Technical Notes and Correspondence

Robust H_∞ Control With Missing Measurements and Time Delays

Fuwen Yang, Zidong Wang, Daniel W. C. Ho, and Mahbub Gani

***Abstract*—In this technical note, the robust H_∞ control problem is investigated for a class of stochastic uncertain discrete time-delay systems with missing measurements. The parameter uncertainties enter into the state matrices, and the missing measurements are described by a binary switching sequence satisfying a conditional probability distribution. The purpose of the problem is to design a full-order dynamic feedback controller such that, for all possible missing observations and admissible parameter uncertainties, the closed-loop system is asymptotically mean-square stable and satisfies the prescribed H_∞ performance constraint. Delay-dependent conditions are derived under which the desired solution exists, and the controller parameters are designed by solving a linear matrix inequality (LMI). A numerical example is provided to illustrate the usefulness of the proposed design method.**

***Index Terms*—H_∞ control, missing measurements, parameter uncertainty, robust control, time-delay systems.**

I. Introduction

Dynamical systems with time delays have received much attention in the past few decades, since time delays exist in many practical systems, such as hydraulic processes, chemical systems, temperature processes, and are often a primary source of instability and performance degradation. A great number of research results have been reported for various time-delay systems, mainly on control problems [13], [14], [20], filtering problems [4], [5], [7], [22], and model reduction problems [21]. In almost all literature mentioned here, however, it is implicitly assumed that the system measurement always contains the real signal probably mixed with external disturbances. Unfortunately, this is not true in many practical applications. For example, due to sensor temporal failure or network transmission delay/loss, at certain time points, the system measurement may contain *noise only*, indicating that the real signal is missing with probability less than 1. Note that in network signal transmissions, the missing measurement is also called dropout or intermittence, see [6], [15].

Manuscript received July 25, 2005; revised February 23, 2006 and June 15, 2006. Recommended by Associate Editor A. Hansson. This work was supported in part by the Engineering and Physical Sciences Research Council (EPSRC) of the U.K. under Grant GR/S27658/01, an International Joint Project sponsored by the Royal Society of the U.K, City University of Hong Kong under Grant 7001992, the Alexander von Humboldt Foundation of Germany, the National Natural Science Foundation of China under Grant 60474049, and the Fujian Provincial Natural Science Foundation of China under Grant A0410012.

F. Yang was with the Department of Electrical Engineering, Fuzhou University, Fuzhou 350002, P. R. China and the Department of Electronic Engineering, King's College London, Strand, London, WC2R 2LS, U.K. He is now with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, U.K. (e-mail: fuwen.yang@brunel.ac.uk).

Z. Wang is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, U.K. (e-mail: Zidong.Wang@brunel.ac.uk).

D. W. C. Ho is with the Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong.

M. Gani is with the Department of Electronic Engineering, King's College London, Strand, London, WC2R 2LS, U.K.

Digital Object Identifier 10.1109/TAC.2007.904250

Typically, there have been two ways to model the probabilistic missing measurements. One way is to describe the missing measurement as a binary switching sequence that is specified by a conditional probability distribution in the output equation. The binary switching sequence is viewed as a Bernoulli distributed white sequence taking on values of 0 and 1. Much work has been done on such a model. In [9], the optimal recursive filter was obtained for the systems with missing measurement whose structure is similar to the Kalman filter. Recently, the variance-constrained filtering problem has been considered in [16], [17] for discrete-time stochastic systems with probabilistic missing measurements subject to norm-bounded parameter uncertainties. In [15], the Kalman filtering problem with intermittent observations has been studied, and the statistical convergence properties of the estimation error covariance have been established. Another way is to regard the systems with missing measurement as jump linear systems, and an incompleteness matrix can be used to quantify the missing data. In [10], [11], the robust filtering problem with missing data and uncertain systems has been investigated by using a jump Riccati equation approach.

On the other hand, the control problem for systems with missing measurements has many engineering applications. For instance, in target tracking control problems, it is always desired to make sure that the tracking error is bounded in the mean square if some measurements are missing. In network congestion control, when there are signals missing during the transmission process, it is hoped that the transmission process can still keep operating with acceptable accuracy. So far, to the best of the authors' knowledge, for discrete-time stochastic systems in the simultaneous presence of time delays, missing measurements and parameter uncertainties, the problem of robust H_∞ control has not received much research attention, and is still open.

In this technical note, we study the robust H_∞ control problem for a class of stochastic uncertain discrete time-delay systems with missing measurements. Similar to [15]–[17], the missing measurements are characterized as a binary switching sequence satisfying a conditional probability distribution. We aim at developing a linear matrix inequality (LMI) approach for designing the full-order dynamic feedback controller such that, for all possible missing observations and admissible parameter uncertainties, the closed-loop system is asymptotically mean-square stable and satisfies the prescribed H_∞ performance constraint. As a by-product, a subsequent optimization problem is also formulated within the LMI framework so as to pursue a suboptimal system performance. An example is provided to illustrate the numerical efficiency of the proposed method.

Notation

The notation used here is fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). The superscript "T" denotes the transpose. $\mathbb{E}\{x\}$ stands for the expectation of x and $\mathbb{E}\{x|y\}$ for the expected value of x conditional on y. $\mathrm{Prob}\{\cdot\}$ means the occurrence probability of the event "$\cdot$". $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$ denotes a block diagonal matrix whose diagonal blocks are given by $M_1, M_2, \ldots, M_n$. $l_2[0,\infty)$ is the space of square integrable vectors. In symmetric block matrices, "$*$" is used as an ellipsis for terms induced by symmetry.

