Robust H_2 Filtering for a Class of Systems With Stochastic Nonlinearities

Fuwen Yang, Zidong Wang, Daniel W. C. Ho, and Xiaohui Liu

***Abstract*—This paper addresses the robust H_2 filtering problem for a class of uncertain discrete-time nonlinear stochastic systems. The nonlinearities described by statistical means in this paper comprise some well-studied classes of nonlinearities in the literature. A technique is developed to tackle the matrix trace terms resulting from the nonlinearities, and the well-known S-procedure technique is adopted to cope with the uncertainties. A unified framework is established to solve the addressed robust H_2 filtering problem by using a linear matrix inequality approach. A numerical example is provided to illustrate the usefulness of the proposed method.**

***Index Terms*—Deterministic uncertainty, linear matrix inequality (LMI), robust H_2 filtering, stochastic nonlinearity.**

I. Introduction

RECENTLY, there have been growing research interests in the robust filtering problems for stochastic systems, see, e.g., [4], [6]–[8], [12] and references therein. In particular, a linear matrix inequality (LMI) approach has been proposed in [6] to deal with the robust steady-state filtering problem with multiplicative noises over an infinite-horizon, and the corresponding finite-horizon filtering problem has been studied in [9] in terms of recursive Riccati-like equations. Also, the mean square stability has been dealt with in [11] for a class of stochastic systems with both Markovian switching and nonlinearities. On the other hand, in [10], an elegant LMI approach has been developed to deal with the analysis problem for a class of systems with stochastic nonlinearities, where the nonlinearities characterized by statistical means were first introduced in [5]. Unfortunately, the robustness issue in the presence of parameter uncertainties has not been addressed.

The main purpose of this paper is: 1) to substantially extend part of the analysis results in [10] to the uncertain systems; 2) to derive the explicit expression of an upper bound for the robust H_2 performance; and 3) to deal with the corresponding robust filter design problem where the upper bound of the H_2 performance is minimized. Specifically, we are interested in designing a filter such that, for all admissible stochastic nonlinearities and deterministic uncertainties, the overall filtering process is exponentially mean-square quadratically stable, and the H_2 filtering performance is achieved as well. The solution to the H_2 filtering problem is enforced within a unified LMI framework. A numerical example is provided to illustrate the design procedures and performances.

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II. System Description and Problem Formulation

Consider the following class of discrete-time systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties:

$$\begin{cases} x_{k+1} = (A + H_1FE)x_k + f(x_k) + B_1w_k \\ y_k = (C + H_2FE)x_k + g(x_k) + D_{11}w_k \\ z_k = L_2x_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^m$ is the measured output, $z_k \in \mathbb{R}^p$ is a combination of the states to be estimated. $w_k \in \mathbb{R}^r$ is a zero mean Gaussian white noise sequence with covariance $\Phi > 0$, and A, B_1, C, D_{11}, L_2, H_1, H_2 and E are known real matrices with appropriate dimensions.

The matrix $F \in \mathbb{R}^{i\times j}$ represents the deterministic norm-bounded parameter uncertainties satisfying $FF^T \leq I$, and $f(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(x_k) : \mathbb{R}^n \to \mathbb{R}^m$ are stochastic nonlinear functions of the states, which are assumed to have the following first moments for all x_k:

$$\mathbb{E}\left\{ \begin{bmatrix} f(x_k) \\ g(x_k) \end{bmatrix} | x_k \right\} = 0 \tag{2}$$

with the covariance given by

$$\mathbb{E}\left\{ \begin{bmatrix} f(x_k) \\ g(x_k) \end{bmatrix} \begin{bmatrix} f(x_j)^T & g(x_j)^T \end{bmatrix} | x_k \right\} = 0, \qquad k \neq j \tag{3}$$

and

$$\mathbb{E}\left\{ \begin{bmatrix} f(x_k) \\ g(x_k) \end{bmatrix} \begin{bmatrix} f(x_k)^T \ g(x_k)^T \end{bmatrix} | x_k \right\} = \sum_{i=1}^{q} \begin{bmatrix} \pi_{1i} \\ \pi_{2i} \end{bmatrix} \begin{bmatrix} \pi_{1i} \\ \pi_{2i} \end{bmatrix}^T x_k^T \Gamma_i x_k \tag{4}$$

where $\pi_{1i} \in \mathbb{R}^{n\times 1}$ and $\pi_{2i} \in \mathbb{R}^{m\times 1}$ $(i = 1, \ldots, q)$ are known column vectors with compatible dimensions of $f(x_k)$ and $g(x_k)$, and Γ_i $(i = 1, \ldots, q)$ are known positive-definite matrices with appropriate dimensions.

