Robust Synchronization of an Array of Coupled Stochastic Discrete-Time Delayed Neural Networks

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***Abstract*—This paper is concerned with the robust synchronization problem for an array of coupled stochastic discrete-time neural networks with time-varying delay. The individual neural network is subject to parameter uncertainty, stochastic disturbance, and time-varying delay, where the norm-bounded parameter uncertainties exist in both the state and weight matrices, the stochastic disturbance is in the form of a scalar Wiener process, and the time delay enters into the activation function. For the array of coupled neural networks, the constant coupling and delayed coupling are simultaneously considered. We aim to establish easy-to-verify conditions under which the addressed neural networks are synchronized. By using the Kronecker product as an effective tool, a linear matrix inequality (LMI) approach is developed to derive several sufficient criteria ensuring the coupled delayed neural networks to be globally, robustly, exponentially synchronized in the mean square. The LMI-based conditions obtained are dependent not only on the lower bound but also on the upper bound of the time-varying delay, and can be solved efficiently via the Matlab LMI Toolbox. Two numerical examples are given to demonstrate the usefulness of the proposed synchronization scheme.**

***Index Terms*—Coupled neural networks, discrete time, Kronecker product, matrix functional, robust synchronization, stochastic perturbation, time-varying delay.**

I. INTRODUCTION

THE last few decades have seen successful applications of delayed neural networks in various areas such as signal processing, fault diagnosis, pattern recognition, image processing, and some optimization problems, and a great deal of research results have been published in the literature; see [1], [11], [12], [21], [30]–[33], [37], [39], [40], [44], [45], and the references therein. When designing the neural networks or in the implementation of neural systems, due to the thermal noise in the electronic devices or the random fluctuations from the release of neurotransmitters, stochastic disturbances are almost inevitable. On the other hand, vital data, such as the signal transmission delays, the synaptic interconnection weights, and the neuron's fire rate, are usually acquired and measured by way of statistical estimation, and this would give rise to estimation errors. It has recently been recognized that both stochastic disturbances and parameter uncertainties are unavoidable when modeling and implementing real neural networks [27], and constitute the main causes for performance degradation of even instability of the system. Therefore, dynamical analysis of various uncertain stochastic neural networks with time delays has quickly become an attractive topic of research in the past few years; see, e.g., [20], [27], [33], and [38]–[40].

In engineering applications such as network-based control, time-series analysis, and system identification, it is often necessary to formulate a discrete-time analog of the continuous-time neural networks. Unfortunately, as pointed out in [28], the discretization cannot preserve the dynamics of the continuous-time counterpart even for a small sampling period. Consequently, much effort has been devoted to the investigation of the dynamical behaviors of delayed discrete-time neural networks (DDNNs) with *neither* modeling errors *nor* stochastic disturbances via Lyapunov approaches [8], [16], [18], [20], [34]. It should be noted that the parameter uncertainties and stochastic disturbances have been largely overlooked due primarily to the difficulty in mathematical analysis, despite the promising application potentials of uncertain stochastic DDNNs in practical areas such as image processing with circuit implementations, economic dispatch, and unit commitment in power systems. Therefore, one of the objectives of this paper is to deal with the uncertainties as well as the stochasticity of the DDNNs.

Synchronization means that two or more systems that are either periodic or chaotic adjust each other to give rise to a common dynamical behavior. Since Pecora and Carroll introduced a method to synchronize two identical chaotic systems with different initial conditions [29], chaos synchronization has gained considerable attention from various research areas such as biological networks, secure communication, and chemical reactions [3], [9], [13], [26]. It has been shown that synchronization could be induced either by coupling or by external forcing. More recently, arrays of coupled systems have stirred much research interest due to the fact that they can exhibit many interesting phenomena such as synchronization and autowaves, and they are important in modeling populations of interacting biological systems [42], [43]. Among them, the synchronization in coupled identical delayed neural networks has been shown to have an important impact on the fundamental science (e.g., the self-organization behavior in the brain). The synchronization hypothesis for brain activities was formulated in 1938 by Russian neurophysiologists [36], and the modern neurophysiological experiments reinforced that synchronous oscillations of neural activities in the brain structures (such as olfactory bulb and cortex, visual cortex, hippocampus, neocortex, and

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thalamo–cortical systems) play a very important role in information processing and coding [7], [10]. For example, the theta rhythm related to the behavior of animals is produced by partial synchronization of neuronal activity in the hippocampal network [41], and an excessive synchronization of the neuronal activity over a wide area in the brain results in the epileptic rhythm [35]. In [6], the Lyapunov functional method and the Hermitian matrix theory have been employed to investigate the global synchronization of the coupled neural networks by defining a distance between any point and the synchronization manifold. For details concerning synchronization of neural networks, please see [4], [22], [42], [43], and the references cited therein.

To the best of the authors' knowledge, up to now, there have been very few results dealing with the robust synchronization problem for an array of uncertain stochastic discrete-time neural networks with time-varying delays. It is, therefore, the main focus of this paper to analyze the globally exponential synchronization problem for an array of delayed neural systems with stochastic disturbances and parameter uncertainties. By employing the matrix functional method in combination with the properties of Kronecker product, several delay-dependent criteria are obtained ensuring that the addressed neural networks are globally robustly synchronized in the mean square. These conditions, which are expressed in the form of linear matrix inequalities (LMIs), can be easily solved by the Matlab LMI Toolbox. It is known that the delay-dependent conditions are generally less conservative than the delay-independent ones especially when the delay in the system is small. The sufficient conditions acquired in this paper are dependent not only on the lower bound but also on the upper bound of the time-varying delay.

The rest of this paper is organized as follows. In Section II, the model studied in this paper is presented, and some definitions and lemmas are introduced. In Section III, the robust synchronization problem is studied and some sufficient conditions in the form of LMIs are developed. In Section IV, two illustrative examples are provided to demonstrate the effectiveness of the proposed criteria and, finally, conclusions are drawn in Section V.

Notations: Throughout this paper, E_m is an m-dimensional identity matrix. $\mathbb{R}^n$ is the n-dimensional Euclidean space and the notation $\|\cdot\|$ refers to the Euclidean vector norm. For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). The Kronecker product of an $n \times m$ matrix X and a $p \times q$ matrix Y is defined by an $np \times mq$ matrix $X \otimes Y$. We use $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ to denote the minimum and maximum eigenvalue of a symmetric matrix. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all $\mathcal{P}$-null sets and is right continuous) and generated by Brownian motion $\{\omega(s) : 0 \leq s \leq t\}$, where we associate Ω with the canonical space generated by $\omega(t)$, and denote $\mathcal{F}$ the associated σ-algebra generated by $\{\omega(t)\}$ with all the probability measure $\mathcal{P}$. For integers a and b with $a < b$, we use $\mathbb{N}[a, b]$ to denote the discrete interval given by $\mathbb{N}[a, b] = \{a, a+1, \ldots, b-1, b\}$. Let $C(\mathbb{N}[\alpha, \beta], \mathbb{R}^n)$ be the set of all functions $\phi : \mathbb{N}[\alpha, \beta] \to \mathbb{R}^n$. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$ and the asterisk $*$ in a matrix is used to denote the term that is induced by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. Model Description and Some Preliminaries