II. Problem Formulation and Preliminaries

Consider the following class of uncertain linear discrete time-delay systems

$$\begin{cases} x_{k+1} = (A+\Delta A)x_k + (A_d+\Delta A_d)x_{k-d} + Bw_k + B_2u_k, \\ z_k = Cx_k + C_d x_{k-d} + Dw_k + B_1u_k \\ x_k = \phi_k, \quad k=-d,-d+1,\ldots,0 \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $z_k \in \mathbb{R}^r$ is the controlled output, $u_k \in \mathbb{R}^m$ is the control input, $w_k \in \mathbb{R}^q$ is the disturbance input belonging to $l_2[0,\infty)$, A, A_d, B, B_1, B_2, C, C_d and D are known real constant matrices with appropriate dimensions. ΔA and ΔA_d are unknown matrices representing parameter uncertainties, $d>0$ is a known time delay, ϕ_k is a real-valued initial function on $[-d,0]$. In this technical note, the parameter uncertainties are assumed to be of the form

$$[\,\Delta A \quad \Delta A_d\,] = HF_k[\,E_1 \quad E_2\,] \tag{2}$$

where H, E_1 and E_2 are known real constant matrices of appropriate dimensions, and F_k represents an unknown real-valued time-varying matrix satisfying $F_kF_k^T \le I$.

The measurements, which may contain missing data, are described by

$$y_k = \gamma_k C_2 x_k + D_2 w_k \tag{3}$$

where the stochastic variable $\gamma_k \in \mathbb{R}$ is a Bernoulli distributed white sequence taking the values of 0 and 1 with

$$\mathrm{Prob}\{\gamma_k = 1\} = \mathbb{E}\{\gamma_k\} := \beta \tag{4}$$

$$\mathrm{Prob}\{\gamma_k = 0\} = 1-\mathbb{E}\{\gamma_k\} := 1-\beta \tag{5}$$

and $\beta \in \mathbb{R}$ is a known positive scalar. $y_k \in \mathbb{R}^p$ is a measured output and w_k is defined in (1), C_2 and D_2 are known real constant matrices of appropriate dimensions.

Remark 1: The increasing use of digital computers in control systems has led to considerable activity in the field of discrete-time and digital control systems. The system (1) encompasses many state space models of uncertain delay systems and can be used to represent many important physical systems; for example, cold rolling mills, wind tunnel and water resources systems, where modeling errors and time delays become concerns for controller design [8]. Note that, for the controlled output variable z_k, we consider both the delay-free term Cx_k and delayed term C_dx_{k-d}, where the constant matrices C and C_d can be viewed as parameters. The reason for including the delayed term is that, in many engineering systems such as temperature control systems, the controlled output (temperature) is always a time delayed signal, see [19]. Hence, the model presented in this technical note is meaningful in practice.

Remark 2: The system measurement mode (3), which can be used to represent missing measurements or uncertain observations, was first introduced in [9], and has been subsequently studied in many papers, see e.g., [15]–[17]. Note that when the real signal is missing (i.e., $\gamma_k = 0$), the system measurement contains *noise only*. Such a case does happen in practice. For example, in target tracking, due to high maneuverability of the tracked target, there may be a nonzero probability that any observation consists of noise alone if the target is absent, i.e., the measurements are not consecutive but contain missing observations. On the other hand, as discussed in the introduction, there are still other ways to model the missing phenomenon, such as those using randomly delayed sensor output and probabilistic jumps.

In this technical note, we consider the following full-order dynamic controller for system (1)–(3):

$$\begin{cases} \hat{x}_{k+1} = A_c\hat{x}_k + B_cy_k \\ u_k = C_c\hat{x}_k \end{cases} \tag{6}$$

where $\hat{x}_k$ is the state estimate, and A_c, B_c and C_c are the parameters to be determined.

Defining

$$\eta_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \tag{7}$$

we have the following closed-loop system by substituting (6) and (3) into (1):

$$\begin{cases} \eta_{k+1} = \tilde{A}\eta_k + (\gamma_k-\beta)\tilde{A}_1\eta_k + \tilde{A}_dZ\eta_{k-d} + \tilde{B}w_k \\ z_k = \tilde{C}\eta_k + C_dZ\eta_{k-d} + Dw_k \end{cases} \tag{8}$$

where

$$\tilde{A} = \begin{bmatrix} A+\Delta A & B_2C_c \\ \beta B_cC_2 & A_c \end{bmatrix} = \begin{bmatrix} A & B_2C_c \\ \beta B_cC_2 & A_c \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} F_k[\,E_1 \quad 0\,] := \bar{A} + \tilde{H}F_k\tilde{E}_1 \tag{9}$$

$$\tilde{A}_d = \begin{bmatrix} A_d+\Delta A_d \\ 0 \end{bmatrix} = \begin{bmatrix} A_d \\ 0 \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} F_kE_2 := \bar{A}_d + \tilde{H}F_k\tilde{E}_2, \quad \tilde{A}_1 = \begin{bmatrix} 0 & 0 \\ B_cC_2 & 0 \end{bmatrix} \tag{10}$$

$$\tilde{B} := \begin{bmatrix} B \\ B_cD_2 \end{bmatrix}, \quad \tilde{C} := [\,C \quad B_1C_c\,], \quad Z := [\,I \quad 0\,]. \tag{11}$$

It should be noticed that the closed-loop system (8) is actually a stochastic parameter system, since it contains the stochastic quantity γ_k. Therefore, in the sequel, we will use the notion of stochastic stability in the mean-square sense.