Remark 1: The nonlinearity description in (2)–(4) covers several classes of well-studied nonlinear systems, for example, the system with state-dependent multiplicative noises and the system whose state's power depends on the sector-bound (or sign) of the nonlinear state function of the state, see [5], [10].

Now consider the following filter for the system (1):

$$\begin{cases} \hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{K}y_k \\ \hat{z}_k = \hat{L}_2\hat{x}_k \end{cases} \tag{5}$$

where $\hat{x}_k$ is the state estimate, $\hat{z}_k$ is an estimate for z_k, and $\hat{A}$, $\hat{K}$ and $\hat{L}_2$ are the filter parameters to be determined.

The augmented system is described by combining (5) with (1) as follows:

$$\begin{cases} \tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + B_e h(x_k) + \tilde{B}w_k \\ e_k := z_k - \hat{z}_k = L_2 x_k - \hat{L}_2\hat{x}_k = C_2\tilde{x}_k \end{cases} \tag{6}$$

where

$$\tilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \quad h(x_k) = \begin{bmatrix} f(x_k) \\ g(x_k) \end{bmatrix}, \quad \tilde{A} = \bar{A} + \Delta A \tag{7}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ \hat{K}C & \hat{A} \end{bmatrix}, \quad B_e = \begin{bmatrix} I & 0 \\ 0 & \hat{K} \end{bmatrix} \tag{8}$$

$$\Delta A = \begin{bmatrix} H_1 \\ \hat{K}H_2 \end{bmatrix} F[E \quad 0] := H_e F E_e \tag{9}$$

$$\tilde{B} = \begin{bmatrix} B_1 \\ \hat{K}D_{11} \end{bmatrix}, \quad C_2 = [L_2 \quad -\hat{L}_2]. \tag{10}$$

In this paper, our objective is to design the filter (5) for the system (1) such that, for all stochastic nonlinearities and all admissible deterministic uncertainties, the augmented system (6) is exponentially mean-square quadratically stable and the estimation error e_k satisfies the H_2 performance constraint, i.e.,

$$J_2 = \lim_{k\to\infty} \mathbb{E}\left\{\|e_k\|^2\right\} < \beta \tag{11}$$

where $\beta > 0$ is a prescribed scalar.

III. Robust H_2 Filter Design

Before proceeding, we denote

$$\tilde{\Gamma}_i := \begin{bmatrix} \Gamma_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_i := \begin{bmatrix} \pi_{1i} \\ \pi_{2i} \end{bmatrix} \begin{bmatrix} \pi_{1i} \\ \pi_{2i} \end{bmatrix}^T. \tag{12}$$

In order to derive the robust H_2 filter, we need the following technical results.

Lemma 1: Consider the system

$$\xi_{k+1} = M\xi_k + B_e f(x_k) \tag{13}$$

where $\xi_k = [x_k^T \hat{x}_k^T]^T \in \mathbb{R}^{2n}$, $x_k \in \mathbb{R}^n$, $\mathbb{E}\{f(x_k)|x_k\} = 0$ and $\mathbb{E}\{f(x_k)f(x_k)^T|x_k\} = \sum_{i=1}^q \Pi_i(x_k^T\Xi_i x_k)$. Here, $\Pi_i = \pi_i\pi_i^T$, π_i $(i = 1,\ldots,q)$ are column vectors, Ξ_i $(i = 1,\ldots,q)$ are known positive-definite matrices with appropriate dimensions. If the system (13) is exponentially mean-square stable, and there exists a symmetric matrix Y satisfying

$$M^T Y M - Y + \sum_{i=1}^q \tilde{\Xi}_i \mathrm{tr}\left(B_e\Pi_i B_e^T Y\right) < 0 \tag{14}$$

where

$$\tilde{\Xi}_i = \begin{bmatrix} \Xi_i & 0 \\ 0 & 0 \end{bmatrix} \tag{15}$$

then $Y \geq 0$.