In this section, to facilitate the readers, we will present the uncertain stochastic discrete-time delayed neural networks in a step-by-step manner. Let us start with the commonly studied discrete-time neural networks described by the following equation:

$$\begin{aligned} x_i(k+1) = &Ax_i(k) + Bf(x_i(k)) \\ &+ Cf(x_i(k-\tau(k))) + I(k), \qquad k = 1, 2, \ldots \end{aligned} \tag{1}$$

where $x_i(k) = (x_{i1}(k), x_{i2}(k), \ldots, x_{in}(k))^T \in \mathbb{R}^n$ is the state vector associated with the network at time k; $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ with a_i $(0 < a_i < 1, i = 1, 2, \ldots, n)$ denoting the rate at which the cell i resets its potential to the resting states when isolated from other cells and inputs; B and C are weight matrix and the delayed weight matrix, respectively; $f(x_i(k)) = (f_1(x_{i1}(k)), f_2(x_{i2}(k)), \ldots, f_n(x_{in}(k)))^T \in \mathbb{R}^n$ is the activation function; $I(k) = (I_1(k), I_2(k), \ldots, I_n(k))^T$ is the external input; and $\tau(k)$ represents the time-varying transmission delay that satisfies

$$\tau_m \leq \tau(k) \leq \tau_M, \qquad k = 1, 2, \ldots \tag{2}$$

where τ_m and τ_M are known positive integers.

As discussed in the previous section, in practice, the network parameters may contain uncertainties due to modeling errors and the neural network may be disturbed by environmental noises that affect the overall dynamical behavior of the neural system. Therefore, model (1) can be modified to the following more realistic one:

$$\begin{aligned} x_i(k+1) = &(A + \Delta A(k))\, x_i(k) + (B + \Delta B(k))\, f(x_i(k)) \\ &+ (C + \Delta C(k))\, f(x_i(k-\tau(k))) + I(k) \\ &+ \sigma_i(k, x_i(k), x_i(k-\tau(k)))\, \omega(k) \end{aligned} \tag{3}$$

where the matrices $\Delta A(k)$, $\Delta B(k)$, and $\Delta C(k)$ represent the parameter uncertainties satisfying the following admissible condition:

$$[\Delta A(k)\ \Delta B(k)\ \Delta C(k)] = M\Xi(k)[W_1\ W_2\ W_3] \tag{4}$$

in which M and W_i $(i = 1, 2, 3)$ are known real constant matrices and $\Xi(k)$ is the unknown matrix subject to

$$\Xi(k)^T \Xi(k) \leq E_n \qquad \forall\, k = 1, 2, \ldots. \tag{5}$$

$\omega(k)$ is a scale Wiener process (Brownian motion) defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$\mathbb{E}\{\omega(k)\} = 0 \quad \mathbb{E}\{\omega^2(k)\} = 1 \quad \mathbb{E}\{\omega(i)\omega(j)\} = 0, \quad (i \neq j) \tag{6}$$

and $\sigma_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the diffusion coefficient vector satisfying

$$(\sigma_i(k,u_i,v_i) - \sigma_j(k,u_j,v_j))^T (\sigma_i(k,u_i,v_i) - \sigma_j(k,u_j,v_j)) \leq \rho_1 \|u_i - u_j\|^2 + \rho_2 \|v_i - v_j\|^2 \quad (7)$$

for all $(k, u_j, v_j) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, where ρ_1 and ρ_2 are known positive constants.

Remark 1: The matrix $\Xi(k)$ in (4) may be time varying and state dependent, i.e., $\Xi(k) = \Xi(k, x(k), x(k-\tau(k)))$, as long as (5) holds. The parameter uncertainty structures (4) and (5) are generally exploited in the research of robust control of uncertain systems; see [38] for example.

The initial conditions associated with model (3) are given by

$$x_i(s) = \varphi_i(s) \in L^2_{\mathcal{F}_0}([-\tau_M, 0], \mathbb{R}^n) \quad (8)$$

where $L^2_{\mathcal{F}_0}([-\tau_M, 0], \mathbb{R}^n)$ is the family of all $\mathcal{F}_0$-measurable $C([-\tau_M, 0], \mathbb{R}^n)$-valued random variable satisfying that $\sup_{-\tau_M \leq s \leq 0} \mathbb{E}\{\|\varphi_i(s)\|^2\} < \infty$.

Note that the model (3) consists of an array of uncertain stochastic DDNNs that have similar structure with possible *couplings*. Generally speaking, the DDNNs in model (3) with different initial conditions (8) will have different dynamical trajectories even if the distance of the two initial functions is very small. This is particularly true when the DDNNs are presented with chaotic dynamical behaviors. Therefore, the synchronization of the *coupled* uncertain stochastic DDNNs becomes an important problem. Motivated by this discussion, in this paper, we will study the global robust synchronization problem for the following array of *coupled* stochastic discrete-time neural networks with time-varying delay:

$$\begin{aligned} x_i(k+1) &= (A + \Delta A(k)) x_i(k) + (B + \Delta B(k)) f(x_i(k)) \\ &\quad + (C + \Delta C(k)) f(x_i(k-\tau(k))) + I(k) \\ &\quad + \sum_{j=1, j\neq i}^{N} G_{ij}^{(1)} D_1 (x_j(k) - x_i(k)) \\ &\quad + \sum_{j=1, j\neq i}^{N} G_{ij}^{(2)} D_2 (h(x_j(k-\tau(k))) - h(x_i(k-\tau(k)))) \\ &\quad + \sigma_i(k, x_i(k), x_i(k-\tau(k))) \omega(k) \end{aligned} \quad (9)$$

where $i = 1, 2, \ldots, N$; $G_{ij}^{(q)}$ is defined as follows: if there is a connection between subnetwork i and subnetwork $j (j \neq i)$, then $G_{ij}^{(q)} = G_{ji}^{(q)} > 0$; otherwise, $G_{ij}^{(q)} = G_{ji}^{(q)} = 0 (j \neq i)$. Let the diagonal element $G_{ii}^{(q)}$ be defined as

$$G_{ii}^{(q)} = -\sum_{j=1, j\neq i}^{N} G_{ij}^{(q)} = -\sum_{j=1, j\neq i}^{N} G_{ji}^{(q)}, \quad (q = 1, 2,\ i, j = 1, 2, \ldots, N) \quad (10)$$

then the network (9) can be rewritten as follows:

$$\begin{aligned} x_i(k+1) &= (A + \Delta A(k)) x_i(k) + (B + \Delta B(k)) f(x_i(k)) \\ &\quad + (C + \Delta C(k)) f(x_i(k-\tau(k))) + I(k) \\ &\quad + \sum_{j=1}^{N} G_{ij}^{(1)} D_1 x_j(k) + \sum_{j=1}^{N} G_{ij}^{(2)} D_2 h(x_j(k-\tau(k))) \\ &\quad + \sigma_i(k, x_i(k), x_i(k-\tau(k))) \omega(k) \end{aligned}$$

where D_1 and $D_2 \in \mathbb{R}^{n\times n}$ represent the linear linking matrix and the nonlinear linking matrix of the neural networks, respectively; $h(x_j(k-\tau(k))) = (h_1(x_{j1}(k-\tau(k))), h_2(x_{j2}(k-\tau(k))), \ldots, h_n(x_{jn}(k-\tau(k))))^T$ is another activation function; $G^{(1)} = (G_{ij}^{(1)})_{N\times N}$ and $G^{(2)} = (G_{ij}^{(2)})_{N\times N}$ that satisfy the diffusive coupling condition (10) denote the linear coupling and the nonlinear coupling configuration of the array, respectively.