Our objective in this technical note is to design a controller of the type (6) for the system (1), such that for all possible missing measurement in (3), the closed-loop system satisfies the following two requirements simultaneously:

Q1) The closed-loop system (8) is asymptotically mean-square stable.

Q2) Under the zero-initial condition, the controlled output z_k satisfies

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|z_k\|^2\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|w_k\|^2\} \tag{12}$$

for all nonzero w_k, where $\gamma>0$ is a prescribed scalar.

It can be easily seen that the requirement Q1) ensures the robust stability of the controlled system, while the requirement Q2) given in (12) guarantees the desired robust performance constraints, in this case, the disturbance rejection attenuation property.

III. Robust H_∞ Controller Design

Lemma 1: (S-procedure) [18] Let $M = M^T < 0$, H and E be real matrices of appropriate dimensions, with F satisfying (2), then

$$M + HFE + E^TF^TH^T < 0 \tag{13}$$

if and only if there exists a positive scalar $\varepsilon>0$ such that

$$M + \frac{1}{\varepsilon}HH^T + \varepsilon E^TE < 0 \tag{14}$$

or equivalently

$$\begin{bmatrix} M & H & \varepsilon E^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon E & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{15}$$

The following theorem provides a delay-dependent sufficient condition for the closed-loop system (8) to be asymptotically mean-square stable and for the controlled output z_k to satisfy the H_∞ disturbance attenuation in (12).

Theorem 1: Given a scalar $\gamma > 0$ and the controller parameters A_c, B_c, and C_c. The closed-loop system (8) is asymptotically mean-square stable and the controlled output z_k satisfies (12) if there exist positive–definite matrices $P = P^T > 0$, $Q = Q^T > 0$ and $W = W^T > 0$ satisfying (16) as shown at the bottom of the page, where $\alpha_1 := [(1-\beta)\beta]^{1/2}$ and $d_1 := d^{1/2}$.

Proof: Denote

$$\xi_k = \eta_{k+1} - \eta_k. \tag{17}$$

Then, it follows from (8) that

$$\eta_{k+1} = (\tilde{A}+\tilde{A}_d Z)\eta_k + (\gamma_k - \beta)\tilde{A}_1\eta_k - \tilde{A}_d Z\sum_{i=k-d}^{k-1}\xi_i + \tilde{B}w_k \tag{18}$$

and

$$\xi_k = (\tilde{A}+\tilde{A}_d Z - I)\eta_k + (\gamma_k - \beta)\tilde{A}_1\eta_k - \tilde{A}_d Z\sum_{i=k-d}^{k-1}\xi_i + \tilde{B}w_k. \tag{19}$$

Let $\Theta_k = [\eta_k^T, \eta_{k-1}^T, \ldots, \eta_{k-d}^T]^T$ and define a Lyapunov functional for the system (18) as

$$V_k(\Theta_k) = \eta_k^T P\eta_k + \sum_{i=k-d}^{k-1}\eta_i^T Z^T QZ\eta_i + \sum_{l=-d+1}^{0}\sum_{i=k-1+l}^{k-1}\xi_i^T Z^T WZ\xi_i \tag{20}$$

where P, Q and W are positive definite matrices, and Z is defined in (11). The difference of the Lyapunov functional (20) along the trajectory of (18) is obtained as follows:

$$\begin{aligned}
\Delta V_k :=& \mathbb{E}\{V_{k+1}(\Theta_{k+1})|\Theta_k\} - V_k(\Theta_k)\\
=& \eta_k^T(\tilde{A}+\tilde{A}_d Z)^T P(\tilde{A}+\tilde{A}_d Z)\eta_k\\
&+ \mathbb{E}\{(\gamma_k-\beta)^2\}\eta_k^T\tilde{A}_1^T P\tilde{A}_1\eta_k\\
&+ \left(\tilde{A}_d Z\sum_{i=k-d}^{k-1}\xi_i\right)^T P\tilde{A}_d Z\sum_{i=k-d}^{k-1}\xi_i\\
&- 2\eta_k^T(\tilde{A}+\tilde{A}_d Z)^T P\tilde{A}_d Z\sum_{i=k-d}^{k-1}\xi_i\\
&- 2\left(\tilde{A}_d Z\sum_{i=k-d}^{k-1}\xi_i\right)^T P\tilde{B}w_k + 2\eta_k^T(\tilde{A}+\tilde{A}_d Z)^T P\tilde{B}w_k\\
&+ w_k^T\tilde{B}^T P\tilde{B}w_k + \eta_k^T(Z^T QZ - P)\eta_k\\
&- \eta_{k-d}^T Z^T QZ\eta_{k-d} + d\mathbb{E}\{\xi_k^T Z^T WZ\xi_k\}\\
&- \sum_{i=k-d}^{k-1}\xi_i^T Z^T WZ\xi_i.
\end{aligned} \tag{21}$$

From (17) and (19), we obtain

$$\sum_{i=k-d}^{k-1}\xi_i = \eta_k - \eta_{k-d} \tag{22}$$

and

$$\xi_k = (\tilde{A}-I)\eta_k + (\gamma_k-\beta)\tilde{A}_1\eta_k + \tilde{A}_d Z\eta_{k-d} + \tilde{B}w_k. \tag{23}$$

