-\infty}^{\infty}\left(e^{\psi(u)}-K\right)^{+} g(\psi)d\psi, && (3.28)\\
&= \int_{\ln K}^{\infty}\left(e^{\psi(u)}-K\right) g(\psi)d\psi, && (3.29)\\
&= \left[\int_{\ln K}^{\infty} e^{\psi(u)} g(\psi)d\psi - K\int_{\ln K}^{\infty} g(\psi)d\psi\right], && (3.30)\\
&= \exp\left(\frac{1}{2y}\left[(b_0+y)^2-b_0^2\right]\right)\Phi\left(\frac{b_0+y-\ln K}{\sqrt{y}}\right)\\
&- K\Phi\left(\frac{b_0-\ln K}{\sqrt{y}}\right), && (3.31)
\end{aligned}
$$

where $g(\psi)$ is the density of $\psi(u)$ given by

$$
g(\psi) = \frac{1}{\sqrt{y}}\phi\left(\frac{\psi(u)-b_0}{\sqrt{y}}\right), \tag{3.32}
$$

where $y = Var(\psi(u)) = \sum_{j=1}^{d} b_j^2$.

Thus, the approximated price of the basket option is given by

$$
C(0,T,K) \approx e^{-rT}\left[\exp\left(\frac{1}{2y}\left[(b_0+y)^2-b_0^2\right]\right)\Phi\left(\frac{b_0+y-\ln K}{\sqrt{y}}\right) - K\Phi\left(\frac{b_0-\ln K}{\sqrt{y}}\right)\right]. \tag{3.33}
$$

□

This closed-form approximation 3.33 for a basket option price is analogous to Black-Scholes representation for the price of a single asset. We demonstrate through examples for two-asset case (which can be generalised to more assets in the basket) that this yields good approximations for a variety of underlying parameter values and very short maturities.

Also, this closed-form estimate can be used as a control variate for pricing basket options. However, the computational complexity of calculating the basket option price increases as the number of assets in the basket increase. To overcome this, we suggest an adaptation to the previously mentioned method, to allow for its use as a control variate for pricing basket options with sufficiently large assets in the basket.

In general,

$$\ln(1+\overline{S_i(T)}) \leq \overline{S_i(T)}, \tag{3.34}$$

where $\overline{S_i(T)} = \omega_i S_i(T)$. Taking sums of (3.34) over i, we can deduce the following inequality

$$\ln\left(\prod_{i=1}^{n} \overline{S_i(T)}\right) < \sum_{i=1}^{n} \ln\left(1+\overline{S_i(T)}\right) < \sum_{i=1}^{n} \overline{S_i(T)} = S(T). \tag{3.35}$$

We can simplify the weighted products of assets as

$$\prod_{i=1}^{n} \overline{S_i(T)} = \left(\prod_{i=1}^{n} \overline{S_i(0)}\right) \exp\left[\left(nr - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\right)T\right] \exp\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}W_j(T)\right), \tag{3.36}$$

and its logarithm can be becomes

$$\ln\left(\prod_{i=1}^{n} \overline{S_i(T)}\right) = \ln\left(\prod_{i=1}^{n} \overline{S_i(0)}\right) + \left(nr - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\right)T + \sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}W_j(T), \tag{3.37}$$

$$= \gamma + \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}W_j(T), \tag{3.38}$$

where $\overline{S_i(T)} = \omega_i S_i(T)$ and $\gamma = \ln\left(\prod_{i=1}^{n} \overline{S_i(0)}\right) + \left(nr - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\right)T$.

We define a new function $\tilde{\psi}(u)$ by simply replacing $S(T)$ in (3.12) with and the strike K of the basket with $\ln K$ such that

$$\tilde{\psi} = \ln\left(\ln\left(\prod_{i=1}^{n} S(T)\right)\right) \tag{3.39}$$

$$= \ln\left(\gamma + \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}W_j(T)\right)$$

$$= \ln\left(\gamma + \sqrt{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}u_j\right) \tag{3.40}$$

where $u_j \sim \mathcal{N}(0,1)\ \forall j$. Since $\tilde{\psi}(u) \in \mathcal{Y}$, we can approximate $\tilde{\psi}$ similar to ψ as

$$\tilde{\psi}(u) = \tilde{b}_0 + \sum_{j=1}^{n} \tilde{b}_j u_j, \tag{3.41}$$

where the parameters $\tilde{b}_j$'s are estimated as b_j's in (3.22) and (3.25) but are estimated by replacing ψ instead of $\tilde{\psi}$ for all $0 \leq j \leq n$.

We can estimate the parameters of $\tilde{b}_j$'s in (3.41) using third order Taylor series approximation of $\tilde{\psi}$ of u_j's about 0 given by

$$\begin{aligned}\tilde{\psi}(u) = \tilde{\psi}(0) + \sum_{j=1}^{n} \frac{\partial \tilde{\psi}}{\partial u_j}\bigg|_{u_j=0} u_j + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{n} \frac{\partial^2 \tilde{\psi}}{\partial u_j u_k}\bigg|_{u_j=u_k=0} u_j u_k \\ + \frac{1}{6}\sum_{j=1}^{d}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^3 \tilde{\psi}}{\partial u_j u_k u_l}\bigg|_{u_j=u_k=u_l=0} u_j u_k u_l. \end{aligned}\tag{3.42}$$

Thus, the coefficients of the parameters of $\tilde{\psi}$ are

$$\tilde{b}_0 = \ln\gamma - \frac{1}{2}\frac{T}{\xi^2}\sum_{j=1}^{d}\sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ij}\sigma_{kj}, \tag{3.43}$$

$$= \ln\gamma - \frac{1}{2}\frac{T}{\xi^2}\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ik}, \tag{3.44}$$

where A is the volatility matrix and

$$\tilde{b}_j = \frac{\sqrt{T}}{\gamma}\sum_{i=1}^{n}\sigma_{ij} + \left(\frac{\sqrt{T}}{\gamma}\right)^3 \sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\sigma_{ij}\sigma_{lj}\sigma_{mj} + \left(\frac{\sqrt{T}}{\gamma}\right)^3 \sum_{k=1}^{d}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\sigma_{ik}\sigma_{mk}\sigma_{lk}, \tag{3.45}$$

for all $1 \leq j \leq n$.

We denote the price at time 0 of an option on the logarithm of the product of weighted assets in the basket with a strike $\ln K$, maturing at time T as $\tilde{C}(0, T, \ln K)$ is given by

$$\tilde{C}(0, T, \ln K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\ln\left(\prod_{i=1}^{n} S(T)\right) - \ln K\right)^{+}\right]. \tag{3.46}$$

Using the lognormal approximation for $\ln\left(\prod_{i=1}^{n} S(T)\right)$ in (3.40), we can obtain the option price $\tilde{C}(0, T, \ln K)$ in a lognormal framework. Thus, the option price $\tilde{C}(0, T, \ln K)$ is given by

$$\tilde{C}(0, T, \ln K) = e^{-rT}\left[\exp\left(\frac{1}{2}\left(\bar{V} + 2\tilde{b}_0\right)\right)\Phi\left(\frac{\tilde{b}_0 + V - \ln\bar{K}}{\sqrt{\bar{V}}}\right) - \bar{K}\Phi\left(\frac{\tilde{b}_0 - \ln\bar{K}}{\sqrt{\bar{V}}}\right)\right], \tag{3.47}$$

where $\bar{K} = \ln K$ and $\bar{V} = \sum_{j=1}^{n}\tilde{b}_j^2$.

Given that we have the price of the option $\tilde{C}(0, T, \ln K)$ in closed-form, we have all the essential ingredients necessary to use this method for pricing basket options, using the first order Hermite polynomial as a control variate for a large number of underlying assets.

3.4 Gaussian Approximation For Short Maturity Basket Options

Despite assets being lognormally distributed, we can find simple Gaussian approximations for the price of a basket option for small maturities. Given an n- asset basket, we can rewrite 3.2 as

$$S(t) = \sum_{i=1}^{n}\omega_i S_i(0)e^{rt}Z_i(t), \tag{3.48}$$

where $Z_i(t)$ is a $\mathbb{Q}$-martingale which can be represented as

$$Z_i(t) = 1 + \int_0^t Z_i(s) \sum_{j=1}^{d} \sigma_{ij} dW_j(s). \tag{3.49}$$

For small maturities t and for any $\epsilon > 0$, we assume that $|Z_i(s) - 1| < \epsilon$ for all i and $0 \leq s < t$. Using this we consider a simplification of the value of the basket, so that 3.48 becomes

$$S(t) \approx \sum_{i=1}^{n} \omega_i S_i(0) e^{rt} \Big(1 + \sum_{j=1}^{d} \sigma_{ij} W_j(t)\Big), \tag{3.50}$$

$$= \sum_{i=1}^{n} \omega_i S_i(0) e^{rt} \Big(1 + P_i(t)\Big), \tag{3.51}$$

where $P_i(t) = \sum_{j=1}^{d} \sigma_{ij} W_j(t)$, is normally distributed with mean 0 and covariance $t \sum_{j=1}^{d} \sigma_{ij}\sigma_{kj}$. This leads to a basket which follows a Gaussian distribution, allowing for closed-form approximation of the price of basket option on it. To understand the accuracy of this Gaussian estimation of the basket, we observe the error process $E(t)$ of the difference between 3.48 and 3.50.

Proposition 3.4.1. *The error process $E(t)$ is a martingale with zero mean and its variance $V_E(t)$ is a nonnegative submartingale given by*

$$V_E(t) = e^{2rt} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{d} \omega_i \omega_k S_i(0) S_k(0) \left[\frac{1}{A_{ik}} \left(\exp\left(tA_{ik}\right) - 1\right) - t \right],$$

which is an increasing function of time, where $A_{ik} = \sum_{j=1}^{d} \sigma_{ij} \sigma_{jk}^T$.

Proof. $E(t)$ is given by

$$E(t) = \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{d} \omega_i S_i(0) e^{rt} \left(Z_i(s) - 1\right) \sigma_{ij} dW_j(s), \tag{3.52}$$

which is an Ito integral with zero mean and its variance $V_E(t)$ is given by

$$V_E(t) = e^{2rt}\mathbb{E}\left[\int_0^t \sum_{i=1}^n\sum_{k=1}^n\sum_{j=1}^d \omega_i\omega_k S_i(0)S_k(0)(Z_i(s)-1)(Z_k(s)-1)ds\right], \tag{3.53}$$

$$= e^{2rt}\int_0^t \sum_{i=1}^n\sum_{k=1}^n\sum_{j=1}^d \omega_i\omega_k S_i(0)S_k(0)\mathbb{E}\left[(Z_i(s)-1)(Z_k(s)-1)\right]ds. \tag{3.54}$$

Using the fact that $\mathbb{E}[Z_i(s)] = \mathbb{E}[Z_k(s)] = 1, \forall i,k$ and $\mathbb{E}[Z_i(s)Z_k(s)] = \exp\left(s\sum_{j=1}^d \sigma_{ij}\sigma_{jk}^T\right)$, we can simplify $V_E(t)$ to obtain

$$V_E(t) = e^{2rt}\int_0^t \sum_{i=1}^n\sum_{k=1}^n\sum_{j=1}^d \omega_i\omega_k S_i(0)S_k(0)\left(\exp\left(s\sum_{j=1}^d \sigma_{ij}\sigma_{jk}^T\right)-1\right)ds, \tag{3.55}$$

$$= e^{2rt}\sum_{i=1}^n\sum_{k=1}^n\sum_{j=1}^d \omega_i\omega_k S_i(0)S_k(0)\left[\frac{1}{\sum_{j=1}^d \sigma_{ij}\sigma_{jk}^T}\left(\exp\left(t\sum_{j=1}^d \sigma_{ij}\sigma_{jk}^T\right)-1\right)-t\right], \tag{3.56}$$

which concludes the first part of the proof. To prove that the variance $V_E(t)$ is an increasing function of time, it sufficient to show that

$$\frac{\partial}{\partial t}\left(\frac{1}{A_{ik}}\left(\exp\left(tA_{ik}\right)-1\right)-t\right) = \left(\exp\left(tA_{ik}\right)-1\right)-1 \geq 0, \quad \forall t \geq 0. \tag{3.57}$$

□

Given that our Gaussian approximation for the value of a basket is valid for small time intervals, we now look to obtain the price of a basket option for very short maturities within this framework.

Proposition 3.4.2. *Given the value of a basket with short maturity t which satisfies 3.51, the price of its corresponding basket option at a time 0 with strike K is given by*

$$C(0,t,K) \approx e^{-rt}\left[V(t)\phi\left(\frac{K-m(t)}{V(t)}\right) + (m(t)-K)\Phi\left(\frac{m(t)-K}{V(t)}\right)\right], \tag{3.58}$$

where $m(t) = \sum_{i=1}^{n} \omega_i S_i(0) e^{rT}$, $V^2(t) = e^{2rt} t \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{d} \omega_i \omega_k S_i(0) S_k(0) \sigma_{ij} \sigma_{kj}$ *and* $\phi(.)$ *is the standard Gaussian density and* $\Phi(.)$ *is its corresponding distribution,*

Proof. We define the variance $V^2(t)$ of $S(t)$ as

$$V^2(t) = \mathbb{E}\left[(S(t) - m(t))^2\right], \tag{3.59}$$

$$= e^{2rt}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{d}\sum_{l=1}^{d} \omega_i \omega_k S_i(0) S_k(0) \sigma_{ij} \sigma_{kl} W_j(t) W_l(t)\right], \tag{3.60}$$

$$= e^{2rt}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{d} \omega_i \omega_k S_i(0) S_k(0) \sigma_{ij} \sigma_{kl} W_j^2(t)\right], \tag{3.61}$$

$$= te^{2rt}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{d} \omega_i \omega_k S_i(0) S_k(0) \sigma_{ij} \sigma_{kl}. \tag{3.62}$$

Using the representation of $S(t)$ in 3.50, the density of $S(t)$ is given by

$$\mathbb{Q}\left(S(t) \in dy\right) = \frac{1}{V(t)}\phi\left(\frac{y - m(t)}{V(t)}\right) dy. \tag{3.63}$$

The price of a basket option at a time 0 with small maturity and strike K si given by

$$C(0, t, K) = e^{-rt}\mathbb{E}\left[(S(t) - K)^+\right], \tag{3.64}$$

$$\approx e^{-rt}\int_K^\infty (y - K)\,\mathbb{Q}\left(S(t) \in dy\right), \tag{3.65}$$

$$= e^{-rt}\int_K^\infty (y - K)\frac{1}{V(t)}\phi\left(\frac{y - m(t)}{V(t)}\right) dy, \tag{3.66}$$

$$= \int_K^\infty (y - K)\frac{1}{V(t)}\phi\left(\frac{y - m(t)}{V(t)}\right) dy. \tag{3.67}$$

Set $z = \dfrac{y - m(t)}{V(t)}$ and we take its derivative with respect to y. We substitute for y to obtain,

$$C(0,t,K) \approx e^{-rt} \int_{\frac{y-m(t)}{V(t)}}^{\infty} (m + zV(t))\, \phi(z) dz - Ke^{-rt} \int_{\frac{y-m(t)}{V(t)}}^{\infty} \phi(z) dz, \tag{3.68}$$

$$= e^{-rt} \left[(m(t) - K)\, \Phi\left(\frac{m(t) - K}{V(t)}\right) + V(t) \phi \left(\frac{m(t) - K}{V(t)}\right) \right]. \tag{3.69}$$

□

Furthermore, 3.50 suggests that using a first order Taylor expansion of the exponential martingale term in an asset under a GBM model might be a good estimate for pricing short maturity basket options, since for small maturities higher orders of the Taylor terms and their sum go to zero.

3.5 Bounds On The Price Of A Basket Option

The option price on the geometric lower bound of a basket of assets as in [13], is the most popular bound on basket option prices and is also used as a benchmark in comparing the performance of other lower bounds. It involves using the arithmetic mean of a sequence being bounded below by its geometric mean. For a basket of n assets, its geometric mean $G(T)$ is given by

$$G(T) = \prod_{i=1}^{n} [S_i(T)]^{\omega_i} \leq \sum_{i=1}^{n} \omega_i S_i(T). \tag{3.70}$$

The geometric mean $G(T)$ is lognormally distributed and the price of a European call option on it can be obtained in closed-form. The price of a European call option on $G(T)$ given by

$C_G(0,T)$, with strike K and maturity T at a time 0 is given by

$$C_G(0,T,K) = e^{-rT}\left[\Theta\beta\exp\left(\frac{1}{2}v\right)\Phi\left(\frac{v-\ln\left(\frac{K}{\Theta\beta}\right)}{\sqrt{v}}\right) - K\Phi\left(\frac{-\ln\left(\frac{K}{\Theta\beta}\right)}{\sqrt{v}}\right)\right], \tag{3.71}$$

where $\Theta = \prod_{i=1}^{n}(S_i(0))^{\omega_i}$, $v = Var(\ln\Theta)$, $\beta = \sum_{i=1}^{n}\omega_i\left(r-\frac{1}{2}\sigma_i^2\right)T$.

3.5.1 Direct Upper and Lower Bounds on Basket Option Price

In this section, we obtain tight (lower and upper) bounds on the price of a basket option by a direct application of the Jensen's inequality due to the convexity of the payoff of the basket option.

Proposition 3.5.1. *A basket option has a direct upper bound $U_B(0,T,K)$ on its price at time 0 prior to time T, which is given by*

$$U_B(0,T,K) = e^{-rT}\sum_{i=1}^{n}\omega_i\mathbb{E}\Big(S_i(T)-K\Big)^+.$$

Proof. The expected payoff of the basket option $C_B(0,T,K)$ at its maturity is bounded above as shown below by

$$\begin{aligned}
C_B(0,T,K) &= \mathbb{E}(S(T)-K)^+, &(3.72)\\
&= \mathbb{E}\left(\sum_{i=1}^{n}\omega_i S_i(T) - K\right)^+, &(3.73)\\
&= \mathbb{E}\left(\sum_{i=1}^{n}\omega_i S_i(T) - K\sum_{i=1}^{n}\omega_i\right)^+, &(3.74)\\
&\leq \sum_{i=1}^{n}\omega_i\mathbb{E}\Big(S_i(T)-K\Big)^+. &(3.75)
\end{aligned}$$

Hence,

$$C_B(0,T,K) = e^{-rT}\mathbb{E}\left(S(T)-K\right)^+ \leq e^{-rT}\sum_{i=1}^{n}\omega_i\Big(S_i(T)-K\Big)^+ = U_B(0,T,K). \tag{3.76}$$

$\square$

The upper bound $U_B(0,T,K)$ on the basket option price is the same as holding n options of different assets with the same strike K. The price of such a fictitious portfolio $U_B(0,T,K)$ is given by

$$U_B(0,T,K) = e^{-rT}\sum_{i=1}^{n}\left[S_i(0)e^{rT}\Phi(h_i^+) - K\Phi(h_i^-)\right], \tag{3.77}$$

where $h_{i\pm} = \dfrac{\ln\left(\frac{S_i(0)e^{rT}}{K}\right) \pm \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}$.

Proposition 3.5.2. *A basket option has a direct lower bound $U_L(0,T,K)$ on its price at time $t < T$, which is given by*

$$U_L(0,T,K) = S(0) - Ke^{-rT},$$

provided $S(0) > Ke^{-rT}$.

3.6 Distributional Bounds on a Basket Option Price

Some of the research into finding suitable bounds on the price of a basket option involve using the properties of its payoff function such as in Rogers-Shi lower bound [35]. In this section, we derive new upper and lower bounds on a basket of assets and their corresponding option price by using the distributional properties of Brownian motions.