Remark 2: The class of dynamical systems such as (9) has been extensively investigated as theoretical models of spatio–temporal phenomena in a variety of problems in nonlinear systems and computation studies [14]. The dynamical behavior is governed by the intrinsic nonlinear dynamics at each subsystem and the diffusion due to the spatial coupling between submodels. Clearly, the diffusive coupling condition (10) ensures that the synchronized state of (9) is just a solution of the individual network (3).

For the activation functions, the following assumptions are made.

Assumption 1 [18]–[20]: There exist constants k_i^-, k_i^+, l_i^-, and l_i^+ such that

$$k_i^- \leq \frac{f_i(u) - f_i(v)}{u - v} \leq k_i^+$$
$$l_i^- \leq \frac{h_i(u) - h_i(v)}{u - v} \leq l_i^+, \qquad i = 1, 2, \ldots, n$$

for any different $u, v \in \mathbb{R}$.

Remark 3: The activation functions are usually assumed to continuous, differentiable, monotonically increasing, and bounded, such as the sigmoid-type of function. However, in many electronic circuits, the input–output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence nonmonotonic functions can be more appropriate to describe the activation. The type of activation functions given in this paper was first introduced in [19] and [20]. As pointed out in [19] and [20], the constants $k_i^-, k_i^+, l_i^-, l_i^+$ in Assumption 1 are allowed to be positive, negative, or zero. Hence, the resulting activation functions may be nonmonotonic, and more general than the usual sigmoid functions and Lipschitz-type conditions. Such a description is very precise/tight in quantifying the lower and upper bounds of the activation functions. As in [15], the nonlinear functions f_i, h_i are said to belong to sectors, i.e., the nonlinearities are bounded by sectors $k_i^-, k_i^+, l_i^-, l_i^+$. For more details about how to obtain the sector bounds of several well-studied nonlinear functions, we refer the readers to [15].

For presentation convenience, we denote

$$\begin{aligned}
A(k) &= A+\Delta A(k)\\
B(k) &= B+\Delta B(k)\\
C(k) &= C+\Delta C(k)\\
J(k) &= \left(I^T(k), I^T(k), \dots, I^T(k)\right)^T\\
x(k) &= \left(x_1^T(k), x_2^T(k), \dots, x_N^T(k)\right)^T\\
F(x(k)) &= \left(f^T(x_1(k)), f^T(x_2(k)), \dots, f^T(x_N(k))\right)^T\\
H(x(k-\tau(k))) &= \left(h^T(x_1(k-\tau(k))), h^T(x_2(k-\tau(k))),\right.\\
&\qquad \left.\dots, h^T(x_N(k-\tau(k)))\right)^T\\
\sigma(k) &= \left(\sigma_1^T(k, x_1(k), x_1(k-\tau(k))),\right.\\
&\qquad \sigma_2^T(k, x_2(k), x_2(k-\tau(k))), \dots,\\
&\qquad \left.\sigma_N^T(k, x_N(k), x_N(k-\tau(k)))\right)^T\\
G^{(1,1)} &= G^{(1)}G^{(1)}\\
G^{(1,2)} &= G^{(1)}G^{(2)}\\
G^{(2,2)} &= G^{(2)}G^{(2)}\\
K_1 &= \mathrm{diag}\left\{k_1^-k_1^+, k_2^-k_2^+, \dots, k_n^-k_n^+\right\}\\
L_1 &= \mathrm{diag}\left\{l_1^-l_1^+, l_2^-l_2^+, \dots, l_n^-l_n^+\right\}\\
K_2 &= \mathrm{diag}\left\{-\left(k_1^-+k_1^+\right)/2, -\left(k_2^-+k_2^+\right)/2,\right.\\
&\qquad \left.\dots, -\left(k_n^-+k_n^+\right)/2\right\}\\
L_2 &= \mathrm{diag}\left\{-\left(l_1^-+l_1^+\right)/2, -\left(l_2^-+l_2^+\right)/2,\right.\\
&\qquad \left.\dots, -\left(l_n^-+l_n^+\right)/2\right\}.
\end{aligned}$$

Using the Kronecker product, we can rewrite system (9) into a more compact form as

$$\begin{aligned}
x(k+1) = &\left(E_N\otimes A(k)+G^{(1)}\otimes D_1\right)x(k)\\
&+(E_N\otimes B(k))F(x(k))\\
&+(E_N\otimes C(k))F(x(k-\tau(k)))+J(k)\\
&+\left(G^{(2)}\otimes D_2\right)H(x(k-\tau(k)))+\sigma(k)\omega(k) \quad (11)
\end{aligned}$$

where E_N is an $N\times N$ identity matrix.

Definition 1: Model (11) is said to be globally robustly exponentially synchronized in the mean square if there exist two constants $\vartheta > 0$ and $0 < \delta < 1$ such that, for all $\phi_i(\cdot) \in L^2_{\mathcal{F}_0}([-\tau_M, 0], \mathbb{R}^n)$ and sufficiently large integer $\kappa > 0$, the inequality

$$\mathbb{E}\left\{\|x_i(k)-x_j(k)\|^2\right\} \le \vartheta\delta^k \max_{s\in\mathbb{N}[-\tau_M,0]} \mathbb{E}\left\{\|\phi_i(s)-\phi_j(s)\|^2\right\}$$

holds for all $k > \kappa$ and all parameter uncertainties satisfying the admissible conditions (4) and (5), where $i, j = 1, 2, \dots, N$.

Lemma 1 [2]: Let $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ depend affinely on x. Then, the following LMI:

$$\begin{bmatrix} Q(x) & S(x)\\ S^T(x) & R(x)\end{bmatrix} > 0$$

holds if and only if one of the following conditions holds:

1) $R(x) > 0$, $Q(x) - S(x)R^{-1}(x)S^T(x) > 0$;
2) $Q(x) > 0$, $R(x) - S^T(x)Q^{-1}(x)S(x) > 0$.