By substituting (22) and (23) into (21), and considering $Z\tilde{A}_1 = 0$ and the relation $\mathbb{E}\{(\gamma_k-\beta)^2\} = (1-\beta)\beta$, we have

$$\Delta V_k \le \zeta_k^T \Pi\zeta_k \tag{24}$$

where

$$\zeta_k = \begin{bmatrix}\eta_k\\ Z\eta_{k-d}\\ w_k\end{bmatrix}, \quad \Pi := \begin{bmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ \Pi_{12}^T & \Pi_{22} & \Pi_{23}\\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33}\end{bmatrix} \tag{25}$$

$$\begin{aligned}
\Pi_{11} =& \tilde{A}^T P\tilde{A} - P + Z^T QZ + (1-\beta)\beta\tilde{A}_1^T P\tilde{A}_1\\
&+ d(\tilde{A}-I)^T Z^T WZ(\tilde{A}-I)
\end{aligned} \tag{26}$$

$$\Pi_{12} = \tilde{A}^T P\tilde{A}_d + d(\tilde{A}-I)^T Z^T WZ\tilde{A}_d \tag{27}$$

$$\Pi_{13} = \tilde{A}^T P\tilde{B} + d(\tilde{A}-I)^T Z^T WZ\tilde{B} \tag{28}$$

$$\Pi_{22} = \tilde{A}_d^T P\tilde{A}_d + d\tilde{A}_d^T Z^T WZ\tilde{A}_d - Q \tag{29}$$

$$\Pi_{23} = \tilde{A}_d^T P\tilde{B} + d\tilde{A}_d^T Z^T WZ\tilde{B} \tag{30}$$

$$\Pi_{33} = \tilde{B}^T P\tilde{B} + d\tilde{B}^T Z^T WZ\tilde{B}. \tag{31}$$

By Schur complement and tedious but straightforward manipulations, we arrive at the conclusion that (16) implies $\Pi < 0$. Therefore, for all nonzero η_k, we have $\Delta V_k < 0$, and it then follows from the Lyapunov–Krasovskii stability theory that the closed-loop system (8) is asymptotically mean-square stable [1].

Next, for any nonzero w_k, it follows from (8) and (24) that

$$\begin{aligned}
\mathbb{E}\{V_{k+1}&(\Theta_{k+1})|\Theta_k\} - \mathbb{E}\{V_k(\Theta_k)\} + \mathbb{E}\{\tilde{z}_k^T\tilde{z}_k\} - \gamma^2\mathbb{E}\{w_k^T w_k\}\\
&\le \mathbb{E}\{\zeta_k^T\Pi\zeta_k + (\tilde{C}\eta_k + C_d Z\eta_{k-d} + Dw_k)^T\\
&\quad \cdot(\tilde{C}\eta_k + C_d Z\eta_{k-d} + Dw_k) - \gamma^2 w_k^T w_k\}\\
&= \mathbb{E}\left\{\zeta_k^T\Sigma\zeta_k^T\right\}
\end{aligned} \tag{32}$$

where

$$\Sigma := \begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \Sigma_{13}\\ \Sigma_{12}^T & \Sigma_{22} & \Sigma_{23}\\ \Sigma_{13}^T & \Sigma_{23}^T & \Sigma_{33}\end{bmatrix} \tag{33}$$

$$\Sigma_{11} = \Pi_{11} + \tilde{C}^T\tilde{C}, \quad \Sigma_{12} = \Pi_{12} + \tilde{C}^T C_d \tag{34}$$

$$\Sigma_{13} = \Pi_{13} + \tilde{C}^T D, \quad \Sigma_{22} = \Pi_{22} + C_d^T C_d \tag{35}$$

$$\Sigma_{23} = \Pi_{23} + C_d^T D, \quad \Sigma_{33} = \Pi_{33} + D^T D - \gamma^2 I. \tag{36}$$

and Π_{11}, Π_{12}, Π_{13}, Π_{21}, Π_{22}, Π_{23}, and Π_{33} are defined in (26)–(31).

$$\begin{bmatrix}
-P & 0 & P\tilde{A} & P\tilde{A}_d & P\tilde{B} & 0 & 0 & 0\\
0 & -I & \tilde{C} & C_d & D & 0 & 0 & 0\\
\tilde{A}^T P & \tilde{C}^T & -P & 0 & 0 & \alpha_1\tilde{A}_1^T P & d_1(\tilde{A}-I)^T Z^T & Z^T\\
\tilde{A}_d^T P & C_d^T & 0 & -Q & 0 & 0 & d_1\tilde{A}_d^T Z^T & 0\\
\tilde{B}^T P & D^T & 0 & 0 & -\gamma^2 I & 0 & d_1\tilde{B}^T Z^T & 0\\
0 & 0 & \alpha_1 P\tilde{A}_1 & 0 & 0 & -P & 0 & 0\\
0 & 0 & d_1 Z(\tilde{A}-I) & d_1 Z\tilde{A}_d & d_1 Z\tilde{B} & 0 & -W^{-1} & 0\\
0 & 0 & Z & 0 & 0 & 0 & 0 & -Q^{-1}
\end{bmatrix} < 0 \tag{16}$$

It follows from (16) and (32) that

$$\mathbb{E}\{V_{k+1}(\Theta_{k+1})|\Theta_k\} - \mathbb{E}\{V_k(\Theta_k)\} + \mathbb{E}\{\tilde{z}_k^T \tilde{z}_k\} - \gamma^2 \mathbb{E}\{w_k^T w_k\} < 0. \quad (37)$$

Now, summing (37) from 0 to ∞ with respect to k yields

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\} + \mathbb{E}\{V_0\} - \mathbb{E}\{V_\infty\}. \quad (38)$$

Since the system (8) is asymptotically mean-square stable, it is not difficult to see that the following inequality

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\} \quad (39)$$

holds under the zero initial condition, which completes the proof. ■

Next, the controller design problem is solved in the following theorem, and the controller parameters are given in terms of the solution to an LMI.