3.6.1 Lower And Upper Distributional Bounds Of A Basket Option

In general, we can obtain an upper bound on the basket of assets 3.2 and the price of a basket option 3.8 by replacing the independent Wiener processes with their joint maximum. Also, we can obtain lower bounds on the basket and its option price by using the joint minimum of the independent Brownian motions in the basket.

Proposition 3.6.1. *The value $S(t)$ of the basket of assets at any time t is bounded above by*

$$S_u(t) = \sum_{i=1}^{n}\omega_i S_i(0)\exp\Big((r-\frac{1}{2}\bar{\sigma}_i^2)t\Big)\exp\Big(M_d(t)\sum_{j=1}^{d}\sigma_{ij}\Big), \tag{3.78}$$

and bounded below by

$$S_l(t) = \sum_{i=1}^{n}\omega_i S_i(0)\exp\Big((r-\frac{1}{2}\bar{\sigma}_i^2)t\Big)\exp\Big(m_d(t)\sum_{j=1}^{d}\sigma_{ij}\Big), \tag{3.79}$$

where $M_d(t) = \max_{1\le j\le d} W_j(t)$, $\quad m_d(t) = \min_{1\le j\le d} W_j(t)$ and provided $\sum_{j=1}^{n}\sigma_{ij}$ is nonnegative for $1 \le i \le n$.

Proof. The price $S(t)$ of a basket of assets at a time t is the solution, which satisfies the SDE in 3.3, which has the following form

$$S(t) = \sum_{i=1}^{n}\omega_i S_i(0)\exp\Big[(r-\frac{1}{2}\sigma_i^2)t\Big]\exp\Big(\sum_{j=1}^{d}\sigma_{ij}W_j(t)\Big), \tag{3.80}$$

where $\sigma_i^2 = \sum_{j=1}^{n}\sigma_{ij}^2$, $W_j(t)$ and $W_k(t)$ are independent Brownian motions for $j \neq k$.
For any asset i, we setup the following inequalities

$$\sum_{j=1}^{d}\sigma_{ij}\min_{1\le j\le d} W_j(t) \le \sum_{j=1}^{d}\sigma_{ij}W_j(t) \le \sum_{j=1}^{d}\sigma_{ij}\max_{1\le j\le d} W_j(t), \tag{3.81}$$

$$\sum_{j=1}^{d}\sigma_{ij}m_d(t) \le \sum_{j=1}^{d}\sigma_{ik}W_j(t) \le \sum_{j=1}^{d}\sigma_{ij}M_d(t), \tag{3.82}$$

$$m_d(t)\sum_{j=1}^{d}\sigma_{ij} \le \sum_{j=1}^{d}\sigma_{ij}W_k(t) \le M_d(t)\sum_{j=1}^{d}\sigma_{ij}, \tag{3.83}$$

where $M_d(t) = \max_{1 \leq j \leq d} W_k(u))$ and $m_d(t) = \min_{1 \leq j \leq d} W_j(t)$.

So that,

$$m_d(t) \sum_{j=1}^{d} \sigma_{ij} \leq \sum_{j=1}^{d} \sigma_{ij} W_k(t) \leq M_d(t) \sum_{j=1}^{d} \sigma_{ij} \quad \textit{for all } i. \tag{3.84}$$

Thus, the price of the asset at time t is bounded by

$$S_l(t) = \sum_{i=1}^{n} Y_i \exp\left(m(t)\sigma_i^* \right) \leq S(t) \leq \sum_{i=1}^{n} Y_i \exp\left(M(t)\sigma_i^* \right) = S_u(t), \tag{3.85}$$

where $\sigma_i^* = \sum_{j=1}^{d} \sigma_{ij}$ and $Y_i = \omega_i S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right) T\right]$. □

These bounds on the value of the basket given by $S_u(t)$ and $S_l(t)$ are analytically intractible and are of a similar problem type as the the basket of assets. To this end, we can estimate options on $S_u(t)$ and $S_l(t)$ by using their respective geometric means, as suggested by Gentle [13]. Given the representation for the distributional upper bound on the value of a basket of n assets at a time T given by $S_u(T)$ and $S_l(T)$ respectively in proposition 3.6.1. Let us call $G_u(T)$ as the geometric mean of 3.78, which then has the following representation

$$G_u(T) = \prod_{i=1}^{n} \left[S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right) T\right] \exp\left(M_d(T) \sum_{j=1}^{d} \sigma_{ij} \right) \right]^{\frac{1}{n}}. \tag{3.86}$$

Given the representation of $G_u(T)$ above, we can use it to estimate the price of an option on $S_u(T)$, which is an the upper bound on the basket option price.

Proposition 3.6.2. *The price $C_{G_u}(0,T)$ at time 0 of an option on the geometric mean $G_u(T)$, maturing at time T with nonnegative strike K is given by*

$$C_{G_u}(0,T) = \alpha_1 \beta_1 e^{-rT} \int_{\tilde{K}}^{\infty} e^{\gamma_1 y} \frac{d}{\sqrt{T}} \phi\left(\frac{y}{\sqrt{T}}\right) \left(\Phi\left(\frac{y}{\sqrt{T}}\right) \right)^{d-1} dy - K e^{-rT} \left[1 - \left(\Phi\left(\frac{\tilde{K}}{\sqrt{T}}\right) \right)^{d} \right]$$

where $\alpha_1 = \prod_{i=1}^{n} S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right)T\right]$, $\gamma_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}$ *and* $\tilde{K} = \frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)$.

Proof. We can simplify the expression for $G_u(T)$ in 3.86 to become

$$G_u(T) = \left(\prod_{i=1}^{n}\Big(S_i(0)\Big)^{\frac{1}{n}}\right)\exp\left[\left(r - \frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right)T\right]\exp\left(M_d(T)\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}\right). \tag{3.87}$$

The price $C_{G_u}(0,T)$ of the option on $G_u(T)$ at a time 0 is given by

$$C_{G_u}(0,T) = e^{-rT}\mathbb{E}\left[(G_n(T) - K)^+\right], \tag{3.88}$$

$$= e^{-rT}\int_{\tilde{K}}^{\infty}\left(\alpha_1\beta_1 e^{\gamma_1 y} - K\right)\frac{d}{\sqrt{T}}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\mathcal{N}\left(\frac{y}{\sqrt{T}}\right)\right)^{d-1} dy, \tag{3.89}$$

$$= \alpha_1\beta_1 e^{-rT}\int_{\tilde{K}}^{\infty} e^{\gamma_1 y}\frac{d}{\sqrt{T}}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{d-1} dy$$
$$- Ke^{-rT}\left[1 - \left(\Phi\left(\frac{\tilde{K}}{\sqrt{T}}\right)\right)^{d}\right]. \tag{3.90}$$

This completes the proof. □

Similarly, given the geometric mean of $S_l(T)$ in 3.79, we can derive the price of an option on it with the same strike K. We can define the geometric mean $G_l(T)$ on 3.79 as

$$G_l(T) = \prod_{i=1}^{n}\left[S_i(0)\exp\left[\left(r - \frac{1}{2}\sigma_i^2\right)T\right]\exp\left(m_d(T)\sum_{j=1}^{d}\sigma_{ij}\right)\right]^{\frac{1}{n}}. \tag{3.91}$$

Proposition 3.6.3. *The price $C_{G_l}(0,T)$ of the option on $G_l(T)$ maturing at T at a time 0 with nonnegative strike K is given by*

$$C_{G_l}(0,T) = e^{-rT}\int_{\tilde{K}}^{\infty} e^{\gamma_1 y}\frac{d}{\sqrt{T}}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(-\frac{y}{\sqrt{T}}\right)\right)^{d-1} dy + Ke^{-rT}\left(\Phi\left(-\frac{\tilde{K}}{T}\right)\right)^{d},$$

where $\alpha_1 = \prod_{i=1}^{n} S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right)T\right]$, $\gamma_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}$ *and* $\tilde{K} = \frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)$.

Hence, we are able to use the bounds 3.78 and 3.79 on the value of the basket to obtain closed-form estimates on the basket option price using their respective geometric mean. Alternatively, we can obtain more accurate option prices on the distributional bounds of the value of the basket using Curran's method [8]. This is achieved by conditioning the price using 3.78 and 3.79 on their respective geometric mean which we carry out in the next proposition.

Proposition 3.6.4. *The estimated option price $C_B^u(0,T)$, on $S_u(T)$ with strike K, at a time 0 prior to its maturity T using Curran's conditioning arguments is given by*

$$C_B^u(0,T) = e^{-rT}\mathbb{E}\left[S_u(T)\mathbb{1}\{G_u(T) \geq K\}\right] + Ke^{-rT}\left[\left(\Phi\left(\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)\right)^d - 1\right], \quad (3.92)$$

where $S_u(T)$ is as defined is 3.78, $\alpha_1 = \prod_{i=1}^{n} S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right)T\right]$, $\gamma_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}$.

Proof. The price at a time 0 an option on $S_u(T)$ prior to the maturity T is given by

$$C_B^u(0,T) = e^{-rT}\mathbb{E}\left[(S_u(T) - K)^+\right], \quad (3.93)$$

$$= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[(S_u(T) - K)^+ \middle| G_u(T) = y\right]\right], \quad (3.94)$$

$$= e^{-rT}\int_0^K \mathbb{E}\left[(S_u(T) - K)^+ \middle| G_u(T) = y\right]\mathbb{Q}\left(G_u(T) \in dy\right)$$
$$+ e^{-rT}\int_K^\infty \mathbb{E}\left[(S_u(T) - K)^+ \middle| G_u(T) = y\right]\mathbb{Q}\left(G_u(T) \in dy\right). \quad (3.95)$$

We use the fact that

$$\int_0^K \mathbb{E}\left[(S_u(T) - K)^+ \middle| G_u(T) = y\right]\mathbb{Q}\left(G_u(T) \in dy\right) \approx 0. \quad (3.96)$$

Substituting 3.105 in 3.104 to obtain,

$$C_B^u(0,T) = e^{-rT}\int_K^\infty \mathbb{E}\left[(S_u(T)-K)^+ \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right) \tag{3.97}$$

$$= e^{-rT}\int_K^\infty \mathbb{E}\left[S_u(T)-K \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right) \tag{3.98}$$

$$= e^{-rT}\mathbb{E}\left[S_u(T)\mathbb{1}G_u(T)\geq K\right] - Ke^{-rT}\mathbb{Q}\left(G_u(T)\geq K\right) \tag{3.99}$$

$$= e^{-rT}\mathbb{E}\left[S_u(T)\mathbb{1}\{G_u(T)\geq K\}\right] - Ke^{-rT}\left[1-\left(\Phi\left(\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)^d\right)\right] \tag{3.100}$$

Thus, we are able to obtain the required results. □

The option price given by $C_B^u(0,T)$ is an upper bound on the basket option price in 3.11. Next, we shall proceed to work out the lower bound on the basket option price using similar conditioning arguments.

Proposition 3.6.5. *The estimated option price $C_B^l(0,T)$, on $S_l(T)$, with strike K at a time 0 prior to its maturity T using Curran's conditioning arguments is given by*

$$C_B^l(0,T) = e^{-rT}\mathbb{E}\left[S_l(T)\mathbb{1}\{G_l(T)\geq K\}\right] + Ke^{-rT}\left[\left(\Phi\left(-\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)\right)^d\right], \tag{3.101}$$

where $S_l(T)$ is as defined is 3.79, $\alpha_1 = \prod_{i=1}^n S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2}\sum_{i=1}^n \frac{\sigma_i^2}{n}\right)T\right]$, $\gamma_1 = \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^d \sigma_{ij}$.

Proof. The price at a time 0 an option on $S_l(T)$ prior to the maturity T is given by

$$C_B^l(0,T) = e^{-rT}\mathbb{E}\left[(S_l(T)-K)^+\right], \tag{3.102}$$

$$= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[(S_l(T)-K)^+\Big|G_l(T)=y\right]\right], \tag{3.103}$$

$$= e^{-rT}\int_0^K \mathbb{E}\left[(S_l(T)-K)^+\Big|G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right)$$

$$+ e^{-rT}\int_K^\infty \mathbb{E}\left[(S_l(T)-K)^+\Big|G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right). \tag{3.104}$$

We use the fact that

$$\int_0^K \mathbb{E}\left[(S_l(T)-K)^+\Big|G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right)\approx 0. \tag{3.105}$$

Substituting 3.105 in 3.104 to obtain,

$$C_B^l(0,T) = e^{-rT}\int_K^\infty \mathbb{E}\left[(S_l(T)-K)^+\Big|G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right) \tag{3.106}$$

$$= e^{-rT}\int_K^\infty \mathbb{E}\left[S_l(T)-K\Big|G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right) \tag{3.107}$$

$$= e^{-rT}\mathbb{E}\left[S_l(T)\mathbb{1}G_l(T)\geq K\right] - Ke^{-rT}\mathbb{Q}\left(G_l(T)\geq K\right) \tag{3.108}$$

$$= e^{-rT}\mathbb{E}\left[S_l(T)\mathbb{1}\{G_l(T)\geq K\}\right] + Ke^{-rT}\left[\left(\Phi\left(-\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)\right)^d\right] \tag{3.109}$$

Thus, we are able to obtain the required results. $\square$

Given the analytic intractibility of the bounds $S_l(t)$ and $S_u(t)$ on the value of a basket at any time t, we can impose integrability conditions on the the volatility parameters which will allow for closed-form evaluation of options on these bounds. These integrability conditions leads to further bounds on the value of the basket given by

$$\bar{S}_l(t) = \sum_{i=1}^{n} Y_i \exp\Big(m_d(t)\sigma_m\Big) \leq S_l(t) \leq S(t) \leq S_u(t) \leq \sum_{i=1}^{n} Y_i \exp\Big(M_d(t)\sigma_M\Big) = \bar{S}_u(t), \tag{3.110}$$

where $\sigma_i^2 = \sum_{j=1}^{d} \sigma_{ij}^2$, $\sigma_M = \max_{1\leq i\leq n} \sigma_i^*$ and $\sigma_m = \min_{1\leq i\leq n} \sigma_i^*$.

In general, the process $M_d(t)$ can be replaced with the maximum of the running maximums or the maximum of the minimum of the absolute value of a Brownian motion and its running maximum of independent Brownian motions. These are themselves upper bounds on the underlying Brownian motions and vice-versa to obtain different bounds on the value of the basket and the option price. With these new upper and lower analytic bounds on the value of the basket given by $S_u(t)$ and $S_l(t)$ respectively, we are able to work out the density and the corresponding distribution of these bounds.

Proposition 3.6.6. *Given the lower bound $\bar{S}_l(t)$ on the value of a basket $S(t)$ at a time t, its density and distribution is given by*

$$\mathbb{Q}\left(\bar{S}_l(t) \in dy\right) = \frac{1}{y}\frac{1}{\sigma_m\sqrt{T}} n\phi\left(\frac{1}{\sigma_m\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\left[1-\Phi\left(\frac{1}{\sigma_m\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\right]^{n-1} dy,$$

and

$$\mathbb{Q}\left(\bar{S}_l(t) \leq y\right) = 1 - \left[1-\Phi\left(\frac{1}{\sigma_m\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\right]^{n},$$

respectively, where $\vartheta = \sum_{i=1}^{n} \omega_i S_i(0) \exp\left[(r-\frac{1}{2}\sigma_i^2)T\right]$.

Proposition 3.6.7. *Given the upper bound $\bar{S}_u(t)$ on the value of a basket $S_u(t)$, its density* *and distribution is given by*

$$\mathbb{Q}\left(\bar{S}_u(t) \in dy\right) = \frac{1}{y}\frac{1}{\sigma_M\sqrt{T}} n\phi\left(\frac{1}{\sigma_M\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\left(\Phi\left(\frac{1}{\sigma_M\sqrt{T}}\ln\left(\frac{y}{\alpha}\right)\right)\right)^{n-1} dy,$$

and

$$\mathbb{Q}\left(\bar{S}_u(t) \leq y\right) = \left(\Phi\left(\frac{1}{\sigma_M\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\right)^{n},$$

respectively, where $\vartheta = \sum_{i=1}^{n} \omega_i S_i(0)\exp\left[(r - \frac{1}{2}\sigma_i^2)T\right]$.

Proposition 3.6.8. *Given a basket of assets, which has the bounds as specified in* (3.110) *such a basket has the following bounds on the price of the basket option at a time* 0 *given by*

$$U_1^n = ne^{-rT}\frac{\sum_{i=1}^{n} Y_i}{\sqrt{T}}\int_{\xi}^{\infty} e^{\sigma_M y}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1} dy - Ke^{-rT}\left[1 - \left(\Phi\left(\frac{\xi}{\sqrt{T}}\right)\right)^{n}\right], \tag{3.111}$$

and

$$L_1^n = ne^{-rT}\frac{\sum_{i=1}^{n} Y_i}{\sqrt{T}}\int_{\tau}^{\infty} e^{\sigma_m y}\phi\left(\frac{y}{\sqrt{T}}\right)\left(1 - \Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1} dy + Ke^{-rT}\left[1 - \Phi\left(\frac{\tau}{\sqrt{T}}\right)\right]^{n}, \tag{3.112}$$

where $Y_i = \omega_i S_i(0)\exp\left[\left(r - \frac{1}{2}\sigma_i^2\right)T\right]$, $\xi = \frac{1}{\sigma_M}\ln\frac{K}{\sum_{i=1}^{n} Y_i}$ *and* $\tau = \frac{1}{\sigma_m}\ln\frac{K}{\sum_{i=1}^{n} Y_i}$.

Theoretically, a basket of assets can contain any number of assets in the basket and would require an increasing number of maximum or minimum of Wiener processes. For a large number of assets in the basket, it can be computationally challenging to work out the bounds on the basket, and thereafter, the bounds on the basket option. For a large number of assets, we can obtain overall bounds on the basket and its corresponding option price using proposition 2.3.4 and 2.3.5 and these bounds are independent of the number of assets in the basket.

Corollary 3.6.1. *The distributional bounds on the price of a basket option using the maximum and minimum of Brownian motions is given by*

$$U_1 = e^{-rT}\left[\frac{\sum_{i=1}^{n} Y_i}{\sqrt{T}}\int_{\xi}^{\infty} e^{\sigma_M y}\phi\left(\frac{y}{\sqrt{T}}\right)\Phi\left(\frac{y}{\sqrt{T}}\right)dy - \frac{K}{2}\left(1-\Phi^2\left(\frac{\xi}{\sqrt{T}}\right)\right)\right] \tag{3.113}$$

and

$$L_1 = e^{-rT}\left[\frac{\sum_{i=1}^{n} Y_i}{\sqrt{T}}\int_{\tau}^{\infty} e^{\sigma_m y}\phi\left(\frac{y}{\sqrt{T}}\right)\Phi\left(-\frac{y}{\sqrt{T}}\right)dy + \frac{K}{2}\Phi^2\left(\frac{\tau}{\sqrt{T}}\right)\right], \tag{3.114}$$

independent of n*, which follows from proposition 3.6.8.*

Proposition 3.6.9. *The upper* U_2 *and lower* L_2 *bounds on the basket of asset at time* 0 *given that the maximum of the running maximums and minimum of the running minimums of* n *independent Wiener processes is given by*

$$\begin{aligned} U_2^n = &\sum_{i=1}^{n} Y_i \frac{2ne^{-rT}}{\sqrt{T}}\int_{\xi^*}^{\infty} \mathbb{1}\{y \geq 0\}e^{\sigma_M y}\phi\left(\frac{y}{\sqrt{T}}\right)\left(2\Phi\left(\frac{y}{\sqrt{T}}\right)-1\right)^{n-1} dy \\ &- Ke^{-rT}\left[1-\left(2\Phi\left(\frac{\xi^*}{\sqrt{T}}\right)-1\right)^{n}\right], \end{aligned}$$

and

$$\begin{aligned} L_2^n = &\sum_{i=1}^{n} Y_i \frac{2ne^{-rT}}{\sqrt{T}}\int_{\tau^*}^{\infty} \mathbb{1}\{y \leq 0\}e^{\sigma_m y}\phi\left(\frac{y}{\sqrt{T}}\right)\left(1-2\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1} dy \\ &+ Ke^{-rT}\left(1-2\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n}, \end{aligned}$$

respectively. Where $Y_i = \omega_i S_i(0)\exp\left[\left(r-\frac{1}{2}\sigma_i^2\right)T\right]$, $\xi^* = \frac{1}{\sigma_M}\ln\frac{K}{\sum_{i=1}^{n} Y_i}$ *and* $\tau^* = \frac{1}{\sigma_m}\ln\frac{K}{\sum_{i=1}^{n} Y_i}$.