Lemma 2: The system (6) is exponentially mean-square quadratically stable if, for all admissible uncertainties, there exists a positive definite matrix P satisfying

$$\tilde{A}^T P \tilde{A} - P + \sum_{i=1}^q \left[\tilde{\Gamma}_i \mathrm{tr}\left(B_e\Pi_i B_e^T P\right)\right] < 0. \tag{16}$$

Lemma 1 and 2 can be easily proved by using the Lyapunov method, hence the proofs are omitted.

Lemma 3: [1], [3], [10] If the system (6) is exponentially mean-square quadratically stable, then

$$\rho\left\{\tilde{A}\otimes\tilde{A} + \sum_{i=1}^q \mathrm{st}\left(B_e\Pi_i B_e^T\right)\mathrm{st}^T(\tilde{\Gamma}_i)\right\} < 1 \tag{17}$$

where $\otimes$ is the Kronecker product of matrices; ρ is the spectral radius of a matrix, and $\mathtt{st}$ stands for the stack that forms a vector out of the columns of matrix.

Lemma 4: (S-procedure) [2]. Let $M = M^T$, H and E be real matrices of appropriate dimensions with F satisfying $FF^T \leq I$, then

$$M + HFE + E^T F^T H^T < 0 \tag{18}$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$M + \frac{1}{\varepsilon}HH^T + \varepsilon E^T E < 0 \tag{19}$$

or equivalently

$$\begin{bmatrix} M & H & \varepsilon E^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon E & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{20}$$

Now, let us proceed to compute the H_2 performance J_2 that is used in the constraint (11). Define the state variance by $Q_k := \mathbb{E}\{\tilde{x}_k\tilde{x}_k^T\}$, and then the evolution of the state variance matrix Q_k can be derived from the system (6) as follows:

$$\begin{aligned} Q_{k+1} &= \tilde{A}Q_k\tilde{A}^T + \mathbb{E}\left[B_e h(x_k) h(x_k)^T B_e^T\right] + \tilde{B}\Phi\tilde{B}^T \\ &= \tilde{A}Q_k\tilde{A}^T + \sum_{i=1}^q \left[B_e\Pi_i B_e^T \mathrm{tr}(Q_k\tilde{\Gamma}_i)\right] + \tilde{B}\Phi\tilde{B}^T. \end{aligned} \tag{21}$$

Rewrite (21) in the form of stack matrices

$$\mathrm{st}(Q_{k+1}) = \Psi\mathrm{st}(Q_k) + \mathrm{st}(\tilde{B}\Phi\tilde{B}^T) \tag{22}$$

where

$$\Psi := \tilde{A}\otimes\tilde{A} + \sum_{i=1}^q \mathrm{st}\left(B_e\Pi_i B_e^T\right)\mathrm{st}^T(\tilde{\Gamma}_i). \tag{23}$$

If the system (6) is exponentially mean-square quadratically stable, it then follows from Lemma 3 that $\rho(\Psi) < 1$, and $Q = \lim_{k\to\infty} Q_k$. Hence, the H_2 performance is

$$J_2 = \lim_{k\to\infty} \mathbb{E}\left\{\|e_k\|^2\right\} = \mathrm{tr}\left[C_2 Q C_2^T\right]. \tag{24}$$

Now, suppose that there exists a symmetric matrix $\hat{P}_k$ such that the following backward recursion is satisfied:

$$\hat{P}_k = \tilde{A}^T\hat{P}_{k+1}\tilde{A} + \sum_{i=1}^q \tilde{\Gamma}_i \mathrm{tr}\left(B_e\Pi_i B_e^T\hat{P}_{k+1}\right) + C_2^T C_2. \tag{25}$$

Similarly, if the system (6) is exponentially mean-square quadratically stable, it then follows from Lemma 3 that (25), in the steady state, becomes

$$\hat{P} = \tilde{A}^T \hat{P} \tilde{A} + \sum_{i=1}^{q} \tilde{\Gamma}_i \mathrm{tr}\left(B_e \Pi_i B_e^T \hat{P}\right) + C_2^T C_2. \tag{26}$$

To this end, we have the following theorem that gives an alternative to compute the H_2 performance.

Theorem 1: If the system (6) is exponentially mean-square quadratically stable, the H_2 performance can be expressed in terms of $\hat{P}$ as follows:

$$J_2 = \mathrm{tr}[\Phi \tilde{B}^T \hat{P} \tilde{B}] \tag{27}$$

where $\hat{P} > 0$ is the solution to (26).