Lemma 2: Let x and y be any vectors in $\mathbb{R}^n$, X, Y, and Ξ be matrices with Ξ satisfying $\Xi^T\Xi \le E_n$, and P be a positive definite matrix. Then, the following inequalities hold:

1) $X\Xi Y + (X\Xi Y)^T \le XX^T + Y^TY$;
2) $2x^Ty \le x^TPx + y^TP^{-1}y$.

Lemma 3 [5]: Let $\alpha \in \mathbb{R}$ and X, Y, P, and Q be matrices with appropriate dimensions. Then, the following statements are true:

1) $(\alpha X)\otimes Y = X\otimes(\alpha Y)$;
2) $(X+Y)\otimes P = X\otimes P + Y\otimes P$;
3) $(X\otimes Y)(P\otimes Q) = (XP)\otimes(YQ)$;
4) $(X\otimes Y)^T = X^T\otimes Y^T$.

Lemma 4: Let e be the N-dimensional vector with all components being 1 and $U = NE_N - ee^T$. For $i = 1, 2, \dots, N$, assume that P is an $n\times n$ matrix, $x = (x_1^T, x_2^T, \dots, x_N^T)^T$ where $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbb{R}^n$, and $y = (y_1^T, y_2^T, \dots, y_N^T)^T$ where $y_i = (y_{i1}, y_{i2}, \dots, y_{in})^T \in \mathbb{R}^n$. Then, we have the following relationships:

1) $UG^{(j)} = G^{(j)}U = NG^{(j)}$, $j = 1, 2$;
2) $(U\otimes P)J(k) = 0$, $J^T(k)(G^{(j)}\otimes P) = 0$, $j = 1, 2$;
3) $x^T(U\otimes P)y = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} u_{ij}(x_i - x_j)^TP(y_i - y_j)$;

where $J(k)$, $G^{(1)}$, and $G^{(2)}$ are defined in model (11).

Remark 4: In this paper, we aim to study the synchronization problem for an array of discrete-time neural networks so that the trajectory of any two solutions of (3) will not diverge out of the given distance as long as the distance of their two initial conditions is controlled. The criteria obtained here are also useful in other fields such as secure communication where the transmitter and receiver are all chaotic or oscillatory systems. The model (11) analyzed here are more general than those in the literature in the sense that we have included, for the first time, the parameter uncertainties and the stochastic disturbances in the synchronization problem.

III. Main Results

In this section, a Lyapunov functional approach combined with the linear matrix inequalities will be employed to investigate the global robust synchronization problem for system (11).

Theorem 1: The dynamical system (11) is globally robustly exponentially synchronized in the mean square if there exist two matrices $P > 0$ and $Q > 0$, three diagonal matrices $R = \mathrm{diag}\{r_1, r_2, \dots, r_n\} > 0$, $S = \mathrm{diag}\{s_1, s_2, \dots, s_n\} > 0$, and $W = \mathrm{diag}\{w_1, w_2, \dots, w_n\} > 0$, and two scalars $\lambda^* > 0$ and $\varepsilon > 0$ such that the LMIs (12) and (13), shown at the bottom of the next page, hold for all $1 \le i < j \le N$, where $\Pi_1^{(i,j)} = -RK_1 + \lambda^*\rho_1E_n + (\tau_M - \tau_m + 1)Q - P + N(G_{ij}^{(1)} - G_{ij}^{(1,1)})D_1^TPD_1 + \varepsilon W_1^TW_1$, $\Pi_2 = -WK_1 - SL_1 - Q + \lambda^*\rho_2E_n$, $\Pi_3^{(i,j)} = -S + N(G_{ij}^{(2)} - G_{ij}^{(2,2)})D_2^TPD_2$, $\Pi_4^{(i,j)} = -(1 + NG_{ij}^{(1)} + NG_{ij}^{(2)})^{-1}P$.

Proof: By Lemma 1, LMI (13) implies

$$\Phi_1^{(i,j)} + \varepsilon^{-1}\hat{P}MM^T\hat{P}^T < 0 \quad (14)$$

where the matrices $\Phi_1^{(i,j)}$ and $\hat{P}$ are shown at the bottom of the page.

Let $x(k)$ be the trajectory of (11) and choose the following matrix functional:

$$\begin{aligned} V(k) &= V_1(k)+V_2(k)+V_3(k) \\ &= x^T(k)(U\otimes P)x(k)+\sum_{i=k-\tau(k)}^{k-1} x^T(i)(U\otimes Q)x(i) \\ &\quad +\sum_{j=-\tau_M+1}^{-\tau_m}\sum_{i=k+j}^{k-1} x^T(i)(U\otimes Q)x(i) \end{aligned} \tag{15}$$

where the matrix U is defined in Lemma 4. Calculating the difference of $V(k)$ along the solutions of (11) and taking the mathematical expectation, we obtain from Lemmas 3 and 4 that

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\}+\mathbb{E}\{\Delta V_2(k)\}+\mathbb{E}\{\Delta V_3(k)\} \tag{16}$$

where

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(k)\} \\ &= \mathbb{E}\{V_1(k+1)-V_1(k)\} \\ &= \mathbb{E}\{x^T(k+1)(U\otimes P)x(k+1)-x^T(k)(U\otimes P)x(k)\} \\ &= \mathbb{E}\Big\{\Big[x^T(k)\Big(E_N\otimes A^T(k)+G^{(1)}\otimes D_1^T\Big) \\ &\qquad + F^T(x(k))\big(E_N\otimes B^T(k)\big) \\ &\qquad + F^T(x(k-\tau(k)))\big(E_N\otimes C^T(k)\big)+J^T(k) \\ &\qquad + H^T(x(k-\tau(k)))\Big(G^{(2)}\otimes D_2^T\Big)+\sigma^T(k)\omega(k)\Big] \\ &\qquad \times\Big[\Big(U\otimes(PA(k))+NG^{(1)}\otimes(PD_1)\Big)x(k) \\ &\qquad\quad +(U\otimes(PB(k)))F(x(k))+(U\otimes P)\sigma(k)\omega(k) \\ &\qquad\quad +(U\otimes(PC(k)))F(x(k-\tau(k))) \\ &\qquad\quad +\Big(NG^{(2)}\otimes(PD_2)\Big)H(x(k-\tau(k)))\Big] \\ &\qquad - x^T(k)(U\otimes P)x(k)\Big\} \\ &= \mathbb{E}\Big\{\Big[x^T(k)\Big(E_N\otimes A^T(k)+G^{(1)}\otimes D_1^T\Big) \\ &\qquad + F^T(x(k-\tau(k)))\big(E_N\otimes C^T(k)\big)+J^T(k) \\ &\qquad + F^T(x(k))\big(E_N\otimes B^T(k)\big) \\ &\qquad + H^T(x(k-\tau(k)))\Big(G^{(2)}\otimes D_2^T\Big)\Big] \\ &\qquad \times[(U\otimes(PC(k)))F(x(k-\tau(k))) \\ &\qquad\quad +\Big(U\otimes(PA(k))+NG^{(1)}\otimes(PD_1)\Big)x(k) \\ &\qquad\quad +(U\otimes(PB(k)))F(x(k)) \\ &\qquad\quad +\Big(NG^{(2)}\otimes(PD_2)\Big)H(x(k-\tau(k)))\Big] \\ &\qquad + \sigma^T(k)(U\otimes P)\sigma(k)-x^T(k)(U\otimes P)x(k)\Big\} \end{aligned} \tag{17}$$