Proposition 3.6.10. *The upper U_3 and lower L_3 bounds on the basket of assets at time* 0 *given that the maximum of the running maximums of n independent Wiener processes by*

$$U_3^n = \sum_{i=1}^{n} Y_i \frac{ne^{-rT}}{\sqrt{T}} \int_{\xi^*}^{\infty} \mathbb{1}\{y \geq 0\} \left(\Phi\left(\frac{3y}{\sqrt{t}}\right) - \Phi\left(\frac{y}{\sqrt{t}}\right) \right)^{n-1} \left(\phi\left(\frac{3y}{\sqrt{t}}\right) - \phi\left(\frac{y}{\sqrt{t}}\right) \right) dy$$
$$+ Ke^{-rT} \left(\Phi\left(\frac{3\xi^*}{\sqrt{T}}\right) - \Phi\left(\frac{\xi^*}{\sqrt{T}}\right) \right)^{n},$$

and

$$L_3^n = \sum_{i=1}^{n} Y_i \frac{ne^{-rT}}{\sqrt{T}} \int_{\tau^*}^{\infty} \mathbb{1}\{y \leq 0\} \left(\Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{3y}{\sqrt{t}}\right) \right)^{n-1} \left(\phi\left(\frac{y}{\sqrt{t}}\right) - \phi\left(\frac{3y}{\sqrt{t}}\right) \right) dy$$
$$+ Ke^{-rT} \left(\Phi\left(\frac{\tau^*}{\sqrt{T}}\right) - \Phi\left(\frac{3\tau^*}{\sqrt{T}}\right) \right)^{n},$$

respectively. Where $Y_i = \omega_i S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right) T\right]$, $\xi^* = \frac{1}{\sigma_M} \ln \frac{K}{\sum_{i=1}^n Y_i}$ *and* $\tau^* = \frac{1}{\sigma_m} \ln \frac{K}{\sum_{i=1}^n Y_i}$.

All of these bounds can easily be evaluated in closed-form for $n = 2, 3$. For quick approximations of the bounds, we can use numerical methods or other approximations for $n > 3$. It is easy to see in the case of a single asset, that the natural bounds of any asset at a time $t < T$ is obtained by substituting the underlying Brownian motion for its running maximum to get its upper bound or its running minimum to get its lower bound. We can also obtain suitable estimates for the bounds by using suitable Taylor series expansion of the exponential component of the integrand.

Alternatively, we can use $\mathcal{N}(y) \leq 1 \, \forall y \in \mathbb{R}$ to obtain reasonable simplifications of these bounds.

3.7 Summary

We have derived closed-form estimates for the price of a basket option in a Gaussian framework and in a lognormal framework using first order Hermite polynomials. The accuracy of the first order Hermite polynomial method in pricing basket option would depend on the choice of order of Taylor series order estimates of ψ. However, we do not focus solely on the accuracy of the estimates of this lognormal approach, but also on whether it is a good control variate for obtaining efficient Monte Carlo estimates. For the Gaussian approximation, we see that the accuracy reduces as the maturity of the option increases, and is only valid for short maturity options and as a control variate for longer dated ones.

We also obtained several analytic bounds on the price of a basket option, namely the direct and distributional bounds are obtained. The direct upper and lower bounds provides tight bounds on basket option price which follow from the convexity of its payoff. The distributional bounds are obtained by replacing the underlying in dependent Brownian motions with their overall maximum (or minimum) to obtain bounds on the basket option price. These bounds are estimated by either using (Curran's) conditioning arguments or imposing suitable integrability conditions on the volatility parameters of the underlying assets in the basket.

Chapter 4

Asian Options

4.1 Overview

In this chapter, we begin by studying the general dynamics of an Asian option by looking at the behaviour of the underlying asset and thereafter the relevant equation governing the price of an Asian option. We then obtain estimates and bounds on the price of an Asian option. We estimate the price of an Asian option using first order Hermite polynomial, using a similar approach to the one used in estimating the price of a basket option. While for the bounds estimation we use the notion of distributional bounds to estimate the upper and lower bounds on the price of an Asian option.

4.2 Asian Option Dynamics

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), \mathbb{Q})$, where $\mathcal{F}$ is the sigma algebra, $\{\mathcal{F}(t)\}_{t\geq 0}$ is the filtration generated by one or more independent Brownian motions and $\mathbb{Q}$ is the risk-neutral measure. We assume that the asset price is adapted to the Brownian filtration $\{\mathcal{F}(t)\}$ and follows a geometric Brownian motion model given by

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{4.1}$$

where $\{W(t)\colon\ t \in [0,T]\}$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$, r is the lending/borrowing rate. The average value of an asset $A(0,T)$ over the time interval $[0,T]$ is given by

$$A(0,T) = \frac{1}{T}\int_0^T S(u)du. \tag{4.2}$$

Alternatively, $A(0,T)$ can be written in discrete form as

$$A(0,T) = \frac{1}{M+1}\sum_{i=1}^{M} S\left(i\frac{T}{m}\right). \tag{4.3}$$

This averaging feature is essentially what makes Asian option less prone to manipulation. Also, Asian options are path-dependent unlike standard options which are markov. The price of an Asian option at any time t, depends not only on the time t and its value $S(t)$ at time t but also the history of $S(t)$ up to time t. If we define an $\{\mathcal{F}(t)\}$-measurable process $Y(t)$ for $0 \leq t \leq T$ such that

$$Y(t) = \int_0^t S(u)du, \tag{4.4}$$

which satisfies the following stochastic differential equation given by

$$dY(t) = S(t)dt. \tag{4.5}$$

The pair $S(t)$ and $Y(t)$ form a 2-dimensional markov process, so that there exists some function $h(t,x,y)$ such that

$$h(t,S(t),Y(t)) = e^{-r(T-t)}\mathbb{E}\left[(A(0,T)-K)^{+}\,|\mathcal{F}(t)\right], \tag{4.6}$$

for $0 \leq t < T$. Furthermore, the price of an Asian option at an earlier time t prior to its maturity T is governed by the following partial differential equation which is given by

$$h_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y) \tag{4.7}$$

for $x \geq 0$ and $y \in \mathbb{R}$, subject to the following boundary conditions

$$h(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+, \tag{4.8}$$

$$\lim_{y \downarrow -\infty} v(t,x,y) = 0, \tag{4.9}$$

$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+ \tag{4.10}$$

for $0 \leq t < T$. This function $h(t, S(t), Y(t))$ can be estimated numerically. The price $C_A(0,T)$ of Asian call option at a time 0 which matures at a time T is given by

$$C_A(0,T) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S(u)du - K\right)^+\right], \tag{4.11}$$

where K is a non-negative strike K and its price $C_A(t,T)$ at an earlier time $t < T$ is given by

$$C_A(t,T) = e^{-r(T-t)}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S(u)du - K\right)^+ \middle| \mathcal{F}(t)\right]. \tag{4.12}$$

4.3 Closed-form Approximation Of Asian Options

Under the GBM framework, we can decompose the price of an asset $S(t)$ into a deterministic component and an exponential martingale which is lognormal, resulting in the lack of closed-form solution of an Asian on the asset $S(t)$. To overcome this, we can approximate the exponential martingale as a polynomial or linear combination of Gaussian random variables, which will allow us to price an Asian options within the Gaussian framework. In general, given a Brownian exponential martingale $Z(t)$, we can approximate the martingale $Z(t)$ using Hermite polynomials to obtain the identity given below,

$$\exp\left(\theta W(t) - \frac{1}{2}\theta^2 t\right) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} h_n(t, W(t)). \tag{4.13}$$

where $h_n(t, W(t))$ is the n-th order Hermite polynomial.

The first few Hermite polynomials are given by $h_0(t, W(t)) = 1, h_1(t, W(t)) = W(t)$ and $h_2(t, W(t)) = W^2(t) - t$. Furthermore, for each n we observe that $h_n(t, W(t))$ is a $\mathbb{Q}$ - martingale. We can obtain a closed-form Gaussian approximation of $A(0, T)$ by using the first order Hermite polynomial approximation for the exponential martingale identity in 4.13, and then estimate the price $A(0, T)$ of the Asian option.

Proposition 4.3.1. *The price of an Asian option has the approximation given by*

$$C_A(0, T) \approx e^{-rT} \left[(\bar{X} - K)\Phi\left(\frac{\bar{X} - K}{\sigma_X}\right) - \sigma_X \phi\left(\frac{K - \bar{X}}{\sigma_X}\right) \right],$$

where $\bar{X} = \frac{S(0)}{rT}\left(e^{rT} - 1\right)$ *and* $\sigma_X = \frac{\sigma S(0)}{rT}\left(\int_0^T (e^{rT} - e^{rs})^2 ds\right)^{\frac{1}{2}}$, *using the first order Hermite polynomial approximation for average value of an asset over the interval* $[0, T]$.

Proof.

$$A(0,T) = \frac{S(0)}{T}\int_0^T e^{ru}\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)du, \tag{4.14}$$

$$\approx \frac{S(0)}{T}\int_0^T e^{ru}\left(1+\sigma W(t)\right)du, \tag{4.15}$$

$$\approx \frac{S(0)}{T}\int_0^T e^{ru}\left[1+\sigma\int_0^u dW(s)\right]du, \tag{4.16}$$

$$= \frac{S(0)}{T}\left[\int_0^T e^{ru}du + \sigma\int_0^T\int_0^u e^{ru}dW(s)du\right]. \tag{4.17}$$

Re-arranging the double integral above we get,

$$A(0,T) \approx \frac{S(0)}{T}\left[\frac{e^{rT}-1}{r} + \sigma\int_0^T\int_s^T e^{ru}du\,dW(s)\right], \tag{4.18}$$

$$= \frac{S(0)}{T}\left[\frac{e^{rT}-1}{r} + \frac{\sigma}{r}\int_0^T (e^{rT}-e^{rs})dW(s)\right], \tag{4.19}$$

$$= A_H^1(0,T). \tag{4.20}$$

This first order Hermite approximation of $A(0,T)$ which we dub $A_H^1(0,T)$, is Gaussian with mean $\frac{S(0)}{rT}(e^{rT}-1)$ and its variance $\frac{\sigma^2 S^2(0)}{r^2T^2}\int_0^T (e^{rT}-e^{rs})^2 ds$, obtained by a direct application of the Wiener-Itô isometry. To obtain the approximate value of the Asian option, we define a standard normal random variable z such that

$$z = \frac{\displaystyle\int_0^T \left(e^{rT}-e^{rs}\right)dW(s)}{\sqrt{\left(\displaystyle\int_0^T (e^{rT}-e^{rs})^2 ds\right)}}, \tag{4.21}$$

and a random variable X given by

$$X = \bar{X} + \sigma_X z, \tag{4.22}$$

where $\bar{X}$ and σ_X are the mean of $A^1_H(0,T)$ and variance of $A^1_H(0,T)$ respectively. We can also define a critical value of z^* such that,

$$z \geq \frac{K - \bar{X}}{\sigma_X} = z^*. \tag{4.23}$$

Using equations 4.21-4.23, the approximate price of an Asian option price is given by

$$C_A(0,T) = e^{-rT}\mathbb{E}\left[(A(0,T) - K)^+\right], \tag{4.24}$$

$$\approx e^{-rT}\int_{z^*}^{\infty} \left(\bar{X} + \sigma_X z - K\right)\phi(z)dz, \tag{4.25}$$

$$= e^{-rT}\left[(\bar{X} - K)\Phi\left(\frac{\bar{X} - K}{\sigma_X}\right) - \sigma_X\phi\left(\frac{K - \bar{X}}{\sigma_X}\right)\right]. \tag{4.26}$$

□

An alternative approach to estimating the price of an Asian option, is by using first order Hermite polynomials method which we have previously used for pricing basket options, see section 3.3.1. We can think of Asian options as continuously monitored basket options by setting our ψ to be

$$\psi(u) = \log\left(\frac{1}{T}\int_0^T S(u)du\right), \tag{4.27}$$

and the coefficients b_0 and b_1 are given by 3.22 and 3.25 such that

$$\psi(u) = b_0 + b_1 u, \tag{4.28}$$

where b_0 is given as in 3.22 and b_1 is as given as in 3.25. Similar to the basket option case, we can estimate ψ using Taylor's series expansion of u around 0. We can estimate the price of the Asian option by

$$C_A(0,T) \approx e^{-rT}\mathbb{E}\left[\left(e^{\psi(u)} - K\right)^+\right].$$

whose solution is given by 3.33.

4.4 Bounds Estimation of Asian Option Price

Owing to the difficulty in pricing Asian options, numerous research has been carried out in finding suitable bounds on its prices (as in the works done by Curran in [8], Rogers-Shi in [35], Xu and Zhang in [39]). Trivially, we can find an upper bound an Asian option given by

$$(A(0,t) - K)^+ \leq A(0,t) \leq A(0,t) + K, \tag{4.29}$$

for some nonnegative strike K. In this section, we provide analytical bounds on the payoff of Asian options using the properties of Brownian motion in the underlying asset, which follows a geometric Brownian Motion (GBM) model. We shall later rely on some of these bounds as control variates for the pricing of Asian options using Monte Carlo simulations.

4.4.1 Direct Bounds on Asian Options

In this approach, we obtain a not so direct upper and lower bounds on the price of an Asian option using the unconditional and conditional version of the Jensen's inequality on the expected payoff of the Asian option price, which is due to the convexity of the payoff function.

Proposition 4.4.1. *The price $C_A(0,T)$ of an Asian option is bounded above by $C_S(0,T)$ which is the time average of options with different maturities over the interval $[0,T]$ and is given by*

$$\frac{1}{T}\int_0^T \Big(S_0\mathcal{N}(d_+) - e^{-rT}K\mathcal{N}(d_-)\Big)du. \tag{4.30}$$

The price $C_A(0,T)$ of an Asian option given by 4.11 has an upper bound given by

$$C_A(0,T) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T \Big(S(u) - K\Big)du\right)^+\right] \tag{4.31}$$

$$\leq e^{-rT}\frac{1}{T}\int_0^T \mathbb{E}\left[(S(u) - K)^+\right]du \tag{4.32}$$

$$= \frac{1}{T}\int_0^T \Big(S_0\mathcal{N}(d_+) - e^{-rT}K\mathcal{N}(d_-)\Big)du \tag{4.33}$$

where $d_{\pm} = \dfrac{\ln\left(S(0)e^{ru}/K\right) \pm \frac{1}{2}\sigma^2 u}{\sigma\sqrt{u}}$. This upper bound given by 4.33 which we shall call the average upper bound (AUB), will be used later on.

Furthermore, this upper bound $C_S(0,T)$ on the price of an Asian option is bounded above by the price at a time 0 of a European call option on underlying asset $S(t)$ is maturing at time T with the same strike K. To see this,

$$C_S(0,T) = e^{-rT}\frac{1}{T}\int_0^T \mathbb{E}\left[(S(u) - K)^+\right]du, \tag{4.34}$$

$$= e^{-rT}\frac{1}{T}\int_0^T \mathbb{E}\left[(e^{-r(T-u)}\mathbb{E}[S(T)|\mathcal{F}_u] - K)^+\right]du, \tag{4.35}$$

$$\leq e^{-rT}\frac{1}{T}\int_0^T \mathbb{E}\left[(\mathbb{E}[S(T)|\mathcal{F}_u] - K)^+\right]du, \tag{4.36}$$

$$= e^{-rT}\frac{1}{T}\int_0^T \mathbb{E}\left[(\mathbb{E}[S(T) - K|\mathcal{F}_u])^+\right]du, \tag{4.37}$$

$$\leq e^{-rT}\frac{1}{T}\int_0^T \mathbb{E}\left[(S(T) - K)^+\right]du, \tag{4.38}$$

$$= e^{-rT}\mathbb{E}\left[(S(T) - K)^+\right]. \tag{4.39}$$

The result 4.39 is generally known and provided for completeness.

Proposition 4.4.2. *The price of an Asian option is bounded below by $C_L(0,T)$, which is* *given by*

$$C_L(0,T) = \left(\frac{S(0)}{rT}\left(1 - e^{-rT}\right) - Ke^{-rT}\right)^+, \tag{4.40}$$

and is nontrivial provided that $S(0) > \dfrac{rTK}{e^{rT} - 1}$.

4.4.2 Distributional Bounds on Asian Options

Given the average value $A(0,t)$ of the asset $S(t)$ over the interval $[0,t]$, we seek to derive an upper bound on it using the distributional properties of the underlying noise process, and eventually an upper bound on the price of an Asian option. We can find an upper bound on $A(0,t)$ which we call $A'(0,t)$, using the properties of the Brownian motion $W(t)$ over the fixed interval $[0,t]$.

Proposition 4.4.3. *There is an upper bound* $A'(0,t)$ *on* $A(0,t)$ *given by*

$$A'(0,t) = \mu(t)\exp\left(\sigma M(t)\right),$$

where $\mu(t) = S(0)\left(\dfrac{\exp\left(r - \frac{1}{2}\sigma^2 t\right) - 1}{\left(r - \frac{1}{2}\sigma^2\right)t}\right)$, $M(t) = \sup\limits_{0\leq u\leq t} W(u)$, *and* $A'(0,t)$ *is real-valued provided* $r \neq \dfrac{1}{2}\sigma^2$ *and* $t > 0$.

Proof.

$$A(0,t) \leq \frac{1}{t}\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]\exp\left(\sigma \sup_{0\leq s\leq u} W(s)\right)du, \tag{4.41}$$

$$= \frac{1}{t}\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]\exp\left(\sigma M(u)\right)du, \tag{4.42}$$

$$\leq \frac{1}{t}\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]\exp\left(\sigma M(t)\right)du, \tag{4.43}$$

$$= \exp\left(\sigma M(t)\right)\frac{1}{t}\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]du, \tag{4.44}$$

$$= S(0)\exp\left(\sigma M(t)\right)\left[\frac{\exp\left(r-\frac{1}{2}\sigma^2\right)t-1}{\left(r-\frac{1}{2}\sigma^2\right)t}\right]. \tag{4.45}$$

$$\therefore A(0,t) \leq \mu(t)\exp\left(\sigma M(t)\right), \tag{4.46}$$

$$= A'(0,t). \tag{4.47}$$

□

Proposition 4.4.4. *The upper bound $A'(0,t)$ on the time average $A(0,t)$ of the asset $S(t)$ has the following distribution and density given by*

$$\mathbb{Q}\left(A'(0,t)\leq y\right) = 2\Phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)-1,\quad y>0,$$

and

$$\mathbb{Q}\left(A'(0,t)\in dy\right) = \mathbb{1}\{y>\mu(t)\}\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy,$$

respectively, provided $\sigma > 0$.