Theorem 2: Given a scalar $\gamma > 0$. The system (8) is asymptotically mean-square stable and the H_∞-norm constraint (12) is achieved for all nonzero w_k, if there exist positive definite matrices $R = R^T > 0$, $S = S^T > 0$, $Y = Y^T > 0$ and $Q = Q^T > 0$, real matrices Q_1, Q_2 and Q_3, and a real scalar $\varepsilon > 0$ such that we have (40) (shown at the bottom of the page) for some given constant matrix $G > 0$, where $\alpha_1 := [(1-\beta)\beta]^{1/2}$ and $d_1 := d^{1/2}$. Moreover, the controller parameters are given by

$$A_c = X_{12}^{-1}\left[Q_1(S-R)^{-1}X_{12} - RB_2Q_3\right] \quad (41)$$

$$B_c = X_{12}^{-1}Q_2 \quad (42)$$

$$C_c = Q_3(S-R)^{-1}X_{12}. \quad (43)$$

where the matrix X_{12} comes from the factorization $I - RS^{-1} = X_{12}Y_{12}^T < 0$.

Proof: We begin with rewriting (16) as

$$M + HFE + E^T F^T H^T < 0 \quad (44)$$

where $M, H,$ and E are given in the unnumbered equation shown at the bottom of the next page.

By applying Lemma 1 to (44), we know that (44) holds if and only if there exists a positive scalar parameter ε such that the LMI (45) that is shown at the bottom of the next page holds.

Recall that our goal is to derive the expression of the controller parameters from (6). To do this, we partition P and P^{-1} as

$$P = \begin{bmatrix} R & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} S & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \quad (46)$$

$$\left[\begin{array}{ccccccccccccc}
-G-G^T+G^TSG & -I & 0 & A+B_2Q_3 & A & A_d & B & 0 & 0 & 0 & 0 & H & 0 \\
* & -R & 0 & RA+\beta Q_2C_2+Q_1 & RA+\beta Q_2C_2 & RA_d & RB+Q_2D_2 & 0 & 0 & 0 & 0 & RH & 0 \\
* & * & -I & C+B_1Q_3 & C & C_d & D & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -S & -S & 0 & 0 & 0 & \alpha_1 C_2^T Q_2^T & d_1(A-I)^T + d_1 Q_3^T B_2^T & Q & 0 & \varepsilon E_1^T \\
* & * & * & * & -R & 0 & 0 & 0 & \alpha_1 C_2^T Q_2^T & d_1(A-I)^T & Q & 0 & \varepsilon E_1^T \\
* & * & * & * & * & -Q & 0 & 0 & 0 & d_1 A_d^T & 0 & 0 & \varepsilon E_2^T \\
* & * & * & * & * & * & -\gamma^2 I & 0 & 0 & d_1 B^T & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -S & -S & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -R & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -Y & 0 & d_1 H & 0 \\
* & * & * & * & * & * & * & * & * & * & -Q & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\varepsilon I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon I
\end{array}\right] < 0 \quad (40)$$

where the partitioning of P and P^{-1} is compatible with that of $\bar{A}$ defined in (9), i.e., $R \in R^{n\times n}, X_{12} \in R^{n\times n}, X_{22} \in R^{n\times n}, S \in R^{n\times n}$, $Y_{12} \in R^{n\times n}, Y_{22} \in R^{n\times n}$. Define

$$T_1 = \begin{bmatrix} S & I \\ Y_{12}^T & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & R \\ 0 & X_{12}^T \end{bmatrix} \tag{47}$$

which imply that $PT_1 = T_2$ and $T_1^T P T_1 = T_1^T T_2$.

Now define the change of controller parameters as follows:

$$Q_1 = (RB_2C_c + X_{12}A_c)Y_{12}^T S, \quad Q_2 = X_{12}B_c, \quad Q_3 = C_c Y_{12}^T S. \tag{48}$$

By applying the congruence transformations $\operatorname{diag}\{T_1, I, T_1, I, I, T_1, I, I, I, I\}$ to (45), we obtain (49) shown at the bottom of the next page.

Also, performing the congruence transformation $\operatorname{diag}\{I, I, I, S, I, I, I, S, I, I, Q, I, I\}$ to (49) and defining $Y = W^{-1}$ lead to (50) shown at the bottom of the next page.

On the other hand, we know from the fact of $(S^{-1} - G)^T S (S^{-1} - G) \geq 0$ that $-S^{-1} \leq -G - G^T + G^T S G$. Hence, if (40) is satisfied, we obtain (50) and then conclude that the condition (16) holds. Therefore, by Theorem 1, the desired result follows immediately.