Proof: Noting that

$$\begin{aligned}
&\lim_{k\to\infty} \mathrm{tr}\{Q_{k+1}\hat{P}_{k+1} - Q_k \hat{P}_k\} \\
&= \lim_{k\to\infty} \mathrm{tr}\left\{\left[\tilde{A} Q_k \tilde{A}^T + \sum_{i=1}^{q} B_e \Pi_i B_e^T \mathrm{tr}(Q_k \tilde{\Gamma}_i) + \tilde{B}\Phi\tilde{B}^T\right] \right.\\
&\qquad \times \hat{P}_{k+1} - Q_k \\
&\qquad \times \left[\tilde{A}^T \hat{P}_{k+1} \tilde{A} + \sum_{i=1}^{q} \tilde{\Gamma}_i \mathrm{tr}\left(B_e \Pi_i B_e^T \hat{P}_{k+1}\right) \right.\\
&\qquad\quad \left.\left. + C_2^T C_2\right]\right\} \\
&= 0
\end{aligned} \tag{28}$$

we have $\mathrm{tr}[C_2 Q C_2^T] = \mathrm{tr}[\Phi \tilde{B}^T \hat{P} \tilde{B}]$, and the proof follows from (24) immediately.

Notice that the model (1) involves parameter uncertainties, hence the exact H_2 performance (27) cannot be obtained by simply solving (26). We now aim to provide an upper bound for the actual H_2 performance.

Suppose that there exists a positive definite matrix P such that the following matrix inequality is satisfied:

$$\tilde{A}^T P \tilde{A} - P + \sum_{i=1}^{q} \tilde{\Gamma}_i \mathrm{tr}\left(B_e \Pi_i B_e^T P\right) + C_2^T C_2 < 0. \tag{29}$$

Theorem 2: If there exists a positive definite matrix P satisfying (29), then the system (6) is exponentially mean-square quadratically stable, and

$$\hat{P} \le P \tag{30}$$

$$\mathrm{tr}[\Phi \tilde{B}^T \hat{P} \tilde{B}] \le \mathrm{tr}[\Phi \tilde{B}^T P \tilde{B}] \tag{31}$$

where $\hat{P}$ satisfies (26).

Proof: Observing that (29) implies (16), it follows directly from Lemma 2 that the system (6) is exponentially mean-square quadratically stable, $\hat{P}$ exists and meets (26). Subtracting (29) from (26) yields

$$\tilde{A}^T (P - \hat{P}) \tilde{A} - (P - \hat{P}) + \sum_{i=1}^{q} \tilde{\Gamma}_i \mathrm{tr}\left[B_e \Pi_i B_e^T (P - \hat{P})\right] < 0 \tag{32}$$

which indicates from Lemma 1 that $P - \hat{P} \ge 0$. Also, (30) implies (31). This completes the proof. ■

The corollary given below follows immediately from Theorem 2 and (11).

Corollary 1: If there exists a positive definite matrix P satisfying (29) and $\mathrm{tr}[\Phi \tilde{B}^T P \tilde{B}] < \beta$, then the system (6) is exponentially mean-square quadratically stable, and (11) is satisfied for some β.

Now we are in a position to present the main results on the addressed robust H_2 filter design. For the purpose of clarity, we only give the sketch of the proof.

Theorem 3: Given a scalar $\beta > 0$. If there exist positive-definite matrices $S > 0$, $R > 0$ and $\Theta > 0$, real matrices Q_i $(i = 1, 2, 3)$, positive scalars $\alpha_i > 0$ $(i = 1, \cdots, q)$ and $\varepsilon_1 > 0$ such that the following linear matrix inequalities are feasible:

$$\begin{aligned}
&[1 \quad 0 \quad \cdots \quad 0]\Theta[1 \quad 0 \quad \cdots \quad 0]^T + \cdots \\
&\quad + [0 \quad \cdots \quad 0 \quad 1]\Theta[0 \quad \cdots \quad 0 \quad 1]^T < \beta
\end{aligned} \tag{33}$$