Similar to the distributional upper bound on the time-average value of an asset, we can also obtain lower bounds on it, using the distributional properties of a Brownian motion.

Proposition 4.4.5. *Given the time-average of an asset $A(0,t)$ over the interval $[0,t]$, we can obtain a lower bound $A_*(0,t)$ on $A(0,t)$ which is given by*

$$A_*(0,t) = S(0)\left(\frac{\exp\left(r - \frac{1}{2}\sigma^2\right)t - 1}{\left(r - \frac{1}{2}\sigma^2\right)t}\right)\exp\left(\sigma m(t)\right),$$

where $m(t) = \inf_{0\le u\le t} W(u)$.

Proof.

$$A(0,t) \ge \frac{1}{t}\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]\exp\left(\sigma \inf_{0\le s\le u} W(s)\right)du, \tag{4.48}$$

$$= \frac{1}{t}\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]\exp\left(\sigma m(u)\right)du, \tag{4.49}$$

$$\ge \frac{1}{t}\exp\left(\sigma m(t)\right)\int_0^t S(0)\exp\left[(r-\frac{1}{2}\sigma^2)u\right]du, \tag{4.50}$$

$$= \mu(t)\exp\left(\sigma m(t)\right), \tag{4.51}$$

$$= A_*(0,t), \tag{4.52}$$

where $\mu(t) = S(0)\left(\frac{\exp\left(r - \frac{1}{2}\sigma^2\right)t - 1}{\left(r - \frac{1}{2}\sigma^2\right)t}\right)$. □

Proposition 4.4.6. *The lower bound $A_*(0,t)$ on the average asset price $A(0,t)$ has the following distribution and density given by*

$$\mathbb{Q}(A_*(0,t) \le y) = 2\mathcal{N}\left(\min\left(0, \frac{1}{\sigma}\ln\left(\frac{y}{\mu(t)}\right)\right)\right),$$

and

$$\mathbb{Q}(A_*(0,t) \in dy) = \mathbb{1}\{y < \mu(t)\}\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy.$$

Having looked at the the distributional bounds on the price of an Asian option, as well as the density/distribution of these bounds. We then proceed to obtain the price of options on these bounds, which naturally will be bounds on the price of an Asian option, provided they have the same nonnegative strike K. That is

$$A_*(0,t) \leq A(0,t) \leq \tilde{A}(0,t),$$

implies

$$C_{A_*}(0,t) \leq C_A(0,t) \leq C_{A'}(0,t),$$

where $C_{A_*}(0,t)$ and $C_{A'}(0,t)$ are the the respective lower and upper bounds on the price of an Asian option $C_A(0,t)$. We can work out these bounds on the price of an Asian option since we know their densities.

Proposition 4.4.7. *The distributional upper bound on the price of an Asian option is given by*

$$C_{A'}(0,t) = 2e^{-rt}\left[\mu(t)\exp\left(\frac{1}{2}\sigma^2 t\right)\left(1-\mathcal{N}\left((-\sigma\sqrt{t})\vee(\upsilon-\sigma\sqrt{t})\right)\right) - K\left(1-\mathcal{N}(0\vee\upsilon)\right)\right],$$

where $\upsilon = \dfrac{1}{\sigma\sqrt{t}}\ln\left(\dfrac{y}{\mu(t)}\right)$.

Proof.

$$C_{A'}(0,t) = e^{-rt}\mathbb{E}\left[(\tilde{A}(0,t)-K)^+\right], \tag{4.53}$$

$$= e^{-rt}\int_{K\vee\mu(t)}^{\infty}(y-K)\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy. \tag{4.54}$$

We can decompose the above expression into I_1 and I_2 such that

$$I_1 = \int_{K\vee\mu(t)}^{\infty} y\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy, \tag{4.55}$$

and

$$I_2 = \int_{K\vee\mu(t)}^{\infty} K\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy. \tag{4.56}$$

We can simplify I_1 to obtain

$$I_1 = \int_{K\vee\mu(t)}^{\infty} \frac{1}{\sigma}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy \tag{4.57}$$

Setting $w = \dfrac{1}{\sigma\sqrt{t}}\ln\left(\dfrac{y}{\mu(t)}\right)$, and taking its derivative with respect y, I_1 becomes,

$$\begin{aligned}
I_1 &= \int_{0\vee\upsilon}^{\infty} \frac{2}{\sigma}\frac{1}{\sqrt{t}}\sigma\sqrt{t}\mu(t)\exp\left(w\sigma\sqrt{t}\right)\phi(w)dw, && (4.58)\\
&= 2\mu(t)\exp\left(\frac{1}{2}\sigma^2 t\right)\int_{0\vee\upsilon}^{\infty}\phi\left(w-\sigma\sqrt{t}\right)dw, && (4.59)\\
&= 2\mu(t)\exp\left(\frac{1}{2}\sigma^2 t\right)\left[1-\mathcal{N}\Big((-\sigma\sqrt{t})\vee(\upsilon-\sigma\sqrt{t})\Big)\right]. && (4.60)
\end{aligned}$$

To obtain I_2,

$$\begin{aligned}
I_2 &= \int_{K\vee\mu(t)}^{\infty} K\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy, && (4.61)\\
&= \frac{2K}{\sigma}\int_{K\vee\mu(t)}^{\infty}\frac{1}{y}\frac{1}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy. && (4.62)
\end{aligned}$$

Using $w = \dfrac{\ln(\frac{1}{\mu(t)}y)}{\sigma\sqrt{t}}$, and taking its derivative with respect to y I_2 becomes,

$$\begin{aligned}
I_2 &= \frac{2K}{\sigma}\int_{0\vee\upsilon}^{\infty}\frac{1}{y}\frac{1}{\sqrt{t}}\phi(w)\mu(t)\sigma\sqrt{t}\exp(w\sigma\sqrt{t})dy, && (4.63)\\
&= 2K\int_{0\vee\upsilon}^{\infty}\phi(w)dw, && (4.64)\\
&= 2K\left[1-\mathcal{N}(0\vee\upsilon)\right]. && (4.65)
\end{aligned}$$

Combining I_1 and I_2 and re-arranging them to obtain,

$$C_{A'}(0,t) = 2e^{-rt}\Bigg[\mu(t)\exp\left(\frac{1}{2}\sigma^2 t\right)\left[1-\mathcal{N}\Big((-\sigma\sqrt{t})\vee(\upsilon-\sigma\sqrt{t})\Big)\right] - K\left[1-\mathcal{N}\left(0\vee\upsilon\right)\right]\Bigg]. \tag{4.66}$$

□

Proposition 4.4.8. *The distributional lower bound on the price of an Asian option is given by*

$$C_{A_*}(0,t) = e^{-rt}\left[2\mu(t)\exp\left(\frac{1}{2}\sigma^2 T\right)\Big(\mathcal{N}(\mu(t)-\sigma\sqrt{t})-\mathcal{N}(K-\sigma\sqrt{t})\Big) - K\Big(1-2\mathcal{N}(\upsilon)\Big)\right],$$

where $\upsilon = \dfrac{1}{\sigma\sqrt{t}}\ln\left(\dfrac{y}{\mu(t)}\right)$.

Proof.

$$\begin{aligned}
C_{A_*}(0,t) &= e^{-rt}\mathbb{E}[\,(A_*(0,t)-K)^+\,], && (4.67)\\
&= e^{-rt}\int_{-\infty}^{\infty}\mathbb{1}\{y>K\}\mathbb{1}\{y<\mu(t)\}(y-K)\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy,\\
&= e^{-rt}\int_{-\infty}^{\infty}\mathbb{1}\{K<y<\mu(t)\}(y-K)\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy, && (4.68)\\
&= e^{-rt}\int_{K}^{\mu(t)}(y-K)\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy. && (4.69)
\end{aligned}$$

Decomposing the above expression for the lower bound on $A(0,t)$ to I_3 and I_4 such that,

$$I_3 = \int_{K}^{\mu(t)} y\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy, \tag{4.70}$$

and

$$I_4 = \int_{K}^{\mu(t)} K\frac{1}{\sigma y}\frac{2}{\sqrt{t}}\phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{y}{\mu(t)}\right)\right)dy. \tag{4.71}$$

Using $w = \dfrac{1}{\sigma\sqrt{t}} \ln\left(\dfrac{y}{\mu(t)}\right)$, and taking its derivative with respect to y, I_3 is simplified to obtain,

$$I_3 = 2\mu(t) \int_v^0 \exp\left(\frac{1}{2}\sigma^2 t\right) \phi(w - \sigma\sqrt{t}) dw. \tag{4.72}$$

Setting $\xi = w - \sigma\sqrt{t}$ and $d\xi = dw$, I_4 is simpified to obtain

$$I_4 = 2K\left(\frac{1}{2} - \mathcal{N}(v)\right). \tag{4.73}$$

Combining I_3 and I_4 and re-arranging them the required result follows. □

Given the respective lower and upper bounds, $A_*(0,t)$ and $A'(0,t)$ on $A(0,t)$ using the overall running maximum or minimum of a Brownian motion, we can also find more conservative bounds using the distributional properties of a Brownian motion satisfying the following inequality

$$A_*(0,t) \le \bar{A}(0,t) \le A(0,t) \le \tilde{A}(0,t) \le A'(0,t), \tag{4.74}$$

such that

$$\tilde{A}(0,t) = \frac{1}{t}\int_0^t \tilde{S}(u)du, \tag{4.75}$$

and

$$\bar{A}(0,t) = \frac{1}{t}\int_0^t \bar{S}(u)du, \tag{4.76}$$

where $\tilde{S}(u) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)u\right]\exp\left(\sigma M(u)\right)$ and $\bar{S}(u) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)u\right]\exp\left(\sigma m(u)\right)$. These bounds $\tilde{A}$ and $\bar{A}$ in 4.75 and 4.76 respectively are tighter, making them more ideal as suitable bounds and could potentially lead to better closed-form bounds on the price of an Asian option.

We begin by defining an $\{\mathcal{F}(t)\}$-adapted processes $\tilde{S}(t)$ and $\bar{S}(t)$ which we dub the fictitious assets which have the following representations

$$\tilde{S}(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)t\right]\exp\left(\sigma M(t)\right), \tag{4.77}$$

and

$$\bar{S}(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)t\right]\exp\left(\sigma m(t)\right). \tag{4.78}$$

Proposition 4.4.9. *Given the representations of $\tilde{S}(t)$ and $\bar{S}(t)$ in 4.77 and 4.78. $\tilde{S}(t)$ and $\bar{S}(t)$ have the following stochastic dynamics*

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \left(r + \frac{1}{2}\sigma^2\left(\mathbb{1}\{W(t) > M(t)\} - 1\right)\right)dt + \sigma\mathbb{1}\{W(t) > M(t)\}dW(t) + \frac{1}{2}\sigma dL_t^M, \tag{4.79}$$

and

$$\frac{d\bar{S}(t)}{\bar{S}(t)} = \left(r + \frac{1}{2}\sigma^2\left(\mathbb{1}\{W(t) < m(t)\} - 1\right)\right)dt + \sigma\mathbb{1}\{W(t) < m(t)\}dW(t) + \frac{1}{2}\sigma dL_t^m, \tag{4.80}$$

respectively.

Proof. By a direct application of Itô's lemma to 4.77 and 4.78 the required result follows. □

We can observe trivially that $\mathbb{Q}\left(\bar{S}(t) \leq S(t) \leq \tilde{S}(t)\right) = 1$, and the resulting fictitious Asian options from each of our fictitious assets is a bound on the actual Asian option price for the same strike K.

So evaluating the price of a fictitious Asian option $C_{\tilde{A}}(0,t)$ with the same strike K as an Asian option in 4.11, will always yield an upper bound on the price of an Asian option and vice-versa. The price of such a fictitious Asian option at a time 0, maturing at T is given by

$$C_{\tilde{A}}(0,T) = e^{-rT}\mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+\right]. \tag{4.81}$$

However, estimating $C_{\tilde{A}}(0,T)$ leaves us with a problem of a similar type to finding a closed-form solution of an Asian option $C_A(0,T)$. This is because, despite knowing the distribution of $M(t)$, the distribution of $\int e^{\sigma M(t)}dt$ is largely unknown for some $\sigma > 0$. To overcome this, we use Jensen's inequality to define a geometric lower bound $\tilde{G}(0,T)$ on $\tilde{A}(0,T)$ which is given by

$$\tilde{G}(0,T) = \exp\left(\frac{1}{T}\int_0^T \ln \tilde{S}(u)du\right), \tag{4.82}$$

or

$$\tilde{G}_n(0,T) = \left(\prod_{i=0}^{n-1} S(t_i)\right)^{\frac{1}{n}}, \tag{4.83}$$

where $S(t_i) = S(0)\exp\left[(r - \frac{1}{2}\sigma^2)t_i + \sigma M(t_i)\right]$ and satisfies the following relation

$$\tilde{G}(0,T) \leq \tilde{A}(0,T). \tag{4.84}$$

Proposition 4.4.10. *The estimated price of a fictitious Asian option $C_{\tilde{A}}(0,T)$ at a time 0, maturing at time T is given by the following*

$$\begin{aligned}
C_{\tilde{A}}(0,T) \approx{}& e^{-rT}\frac{\alpha_1}{2\pi}\int_{\varsigma^*}^{\infty} S(t_0)e^{y}\int_{-\infty}^{\infty} e^{-ipy}\int_{z_{n-1}}^{\infty} \ldots \int_0^{\infty} e^{ip(z_1+z_2+\ldots+z_n)}\prod_{k=1}^{n} q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) \\
&\times dz_k dp dy - Ke^{-rT}\frac{1}{2\pi}\int_{\varsigma*}^{\infty}\int_{-\infty}^{\infty} e^{-ipy}\int_{z_{n-1}}^{\infty}\ldots\int_0^{\infty} e^{ip(z_1+z_2+\ldots+z_n)} \\
&\times \prod_{k=1}^{n} q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,
\end{aligned} \tag{4.85}$$

where $\alpha_1 = S(0)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)\frac{\sum_{k=1}^{n-1} t_k}{n}\right)$, $\beta = \frac{\sigma}{n}$, and $\varsigma* = \frac{1}{\beta}\ln\left(\frac{K}{\alpha_1 S(t_0)}\right)$.

Proof. By definition,

$$\tilde{G}_n(0,T) = S(t_0)\left(\exp\left[(r-\frac{1}{2}\sigma^2)\frac{\sum_{k=0}^{n-1}t_k}{n}\right]\right)^{\frac{1}{n}}\exp\left(\frac{\sigma}{n}\sum_{k=0}^{n-1}(M(t_{k+1})-M(t_k))\right),$$
$$= \alpha_1 S(t_0)\exp\left(\sum_{k=0}^{n-1}c_k M(t_k)\right), \tag{4.86}$$

where $c_0 = 1-n$, $c_k = 1$ for $1 \leq k \leq n-1$ and α_1 and β are as defined above.

The price of the option on the geometric average $\tilde{G}(0,T)$ is given by

$$C_{\tilde{G}}(0,T) = \mathbb{E}\left[\left(\alpha_1 S(t_0)\exp\left(\sum_{k=0}^{n-1}c_k M(t_k)\right) - K\right)^+\right]. \tag{4.87}$$

Using the density of $\sum_{k=0}^{n-1}c_k M(t_k)$ we have derived in 2.83, we are able to obtain the required result. □

In general, it is not particularly clear if the estimate $C_{\tilde{G}}(0,T)$ for the price of the fictitious Asian option $C_{\tilde{A}}(0,T)$ is really an upper bound on the price of an Asian option.

Corollary 4.4.1. *A sufficient condition for the estimate $C_{\tilde{G}}(0,T)$ of the price of the fictitious Asian option $C_{\tilde{A}}(0,T)$ to be an upper bound on the price of an Asian option is*

$$1 + \ln\left(\frac{\tilde{S}(T)}{\tilde{G}(0,T)}\right) \geq \frac{S(T)}{A(0,T)}. \tag{4.88}$$

Proof. To prove this, we begin by assuming that $C_{\tilde{G}}(0,T) \geq C_A(0,T)$. This implies that for any time $0 \leq t \leq T$,

$$\tilde{G}(0,t) \geq A(0,t), \tag{4.89}$$

for any nonnegative strike K. By taking logs of the above inequality and then taking its derivative with respect to t, we obtain

$$-\frac{1}{t}\int_0^t \ln \tilde{S}(u)du + \ln \tilde{S}(t) \geq \frac{-A(0,t)+S(t)}{A(0,t)}, \tag{4.90}$$

$$= -1 + \frac{S(t)}{A(0,t)}. \tag{4.91}$$

Making further simplifications we can rewrite the above expression as

$$-\ln\left[\exp\left(\frac{1}{t}\int_0^t \ln \tilde{S}(u)du\right)\right] + \ln \tilde{S}(t) \geq -1 + \frac{S(t)}{A(0,t)}, \tag{4.92}$$

which then becomes,

$$-\ln \tilde{G}(0,t) + \ln \tilde{S}(t) \geq -1 + \frac{S(t)}{A(0,t)}. \tag{4.93}$$

We thus obtain

$$\ln \tilde{S}(t) - \ln \tilde{G}(0,t) \geq -1 + \frac{S(t)}{A(0,t)}, \tag{4.94}$$

which gives us the required result. □

Alternatively, we can derive the upper bound on the price of an Asian option using similar conditioning arguments as done by Curran in [8], to obtain the price of an Asian option. We can achieve this, by conditioning the price of a fictitious Asian option $C_{\tilde{A}}(0,t)$ on the geometric mean $\tilde{G}(0,t)$ of $\tilde{A}(0,t)$.

Proposition 4.4.11. *The price at a time* 0 *of the fictitious Asian option* $C_{\tilde{A}}(0,T)$*, maturing at time* T *with a nonnegative strike* K *using conditioning arguments on its geometric mean* $\tilde{G}(0,T)$ *is given by*

$$C_{\tilde{A}}(0,T) = e^{-rT}\mathbb{E}\left[\tilde{A}(0,T)\mathbb{1}\{\tilde{G}(0,T) \geq K\}\right] - Ke^{-rT}\frac{1}{2\pi}\int_{\Upsilon^*}^{\infty}\int_{-\infty}^{\infty}\int_{z_{n-1}}^{\infty}\cdots\int_0^{\infty} e^{ip(z_1+z_2+\ldots+z_n-y)}$$
$$\times \prod_{k=1}^{n} q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,$$

where $\Upsilon = \ln\left(\dfrac{K}{\alpha_1 S(t_0)}\right)$ *and* α_1 *is as defined in proposition 4.4.10.*

Proof.

$$C_{\tilde{A}}(0,T) = e^{-rT}\mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+\right], \tag{4.95}$$

$$= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+ \middle| \tilde{G}(0,T) = z\right]\right], \tag{4.96}$$

$$= e^{-rT}\int_0^K \mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+ \middle| \tilde{G}(0,T) = z\right]\mathbb{Q}\left(\tilde{G}(0,T) \in dz\right)$$

$$+ e^{-rT}\int_K^\infty \mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+ \middle| \tilde{G}(0,T) = z\right]\mathbb{Q}\left(\tilde{G}(0,T) \in dz\right). \tag{4.97}$$

We set I_1 to become

$$I_1 = e^{-rT}\int_0^K \mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+ \middle| \tilde{G}(0,T) = z\right]\mathbb{Q}\left(\tilde{G}(0,T) \in dz\right). \tag{4.98}$$

We can estimate I_1 using its geometric mean $\tilde{G}(0,T)$ and thus, yielding that $I_1 = 0$.