$$\begin{aligned} &\mathbb{E}\{\Delta V_2(k)\} \\ &= \mathbb{E}\{V_2(k+1)-V_2(k)\} \\ &= \mathbb{E}\Big\{\sum_{i=k+1-\tau(k+1)}^{k} x^T(i)(U\otimes Q)x(i) \\ &\qquad - \sum_{i=k-\tau(k)}^{k-1} x^T(i)(U\otimes Q)x(i)\Big\} \\ &= \mathbb{E}\Big\{x^T(k)(U\otimes Q)x(k)-x^T(k-\tau(k))(U\otimes Q) \\ &\qquad \times x(k-\tau(k))+\sum_{i=k+1-\tau(k+1)}^{k-\tau_m} x^T(i)(U\otimes Q)x^T(i) \end{aligned}$$

$$P<\lambda^* E_n \tag{12}$$

$$\Phi^{(i,j)} = \begin{bmatrix} \Pi_1^{(i,j)} & 0 & -RK_2+\varepsilon W_1^TW_2 & \varepsilon W_1^TW_3 & -NG_{ij}^{(1,2)}D_1^TPD_2 & A^TP & 0 \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 & 0 & 0 \\ * & * & -R+\varepsilon W_2^TW_2 & \varepsilon W_2^TW_3 & 0 & B^TP & 0 \\ * & * & * & -W+\varepsilon W_3^TW_3 & 0 & C^TP & 0 \\ * & * & * & * & \Pi_3^{(i,j)} & 0 & 0 \\ * & * & * & * & * & \Pi_4^{(i,j)} & PM \\ * & * & * & * & * & * & -\varepsilon E_n \end{bmatrix} < 0 \tag{13}$$

$$\Phi_1^{(i,j)} = \begin{bmatrix} \Pi_1^{(i,j)} & 0 & -RK_2+\varepsilon W_1^TW_2 & \varepsilon W_1^TW_3 & -NG_{ij}^{(1,2)}D_1^TPD_2 & A^TP \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 & 0 \\ * & * & -R+\varepsilon W_2^TW_2 & \varepsilon W_2^TW_3 & 0 & B^TP \\ * & * & * & -W+\varepsilon W_3^TW_3 & 0 & C^TP \\ * & * & * & * & \Pi_3^{(i,j)} & 0 \\ * & * & * & * & * & \Pi_4^{(i,j)} \end{bmatrix} \quad \hat{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P \end{bmatrix}.$$

$$\begin{aligned}&+\sum_{i=k-\tau_m+1}^{k-1} x^T(i)(U\otimes Q)x(i)\\&-\left.\sum_{i=k-\tau(k)+1}^{k-1} x^T(i)(U\otimes Q)x(i)\right\}\\ \le \mathbb{E}&\Bigg\{x^T(k)(U\otimes Q)x(k)-x^T(k-\tau(k))(U\otimes Q)\\&\times x(k-\tau(k))+\sum_{i=k+1-\tau_M}^{k-\tau_m} x^T(i)(U\otimes Q)x(i)\Bigg\} \quad (18)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\{\Delta V_3(k)\}&=\mathbb{E}\{V_3(k+1)-V_3(k)\}\\&=\mathbb{E}\Bigg\{\sum_{j=-\tau_M+1}^{-\tau_m}\sum_{i=k+1+j}^{k} x^T(i)(U\otimes Q)x(i)\\&\qquad-\sum_{j=-\tau_M+1}^{-\tau_m}\sum_{i=k+j}^{k-1} x^T(i)(U\otimes Q)x(i)\Bigg\}\\&=\mathbb{E}\Bigg\{\sum_{j=-\tau_M+1}^{-\tau_m}\big[x^T(k)(U\otimes Q)x(k)\\&\qquad-x^T(k+j)(U\otimes Q)x(k+j)\big]\Bigg\}\\&=\mathbb{E}\Bigg\{(\tau_M-\tau_m)x^T(k)(U\otimes Q)x(k)\\&\qquad-\sum_{i=k+1-\tau_M}^{k-\tau_m} x^T(i)(U\otimes Q)x(i)\Bigg\}. \quad (19)\end{aligned}$$

Substituting (17)–(19) into (16), one derives

$$\begin{aligned}&\mathbb{E}\{\Delta V(k)\}\\&\le \mathbb{E}\Big\{x^T(k)\Big[U\otimes\big(A^T(k)PA(k)\big)+NG^{(1)}\\&\qquad\otimes\big(D_1^TPA(k)+A^T(k)PD_1\big)+NG^{(1,1)}\\&\qquad\otimes\big(D_1^TPD_1\big)-U\otimes P\\&\qquad+(\tau_M-\tau_m+1)U\otimes Q\Big]x(k)\\&\quad-x^T(k-\tau(k))(U\otimes Q)x(k-\tau(k))\\&\quad+2x^T(k)\Big[U\otimes\big(A^T(k)PB(k)\big)\\&\qquad+NG^{(1)}\otimes\big(D_1^TPB(k)\big)\Big]F(x(k))\\&\quad+2x^T(k)\Big[U\otimes\big(A^T(k)PC(k)\big)\\&\qquad+NG^{(1)}\otimes\big(D_1^TPC(k)\big)\Big]\\&\quad\times F(x(k-\tau(k)))\\&\quad+2x^T(k)\Big[NG^{(2)}\otimes\big(A^T(k)PD_2\big)\\&\qquad+NG^{(1,2)}\otimes\big(D_1^TPD_2\big)\Big]\\&\quad\times H(x(k-\tau(k)))+F^T(x(k))\\&\quad\times\big(U\otimes\big(B^T(k)PB(k)\big)\big)F(x(k))\\&\quad+F^T(x(k-\tau(k)))\big(U\otimes\big(C^T(k)PC(k)\big)\big)\\&\quad\times F(x(k-\tau(k)))+2F^T(x(k))\\&\quad\times\big(U\otimes\big(B^T(k)PC(k)\big)\big)F(x(k-\tau(k)))\\&\quad+2F^T(x(k))\Big(NG^{(2)}\otimes\big(B^T(k)PD_2\big)\Big)\\&\quad\times H(x(k-\tau(k)))+2F^T(x(k-\tau(k)))\\&\quad\times\Big(NG^{(2)}\otimes\big(C^T(k)PD_2\big)\Big)H(x(k-\tau(k)))\\&\quad+H^T(x(k-\tau(k)))\Big(NG^{(2,2)}\otimes\big(D_2^TPD_2\big)\Big)\\&\quad\times H(x(k-\tau(k)))+\sigma^T(k)(U\otimes P)\sigma(k)\Big\}. \quad (20)\end{aligned}$$