Furthermore, if the LMI (40) is feasible, then we have $\begin{bmatrix} -S & -S \\ -S & -R \end{bmatrix} < 0$, i.e., $\begin{bmatrix} S^{-1} & I \\ I & R \end{bmatrix} > 0$. It follows directly from $XX^{-1} = I$ that $I - RS^{-1} = X_{12}Y_{12}^T < 0$. Hence, one can always find square and nonsingular X_{12} and Y_{12} [12]. Therefore, (41)–(43) are obtained from (48), which completes the proof. ■

Remark 3: It is worth pointing out that an additional matrix $G > 0$ is introduced in the condition (40). When $G > 0$ is fixed, the addressed robust H_∞ controller can be obtained by solving the LMI (40) in Theorem 2. Note that the LMI (40) is a delay-dependent sufficient condition. Since the Schur complement and the S-procedure do not bring the conservatism, the overall conservatism actually results from the introduction of the matrix G and the use of the Lyapunov stability theory. Although we have tried to construct proper Lyapunov functional to increase the design flexibility, how to further reduce the conservatism is one of our future research topics.

Up to now, the controller has been designed to satisfy the requirements Q1) and Q2). As a by-product, the results in Theorem 2 also suggest the following optimization problem:

P1) *The optimal H_∞ control problem for uncertain stochastic time-delay systems with missing measurements is defined by*

$$\min_{S>0,\, R>0,\, Y>0,\, Q>0,\, Q_1,\, Q_2,\, Q_3,\, \varepsilon} \gamma \quad \text{subject to (40) with given } G. \tag{51}$$

Remark 4: In many engineering applications, the performances constraints are often specified *a priori*. For example, in Theorem 2, the controller is designed after H_∞ performance is prescribed. In fact, however, we can obtain an improved performance by optimization method. The aim of problem P1) is to exploit the design freedom to meet the optimal H_∞ performance.

IV. An Illustrative Example

In this section, we shall present an example to demonstrate the effectiveness and applicability of the proposed method. For this purpose, we consider the system described by (1) with parameters as follows:

$$A = \begin{bmatrix} -0.3 & 0 & -0.3 \\ 0 & 0.6 & 0.2 \\ 0.5 & 0 & 0.7 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5 \\ 0 \\ 0.2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \; H = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

$$E_1 = [0.1 \quad 0 \quad 0], \; E_2 = [0.1 \quad 0 \quad 0], \; C = [1 \quad 1 \quad 2]$$

$$C_d = [0.1 \quad 0 \quad 0], \; D = 0.1, \; B_1 = 1.$$

$$M = \begin{bmatrix} -P & 0 & P\bar{A} & P\bar{A}_d & P\tilde{B} & 0 & 0 & 0 \\ 0 & -I & \tilde{C} & C_d & D & 0 & 0 & 0 \\ \bar{A}^T P & \tilde{C}^T & -P & 0 & 0 & \alpha_1 \tilde{A}_1^T P & d_1(\bar{A} - I)^T Z^T & Z^T \\ \bar{A}_d^T P & C_d^T & 0 & -Q & 0 & 0 & d_1 \bar{A}_d^T Z^T & 0 \\ \tilde{B}^T P & D^T & 0 & 0 & -\gamma^2 I & 0 & d_1 \tilde{B}^T Z^T & 0 \\ 0 & 0 & \alpha_1 P\tilde{A}_1 & 0 & 0 & -P & 0 & 0 \\ 0 & 0 & d_1 Z(\bar{A} - I) & d_1 Z\bar{A}_d & d_1 Z\tilde{B} & 0 & -W^{-1} & 0 \\ 0 & 0 & Z & 0 & 0 & 0 & 0 & -Q^{-1} \end{bmatrix}$$

$$H = [P\tilde{H} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad d_1 Z\tilde{H} \quad 0]^T$$

$$E = [0 \quad 0 \quad \tilde{E}_1 \quad \tilde{E}_2 \quad 0 \quad 0 \quad 0 \quad 0].$$

$$\begin{bmatrix} -P & 0 & P\bar{A} & P\bar{A}_d & P\tilde{B} & 0 & 0 & 0 & P\tilde{H} & 0 \\ 0 & -I & \tilde{C} & C_d & D & 0 & 0 & 0 & 0 & 0 \\ \bar{A}^T P & \tilde{C}^T & -P & 0 & 0 & \alpha_1 \tilde{A}_1^T P & d_1(\bar{A} - I)^T Z^T & Z^T & 0 & \varepsilon \tilde{E}_1^T \\ \bar{A}_d^T P & C_d^T & 0 & -Q & 0 & 0 & d_1 \bar{A}_d^T Z^T & 0 & 0 & \varepsilon \tilde{E}_2^T \\ \tilde{B}^T P & D^T & 0 & 0 & -\gamma^2 I & 0 & d_1 \tilde{B}^T Z^T & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 P\tilde{A}_1 & 0 & 0 & -P & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 Z(\bar{A} - I) & d_1 Z\bar{A}_d & d_1 Z\tilde{B} & 0 & -W^{-1} & 0 & d_1 Z\tilde{H} & 0 \\ 0 & 0 & Z & 0 & 0 & 0 & 0 & -Q^{-1} & 0 & 0 \\ \tilde{H}^T P & 0 & 0 & 0 & 0 & 0 & d_1 \tilde{H}^T Z^T & 0 & -\varepsilon I & 0 \\ 0 & 0 & \varepsilon \tilde{E}_1 & \varepsilon \tilde{E}_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{45}$$

The output measurements with random missing measurements described by (3) have the parameters as follows:

$$C_2 = [\,0.1 \quad 0 \quad 0\,], \; D_2 = 0.1$$

and $\beta = 0.9$.