We set I_2 to become,

$$I_2 = e^{-rT}\int_K^\infty \mathbb{E}\left[\left(\tilde{A}(0,T) - K\right)^+ \middle| \tilde{G}(0,T) = z\right]\mathbb{Q}\left(\tilde{G}(0,T) \in dz\right). \tag{4.99}$$

Given $\tilde{G}(0,T) \geq K$, we can drop the max function so that I_2 becomes,

$$I_2 = e^{-rT}\int_K^\infty \mathbb{E}\left[\left(\tilde{A}(0,T) - K\right) \middle| \tilde{G}(0,T) = z\right]\mathbb{Q}\left(\tilde{G}(0,T) \in dz\right), \tag{4.100}$$

$$= e^{-rT}\mathbb{E}\left[\tilde{A}(0,T)\mathbb{1}\{\tilde{G}(0,T) \geq K\}\right] - Ke^{-rT}\mathbb{Q}\left(\tilde{G}(0,T) \geq K\right). \tag{4.101}$$

To complete the proof we need to show that

$$\mathbb{Q}\left(\tilde{G}(0,T) \geq K\right) = \frac{1}{2\pi}\int_{\Upsilon^*}^\infty \int_{-\infty}^\infty \int_{z_{n-1}}^\infty \cdots \int_0^\infty e^{ip(z_1+z_2+\ldots+z_n-y)}$$

$$\times \prod_{k=1}^n q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy.$$

Therefore,

$$\mathbb{Q}\left(\tilde{G}(0,T) \geq K\right) = \mathbb{Q}\left(\sum_{k=0}^{n-1} c_k M(t_k) \geq \ln\left(\frac{K}{\alpha_1 S(t_0)}\right)\right) \tag{4.102}$$

$$= \frac{1}{2\pi}\int_{\Upsilon^*}^{\infty}\int_{-\infty}^{\infty}\int_{z_{n-1}}^{\infty}\cdots\int_{0}^{\infty} e^{ip(z_1+z_2+\ldots+z_n-y)}$$

$$\times \prod_{k=1}^{n} q\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy. \tag{4.103}$$

Substituting the expression for $\mathbb{Q}\left(\tilde{G}(0,T) \geq K\right)$ into I_2, we are able to obtain the required result.

$\square$

Similar to the conservative upper bound $\tilde{A}(0,t)$ on the average value of an asset over the interval $[0,t]$, we can construct conservative lower bounds by assuming the existence of a fictitious asset $\bar{S}(t)$ whose value at any time t is given by

$$\bar{S}(u) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)u\right]\exp\left(\sigma m(u)\right), \tag{4.104}$$

where $m(t) = \inf_{0\leq s\leq t} \mathrm{W(s)}$. The time-average of the fictitious asset $\{\bar{S}(u)\}_{0\leq u\leq t}$ over the interval $[0,t]$ is given by

$$\bar{A}(0,t) = \frac{1}{t}\int_0^t \bar{S}(u)du. \tag{4.105}$$

Furthermore, $\bar{A}(0,t)$ satisfies the following inequality

$$A_*(0,t) \leq \bar{A}(0,t) \leq A(0,t), \tag{4.106}$$

which leads to tighter and possibly closed-form bounds on $A(0,t)$ than $A_*(0,t)$. The price at a time 0 of the fictitious Asian option whose underlying is the time-average of $\bar{S}(t)$, maturing at a time T is given by

$$C_{\bar{A}}(0,T) = e^{-rT}\mathbb{E}\left[\left(\bar{A}(0,T) - K\right)^{+}\right], \tag{4.107}$$

where K is the nonnegative strike price. Just as with the case of the fictitious Asian upper bound, we find that estimating 4.107 is difficult because the distribution of $\bar{A}$ is generally unknown. To this effect, we can estimate its price using the geometric lower bound $\bar{G}(0,T)$ on $\bar{A}(0,T)$ and this is given by the following approximation

$$\bar{G}_n(0,T) = \left(\prod_{k=0}^{n-1}\bar{S}(t_k)\right)^{\frac{1}{n}}, \tag{4.108}$$

where $\bar{S}(t_k) = S(t_0)\exp\left(\sigma m(t_k)\right)\exp\left((r - \frac{1}{2}\sigma^2)t_k\right)$ for $1 \le k \le n$.

This is useful because knowing the distribution of the underlying $\bar{S}(t)$ is sufficient to estimate the distribution of $\bar{G}(0,T)$. With this approximation in mind, we can then proceed to work out the price of the fictitious Asian option whose underlying is $\bar{S}(t)$.

Proposition 4.4.12. *The estimated price of a fictitious Asian option $C_{\bar{A}}(0,T)$ at a time 0, maturing at time T is given by the following*

$$\begin{aligned}C_{\bar{A}}(0,T) \approx\ & e^{-rT}\frac{\alpha_1}{2\pi}\int_{\varsigma^*}^{\infty} S(t_0)e^{y}\int_{-\infty}^{\infty} e^{-ipy}\int_{z_{n-1}}^{\infty}\cdots\int_{0}^{\infty} e^{ip(z_1+z_2+\ldots+z_n)}\prod_{k=1}^{n} q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right)\\ & \times dz_k dp dy - Ke^{-rT}\frac{1}{2\pi}\int_{\varsigma*}^{\infty}\int_{-\infty}^{\infty} e^{-ipy}\int_{z_{n-1}}^{\infty}\cdots\int_{0}^{\infty} e^{ip(z_1+z_2+\ldots+z_n)}\\ & \times \prod_{k=1}^{n} q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,\end{aligned}$$

where $\alpha_1 = S(0)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)\frac{\sum_{k=1}^{n-1} t_k}{n}\right)$, $\beta = \frac{\sigma}{n}$, *and* $\varsigma* = \frac{1}{\beta}\ln\left(\frac{K}{\alpha_1 S(t_0)}\right)$.

Proof. We can the rewrite $C_{\bar{G}}(0,T)$ as

$$C_{\bar{G}}(0,T) = e^{-rT}\int_K^\infty \left(\bar{G}_n(0,T) - K\right)\mathbb{Q}\left(\bar{G}_n(0,T) \in dy\right), \tag{4.109}$$

$$= e^{-rT}\int_K^\infty \left(\left(\prod_{k=0}^{n-1}\bar{S}(t_k)\right)^{\frac{1}{n}} - K\right)\mathbb{Q}\left(\bar{G}_n(0,T) \in dy\right), \tag{4.110}$$

$$\begin{aligned}
&= e^{-rT}\int_{\varsigma^*}^\infty \left(\alpha_1 S(t_0)\exp\left(\sum_{k=0}^{n-1} c_k m(t_k)\right) - K\right) \\
&\times \frac{1}{2\pi}\int_{\varsigma*}^\infty \int_{-\infty}^\infty e^{-ipy}\int_{z_{n-1}}^\infty \dots \int_0^\infty e^{ip(z_1+z_2+\dots+z_n)} \\
&\times \prod_{k=1}^n q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,
\end{aligned} \tag{4.111}$$

where $c_0 = 1 - n$, $c_k = 1$ for $1 \leq k \leq n-1$ and α_1 and β are as defined above.

□

We can also derive a lower bound on the price of an Asian option using similar conditioning arguments as previously carried out for fictitious upper bound $C_{\bar{A}}(0,T)$. We can achieve this by conditioning the price of average value $\bar{A}(0,T)$ of the fictitious asset on its geometric mean $\bar{G}(0,T)$.

Proposition 4.4.13. *The price at a time* 0 *of the fictitious Asian option on* $\bar{S}(t)$ *maturing at time* T *with a nonnegative strike* K *conditioned on its geometric mean is given by*

$$\begin{aligned}
C_{\bar{A}}(0,T) = e^{-rT}\mathbb{E}\left[\bar{A}(0,T)\mathbb{1}\{\bar{G}(0,T) \geq K\}\right] - Ke^{-rT}\frac{1}{2\pi}\int_\Upsilon^\infty \int_{-\infty}^\infty \int_{z_{n-1}}^\infty \dots \int_0^\infty e^{ip(z_1+z_2+\dots+z_n-y)} \\
\times \prod_{k=1}^n q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy,
\end{aligned}$$

where $\Upsilon = \ln\left(\dfrac{K}{\alpha_1 S(t_0)}\right)$.

Proof.

$$C_{\bar{A}}(0,T) = e^{-rT}\mathbb{E}\left[\left(\bar{A}(0,T)-K\right)^{+}\right], \tag{4.112}$$

$$= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[\left(\bar{A}(0,T)-K\right)^{+}\middle|\bar{G}(0,T)=z\right]\right], \tag{4.113}$$

$$= e^{-rT}\int_0^K \mathbb{E}\left[\left(\bar{A}(0,T)-K\right)^{+}\middle|\bar{G}(0,T)=z\right]\mathbb{Q}\left(\bar{G}(0,T)\in dz\right)$$

$$+\, e^{-rT}\int_K^\infty \mathbb{E}\left[\left(\bar{A}(0,T)-K\right)^{+}\middle|\bar{G}(0,T)=z\right]\mathbb{Q}\left(\bar{G}(0,T)\in dz\right). \tag{4.114}$$

We set I_1 to become

$$I_1 = e^{-rT}\int_0^K \mathbb{E}\left[\left(\bar{A}(0,T)-K\right)^{+}\middle|\bar{G}(0,T)=z\right]\mathbb{Q}\left(\bar{G}(0,T)\in dz\right). \tag{4.115}$$

We can estimate I_1 using its geometric mean $\bar{G}(0,T)$ and $I_1 = 0$.

We set I_2 to become,

$$I_2 = e^{-rT}\int_K^\infty \mathbb{E}\left[\left(\bar{A}(0,T)-K\right)^{+}\middle|\bar{G}(0,T)=z\right]\mathbb{Q}\left(\bar{G}(0,T)\in dz\right). \tag{4.116}$$

Given $\tilde{G}(0,T)\geq K$, we can drop the max function so that I_2 becomes,

$$I_2 = e^{-rT}\int_K^\infty \mathbb{E}\left[\left(\bar{A}(0,T)-K\right)\middle|\bar{G}(0,T)=z\right]\mathbb{Q}\left(\bar{G}(0,T)\in dz\right), \tag{4.117}$$

$$= e^{-rT}\mathbb{E}\left[\bar{A}(0,T)\mathbb{1}\{\bar{G}(0,T)\geq K\}\right] - Ke^{-rT}\mathbb{Q}\left(\bar{G}(0,T)\geq K\right). \tag{4.118}$$

Finally,

$$\mathbb{Q}\left(\bar{G}(0,T)\geq K\right) = \mathbb{Q}\left(\sum_{k=0}^{n-1} c_k m(t_k) \geq \ln\left(\frac{K}{\alpha_1 S(t_0)}\right)\right), \tag{4.119}$$

$$= \frac{1}{2\pi}\int_\Upsilon^\infty \int_{-\infty}^\infty \int_{z_{n-1}}^\infty \cdots \int_0^\infty e^{ip(z_1+z_2+\ldots+z_n-y)}$$

$$\times \prod_{k=1}^{n} q'\left(t_k - t_{k-1}, z_k - z_{k-1}\right) dz_k dp dy. \tag{4.120}$$

Using this we achieve the required results. □

4.5 Summary

In this chapter, we obtained closed-form distributional bounds $C_{A'}$ and C_{A_*} on the price of Asian option. These bounds are obtained by using the overall minimum and maximum value attained by the Brownian motion over a finite interval. Furthermore, these bounds can be improved by simply replacing the Brownian motion with the running minimum or maximum of a Brownian motion and these bounds can then be estimated in closed-form using their geometric averages. These bounds allow us to determine whether the Asian option will expire in or out-of-the-money. For example, if the upper bound expires out-of-the-money, then the Asian option will certainly expire out-of-the-money. Also, if the lower bound expires in-the-money, then the Asian option will expire in-the-money.

In the next chapter, we use some of our estimates and bounds on the price of an Asian option as control variates to obtain efficient Asian option prices.

Chapter 5

Control Variates For Basket and Asian Options

5.1 Overview

Monte Carlo methods remains the most accurate way of pricing basket options, but is inefficient due to the high variance of its estimates. In this chapter, our motivation is to develop a computational scheme for the efficient pricing of basket options using control variates. We begin by observing the computational scheme involved in the Monte Carlo approach for pricing basket and Asian options and the general theory behind control variates analysis. Finally, we present our computational schemes for the pricing of basket option using our control variates for variance reduction purposes to obtain efficient Monte Carlo estimates.

5.2 Monte Carlo Approach For Pricing Basket Options

Basket options are difficult to price because the weighted sum of the assets in the basket has no known distribution. Monte Carlo simulations are the most accurate way to price basket options. Consider a basket of assets, which follow a GBM model as in 3.2 subject to the aforementioned weighting constraints. Consider a basket option with n underlying assets in the basket, a strike K and the underlying follows a GBM model. To price the basket option, we use the following computational scheme.

Despite the high accuracy of the Monte Carlo approach, this method has the main drawbacks of being computationally intensive and high variance of its estimates. To overcome this, variance reduction techniques such as importance sampling, antithetic and control variates are employed, to provide efficient Monte Carlo estimates.

In the next section, we will review the general method of control variates in reducing variance of Monte Carlo estimates to produce efficient estimates of the price of a basket option.

5.3 Control Variate Algorithms For Basket Options

In this section, we look at the control variates we shall be using for pricing basket call option and specify their numerical schemes for their implementation as a control variate.

Algorithm 1: Monte Carlo simulation of basket option price

n: Number of assets in the basket

N: Number of Monte Carlo simulations

d: Number of independent Brownian motions

σ_i: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i^{(k)}(T)$: Simulation k of the asset i at time T, for $i = 1, ..., n$ and $k = 1, ..., N$

ω_k: Weighting of asset i

for $k = 1, ..., N$

for $i = 1, ..., n$

for $j = 1, ..., d$

Generate $Z_{ij} \sim \mathcal{N}(0, 1)$, $\forall i, j$

Set $S_i^{(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sum_{j=1}^{d} \sigma_{ij} Z_{ij} \sqrt{T}\right)$

Set $S^{(k)}(T) = \sum_{i=1}^{n} \omega_i S_i^{(k)}(T)$

Set $A_P^{(k)}(0) = e^{-rT} \max\left(S^{(k)}(T) - K, 0\right)$

Set $\hat{A}_P(0) = \frac{1}{N} \sum_{k=1}^{N} A_P^{(k)}(0)$

5.3.1 Geometric Lower Bound As a Control Variate For Basket Option Price

The value of a basket of assets is bound below by its Geometric mean, which can be written as

$$G(T) = \prod_{i=1}^{n} (S_i(T))^{\omega_i} \leq \sum_{i=1}^{n} \omega_i S_i(T) = S(T). \tag{5.1}$$

Gentle [13] proposed the use of the geometric mean of the value of the basket to approximate the basket option price. Also, such an approximate solution was possible in closed-form because the geometric mean of lognormal random variables is itself lognormal and thus has a known distribution. The approximate price $\hat{C_G}(0,T,K)$ of the basket option using the geometric mean is given by

$$\hat{C_G}(0,T,K) = \Theta\lambda \exp\left(\frac{1}{2}v\right) \Phi\left(\frac{v - \ln\left(\frac{K}{\Theta\lambda}\right)}{\sqrt{v}}\right) - K\Phi\left(\frac{-\ln\left(\frac{K}{\Theta\lambda}\right)}{\sqrt{v}}\right), \tag{5.2}$$

where $\Theta = \prod_{i=1}^{n} (S_i(0))^{\omega_i}$, $v = Var(\ln\Theta)$, $\lambda = \sum_{i=1}^{n} \omega_i \left(r - \frac{1}{2}\sigma_i^2\right) T$ and K is the strike price. We can use the payoff $\hat{C_G}(0,T,K)$ as a control variate to obtain an efficient estimate of the price of the basket option price. We specify the estimator C_G as

$$G_{CV} = e^{-rT}\left[A_P - (G_P - C_G)\right], \tag{5.3}$$

where G_{CV} is the Geometric control variate estimator, $A_P = \sum_i \omega_i \left(S_i(T) - K\right)^+$ and $G_P = (\prod_i (S_i(T))^{\omega_i} - K)^+$.

Algorithm 2: Monte Carlo Simulation of Basket Option Price Using Geometric lower bound as control variate

n: Number of assets in the basket

N: Number of Monte Carlo simulations

d: Number of independent Brownian motions

σ_i: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i^{(k)}(T)$: Simulation k of the asset i at time T, for $i = 1, ..., n$ and $k = 1, ..., N$

ω_k: Weighting of asset i

λ_G: Control variate parameter

for $k = 1, ..., N$

  for $i = 1, ..., n$

    for $j = 1, ..., d$

      Generate $Z_{ij} \sim \mathcal{N}(0, 1)$, $\forall i, j$

      Set $S_i^{(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sum_{j=1}^{d} \sigma_{ij} Z_{ij} \sqrt{T}\right)$

    Set $S^{(k)}(T) = \sum_{i=1}^{n} \omega_i S_i^{(k)}(T)$

    Set $C^{(k)}(0) = e^{-rT} \max\left(S^{(k)}(T) - K, 0\right)$

    Set $G^{(k)}(0) = \max\left(\prod_{i=1}^{n} S_i^{(k)}(T)^{\omega_i} - K, 0\right)$

Calculate $C(0) = \frac{1}{N} \sum_{k=1}^{N} C^{(k)}(0)$ and set $\lambda_G = \frac{\mathrm{Cov}(G^{(k)}(0), A_P^{(k)}(0))}{\mathrm{Var}(G)}$

Set $G_{CV}^{(k)} = A_P^{(k)}(0) - \lambda_G \left(G - \hat{C}_G^{(k)}\right)$ and calculate $G_{CV} = \frac{1}{N} G_{CV}^{(k)}$

5.3.2 Direct Upper Bound As a Control Variate For Basket Option Price

We specify the upper bound control variate U_{CV} as

$$U_{CV} = e^{-rT}\left[A_P - \left(U_B - \sum_{i=1}^{n} \omega_i \hat{C}_i\right)\right] \tag{5.4}$$

where $\hat{C}_i = (S_i(T) - K)^+$ is the random payout of an option on the underlying asset i and $U_B = \sum_{i=1}^{N} \mathbb{E}\left(S_i(T) - K\right)^+$.

5.3.3 First Order Hermite Polynomial As a Control Variate For Basket Option Price

First of all, we estimate the parameters $\hat{b}_0$ and $\hat{b}_1$ of the first order Hermite polynomial. We can then work out the closed-form lognormal approximation H_0 of the payoff of the basket option. We specify the Hermite polynomial Control variate H_{CV} as

$$H_{CV} = e^{-rT}\left[A_P - (H_P - C_H)\right], \tag{5.5}$$

where C_H is the closed-form Hermite approximation of a basket in 3.33, and $H_P^{(k)} = \Big(\exp\big(b_0 + \sum_{j=1}^{d} b_j u_j^{(k)}\big) - K\Big)^+$ and $u_j^{(i)} \sim \mathcal{N}(0,1)\ \forall j$.