Considering (7) and the condition (12), by Lemma 4 and noticing the fact $u_{ij}=-1(i\neq j)$, it is easy to see that

$$\begin{aligned}&\sigma^T(k)(U\otimes P)\sigma(k)\\&\quad=\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(\sigma_i(k)-\sigma_j(k))^T P(\sigma_i(k)-\sigma_j(k))\\&\quad\le\lambda_{\max}(P)\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(\sigma_i(k)-\sigma_j(k))^T(\sigma_i(k)-\sigma_j(k))\\&\quad\le\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big(\lambda^*\rho_1\|x_i(k)-x_j(k)\|^2\\&\qquad\qquad+\lambda^*\rho_2\|x_i(k-\tau(k))-x_j(k-\tau(k))\|^2\Big).\end{aligned} \quad (21)$$

On the other hand, from Assumption 1, we have

$$\begin{aligned}&\big[f_m(x_{im}(k))-f_m(x_{jm}(k))-k_m^+(x_{im}(k)-x_{jm}(k))\big]\\&\quad\times\big[f_m(x_{im}(k))-f_m(x_{jm}(k))\\&\qquad-k_m^-(x_{im}(k)-x_{jm}(k))\big]\le 0\\&\big[h_m(x_{im}(k-\tau(k)))-h_m(x_{jm}(k-\tau(k)))\\&\quad-l_m^+(x_{im}(k-\tau(k))-x_{jm}(k-\tau(k)))\big]\\&\quad\times\big[h_m(x_{im}(k-\tau(k)))-h_m(x_{jm}(k-\tau(k)))\\&\qquad-l_m^-(x_{im}(k-\tau(k))-x_{jm}(k-\tau(k)))\big]\le 0\\&\big[f_m(x_{im}(k-\tau(k)))-f_m(x_{jm}(k-\tau(k)))\\&\quad-k_m^+(x_{im}(k-\tau(k))-x_{jm}(k-\tau(k)))\big]\\&\quad\times\big[f_m(x_{im}(k-\tau(k)))-f_m(x_{jm}(k-\tau(k)))\\&\qquad-k_m^-(x_{im}(k-\tau(k))-x_{jm}(k-\tau(k)))\big]\le 0\end{aligned}$$

where $m=1,2,\ldots,n$ and $1\le i<j\le N$, which are equivalent to

$$\begin{aligned}&\begin{bmatrix}x_i(k)-x_j(k)\\ f(x_i(k))-f(x_j(k))\end{bmatrix}^T\\&\quad\times\begin{bmatrix}k_m^-k_m^+e_me_m^T & -\frac{k_m^-+k_m^+}{2}e_me_m^T\\ -\frac{k_m^-+k_m^+}{2}e_me_m^T & e_me_m^T\end{bmatrix}\\&\quad\times\begin{bmatrix}x_i(k)-x_j(k)\\ f(x_i(k))-f(x_j(k))\end{bmatrix}\le 0 \quad (22)\\&\begin{bmatrix}x_i(k-\tau(k))-x_j(k-\tau(k))\\ h(x_i(k-\tau(k)))-h(x_j(k-\tau(k)))\end{bmatrix}^T\end{aligned}$$

$$\times \begin{bmatrix} l_m^- l_m^+ e_m e_m^T & -\frac{l_m^- + l_m^+}{2} e_m e_m^T \\ -\frac{l_m^- + l_m^+}{2} e_m e_m^T & e_m e_m^T \end{bmatrix} \times \begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ h(x_i(k-\tau(k))) - h(x_j(k-\tau(k))) \end{bmatrix} \leq 0 \quad (23)$$

$$\begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))) \end{bmatrix}^T \times \begin{bmatrix} k_m^- k_m^+ e_m e_m^T & -\frac{k_m^- + k_m^+}{2} e_m e_m^T \\ -\frac{k_m^- + k_m^+}{2} e_m e_m^T & e_m e_m^T \end{bmatrix} \times \begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))) \end{bmatrix} \leq 0 \quad (24)$$

where e_m denotes the n-dimensional unit column vector having "1" element on its mth row and zeros elsewhere.

Multiplying both sides of (22), (23), and (24) by r_m, s_m, and w_m, respectively, and summing up from 1 to n with respect to m, we obtain

$$\begin{bmatrix} x_i(k) - x_j(k) \\ f(x_i(k)) - f(x_j(k)) \end{bmatrix}^T \begin{bmatrix} RK_1 & RK_2 \\ RK_2 & R \end{bmatrix} \times \begin{bmatrix} x_i(k) - x_j(k) \\ f(x_i(k)) - f(x_j(k)) \end{bmatrix} \leq 0 \quad (25)$$

$$\begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ h(x_i(k-\tau(k))) - h(x_j(k-\tau(k))) \end{bmatrix}^T \begin{bmatrix} SL_1 & SL_2 \\ SL_2 & S \end{bmatrix} \times \begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ h(x_i(k-\tau(k))) - h(x_j(k-\tau(k))) \end{bmatrix} \leq 0 \quad (26)$$

$$\begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))) \end{bmatrix}^T \begin{bmatrix} WK_1 & WK_2 \\ WK_2 & W \end{bmatrix} \times \begin{bmatrix} x_i(k-\tau(k)) - x_j(k-\tau(k)) \\ f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))) \end{bmatrix} \leq 0. \quad (27)$$

It follows from (20), (21), (25)–(27), and Lemma 4 that (28) shown at the bottom of the page holds, where $\aleph(k) = [A(k)\ 0\ B(k)\ C(k)\ 0]$, $\mathcal{D}_1 = [D_1\ 0\ 0\ 0\ 0]$, $\mathcal{D}_2 = [0\ 0\ 0\ 0\ D_2]$, $\xi_{ij} = ((x_i(k) - x_j(k))^T (x_i(k-\tau(k)) - x_j(k-\tau(k)))^T (f(x_i(k)) - f(x_j(k)))^T (f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))))^T (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k))))^T)^T$, and