Now, letting $d = 3$ and $G = \mathrm{diag}\{0.05,\ 0.05,\ 0.05\}$, we solve the problem (P1) in the previous section by using the LMI ToolBox to minimize $\gamma > 0$. As a result, the optimal H_∞ performance is calculated as $\gamma_{\rm opt} = 3.3339$, and the corresponding controller parameters are

$$\begin{bmatrix}
-S^{-1} & -I & 0 & AS^{-1}+B_2Q_3S^{-1} & A & A_d & B & 0 & 0 & 0 & 0 & H & 0\\
* & -R & 0 & (RA+\beta Q_2C_2+Q_1)S^{-1} & RA+\beta Q_2C_2 & RA_d & RB+Q_2D_2 & 0 & 0 & 0 & 0 & RH & 0\\
* & * & -I & (C+B_1Q_3)S^{-1} & C & C_d & D & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -S^{-1} & -I & 0 & 0 & 0 & \alpha_1 S^{-1}C_2^TQ_2^T & d_1S^{-1}(A-I)^T+d_1S^{-1}Q_3^TB_2^T & S^{-1} & 0 & \varepsilon S^{-1}E_1^T\\
* & * & * & * & -R & 0 & 0 & 0 & \alpha_1 C_2^TQ_2^T & d_1(A-I)^T & I & 0 & \varepsilon E_1^T\\
* & * & * & * & * & -Q & 0 & 0 & 0 & d_1A_d^T & 0 & 0 & \varepsilon E_2^T\\
* & * & * & * & * & * & -\gamma^2 I & 0 & 0 & d_1B^T & 0 & 0 & 0\\
* & * & * & * & * & * & * & -S^{-1} & -I & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -R & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -W^{-1} & 0 & d_1H & 0\\
* & * & * & * & * & * & * & * & * & * & -Q^{-1} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -\varepsilon I & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon I
\end{bmatrix} < 0. \tag{49}$$

$$\begin{bmatrix}
-S^{-1} & -I & 0 & A+B_2Q_3 & A & A_d & B & 0 & 0 & 0 & 0 & H & 0\\
* & -R & 0 & RA+\beta Q_2C_2+Q_1 & RA+\beta Q_2C_2 & RA_d & RB+Q_2D_2 & 0 & 0 & 0 & 0 & RH & 0\\
* & * & -I & C+B_1Q_3 & C & C_d & D & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -S & -S & 0 & 0 & 0 & \alpha_1 C_2^TQ_2^T & d_1(A-I)^T+d_1Q_3^TB_2^T & Q & 0 & \varepsilon E_1^T\\
* & * & * & * & -R & 0 & 0 & 0 & \alpha_1 C_2^TQ_2^T & d_1(A-I)^T & Q & 0 & \varepsilon E_1^T\\
* & * & * & * & * & -Q & 0 & 0 & 0 & d_1A_d^T & 0 & 0 & \varepsilon E_2^T\\
* & * & * & * & * & * & -\gamma^2 I & 0 & 0 & d_1B^T & 0 & 0 & 0\\
* & * & * & * & * & * & * & -S & -S & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -R & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -Y & 0 & d_1H & 0\\
* & * & * & * & * & * & * & * & * & * & -Q & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -\varepsilon I & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon I
\end{bmatrix} < 0. \tag{50}$$

obtained as follows:

$$A_c = \begin{bmatrix} 8.5824 & 12.2458 & 19.7691 \\ -6.6224 & -7.7871 & -12.6333 \\ 3.9109 & 5.3596 & 8.6677 \end{bmatrix}$$
$$B_c = \begin{bmatrix} -20.2854 \\ -1.4694 \\ -0.1955 \end{bmatrix}, \ C_c = [0.0115 \quad 0.0078 \quad 0.0216].$$

V. Conclusions

The problem of robust H_∞ control has been considered in this technical note for stochastic uncertain discrete time-delay systems with missing measurements. The robust H_∞ controller has been designed in terms of a feasible LMI, which guarantees that the closed-loop system is asymptotically mean-square stable, and the controlled output satisfies the H_∞ performance constraint, for all possible missing observations and all admissible parameter uncertainties.

References

[1] K. J. Astrom, *Introduction to Stochastic Control Theory*. New York: Academic, 1970.

[2] A. E. Bouhtouri, D. Hinrichsen, and A. J. Pritchard, "H_∞-type control for discrete-time stochastic systems," *Int. J. Robust Nonlinear Contr.*, vol. 9, no. 13, pp. 923–948, 1999.

[3] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.

[4] H. Gao, J. Lam, C. Wang, and Y. Wang, "Delay-dependent output-feedback stabilisation of discrete-time systems with time-varying state delay," *IEE Proc. Contr. Theory Applicat.*, vol. 151, no. 6, pp. 691–698, 2004.

[5] H. Gao and C. Wang, "A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems," *IEEE Trans. Signal Processing*, vol. 52, pp. 1631–1640, Jun. 2004.

[6] Q. Ling and M. D. Lemmon, "Optimal dropout compensation in networked control systems," in *Proc. IEEE Conf. Decision Contr.*, Hawaii, USA, Dec. 2003, pp. 670–675.