Algorithm 3: Monte Carlo Simulation of Basket Option Price Using Direct Upper Bound As a Control Variate

n: Number of assets in the basket

N: Number of Monte Carlo simulations

d: Number of independent Brownian motions

σ_i: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i^{(k)}(T)$: Simulation k of the asset i at time T, for $i = 1, ..., n$ and $k = 1, ..., N$

ω_k: Weighting of asset i

λ_U: Control variate parameter for the direct upper bound

for $k = 1, ..., N$

for $i = 1, ..., n$

for $j = 1, ..., d$

Generate $Z_{ij} \sim \mathcal{N}(0, 1)$, $\forall i, j$

Set $S_i^{(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sum_{j=1}^{d} \sigma_{ij} Z_{ij} \sqrt{T}\right)$

Set $S^{(k)}(T) = \sum_{i=1}^{n} \omega_i S_i^{(k)}(T)$ and $\hat{C}^{(k)}(0) = \sum_{i=1}^{n} \omega_i \max\left(S^{(k)}(T) - K, 0\right)$

Set $\lambda_U = \dfrac{\mathrm{Cov}(A_P^{(k)}(0), \hat{C}^{(k)}(0))}{\mathrm{Var}(\hat{C}^{(k)}(0))}$ and $U_{CV}^{(k)} = e^{-rT}\left[A_P^{(k)}(0) - \lambda_U\left(U_B - \hat{C}^{(k)}(0)\right)\right]$

Calculate $U_{CV} = \dfrac{1}{N} U_{CV}^{(k)}$

Algorithm 4: Monte Carlo Simulation of Basket Option Price Using First order Hermite polynomial as control variate

n: Number of assets in the basket

N: Number of Monte Carlo simulations

d: Number of independent Brownian motions

σ_i: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i^{(k)}(T)$: Simulation k of the asset i at time T, for $i = 1, ..., n$ and $k = 1, ..., N$

ω_k: Weighting of asset i

λ_H: Control variate parameter for the Hermite polynomial approximation

for $k = 1, ..., N$

  for $i = 1, ..., n$

    for $j = 1, ..., d$

      Generate $Z_{ij} \sim \mathcal{N}(0,1)$, $\forall i, j$

      Set $S_i^{(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sum_{j=1}^{d} \sigma_{ij} Z_{ij} \sqrt{T}\right)$

    Set $S^{(k)}(T) = \sum_{i=1}^{n} \omega_i S_i^{(k)}(T)$ and $A_P^{(k)}(0) = \max\left(S^{(k)}(T) - K, 0\right)$

  Set $\lambda_H = \dfrac{\mathrm{Cov}(C(0), H_P(0))}{\mathrm{Var}(H_P(0))}$ and set $H_P^{(k)}$ as defined above

  Set $H_{CV}^{(k)} = e^{-rT}\left[A_P^{(k)}(0) - \lambda_H \left(C_H - H_P^{(k)}\right)\right]$

Calculate $H_{CV} = \dfrac{1}{N} H_{CV}^{(k)}$

5.3.4 Maximum of a Brownian Motion As a Control Variate For Basket Option Price

For the purpose of the maximum of Brownian motion control variate estimate, we replace U_1^n with U_1, which is given by

$$\widehat{U_1} \approx e^{-rT}\left[\sum_{i=1}^{N} Y_i \exp\left(\frac{1}{2}\sigma_M^2 T\right)\Phi\left(\frac{\sigma_M T-\xi}{\sqrt{T}}\right)-\frac{K}{2}\left(1-\Phi^2\left(\frac{\xi}{\sqrt{T}}\right)\right)\right] \tag{5.6}$$

where we have approximated U_1 by using the fact that $\mathcal{N}(y) \leq 1$, for any $y \in \mathbb{R}$. We specify the maximum of Brownian motion control variate estimator by

$$UBM_{CV} = e^{-rT}\left[A_P - \lambda_{UBM}\left(\widehat{U_1} - U_1^{(i)}\right)\right] \tag{5.7}$$

where $\widehat{U_1}$ is as defined in (5.6) and $\lambda_{UBM} = \dfrac{cov(A_P, U_1)}{var(U_1)}$.

5.3.5 Minimum of a Brownian Motion Distributional Bounds As a Control Variate

Similar to case of the maximum of a Brownian motion as a control variate, we approximate the distributional lower bound L_1^n in 3.112 with $\widehat{L_1}$ which is given by

$$\widehat{L_1} \approx e^{-rT}\left[\sum_{i=1}^{N} Y_i \exp\left(\frac{1}{2}\sigma_m^2 T\right)\Phi\left(\frac{\sigma_m T-\tau}{\sqrt{T}}\right)+\frac{K}{2}\left(1-\Phi^2\left(\frac{\xi}{\sqrt{T}}\right)\right)\right]. \tag{5.8}$$

We specify the control variate for the distributional lower bound using the minimum of a Brownian motion as

$$LBM_{CV} = e^{-rT}\left[A_P - \lambda_{LBM}(\widehat{L_1} - L_1^{(i)})\right] \tag{5.9}$$

where $\lambda_{LBM} = \dfrac{cov(A_P, L_1)}{var(L_1)}$.

Algorithm 5: Monte Carlo Simulation of Basket Option Price Using Maximum of Brownian motion as control variate

n: Number of assets in the basket

N: Number of Monte Carlo simulations

d: Number of independent Brownian motions

σ_i: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i^{(k)}(T)$: Simulation k of the asset i at time T, for $i = 1, ..., n$ and $k = 1, ..., N$

ω_k: Weighting of asset i

λ_{UBM}: Control variate parameter for the distributional upper bound

for $k = 1, ..., N$

for $i = 1, ..., n$

for $j = 1, ..., d$

Generate $Z_{ij} \sim \mathcal{N}(0, 1)$, $\forall i, j$

Set $S_i^{(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sum_{j=1}^{d} \sigma_{ij} Z_{ij} \sqrt{T}\right)$

Set $S_i^{*(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sigma_M |Z_{ij}| \sqrt{T}\right)$

Set $S^{(k)}(T) = \sum_{i=1}^{n} \omega_i S_i^{(k)}(T)$ and $A_P^{(k)}(0) = \max\left(S^{(k)}(T) - K, 0\right)$

Set $\sigma_M = \max \sum_{j=1}^{d} \sigma_{ij}$

Set $U_1^{(k)}(0) = \max\left(\sum_{i=1}^{n} \omega_i S_i^{(k)}(T) - K, 0\right)$ and $\lambda_{UBM} = \dfrac{\text{Cov}(A_P(0), U_1(0))}{\text{Var}(U_1)}$

Set $UBM_{CV}^{(k)} = e^{-rT}\left[A_P^{(k)}(0) - \lambda_{UBM}\left(\hat{U}_1 - U_1^{(k)}(0)\right)\right]$

Calculate $UBM_{CV} = \dfrac{1}{N} UBM_{CV}^{(k)}$

Algorithm 6: Monte Carlo Simulation of Basket Option Price Using Minimum of Brownian motion As a Control Variate

n: Number of assets in the basket

N: Number of Monte Carlo simulations

d: Number of independent Brownian motions

σ_i: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i^{(k)}(T)$: Simulation k of the asset i at time T, for $i = 1, ..., n$ and $k = 1, ..., N$

ω_k: Weighting of asset i

λ_{LBM}: Control variate parameter for the distributional lower bound

for $k = 1, ..., N$

for $i = 1, ..., n$

for $j = 1, ..., d$

Generate $Z_{ij} \sim \mathcal{N}(0, 1)$, $\forall i, j$

Set $S_i^{(k)}(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sum_{j=1}^{d} \sigma_{ij} Z_{ij} \sqrt{T}\right)$

Set $S'^{(k)}_i(T) = S_i(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T - \sigma_M |Z_{ij}| \sqrt{T}\right)$

Set $S^{(k)}(T) = \sum_{i=1}^{n} \omega_i S_i^{(k)}(T)$ and $A_P^{(k)}(0) = \max\left(S^{(k)}(T) - K, 0\right)$

Set $L_1^{(k)}(0) = \max\left(\sum_{i=1}^{n} \omega_i S_i^{(k)}(T) - K, 0\right)$

$\sigma_m = \min \sum_{j=1}^{d} \sigma_{ij}$ and $\lambda_{LBM} = \dfrac{\mathrm{Cov}(A_P(0), L_1(0))}{\mathrm{Var}(L_1(0))}$

Set $LBM_{CV}^{(k)} = e^{-rT}\left[A_P^{(k)}(0) - \lambda_{LBM}\left(\hat{L}_1 - L_1^{(k)}(0)\right)\right]$

Calculate $LBM_{CV} = \dfrac{1}{N} LBM_{CV}^{(k)}$

5.4 Monte Carlo Approach For Pricing Asian Options

Given that Asian options are difficult to price analytically because they do not follow any known distribution, Monte Carlo approach remains the most accurate way of pricing them. In this section, we develop computational schemes for achieving efficient Asian option prices using control variates.

Algorithm 7: Monte Carlo Simulation of Asian Option Price

$S(0)$: Initial value of the asset

M: Number of partitions of the time interval $[0, T]$

N: Number of Monte Carlo simulations

σ: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i(t_j)$: Simulation i of the asset at time t_j, for $i = 1, ..., N$ and $j = 1, ..., M$

Δ: partition size $t_j - t_{j-1}$

$A_i(T)$: Average of the asset $S(t)$ over the interval $[0, T]$ for simulation i

for $i = 1, ...N$

  for $j = 1, ..., M$

    Generate $Z_{ij} \sim \mathcal{N}(0, 1)$

    $S_i(t_j) = S(t_{j-1}) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t_j + \sigma Z_{ij}\sqrt{\Delta}\right)$

  Set $A_i(T) = \displaystyle\sum_{j=0}^{M} \frac{S(t_j)}{M+1}$

Calculate $A(T) = e^{-rT} \dfrac{1}{N} \sum_{i=1}^{N} A_i(T)$

The Monte Carlo approach despite being accurate is undesirable, which is due to high variance of the Monte Carlo estimates. In the next section we look at numerical schemes for computing efficient prices of an Asian option using control variates.

5.5 Control Variates For Pricing Asian Options

We present our choice of control variates which we will employ for variance reduction purposes of the Monte Carlo estimates. These control variates are the geometric lower bound, average upper bound and the distributional lower and upper bounds.

5.5.1 Geometric Lower Bound As a Control Variate

In general, the geometric average of a sequence is less than or equal to its arithmetic average. This principle also applies to Asian options. This approach was first used by [20], noting that though it gives a lower bound on the price of an Asian option but the value is close enough to be an approximation on it. Using $G(0,T)$ to denote the geometric average continuously sampled over the interval $[0,T]$ satisfying,

$$G(0,T) \leq A(0,T), \tag{5.10}$$

where $A(0,T)$ is as previously defined. We define $G(0,T)$ as

$$G(0,T) = \exp\left[\frac{1}{T}\int_0^T \ln S(t)dt\right] \tag{5.11}$$

. To see that $G(0,T)$ is lognormal, we see that

$$\int_0^T \ln S(t)dt = (\ln S(0))T + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T^2 + \sigma\int_0^T W(t)dt, \tag{5.12}$$

$$= (\ln S(0))T + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T^2 + \sigma\int_0^T (T-u)dW(u). \tag{5.13}$$

$G(0,T)$ is lognormal and the price $C_G(0,T)$ of an option on $G(0,T)$ at a time 0 is given by,

$$C_G(0,T) = e^{-rT}\mathbb{E}\left[(G(0,T)-K)^+\right], \tag{5.14}$$

$$= e^{-rT}\int_{G(0,T)\geq K}(G_T - K)\frac{1}{\sqrt{\frac{2}{3}\pi\sigma^2 T}}\exp -\frac{1}{2}\frac{3y^2}{\sigma^2 T}dy, \tag{5.15}$$

$$= e^{-rT}\int_{y^*}^{\infty}\left(\exp\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T + y\right) - K\right)$$

$$\times \frac{1}{\sqrt{\frac{2}{3}\pi\sigma^2 T}}\exp -\frac{1}{2}\frac{3y^2}{\sigma^2 T}dy, \tag{5.16}$$

$$= S(0)\exp\left[(b-r)T\right]\Phi(d_1) - Ke^{-rT}\Phi(d_2). \tag{5.17}$$

where $y^* = \ln\left(\frac{K}{S(0)}\right) - \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T$, $d_1 = \frac{\ln(S(0)/K) + (b + \frac{1}{2}\sigma_G^2)T}{\sigma_G\sqrt{T}}$, $d_2 = d_1 - \sigma_G\sqrt{T}$, $\sigma_G = \frac{\sigma}{\sqrt{3}}$ and $b = \frac{1}{2}\left(r - \frac{1}{2}\sigma_G^2\right)$. This proof is available in books and is shown merely for completeness.

The numerical scheme for the geometric lower bound as a control variate is as follows:

5.5.2 Average Upper Bound As a Control Variate

Direct upper bound method allows us to use the fact that we can bound the Asian option price above using the price of a standard call option on the Asian options underlying with the same strike. The numerical scheme for the direct upper bound as a control variate is as outlined in algorithm 9.

Algorithm 8: Monte Carlo Simulation of Asian Option Using Geometric Lower Bound as a Control Variate.

$S(0)$: Initial value of the asset

M: Number of partitions of the time interval $[0, T]$

N: Number of Monte Carlo simulations

σ: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i(t_j)$: Simulation i of the asset at time t_j, for $i = 1, ..., N$ and $j = 1, ..., M$

Δ: partition size $t_j - t_{j-1}$

$A_i(T)$: Average of the asset $S(t)$ over the interval $[0, T]$ for simulation i

$G_i(T)$: is the geometric average of $A_i(T)$ for simulation i

for $i = 1, ...N$

  for $j = 1, ..., M$

    Generate $Z_{ij} \sim \mathcal{N}(0, 1)$

    $S_i(t_j) = S(t_{j-1}) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t_j + \sigma Z_{ij}\sqrt{\Delta}\right)$

  Set $G_i(T) = \exp\left[\ln S(0) + \left(r - \frac{1}{2}\sigma^2\right) T + \frac{2}{3}\sigma\sqrt{\Delta}\tilde{Z}_{iM}\right]$

  Set $A(T) = \dfrac{1}{N} \sum_{i=1}^{N} \displaystyle\sum_{j=0}^{M} \dfrac{S(t_j)}{M+1}$, $C_A^i = (A_i(T) - K)^+$ and $\hat{C}_G^i = (G_i(T) - K)^+$

  Set and $\lambda_G = \dfrac{\mathrm{Cov}(C_A, C_G)}{\mathrm{Var}(C_G)}$ and $G_i^{CV} = C_A^i + \lambda_G(C_G - C_G^i)$

Calculate $G^{CV} = \frac{1}{N} \sum_{i=1}^{N} G_i^{CV}$

Algorithm 9: Monte Carlo Simulation of Asian Option Using Average Upper Bound as a Control Variate.

$S(0)$: Initial value of the asset

M: Number of partitions of the time interval $[0, T]$

N: Number of Monte Carlo simulations

σ: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i(t_j)$: Simulation i of the asset at time t_j, for $i = 1, ..., N$ and $j = 1, ..., M$

Δ: partition size $t_j - t_{j-1}$

$A_i(T)$: Average of the asset $S(t)$ over the interval $[0, T]$ for simulation i

for $i = 1, ...N$

  for $j = 1, ..., M$

    Generate $Z_{ij} \sim \mathcal{N}(0, 1)$

    $S_i(t_j) = S(t_{j-1}) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t_j + \sigma Z_{ij}\sqrt{\Delta}\right)$

  Set $C_S^i(T) = \frac{1}{M+1} \sum_{j=0}^{M} (S^i(T) - K)^+$

Set $A(T) = \dfrac{1}{N} \sum_{i=1}^{N} \displaystyle\sum_{j=0}^{M} \dfrac{S(t_j)}{M+1}$, $C_A^i = e^{-rT}(A_i(T) - K)^+$ and C_S is as defined in 4.33

  Set $\lambda_S = \dfrac{\text{Cov}(\hat{C}_S, C_A)}{\text{Var}(\hat{C}_S)}$ and $G_i^{CV} = C_A^i - \lambda_G(C_S - C_S^i)$

Calculate $C_S^{CV} = \frac{1}{N} \sum_{i=1}^{N} G_i^{CV}$

5.5.3 Distributional Upper Bound As a Control Variate

The numerical scheme for the distributional upper bound is outlined algorithm 10

5.5.4 Distributional Lower Bound As a Control Variate

The numerical scheme for the distributional lower bound is outlined algorithm 11.

5.6 Summary

Given that Monte Carlo methods are the most accurate way of pricing basket and Asian options, variance reduction methods i.e. control variates are used to obtain efficient estimates of basket option prices. Control variate methods involve using a random variable whose distribution is similar to the basket or average value of the asset and admits a closed-form solution, to reduce the variance of the option. However, for optimal variance reduction to be achieved we specify a critical value of λ. The value of λ varies for different control variate methods, but is generally close to 1 for assets following a GBM model [11]. We specified the numerical schemes for computing Monte Carlo and control variate estimates for the price of a basket/Asian option. In the next chapter, we will simulate the price of a basket/Asian option using our numerical schemes and illustrate the accuracy and efficiency of our control variates.

Algorithm 10: Monte Carlo Simulation of Asian Option Using Distributional Upper Bound as a Control Variate.

$S(0)$: Initial value of the asset

M: Number of partitions of the time interval $[0, T]$

N: Number of Monte Carlo simulations

σ: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i(t_j)$: Simulation i of the asset at time t_j, for $i = 1, ..., N$ and $j = 1, ..., M$

Δ: partition size $t_j - t_{j-1}$

$A_i(T)$: Average of the asset $S(t)$ over the interval $[0, T]$ for simulation i

for $i = 1, ...N$

for $j = 1, ..., M$

Generate $Z_{ij} \sim \mathcal{N}(0, 1)$

$S_i(t_j) = S(t_{j-1}) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t_j + \sigma Z_{ij}\sqrt{\Delta}\right)$

Set $A_u(i) = \dfrac{S_0 \exp\left(\sigma\sqrt{T}|Z_{ij}|\right)}{\left(r - \frac{1}{2}\sigma^2\right) T} \left[\exp\left((r - \frac{1}{2}\sigma^2)T\right) - 1\right]$

Set $\hat{C}^i_{\tilde{A}} = e^{-rT}(A_u(i) - K)^+$ and $\lambda_{AU} = \dfrac{\mathrm{Cov}(\hat{C}_A, \hat{C}_{\tilde{A}})}{\mathrm{Var}(\hat{C}_{\tilde{A}})}$

Set $C^{CV}_i = C^i_A - \lambda_{AU}(C^i_{\tilde{A}} - C_{\tilde{A}})$

Calculate $\hat{C}_{CV} = \dfrac{1}{N} \sum_{i=1}^{N} C^{CV}_i$

Algorithm 11: Monte Carlo Simulation of Asian Option Using Distributional Lower Bound as a Control Variate.