$$\begin{bmatrix} -\Theta & \Phi^{\frac{1}{2}} B_1^T S & \Phi^{\frac{1}{2}}\left(B_1^T R + D_{11}^T Q_2^T\right) \\ S B_1 \Phi^{\frac{1}{2}} & -S & -S \\ (R B_1 + Q_2 D_{11})\Phi^{\frac{1}{2}} & -S & -R \end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix} -\alpha_i & \pi_{1i}^T S & \pi_{1i}^T R + \pi_{2i}^T Q_2^T) \\ S\pi_{1i} & -S & -S \\ R\pi_{1i} + Q_2 \pi_{2i} & -S & -R \end{bmatrix} < 0 \ (i = 1, \cdots, q) \tag{35}$$

$$\begin{bmatrix} G & G_{11}^T \\ G_{11} & G_{22} \end{bmatrix} < 0 \tag{36}$$

where

$$G = \begin{bmatrix} -S & -S & A^T S & A^T R + C^T Q_2^T + Q_1^T \\ * & -R & A^T S & A^T R + C^T Q_2^T \\ * & * & -S & -S \\ * & * & * & -R \end{bmatrix} \tag{37}$$

$$G_{11} = \begin{bmatrix} \Gamma_1^{\frac{1}{2}} & \Gamma_1^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma_q^{\frac{1}{2}} & \Gamma_q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ L_2 - Q_3 & L_2 & 0 & 0 \\ 0 & 0 & H_1^T S & H_1^T R + H_2^T Q_2^T \\ \varepsilon_1 E & \varepsilon_1 E & 0 & 0 \end{bmatrix} \tag{38}$$

$$G_{22} = \begin{bmatrix} -\alpha_1 I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1 I & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\alpha_q I & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_q I & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_1 I & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} \tag{39}$$

then there exists a filter of the form (5) such that the system (6) is exponentially mean-square quadratically stable and (11) is satisfied for all stochastic nonlinearities and all admissible deterministic uncertainties. Moreover, if the LMI's (33)–(36) are feasible, the desired filter parameters can be determined by

$$\begin{aligned}
&\hat{A} = X_{12}^{-1} Q_1 (S - R)^{-1} X_{12}, \quad \hat{K} = X_{12}^{-1} Q_2, \\
&\hat{L}_2 = Q_3 (S - R)^{-1} X_{12}
\end{aligned} \tag{40}$$

where the matrix X_{12} comes from the factorization $I - RS^{-1} = X_{12} Y_{12}^T < 0$.

Proof: Partition P and P^{-1} in (29) as

$$P = \begin{bmatrix} R & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} S^{-1} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \tag{41}$$

and construct

$$T_1 = \begin{bmatrix} S^{-1} & I \\ Y_{12}^T & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & R \\ 0 & X_{12}^T \end{bmatrix} \tag{42}$$

which imply that $PT_1 = T_2$ and $T_1^T P T_1 = T_1^T T_2$.

We define the change of filter parameters as follows:

$$Q_1 = X_{12}\hat{A}Y_{12}^T S, \quad Q_2 = X_{12}\hat{K}, \quad Q_3 = \hat{L}_2 Y_{12}^T S. \tag{43}$$

Using the congruence transformation technique, it is easily seen that (34) is equivalent to

$$\begin{bmatrix} -\Theta & \Phi^{\frac{1}{2}}\tilde{B}^T P \\ P\tilde{B}\Phi^{\frac{1}{2}} & -P \end{bmatrix} < 0 \tag{44}$$

which, together with (33), shows that $\mathrm{tr}[\Phi\tilde{B}^T P\tilde{B}] < \beta$. Applying similar technique to (35), we obtain

$$\mathrm{tr}\left(B_e \Pi_i B_e^T P\right) < \alpha_i, \qquad i = 1, \cdots, q \tag{45}$$

and therefore

$$\tilde{A}^T P\tilde{A} - P + \sum_{i=1}^{q} \alpha_i \tilde{\Gamma}_i + C_2^T C_2 < 0 \tag{46}$$

which infers that (29) is satisfied. Now, we need to prove that (36) is equivalent to (46). By using the Schur complement to (46), we have

$$\begin{bmatrix} -P & \tilde{A}^T & \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & \cdots & \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & C_2^T \\ \tilde{A} & -P^{-1} & 0 & \cdots & 0 & 0 \\ \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & 0 & -\alpha_1 I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & 0 & 0 & \cdots & -\alpha_q I & 0 \\ C_2 & 0 & 0 & \cdots & 0 & -I \end{bmatrix} < 0 \tag{47}$$