[7] H. Liu, F. Sun, K. He, and Z. Sun, "Design of reduced-order H_∞ filter for Markovian jumping systems with time delay," *IEEE Trans. Circuits Syst. -II*, vol. 51, no. 11, pp. 607–612, Nov. 2004.

[8] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications*. Amsterdam, The Netherlands: North-Holland, 1987.

[9] N. Nahi, "Optimal recursive estimation with uncertain observation," *IEEE Trans. Inf. Theory*, vol. 15, pp. 457–462, Apr. 1969.

[10] A. V. Savkin and I. R. Petersen, "Robust filtering with missing data and a deterministic description of noise and uncertainty," *Int. J. Syst. Sci.*, vol. 28, no. 4, pp. 373–390, 1997.

[11] A. V. Savkin, I. R. Petersen, and S. O. R. Moheimani, "Model validation and state estimation for uncertain continuous-time systems with missing discrete-continuous data," *Comput. Elect. Eng.*, vol. 25, no. 1, pp. 29–43, 1999.

[12] C. Scherer, P. Gahient, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Automat. Control*, vol. 42, pp. 896–911, 1997.

[13] P. Shi, E. Boukas, R. Agarwal, and Y. Shi, "Optimal guaranteed cost control of uncertain discrete time-delay systems," *J. Comput. Appl. Math.*, vol. 157, no. 2, pp. 435–451, 2003.

[14] P. Shi, E. Boukas, and Y. Shi, "On stabilization of discrete-time Markovian jump systems with delay in state," *Stoch. Anal. Applicat.*, vol. 21, no. 4, pp. 935–951, 2003.

[15] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. Automat. Control*, vol. 49, pp. 1453–1464, Sep. 2004.

[16] Z. Wang, D. W. C. Ho, and X. Liu, "Variance-constrained filtering for uncertain stochastic systems with missing measurements," *IEEE Trans. Automat. Control*, vol. 48, pp. 1254–1258, Jul. 2003.

[17] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust finite-horizon filtering for stochastic systems with missing measurements," *IEEE Signal Processing Lett.*, vol. 12, pp. 437–440, Jun. 2005.

[18] L. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *Int. J. Contr.*, vol. 63, no. 4, pp. 741–750, Mar. 1996.

[19] L. Xie, E. Fridman, and U. Shaked, "Robust H_∞ control of distributed delay systems with application to combustion control," *IEEE Trans. Automat. Control*, vol. 46, pp. 1930–1935, Jul. 2001.

[20] D. Yue and J. Lam, "Non-fragile guaranteed cost control for uncertain descriptor systems with time-varying state and input delays," *Optim. Contr.: Applicat. Meth.*, vol. 26, no. 2, pp. 85–105, 2005.

[21] L. Zhang and B. Huang, "H_2 approximation of multiple input/output delay systems," *J. Process Contr.*, vol. 14, no. 6, pp. 627–634, Aug. 2004.

[22] X. Zhu, Y. C. Soh, and L. Xie, "Robust Kalman filter design for discrete time-delay systems," *Circuits, Syst. Signal Processing*, vol. 21, no. 3, pp. 319–335, 2002.

An Architecture for Design and Analysis of High-Performance Robust Antiwindup Compensators

Andrés Marcos, Matthew C. Turner, and Ian Postlethwaite

***Abstract*—In this note, a general framework for the design and analysis of high-performance robust antiwindup (AW) compensators is presented. The proposed framework combines the Weston–Postlethwaite AW scheme with ideas from residual generation and from robust control architectures based on high-performance nominal controllers. It is shown that the framework is well connected to the Youla controller parameterization and to fault tolerant/detection schemes. Furthermore, the proposed framework provides a transparent analysis of the interactions between the different design parameters which allows for a clearer design tradeoff between robust stability and robust performance for the saturated and unsaturated closed loops.**

***Index Terms*—Antiwindup (AW), high-performance systems, robust control.**

I. Introduction

Antiwindup (AW) compensation is a common approach used by control engineers to cope with actuator saturation, with many methods available to assist with its design (see, for example, [2], [8], [17], and references therein). With few exceptions, most available methods tackle the problems of stability and performance by (implicitly) assuming that the AW design inherits the robustness properties of the robust linear system [11], [18]. This makes some intuitive sense and if the uncertainty present in the real system is sufficiently small, standard AW techniques can be applied with some confidence. On the other hand, in [16] it was argued that robust stability of the linear system was only a *necessary* condition for robust stability of the saturated closed-loop system. Indeed, in that reference an example was given of a saturated closed-loop system which behaved well using a "good" static AW design but was actually destabilized when uncertainty was introduced.

Recently, several researchers [5], [6], [16] have approached the *robust* AW problem by trying to incorporate robustness directly into the

Manuscript received December 21, 2005; revised December 7, 2006. Recommended by Associate Editor S. Tarbouriech.

A. Marcos is with the Advance Projects Division, Deimos Space S.L., Madrid, 28760, Spain (e-mail: andres.marcos@deimos-space.com).

M. C. Turner and I. Postlethwaite are with the Department of Engineering, University of Leicester, University Road, Leicester LE1 7RH, U.K. (e-mail: mct6@le.ac.uk, ixp@le.ac.uk).

Color versions of one or more of the figures in this note are available at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2007.904276