$S(0)$: Initial value of the asset

M: Number of partitions of the time interval $[0, T]$

N: Number of Monte Carlo simulations

σ: Volatility of asset i

T: Maturity of the basket option

r: Risk-free rate

$S_i(t_j)$: Simulation i of the asset at time t_j, for $i = 1, ..., N$ and $j = 1, ..., M$

Δ: partition size $t_j - t_{j-1}$

$A_i(T)$: Average of the asset $S(t)$ over the interval $[0, T]$ for simulation i

for $i = 1, ...N$

  for $j = 1, ..., M$

    Generate $Z_{ij} \sim \mathcal{N}(0, 1)$

    $S_i(t_j) = S(t_{j-1}) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t_j + \sigma Z_{ij}\sqrt{\Delta}\right)$

  Set $A_l(i) = \dfrac{S_0 \exp\left(-\sigma\sqrt{T}|Z_{ij}|\right)}{\left(r - \frac{1}{2}\sigma^2\right) T} \left[\exp\left((r - \frac{1}{2}\sigma^2)T\right) - 1\right]$

  Set $\hat{C}^i_{\bar{A}} = e^{-rT}(A_l(i) - K)^+$ and $\lambda_{AL} = \dfrac{\mathrm{Cov}(\hat{C}_A, \hat{C}_{\bar{A}})}{\mathrm{Var}(\hat{C}_{\bar{A}})}$

  Set ${C'}^{CV}_i = C^i_A - \lambda_{AL}(C^i_{\bar{A}} - C_{\bar{A}})$

Calculate $\hat{C'}_{CV} = \dfrac{1}{N} \sum_{i=1}^N {C'}^{CV}_i$

Chapter 6

Numerical Experiments and Results For Basket and Asian Options

6.1 Overview

In this chapter, we carry out numerical experiments to determine the efficiency of our control variates using Monte Carlo methods for basket and Asian call options. To obtain the basket/Asian option price using plain Monte Carlo, we use 10^7 simulations and this price is used as the benchmark.More specifically, for Asian options we use $N = 10^4$ sample paths and $M = 1000$ time steps while for basket options we have 10^7 sample paths. All computations have been carried out on matlab using an Apple M1 Pro MacBook Pro 2021 with 16GB of unified memory, and 8-core CPU (with 6 performance cores and 2 efficiency cores) and 14-core GPU.

For basket call options, simulations are carried out using two and five-assets basket for all our control variates i.e direct upper bound (DUB), first order Hermite polynomial and distributional bounds, which are benchmarked against the geometric lower bound (LB) control variate. We then compare the results obtained with the variance reductions as well as their computation times. For pricing Asian call options the control variates employed are the first order Hermite polynomial approximation, the geometric lower bound (GLB), the average of European options for different maturities (AUB) and a basket of assets with the same maturity T and strike K, but with variable initial values and times to maturity which we shall call the future-valued basket (FVB). To see the future-valued approach, we consider a finite partition of the time interval $[0, T]$ in to m intervals. We begin with an asset whose initial value is $S(0)$ at time 0, and its values $S(t_j)$ at the time intervals t_j such that $t_j = j\dfrac{T}{m}$, is $S(0)e^{rt_j}$ and work out the price of a European call option on each of these constructed assets maturing at the same time T. We can form a portfolio of assets whose payoff $V(T, T)$ at a time T is the average of these assets and use this as a control variate for pricing Asian options which is given by

$$V(T, T) = \frac{1}{m+1} \sum_{i=1}^{M+1} (S(t_j, T) - K)^+ , \tag{6.1}$$

where $S(t_j, T)$ is the price at time T of an asset whose initial value is $S(t_j)$.

6.2 Numerical Results For Basket Options

We use 522 observations of daily prices from January 1, 2018 to December 31, 2019 of five market indices namely FTSE 100, FTSE 250, S&P 500, NIKKEI 225 and IMOEX (from Thomson Reuters Datastream) to obtain daily volatility and covariance estimates.

	FTSE 100	**FTSE 250**	**S&P 500**	**NIKKEI 225**	**IMOEX**
FTSE 100	7.18	6.90	3.48	1.86	4.40
FTSE 250	6.90	9.29	3.67	2.58	3.75
S&P 500	3.48	3.67	8.60	7.00	3.89
NIKKEI 225	1.86	2.58	7.00	9.57	1.57
IMOEX	4.40	3.75	3.89	1.57	16.31

Table 6.1: Daily covariance estimates of market indices ($\times 10^{-5}$).

These values of covariances were used to generate simulated basket option prices, for various values of maturity T and strike K.

6.2.1 Lognormal Approximations using Hermite Polynomial

To obtain the closed-form approximations using Hermite polynomials, we estimate $\psi(u)$ in 3.14 using Taylor series expansions of u about 0. Using third order Taylor series and the covariance results of the FTSE 100 and FTSE 250, we estimate the Hermite polynomials coefficients in 3.22 and 3.25 to obtain the closed-form approximation of the two-asset basket option for different maturities T and across different positions of moneyness. We also estimate the absolute error percentage in the price of the Hermite polynomial method and this is calculated as

$$\%AbsError = \frac{|HPPrice - MCPrice|}{MCPrice}, \tag{6.2}$$

where $MCPrice$ is the price estimate obtained using Monte Carlo simulations while $HPPrice$ is the first order Hermite polynomial estimate for the price of a basket option as given by 3.33. To improve the accuracy of basket option price, we make use of a modified strike $\hat{K}$ as suggested by [13]. The idea is to pick a strike whose expected difference from our first order Hermite polynomial price estimate is similar to that of the expected difference of the forward price and strike of the basket. This modified strike $\hat{K}$ is given by

$$\hat{K} = e^{\mathbb{E}\psi(u)} - S(0)e^{rT} + K, \tag{6.3}$$

where $\psi(u)$ is as defined in 3.19, with all parameters suitably calibrated using third order Taylor series. The Monte Carlo price of a basket option is obtained using a sample of 10^7 simulations.

Table 6.2 shows the estimated price of a two-asset basket option using Hermite polynomials for a variety of maturity periods and strikes and the corresponding errors in the estimates. Next, we look at the result from our Monte Carlo simulation for basket options.

6.2.2 Two-asset Basket Case

In this section, we estimate the price of a two-asset basket call option using control variates in which we compare the antithetic Monte Carlo (MC) estimate with different control variates (such as the geometric lower bound (LB), Hermite polnomial (HP) and direct upper bound (UB)) for different criteria. The prices are estimated using daily volatility estimates of

T	K	HP Price	MC Price	% Abs Error	CI L	CI U
	60	20.799	20.361	0.438	20.342	20.385
0.5	80	6.740	4.648	2.091	4.629	4.657
	100	1.348	0.315	1.033	0.309	0.319
	60	21.046	20.987	0.059	20.956	21.017
1	80	6.859	6.658	0.201	6.637	6.679
	100	1.362	1.257	0.105	1.248	1.267
	60	21.545	22.481	0.936	22.431	22.522
2	80	7.108	9.591	2.483	9.567	9.629
	100	1.416	3.345	1.929	3.326	3.365

Table 6.2: Hermite polynomial approximation for a two-asset basket option.

FTSE 100 and FTSE 250. For the case of two-assets in a basket, we simulate the price of the basket option for different positions of moneyness K and maturities T using daily volatilities obtained from the real world market indices of FTSE 100 and FTSE 250.

In table 6.2, we look at the results from the Hermite polynomial closed-form approximation for a basket option of two-assets for different maturities as well as strikes. We observe that the accuracy is higher when the basket of assets are in-the-money and out-of-the-money across the different maturity times. Also, for a fixed maturity T, the basket option price is a decreasing function of the strike K and for a fixed strike K, the price of the basket is a decreasing function of the maturity T.

In tables 6.3, 6.4 and 6.5, we compare the price, variance and computation times of our Monte

Carlo estimate of a two-asset basket option with the results obtained using the Hermite polynomial closed-form, geometric lower bound (also known as Gentle's estimate), direct upper bound, distributional upper bound and distributional lower bound as control variates. The variances are normalized with respect to the Monte Carlo variance of the estimate in all numerical experiments. These are respectively represented as MC, HP, LB, UB, UBM and LBM in the tables. We also use CI L and CI U to denote the respective lower and upper confidence intervals of the Monte Carlo estimates throughout the chapter.

We observe that the price estimates using the direct upper bound and the Hermite polynomial as a control variate are highly accurate and are quite similar to those obtained using Monte Carlo and the Geometric lower bound as a control variate. The variances of our new control variates are significantly lower than the Monte Carlo variances. In fact, the Geometric lower bound and the direct upper bound control variates have the lowest variances and are similar on average.

Furthermore, we see that among our control variates the Hermite polynomial and direct upper bound control variates have faster computation times than the Geometric lower bound and Monte Carlo technique. However, the Hermite polynomial control variate record the fastest computation time across all strikes K and maturities T when compared to other control variates.

6.2.3 Five-asset Basket Case

For a basket of five-assets, we compare the Monte Carlo estimate with estimates obtained using different control variates (geometric lower bound and direct upper bound). We obtain daily volatility/covariance estimates from five market indices namely FTSE 100, FTSE 250, S&P 500, NIKKEI 225 and IMOEX. We compare the results of the control variates (Geometric lower bound and direct upper bound) techniques with standard Monte Carlo estimate for the basket option price.

Tables 6.5-6.7 show the basket option price, the variance (normalized relative to the variance of Monte Carlo estimate) and the computation times across different maturities for the same range of strikes $K = 60, 80, 100$.

Similar to the two-asset case, we observe that basket option prices are non-decreasing functions of time for any fixed position of moneyness and are inversely related with the strike, for a fixed time.

We observe very similar basket option prices for the different techniques implemented.

The variance estimates from the upper bound control variate are lower variance than the Monte Carlo estimates, and almost identical variance to the Geometric lower bound control variance in all cases. The upper bound control variate has faster computational times when compared to the classical Monte Carlo and Geometric lower bound control variate methods.

6.3 Numerical Results For Asian Options

We simulate the price of an Asian option using Monte Carlo methods. For our numerical experiment, we set the initial price of the asset to be $S(0) = 80$ and simulate the price of the option using strikes $K = 60, 80$ and 100, for the respective in-the-money, at-the-money and out-of-the-money positions of moneyness across different maturity times $T = 0.5, 1, 2$. We simulate the Monte Carlo price(s) for the Asian option using $N = 10,000$ sample paths and

$M = 1000$ time steps In the next section we observe the accuracy of the first order Hermite polynomial approximation for the price of an Asian option and then the results from our control variates.

6.3.1 First Order Hermite Polynomial Approximation For Asian Options

We observe the accuracy of our closed-form estimate of the price of an Asian option using first order Hermite polynomial for different maturities for different maturity times T and across different strikes K. We have computed the actual Monte Carlo estimate of the price of an Asian option using using 1000 time steps and 10000 sample paths. We have used an initial asset price $S(0) = 80$, short rate $r = 0.01$ and volatility $\sigma = 20\%$. The Absolute error percentage in the first order Hermite polynomial approximation of the Asian option price is calculated using 6.2, where in this context $HPPrice$ is the first order Hermite polynomial estimate for the price of an Asian option as given by 4.26 and $MCPrice$ is the Monte Carlo price of the Asian option.

Table 6.10 shows the estimated price of an Asian option using Hermite polynomials for different maturity periods and strikes and the corresponding error percentages in the estimates. The error percentages are calculated using the ratio of the difference between the estimated Asian option price and the Monte Carlo price to the Monte Carlo price of the Asian option. We can observe by inspection that for a fixed maturity T, the error percentage increases from the in-the-money case to the out-of-the-money case. Thus implying that the first order Hermite polynomial is good as an approximation for the price of an Asian option except in the out-of-the-money case.

6.3.2 Control Variate Analysis For Asian Options

In this section, we estimate the price of an Asian call option using different control variate methods. We look at the numerical results obtained from implementation of control variate methods, which we compare to the standard Monte Carlo approach (MC). The control variates we have used for this analysis are Hermite polynomial (HP), average upper bound (AUB), the future-valued basket FVB and the geometric lower bound (GLB) (also known as Kemna-Vorst estimate) which we shall use as a benchmark for our control variates. To price the assets, we have used a volatility of 20%, which we have obtained from the FTSE 100.

In tables 6.11, 6.12 and 6.13 we compare the price, variance and computation times of our antithetic Monte Carlo estimate of an Asian call option with the results obtained using the Hermite polynomial (HP), average upper bound (AUB), the future-valued basket FVB and the geometric lower bound (GLB) as control variates. The variances is normalized with respect to the Monte Carlo variance of the estimate for all numerical experiments.

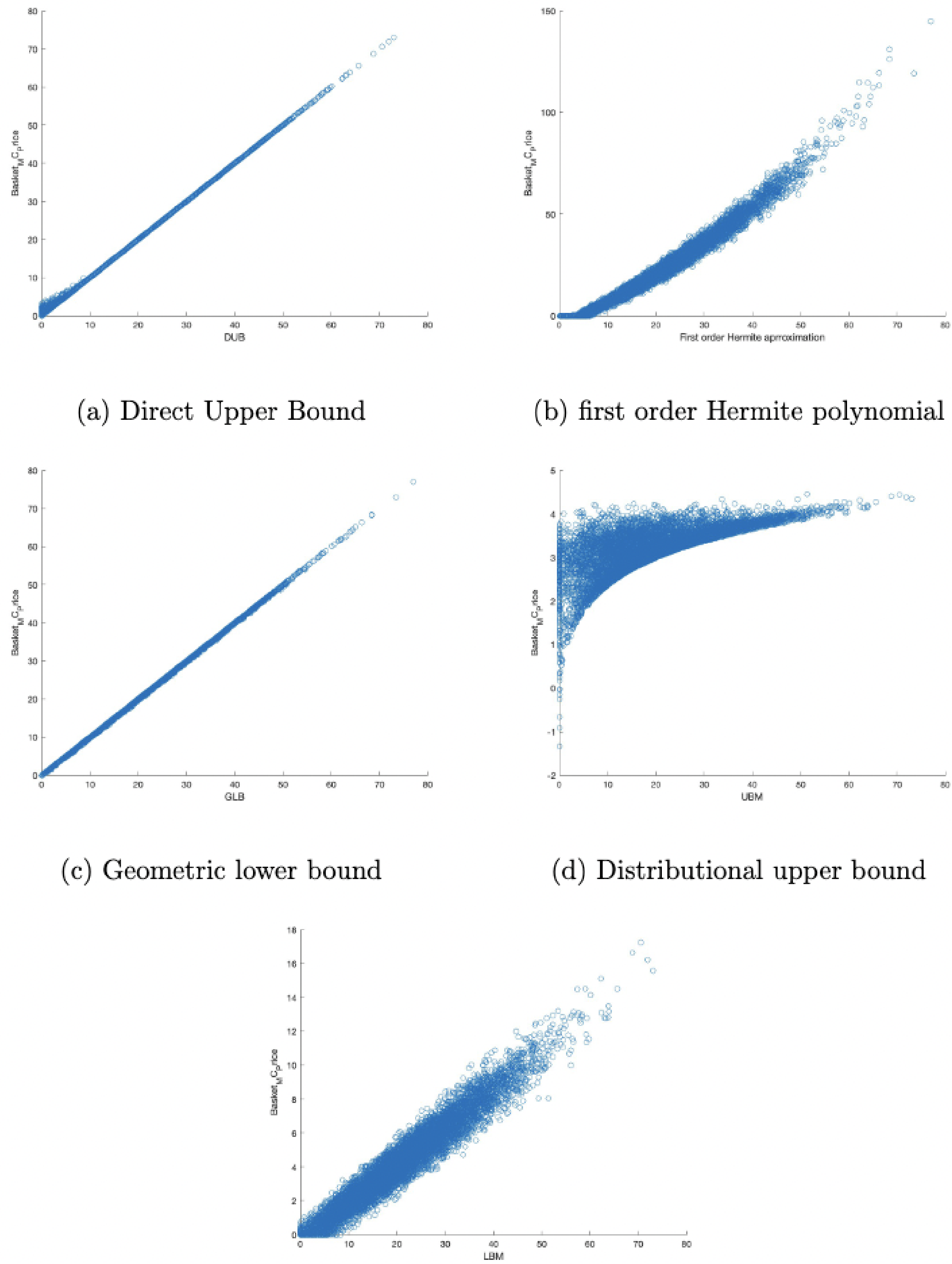
We observe that the price estimates using the average upper bound and the Hermite polynomial as a control variate are highly accurate and are quite similar to those obtained using Monte Carlo and the geometric lower bound as a control variate. The variances of our new control variates are significantly lower than the Monte Carlo variances. In fact, the Geometric lower bound and the direct upper bound control variates have the lowest variances and are similar on average.

Furthermore, we see that besides variance reduction using control variates also reduce computation times of the estimates as well. The Hermite polynomial and average upper bound control variates have faster computation times than the geometric lower bound and Monte Carlo technique. However, the Hermite polynomial control variate record the fastest computation time across all strikes K and maturities T.

We observe that, similar to the basket option pricing case, the price of Asian options are non-decreasing functions of maturity T for any fixed position of moneyness and are inversely related with the strike K, for a fixed T.

Also, our numerical experiments yield very similar results for Asian option prices for the different control variate techniques.

The normalised variance estimates from the average upper bound (AUB), Hermite polynomial (HP), and geometric lower bound upper bound (GLB) control variate are significantly lower than the normalised variance from the Monte Carlo estimates. The geometric lower bound gives the best overall variance reduction, but is very similar to those obtained from the average upper bound and Hermite polynomial control variates in most cases. Furthermore, these three control variates namely the average upper bound, Hermite polynomial and geometric lower bound have faster computation times than the standard Monte Carlo and the future-valued basket control variate. The future-valued basket control variate yields the least variance reduction of usually between $10 - 30\%$ and generally has slower computation times than the standard Monte Carlo simulation.



(a) Direct Upper Bound

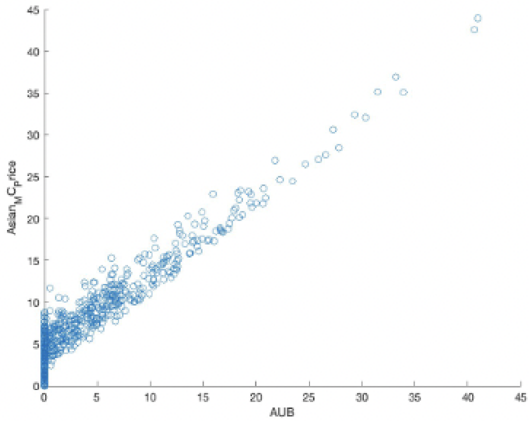
(b) first order Hermite polynomial

(c) Geometric lower bound

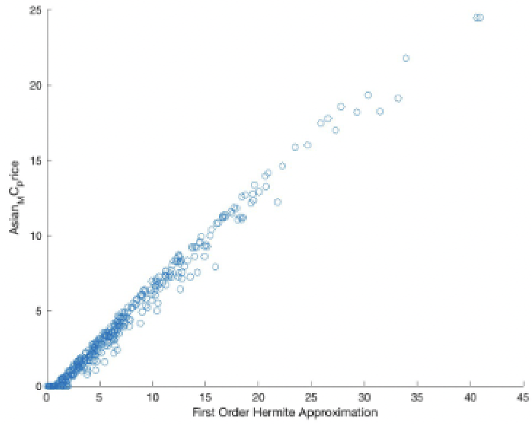
(d) Distributional upper bound

(e) Distributional lower bound

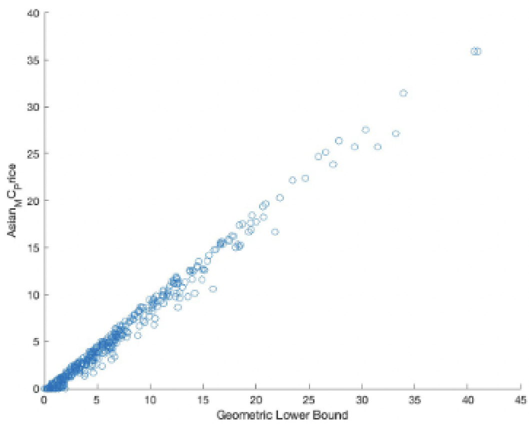
Figure 6.1: Correlation plots between MC estimates and control variates of a basket option



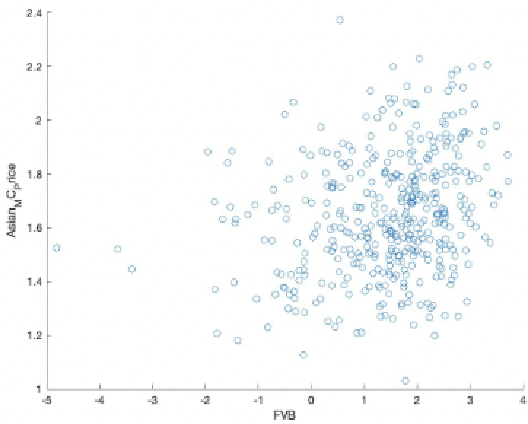
(a) AUB



(b) First order Hermite approximation



(c) Geometric lower bound



(d) FVB

Figure 6.2: Correlation plots between Asian option prices and AUB (a), first order Hermite polynomial approximation (b), geometric lower bound (c) and FVB (d) respectively

6.4 Summary

In this chapter, we observed through numerical experiments the accuracy of our different control variate methods for pricing basket and Asian call options. For the basket call option, we observe the accuracy of the Hermite polynomial approximation of a two-asset basket. Results show that this method is fairly accurate if the basket is deep-in-the-money or at-the-money. The observed error is found to be the lowest for $T = 1$, when compared to other maturity times for a fixed position of moneyness. The control variate analysis, we find that all our control variates yield significant variance reduction. Furthermore, the lower and upper distributional bounds achieve more variance reduction than the benchmark control variate in most cases.