Rewrite (47) in the form of (18) as follows:

$$M + HFE + E^T F^T H^T < 0 \tag{48}$$

where

$$M = \begin{bmatrix} -P & \bar{A}^T & \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & \cdots & \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & C_2^T \\ \bar{A} & -P^{-1} & 0 & \cdots & 0 & 0 \\ \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & 0 & -\alpha_1 I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & 0 & 0 & \cdots & -\alpha_q I & 0 \\ C_2 & 0 & 0 & \cdots & 0 & -I \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & H_e^T & 0 & \cdots & 0 & 0 \end{bmatrix}^T$$

$$E = [E_e \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0].$$

Applying Lemma 4 to (48), it follows that (48) holds if and only if there exists a positive scalar parameter ε_1 such that the following LMI holds

$$\begin{bmatrix} -P & \bar{A}^T & \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & \cdots & \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & C_2^T & 0 & \varepsilon_1 E_e^T \\ \bar{A} & -P^{-1} & 0 & \cdots & 0 & 0 & H_e & 0 \\ \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & 0 & -\alpha_1 I & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & 0 & 0 & \cdots & -\alpha_q I & 0 & 0 & 0 \\ C_2 & 0 & 0 & \cdots & 0 & -I & 0 & 0 \\ 0 & H_e^T & 0 & \cdots & 0 & 0 & -\varepsilon_1 I & 0 \\ \varepsilon_1 E_e & 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} < 0. \tag{49}$$

Applying the congruence transformation $\mathrm{diag}\{I, P, I, \cdots, I, I, I, I\}$ to (49), we get

$$\begin{bmatrix} -P & \bar{A}^T P & \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & \cdots & \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & C_2^T & 0 & \varepsilon_1 E_e^T \\ P\bar{A} & -P & 0 & \cdots & 0 & 0 & PH_e & 0 \\ \alpha_1\tilde{\Gamma}_1^{\frac{1}{2}} & 0 & -\alpha_1 I & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_q\tilde{\Gamma}_q^{\frac{1}{2}} & 0 & 0 & \cdots & -\alpha_q I & 0 & 0 & 0 \\ C_2 & 0 & 0 & \cdots & 0 & -I & 0 & 0 \\ 0 & H_e^T P & 0 & \cdots & 0 & 0 & -\varepsilon_1 I & 0 \\ \varepsilon_1 E_e & 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} < 0. \tag{50}$$

Further applying the congruence transformations $\mathrm{diag}\{T_1, T_1, I, \cdots, I, I, I, I\}$ to (50), we obtain

$$\begin{bmatrix} J_{11} & J_{12}^T \\ J_{12} & J_{22} \end{bmatrix} < 0 \tag{51}$$

where

$$J_{11} = \begin{bmatrix} -S^{-1} & -I & S^{-1}A^T & S^{-1}\left(A^T R + C^T Q_2^T + Q_1^T\right) \\ * & -R & A^T & A^T R + C^T Q_2^T \\ * & * & -S^{-1} & -I \\ * & * & * & -R \end{bmatrix} \tag{52}$$

$$J_{12} = \begin{bmatrix} \Gamma_1^{\frac{1}{2}} S^{-1} & \Gamma_1^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma_q^{\frac{1}{2}} S^{-1} & \Gamma_q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (L_2 - Q_3)S^{-1} & L_2 & 0 & 0 \\ 0 & 0 & H_1^T & H_1^T R + H_2^T Q_2^T \\ \varepsilon_1 E S^{-1} & \varepsilon_1 E & 0 & 0 \end{bmatrix} \tag{53}$$

J_{22} is defined in (54) at the top of the next page. Also, performing the congruence transformation $\mathrm{diag}\{S, I, S, I, I, I, \cdots, I, I, I, I\}$ to (51) results in (36). Hence, from Corollary 1, we conclude that (6) is exponentially mean-square quadratically stable and (11) is satisfied.