$$\Phi_2^{(i,j)} = \begin{bmatrix} \Pi_5^{(i,j)} & 0 & -RK_2 & 0 & -NG_{ij}^{(1,2)} D_1^T P D_2 \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 \\ * & * & -R & 0 & 0 \\ * & * & * & -W & 0 \\ * & * & * & * & -S - NG_{ij}^{(2,2)} D_2^T P D_2 \end{bmatrix}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\leq \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathbb{E}\Big\{ (x_i(k) - x_j(k))^T \Big[\Big(A^T(k) P A(k) - P - NG_{ij}^{(1)} \left(D_1^T P A(k) + A^T(k) P D_1 \right) \\
&\qquad - NG_{ij}^{(1,1)} D_1^T P D_1 + (\tau_M - \tau_m + 1) Q + \lambda^* \rho_1 E_n - RK_1 \Big) (x_i(k) - x_j(k)) \\
&\qquad + 2 \left(A^T(k) P B(k) - NG_{ij}^{(1)} D_1^T P B(k) - RK_2 \right) (f(x_i(k)) - f(x_j(k))) \\
&\qquad + 2 \left(A^T(k) P C(k) - NG_{ij}^{(1)} D_1^T P C(k) \right) (f(x_i(k-\tau(k))) - f(x_j(k-\tau(k)))) \\
&\qquad + 2 \left(-NG_{ij}^{(2)} A^T(k) P D_2 - NG_{ij}^{(1,2)} D_1^T P D_2 \right) \\
&\qquad \times (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k)))) \Big] + (x_i(k-\tau(k)) - x_j(k-\tau(k)))^T \\
&\quad \times [(\lambda^* \rho_2 E_n - Q - WK_1 - SL_1)(x_i(k-\tau(k)) - x_j(k-\tau(k))) \\
&\qquad - 2SL_2 (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k)))) \\
&\qquad - 2WK_2 (f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))))] + (f(x_i(k)) - f(x_j(k)))^T \\
&\quad \times \left[\left(B^T(k) P B(k) - R \right) (f(x_i(k)) - f(x_j(k))) + 2 B^T(k) P C(k) \right. \\
&\qquad \times (f(x_i(k-\tau(k))) - f(x_j(k-\tau(k)))) - 2NG_{ij}^{(2)} B^T(k) P D_2 \\
&\qquad \times (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k))))] + (f(x_i(k-\tau(k))) - f(x_j(k-\tau(k))))^T \\
&\quad \times \Big[\left(C^T(k) P C(k) - W \right) (f(x_i(k-\tau(k))) - f(x_j(k-\tau(k)))) - 2NG_{ij}^{(2)} C^T(k) P D_2 \\
&\qquad \times (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k)))) \Big] - (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k))))^T \\
&\qquad \times \left(S + NG_{ij}^{(2,2)} D_2^T P D_2 \right) (h(x_i(k-\tau(k))) - h(x_j(k-\tau(k)))) \Big\} \\
&= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathbb{E}\left\{ \xi_{ij}^T \left[\Phi_2^{(i,j)} + \aleph^T(k) P \aleph(k) - NG_{ij}^{(1)} \left(\mathcal{D}_1^T P \aleph(k) + \aleph^T(k) P \mathcal{D}_1 \right) - NG_{ij}^{(2)} \left(\mathcal{D}_2^T P \aleph(k) + \aleph^T(k) P \mathcal{D}_2 \right) \right] \xi_{ij} \right\}
\end{aligned} \quad (28)$$

in which $\Pi_5^{(i,j)} = -RK_1 + \lambda^*\rho_1 E_n + (\tau_M - \tau_m + 1)Q - P - NG_{ij}^{(1,1)} D_1^T P D_1$.

Lemma 2 indicates that the following inequalities hold for all $1 \le i < j \le N$:

$$\begin{aligned} &-NG_{ij}^{(1)} \xi_{ij}^T \left(\mathcal{D}_1^T P \aleph(k) + \aleph^T(k) P \mathcal{D}_1\right) \xi_{ij} \\ &\quad \le NG_{ij}^{(1)} \xi_{ij}^T \left(\mathcal{D}_1^T P \mathcal{D}_1 + \aleph^T(k) P \aleph(k)\right) \xi_{ij} \qquad (29) \\ &-NG_{ij}^{(2)} \xi_{ij}^T \left(\mathcal{D}_2^T P \aleph(k) + \aleph^T(k) P \mathcal{D}_2\right) \xi_{ij} \\ &\quad \le NG_{ij}^{(2)} \xi_{ij}^T \left(\mathcal{D}_2^T P \mathcal{D}_2 + \aleph^T(k) P \aleph(k)\right) \xi_{ij}. \qquad (30) \end{aligned}$$

From (28)–(30), we have

$$\begin{aligned} &\mathbb{E}\{\Delta V(k)\} \\ &\le \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathbb{E}\Big\{ \xi_{ij}^T \Big[\Phi_3^{(i,j)} + \Big(1 + NG_{ij}^{(1)} + NG_{ij}^{(2)}\Big) \\ &\qquad \times \aleph^T(k) P \aleph(k) \Big] \xi_{ij} \Big\} \qquad (31) \end{aligned}$$

where

$$\Phi_3^{(i,j)} = \begin{bmatrix} \Pi_6^{(i,j)} & 0 & -RK_2 & 0 & -NG_{ij}^{(1,2)} D_1^T P D_2 \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 \\ * & * & -R & 0 & 0 \\ * & * & * & -W & 0 \\ * & * & * & * & \Pi_3^{(i,j)} \end{bmatrix}$$

and $\Pi_6^{(i,j)} = -RK_1 + \lambda^* \rho_1 E_n + (\tau_M - \tau_m + 1)Q - P + N(G_{ij}^{(1)} - G_{ij}^{(1,1)}) D_1^T P D_1$.

In the following, we will show that (14) implies

$$\Phi_3^{(i,j)} + \Big(1 + NG_{ij}^{(1)} + NG_{ij}^{(2)}\Big) \aleph^T(k) P \aleph(k) < 0$$

or, equivalently

$$\Phi_4^{(i,j)} + \Delta\Phi_4 < 0 \qquad (32)$$

where

$$\Phi_4^{(i,j)} = \begin{bmatrix} \Pi_6^{(i,j)} & 0 & -RK_2 & 0 & -NG_{ij}^{(1,2)} D_1^T P D_2 & A^T P \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 & 0 \\ * & * & -R & 0 & 0 & B^T P \\ * & * & * & -W & 0 & C^T P \\ * & * & * & * & \Pi_3^{(i,j)} & 0 \\ * & * & * & * & * & \Pi_4^{(i,j)} \end{bmatrix}$$

and

$$\Delta\Phi_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & (\Delta A)^T P \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & (\Delta B)^T P \\ * & * & * & 0 & 0 & (\Delta C)^T P \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}.$$

Letting $\hat{\aleph} = [\Delta A \;\; 0 \;\; \Delta B \;\; \Delta C \;\; 0 \;\; 0]$ and $\hat{W} = [W_1 \;\; 0 \;\; W_2 \;\; W_3 \;\; 0 \;\; 0]$, we have from Lemma 2 that

$$\begin{aligned} \Delta\Phi_4 &= \hat{P}\hat{\aleph} + (\hat{P}\hat{\aleph})^T = \hat{P} M \Xi(k) \hat{W} + \Big(\hat{P} M \Xi(k) \hat{W}\Big)^T \\ &\le \varepsilon^{-1} \hat{P} M M^T \hat{P}^T + \varepsilon \hat{W}^T \hat{W}. \qquad (33) \end{aligned}$$

Substituting (33) into (32), one can conclude from (14) that there exists a scalar constant $\lambda^{**} > 0$ such that

$$\mathbb{E}\{\Delta V(k)\} \le -\lambda^{**} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathbb{E}\Big\{ \|x_i(k) - x_j(k)\|^2 \Big\}. \qquad (34)$$

Moreover, along the similar line of proof of Theorem 1 in [17], we can derive that there exist two scalars $\vartheta > 0$ and $\delta < 1$ such that

$$\mathbb{E}\Big\{ \|x_i(k) - x_j(k)\|^2 \Big\} \le \vartheta \delta^k \max_{s \in \mathbb{N}[-\tau_M, 0]} \mathbb{E}\Big\{ \|x_i(s) - x_j(s)\|^2 \Big\}$$

holds for all $k > \kappa$ and all parameter uncertainties satisfying the admissible conditions (4) and (5), where $\kappa > 0$ is a sufficiently large integer. According to Definition 1, we conclude that model (11) is globally robustly exponentially synchronized in the mean square, and the proof is then completed. ■

Remark 5: In this paper, the synchronization problem is studied for an array of stochastic discrete-time neural networks with time-varying delay and parameter uncertainties. Note that the obtained criteria in (12) and (13) are dependent on not only the upper bound but also the lower bound of the time-varying delay, hence less conservative than the traditional delay-independent ones. Also, the LMI-based criteria can be checked efficiently via the Matlab LMI Toolbox.