For Asian call options, we observe that the first order Hermite polynomial approximation of an Asian call option gives good estimates when the average value of the underlying asset is deep-in-the-money or at-the-money. However, for the out-of-the-money case, the estimates obtained for the price of an Asian option are largely unreliable. This is also similar to the results obtained for basket options and with those observed by Milevsky [28] and Curran [8], where the authors stated that closed-form approximation of options on the sum of lognormal assets where inaccurate when the asset was out-of-the-money. The results from the control variates yield significant variance reduction with the exception of the future-valued basket. Similar findings are recorded with the computation times of the control variates.

K	Method	Price	Variance	CI L	CI U	Time
60	MC	20.3633	1.0000	20.3415	20.3850	0.0567
	LB	20.3615	0.0001	20.3612	20.3617	0.2320
	UB	20.3618	0.0002	20.3614	20.3620	0.0733
	HP	20.3609	0.0225	20.3586	20.3631	0.0068
	UBM	20.3581	0.4601	20.3435	20.3626	0.4486
	LBM	20.3094	0.0596	20.3042	20.3147	0.0469
80	MC	4.6431	1.0000	4.6285	4.6567	0.0152
	LB	4.6387	0.0001	4.6385	4.6389	0.2362
	UB	4.6392	0.0047	4.6382	4.6402	0.0769
	HP	4.6392	0.0285	4.6373	4.6409	0.0869
	UBM	4.6374	0.4327	4.6289	4.6403	0.2613
	LBM	4.6464	0.0784	4.6424	4.6504	0.0865
100	MC	0.3122	1.0000	0.3085	0.3159	0.0541
	LB	0.3145	0.0004	0.3144	0.3146	0.2355
	UB	0.3147	0.0454	0.3139	0.3155	0.0782
	HP	0.3144	0.1912	0.3127	0.3161	0.0875
	UBM	0.3138	0.5194	0.3111	0.3154	0.2853
	LBM	0.3127	0.1781	0.3112	0.3142	0.0867

Table 6.3: Basket option prices for a two-asset basket, T=0.5

K	**Method**	**Price**	**Variance**	**CI L**	**CI U**	**Time**
60	MC	20.9871	1.0000	20.9569	21.0179	0.0335
	LB	20.9849	0.0002	20.9845	20.9853	0.2036
	UB	20.9844	0.0005	20.9837	20.9852	0.0341
	HP	20.9847	0.0194	20.9813	20.9881	0.0435
	UBM	20.9844	0.4363	20.9701	20.9997	0.2325
	LBM	20.9791	0.0607	20.9716	20.9865	0.0495
80	MC	6.6758	1.0000	6.6549	6.6966	0.0501
	LB	6.6595	0.0003	6.6592	6.6599	0.2364
	UB	6.6599	0.0007	6.6585	6.6613	0.0751
	HP	6.6609	0.0185	6.6581	6.6637	0.0857
	UBM	6.6631	0.4791	6.6501	6.6771	0.2541
	LBM	6.6174	0.0771	6.6115	6.6233	0.0872
100	MC	1.2449	1.0000	1.2356	1.2542	0.0500
	LB	1.2487	0.0006	1.2484	1.2489	0.2356
	UB	1.2483	0.0221	1.2469	1.2497	0.0771
	HP	1.2486	0.0361	1.2469	1.2500	0.0868
	UBM	1.2322	0.5101	1.2256	1.2338	0.2620
	LBM	1.2359	0.1186	1.2325	1.2391	0.0869

Table 6.4: Basket option prices for a two-asset basket, $T = 1$

K	Method	Price	Variance	CI L	CI U	Time
60	MC	22.4924	1.0000	22.4509	22.5238	0.0545
	LB	22.4834	0.0004	20.4819	22.4845	0.2401
	UB	22.4832	0.0009	22.4819	22.4845	0.0810
	HP	22.4839	0.0311	22.4770	22.4883	0.0906
	UBM	22.2479	0.5278	22.2210	22.2747	0.3034
	LBM	22.2396	0.0645	22.2291	22.2502	0.0965
80	MC	9.5914	1.0000	9.5601	9.6226	0.0574
	LB	9.6002	0.0006	9.5994	9.6009	0.2423
	UB	9.6019	0.0037	9.5596	9.6039	0.0814
	HP	9.5974	0.0421	9.5930	9.6017	0.0928
	UBM	9.4249	0.4424	9.4044	9.4374	0.1918
	LBM	9.4150	0.0777	9.4062	9.4238	0.0566
100	MC	3.3437	1.0000	3.3242	3.3632	0.0375
	LB	3.3377	0.0008	3.3371	3.3383	0.2108
	UB	3.3467	0.0119	3.3445	3.3488	0.0396
	HP	3.3531	0.0718	3.3479	3.3583	0.0525
	UBM	3.3406	0.4914	3.3374	3.3563	0.2172
	LBM	3.2906	0.1152	3.2841	3.2971	0.0601

Table 6.5: Basket option prices for a two-asset basket, $T = 2$

K	Method	Price	Variance	CI L	CI U	Time
60	MC	20.3111	1.0000	20.2937	20.3285	0.1401
	LB	20.3107	0.0021	20.3099	20.3107	0.3675
	UB	20.3082	0.0035	20.3072	20.3092	0.2131
	HP	20.3109	0.3406	20.3007	20.3186	0.2016
	UBM	20.3027	0.5521	20.2897	20.3157	0.4189
	LBM	20.3107	0.2322	20.3024	20.3190	1.6879
80	MC	3.7339	1.0000	3.7228	3.7451	0.1352
	LB	3.7341	0.0051	3.7334	3.7342	0.3754
	UB	3.7237	0.0635	3.7208	3.7265	0.2188
	HP	3.7371	0.3438	3.7284	3.7404	0.2018
	UBM	3.7371	0.5701	3.7297	3.7414	0.3795
	LBM	3.7371	0.2844	3.7312	3.7431	1.6751
100	MC	0.0931	1.0000	0.0914	0.0948	0.1446
	LB	0.0941	0.0227	0.0939	0.0941	0.3624
	UB	0.0909	1.4135	0.0892	0.0917	0.2145
	HP	0.0945	0.3987	0.0929	0.0948	0.1972
	UBM	0.0945	0.6239	0.0931	0.0949	0.3864
	LBM	0.0946	0.3917	0.0934	0.0948	1.6667

Table 6.6: Basket option prices for a five-asset basket, $T = 0.5$

K	Method	Price	Variance	CI L	CI U	Time
60	MC	20.7163	1.0000	20.6920	20.7407	0.1546
	LB	20.7216	0.0047	20.7199	20.7216	0.3869
	UB	20.7114	0.0086	20.7096	20.7136	0.2363
	HP	20.7188	0.3712	20.7051	20.7291	0.2222
	UBM	20.7141	0.5647	20.7057	20.7221	0.4474
	LBM	20.7217	0.2357	20.7101	20.7291	1.7071
80	MC	5.3851	1.0000	5.3688	5.4015	0.1434
	LB	5.3925	0.0103	5.3908	5.3925	0.3682
	UB	5.3764	0.0585	5.3724	5.3804	0.2163
	HP	5.3941	0.3191	5.3834	5.3991	0.2019
	UBM	5.3853	0.5441	5.3734	5.3911	0.3795
	LBM	5.3902	0.2764	5.3814	5.3963	1.7035
100	MC	0.5749	1.0000	0.5694	0.5893	0.1673
	LB	0.5698	0.0309	0.5689	0.5699	0.3891
	UB	0.5612	0.5094	0.5572	0.5651	0.6194
	HP	0.5656	0.4133	0.5609	0.5704	0.2286
	UBM	0.5654	0.7069	5609	0.5698	0.6194
	LBM	0.5669	0.3771	0.5635	0.5701	1.7383

Table 6.7: Basket option prices for five-asset basket, $T = 1$

K	Method	Price	Variance	CI L	CI U	Time
60	MC	22.7791	1.0000	21.7453	21.8128	0.1579
	LB	21.7702	0.0103	21.7667	21.7702	0.3845
	UB	21.7462	0.0126	21.7424	21.7501	0.2341
	HP	21.7692	0.2846	21.7505	21.7723	0.2189
	UBM	21.7814	0.5704	21.7907	22.0133	0.2948
	LBM	21.7698	0.2427	21.7532	21.7764	1.7223
80	MC	7.8575	1.0000	7.8320	7.8879	0.1574
	LB	7.8367	0.0212	7.8333	7.8367	0.3814
	UB	7.8232	0.0529	7.8177	7.8288	0.2303
	HP	7.8402	0.3803	7.8296	7.8504	0.2166
	UBM	7.8424	0.5132	7.8269	7.8593	0.3953
	LBM	7.8402	0.2925	7.8338	7.8596	1.7361
100	MC	1.9530	1.0000	1.9659	1.9560	0.0129
	LB	1.9419	0.0428	1.9393	1.9419	0.3791
	UB	1.9294	0.3107	1.9231	1.9357	0.2309
	HP	1.9454	0.4719	1.9351	1.9508	0.2153
	UBM	1.9394	0.6037	1.9293	1.9421	0.4001
	LBM	1.9454	0.3255	1.9351	1.9511	1.7026

Table 6.8: Basket option price for a five-asset basket, $T = 2$

Control Variates	**Optimal** λ	R^2
LB	0.9998	0.9941
UB	0.9542	0.9001
HP	0.6892	0.9759
UBM	0.7042	0.6818
LBM	2.6687	0.9383

Table 6.9: R^2 and optimal Values of λ for the Basket Option Control Variates

T	K	HP Price	MC Price	% Abs Error	CI L	CI U
	60	20.1009	20.0912	0.1132	20.0911	20.0912
0.5	80	2.6265	2.6217	0.0254	2.6216	2.6217
	100	0.0016	0.0071	0.0018	0.0071	0.0072
	60	20.2351	20.1870	0.0902	20.1868	20.1871
1	80	3.7673	3.7378	0.0313	3.7377	3.7379
	100	0.0474	0.1082	0.0740	0.1083	0.1083
	60	20.6727	20.4561	0.2007	20.4558	20.4563
2	80	5.4275	5.3826	0.0804	5.3824	5.3827
	100	0.3738	0.6527	0.2574	0.6256	0.6527

Table 6.10: Hermite polynomial approximation for the price of an Asian option.

K	Method	Price	Variance	CI L	CI U	Time
60	MC	20.1481	1.0000	20.1476	20.1488	0.5891
	AUB	20.2004	0.0003	20.2004	20.2004	0.4885
	HP	20.0981	0.0043	20.0981	20.0982	0.0567
	GLB	20.1001	0.0003	20.1001	20.1001	0.0595
	FVB	20.1087	0.8319	20.1083	20.1092	0.4879
80	MC	2.6502	1.0000	20.6498	2.6507	0.5422
	AUB	2.6567	0.0127	2.6566	2.6568	1.9188
	HP	2.6387	0.0031	2.6387	2.6388	0.0534
	GLB	2.6211	0.0003	2.6211	2.6211	0.0489
	FVB	2.6393	0.6845	2.6390	2.6397	0.8744
100	MC	0.0066	1.0000	0.0064	0.0068	0.5617
	AUB	0.0059	0.2379	0.0058	0.0059	0.5401
	HP	0.0063	0.1113	0.0062	0.0063	0.0596
	GLB	0.0068	0.0131	0.0068	0.0068	0.0526
	FVB	0.0057	0.5516	0.0055	0.0059	0.6574

Table 6.11: Asian option prices, $T = 0.5$.

K	**Method**	**Price**	**Variance**	**CI L**	**CI U**	**Time**
60	MC	20.1882	1.0000	20.1872	20.1892	0.5333
	AUB	20.4183	0.0019	20.4183	20.4184	0.3829
	HP	20.2029	0.0073	20.2027	20.2031	0.0305
	GLB	20.2731	0.0005	20.2731	20.2731	0.0279
	FVB	20.1121	0.8461	20.1114	20.1129	0.4261
80	MC	3.7539	1.0000	3.7532	3.7544	0.5342
	AUB	3.7822	0.0132	3.7821	3.7823	1.9545
	HP	3.7329	0.0086	3.7328	3.7329	0.0531
	GLB	3.7562	0.0007	3.7537	3.7537	0.0471
	FVB	3.7362	0.8612	3.7356	3.7367	0.8409
100	MC	0.1131	1.0000	0.1129	0.1133	0.5478
	AUB	0.1195	0.4204	0.1194	0.1194	0.1195
	HP	0.1081	0.0567	0.1081	0.1082	0.0529
	GLB	0.1099	0.0101	0.1099	0.1099	0.0474
	FVB	0.1315	0.5456	0.5454	0.5458	0.7134

Table 6.12: Asian option prices, $T = 1$.

K	Method	Price	Variance	CI L	CI U	Time
60	MC	20.5549	1.0000	20.5543	20.5553	0.5429
	AUB	20.9419	0.0038	20.9418	20.9421	0.4025
	HP	20.4576	0.0104	20.4575	20.4576	0.0564
	GLB	20.4945	0.0011	20.4944	20.4946	0.0498
	FVB	20.1228	0.8114	20.1225	20.1231	0.5233
80	MC	5.4036	1.0000	5.4029	5.4042	0.5601
	AUB	5.4911	0.0131	5.4909	5.4912	1.8892
	HP	5.3762	0.0119	5.3761	5.3765	0.0491
	GLB	5.3883	0.0014	5.3883	5.3889	0.0455
	FVB	5.3176	0.7862	5.3173	5.3179	0.8124
100	MC	0.6302	1.0000	0.6299	0.6306	0.5442
	AUB	0.6255	0.1394	0.6254	0.6257	0.4343
	HP	0.6255	0.0081	0.6255	0.6256	0.0491
	GLB	0.6399	0.0075	0.6398	0.6399	0.0481
	FVB	0.6865	0.8323	0.6864	0.6866	0.7451

Table 6.13: Asian option prices, $T = 2$.

Control Variates	**Optimal λ**	R^2
AUB	1.0225	0.9851
HP	1.0532	0.9975
GLB	1.0224	0.9989
FVB	0.0422	0.0248

Table 6.14: R^2 and optimal values of λ for the Asian Option Control Variates

Chapter 7

Summary of Contributions and Future Research

We proposed new control variates for the pricing of basket/Asian options which are based off closed-form estimates and bounds on the aforementioned options. The contributions of this thesis can be summarized into two parts. The first part summarises results pertaining to pricing basket options and the second part summarises relevant results to Asian option prices.

The summary of contributions on basket options as follows:

We derived new closed-form estimates for the price of a basket option using first order Hermite polynomials. This method is fairly accurate for short maturity times for deep-in-the-money and at-the-money positions of moneyness. This is particularly useful for low-dimensional portfolios and becomes more computationally involved as the number of assets in the basket increases. For a large number of assets in the basket, suitable adaptations to this method are made to allow for its use as a control variate. We also derived closed-form Gaussian representation for the price of a basket option for very short maturity times and its accuracy decreases as maturity of the option increases.

We derived closed-form bounds on the price of a basket option using the distributional properties of a Brownian motion. This was achieved by replacing the underlying Wiener processes in the assets with their joint maximum to obtain an upper bound on the basket of assets and vice-versa. These distributional bounds generally lead to a similar problem-type as the basket of assets and are evaluated in two ways. Firstly, they can be estimated using their geometric average allowing for closed-form representation. The second approach involves imposing integrability conditions on the volatility parameters of the bounds on the basket, leading to a closed-form estimates of the bounds on the basket option price.

Direct bounds on the basket option price are also obtained due to the convexity of the payoff function on the option, resulting from a direct application of the Jensen's inequality. Furthermore, the direct upper bound is the weighted sum of options on the individual assets in the basket with the same strike.

In our numerical experiments we simulate the price of a basket option using the first order Hermite polynomial estimate, the distributional bounds, direct upper bound and geometric lower bound as control variates. The geometric lower bound which we refer to as the standard control variate, is used as a benchmark to compare the efficiency of the remaining control variates. The results from the numerical experiments indicate that all control variates yield significant variance reduction. The distributional bounds yield the highest variance reduction compared to all other control variates even outperforming the standard control variate. The distributional bounds also have the fastest computation times compared to all other control variates.

We can summarize our contributions on Asian options as follows:

Closed-form estimates for the price of an Asian option using first order Hermite polynomial approximation were obtained for the average value of the underlying asset over the required time period. Similar to the basket options case, good estimates were also obtained for the in-the-money and at-the-money case across different maturities, except for the out-of-the-money case. We obtained closed-form bounds on the Asian option price by substituting the Brownian term with its overall minimum/maximum over the entire time interval. This leads to relatively large bounds on the Asian option price. To overcome this, we define two fictitious assets, obtained by simply replacing the Brownian term in the underlying asset with its running maximum or minimum. These new bounds on the Asian option prices are estimated using their geometric averages and conditioning arguments. Direct bounds on the Asian option price are also obtained similar to those calculated for basket options. We also introduce a control variate which we dub the future-value basket (FVB),which allows us to capture the dynamics and correlations of the average value of the asset over finite intervals.

Our numerical experiments are carried out using the average upper bound (AUB), geometric lower bound (GLB), first order Hermite polynomial approximation (HPA) and the future-valued basket (FVB) as control variates. Significant variance reduction is obtained for all control variates except for the future-valued basket (FVB). The HPA and GLB have the highest and very similar variance reduction but the GLB has slightly faster computation times than the HPA. The FVB recorded the slowest computation times and very little variance reduction. It is worth mentioning that all the distributional bounds on the price of an Asian option while yielding suitable closed-form bounds on the Asian option price, were unsuitable as control variates due to high bias in the estimated price or the high variance of results.

We can observe from our numerical experiments of both basket and Asian options, that estimates or bounds obtained in a Gaussian or log-normal framework serve as better control variates than those obtained from other distributions such as a 'half-Gaussian (such as the ones implied by the running maximum and the running minimum of a Brownian motion)', when asset prices follow a geometric Brownian motion model.

For future work and research, asset price model can follow generalisations of the Black-Scholes model with non-constant volatility or to include jumps. It might be worth exploring the role of information in the price determination of basket/Asian options given an explicit representation of the market information as in the Brody-Hughston-Macrina framework. It is unknown as to whether having an analytic form for the market filtration will provide a clearer image of what the true distribution of the basket/Asian option should be.

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