Furthermore, if the LMIs (33)–(36) are feasible, then we have $\begin{bmatrix} -S & -S \\ -S & -R \end{bmatrix} < 0$, i.e., $\begin{bmatrix} S^{-1} & I \\ I & R \end{bmatrix} > 0$. It follows directly from $XX^{-1} = I$ that $I - RS^{-1} = X_{12}Y_{12}^T < 0$. Hence, one can always find square and nonsingular X_{12} and Y_{12}, and (40), is then obtained from (43), which concludes the proof. ■

In view of (40), we could make the linear transformation on the state estimate

$$\bar{x}_k = X_{12}\hat{x}_k \tag{55}$$

and then have a new representation form of the filter as follows:

$$\begin{cases} \bar{x}_{k+1} = \breve{A}\bar{x}_k + \breve{K}y_k \\ \hat{z}_k = \breve{L}_2\bar{x}_k \end{cases} \tag{56}$$

where

$$\breve{A} = Q_1(S-R)^{-1}, \; \breve{K} = Q_2, \; \breve{L}_2 = Q_3(S-R)^{-1}. \tag{57}$$

We can now see from (56) that, the filter parameters can be obtained directly by solving LMIs (33)–(36) without solving $I - RS^{-1} = X_{12}Y_{12}^T$ for X_{12} in (40).

$$J_{22} = \begin{bmatrix} -\alpha_1 I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1 I & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\alpha_q I & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_q I & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_1 I & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} \tag{54}$$

Finally, we like to point out that, the upper bound β can be readily minimized within the developed LMI framework, which is illustrated in the next section.

IV. Illustrative Example

Consider a discrete-time system described by (1) with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties as follows:

$$\begin{aligned} A &= \begin{bmatrix} -0.5 & 0 & -0.1 \\ 0.6 & 0.3 & 0.2 \\ 0.1 & 0.4 & 0.1 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0.3 \\ 0 \\ 0.2 \end{bmatrix} \\ D_{11} &= 1 \\ C &= [1 \quad -0.6 \quad 2] \\ L_2 &= [1 \quad 0 \quad 2] \\ H_1 &= \begin{bmatrix} 0.5 \\ 0.6 \\ 0 \end{bmatrix} \\ H_2 &= 0.6, \ E = [0.8 \quad 0 \quad 0] \end{aligned}$$

where w_k is a zero mean Gaussian white noise sequence with covariance $\Phi = 1$. Denote $I_1 = [1\ 0\ 0]^T$, and assume that the stochastic nonlinear functions $f(x_k)$ and $g(x_k)$ satisfy

$$\begin{aligned} \mathbb{E}\{f(x_k)|x_k\} &= 0 \\ \mathbb{E}\{g(x_k)|x_k\} &= 0 \\ \mathbb{E}\left\{f(x_k)f(x_k)^T|x_k\right\} &= I_1 I_1^T x_k^T \mathrm{diag}\{0.5, 0.8, 0.6\} x_k \\ &\quad + (0.1 I_1)\big(0.1 I_1^T\big)\mathrm{diag}\{1, 0.5, 0.8\} x_k \\ \mathbb{E}\big\{g(x_k)g(x_k)^T|x_k\big\} &= x_k^T \mathrm{diag}\{0.5, 0.8, 0.6\} x_k + 0.1 x_k^T \\ &\quad \cdot \mathrm{diag}\{1, 0.5, 0.8\} x_k. \end{aligned}$$

In this example, we wish to achieve the smallest β. Therefore, we employ Matlab LMI ToolBox (function mincx) to minimize β in (33) by using Theorem 3 with $q = 2$. we obtain the minimum H_2 performance $\beta_{\min} = 0.6415$, and

$$\begin{aligned} \breve{A} &= \begin{bmatrix} -0.7779 & 0.1843 & 0.1091 \\ 0.5397 & 0.1935 & -0.0261 \\ -0.4353 & 1.5981 & 0.0363 \end{bmatrix} \\ \breve{K} &= \begin{bmatrix} -0.3821 \\ 0.1830 \\ -1.9477 \end{bmatrix} \\ \breve{L}_2 &= [-0.2072 \quad -0.1303 \quad -0.1473]. \end{aligned}$$

V. Conclusion

A robust H_2 filter has been designed for a class of uncertain discrete time nonlinear stochastic systems. A key technology has been used to convert the matrix trace terms into the linear matrix inequalities. The filter has been obtained under a unified flexible LMI framework, and sufficient conditions for the solvability of the H_2 filtering problem have been given in terms of a set of feasible LMIs. Our results can be extended to the robust H_2 output control problem.

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