In the following, we will show that the main results given in Theorem 1 are general enough to cover many special cases. Let us first consider the case when system (11) has no stochastic disturbances and the model can be reduced to

$$\begin{aligned} x(k+1) = &\Big(E_N \otimes A(k) + G^{(1)} \otimes D_1\Big) x(k) \\ &+ (E_N \otimes B(k)) F(x(k)) \\ &+ (E_N \otimes C(k)) F(x(k - \tau(k))) + J(k) \\ &+ \Big(G^{(2)} \otimes D_2\Big) H(x(k - \tau(k))). \qquad (35) \end{aligned}$$

In this case, if $D_1 = 0$ in system (35), i.e., the system has only the nonlinear coupling term, model (35) can be further simplified to

$$\begin{aligned} x(k+1) = &(E_N \otimes A(k)) x(k) + (E_N \otimes B(k)) F(x(k)) \\ &+ (E_N \otimes C(k)) F(x(k - \tau(k))) + J(k) \\ &+ (G \otimes D) H(x(k - \tau(k))). \qquad (36) \end{aligned}$$

The following corollary is easily accessible from Theorem 1.

Corollary 1: The dynamical system (36) is globally robustly exponentially synchronized if there exist two matrices $P > 0$ and $Q > 0$, three diagonal matrices $R > 0$, $S > 0$, and $W > 0$, and one scalar $\varepsilon > 0$ such that the LMI (37) shown at the bottom of the page holds for all $1 \leq i < j \leq N$, where $\Lambda_1 = -RK_1 + (\tau_M - \tau_m + 1)Q - P + \varepsilon W_1^T W_1$, $\Lambda_2 = -WK_1 - SL_1 - Q$, $\Lambda_3^{(i,j)} = N(-\widetilde{G}_{ij} + G_{ij})D^T PD - S$, and $\widetilde{G} = GG = (\widetilde{G}_{ij})_{N\times N}$.

If $D_2 = 0$ in system (35), i.e., the system has only the linear coupling terms, model (35) becomes

$$x(k+1) = (E_N \otimes A(k) + G \otimes D)\, x(k) + (E_N \otimes B(k))\, F(x(k)) + (E_N \otimes C(k))\, F(x(k-\tau(k))) + J(k) \quad (38)$$

and the following corollary is readily obtained from Theorem 1.

Corollary 2: The dynamical system (38) is globally robustly exponentially synchronized if there exist two matrices $P > 0$ and $Q > 0$, two diagonal matrices $R > 0$ and $W > 0$, and one scalar $\varepsilon > 0$ such that the LMI (39) shown at the bottom of the page holds for all $1 \leq i < j \leq N$, where $\Lambda_4^{(i,j)} = -RK_1 + (\tau_M - \tau_m + 1)Q - P + N(G_{ij} - \widetilde{G}_{ij})D^T PD + \varepsilon W_1^T W_1$, and $\widetilde{G} = GG = (\widetilde{G}_{ij})_{N\times N}$.

Next, let us consider the uncertainty-free case, i.e., there are no uncertainties, and system (11) can be specialized to the following model:

$$\begin{aligned} x(k+1) = {} & \left(E_N \otimes A + G^{(1)} \otimes D_1\right) x(k) \\ & + (E_N \otimes B) F(x(k)) \\ & + (E_N \otimes C) F(x(k-\tau(k))) + J(k) \\ & + \left(G^{(2)} \otimes D_2\right) H(x(k-\tau(k))) + \sigma(k)\omega(k). \end{aligned} \quad (40)$$

In this case, we have the following corollary.

Corollary 3: The dynamical system (40) is globally exponentially synchronized in the mean square if there exist two matrices $P > 0$ and $Q > 0$, three diagonal matrices $R > 0$, $S > 0$, and $W > 0$, and one scalar $\lambda^* > 0$ such that the following LMIs hold for all $1 \leq i < j \leq N$:

$$P < \lambda^* E_n \quad (41)$$

$$\Psi_3^{(i,j)} = \begin{bmatrix} \Pi_6^{(i,j)} & 0 & -RK_2 & 0 & -NG_{ij}^{(1,2)} D_1^T PD_2 & A^T P \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 & 0 \\ * & * & -R & 0 & 0 & B^T P \\ * & * & * & -W & 0 & C^T P \\ * & * & * & * & \Pi_3^{(i,j)} & 0 \\ * & * & * & * & * & \Pi_4^{(i,j)} \end{bmatrix} < 0 \quad (42)$$

where Π_2, $\Pi_3^{(i,j)}$, $\Pi_4^{(i,j)}$, and $\Pi_6^{(i,j)}$ are defined in Theorem 1.

Moreover, if $D_1 = 0$ in model (40), i.e., the system has only the nonlinear coupling term, model (40) can be further refined to

$$\begin{aligned} x(k+1) = {} & (E_N \otimes A)x(k) + (E_N \otimes B)F(x(k)) \\ & + (E_N \otimes C)F(x(k-\tau(k))) + J(k) \\ & + (G \otimes D)H(x(k-\tau(k))) + \sigma(k)\omega(k) \end{aligned} \quad (43)$$

and we have the following corollary directly.

Corollary 4: The dynamical system (43) is globally exponentially synchronized in the mean square if there exist two matrices $P > 0$ and $Q > 0$, three diagonal matrices $R > 0$, $S > 0$, and $W > 0$, and one scalar $\lambda^* > 0$ such that the following LMIs hold for all $1 \leq i < j \leq N$:

$$P < \lambda^* E_n \quad (44)$$

$$\Psi_4^{(i,j)} = \begin{bmatrix} \Lambda_5 & 0 & -RK_2 & 0 & 0 & A^T P \\ * & \Pi_2 & 0 & -WK_2 & -SL_2 & 0 \\ * & * & -R & 0 & 0 & B^T P \\ * & * & * & -W & 0 & C^T P \\ * & * & * & * & \Lambda_3^{(i,j)} & 0 \\ * & * & * & * & * & -(1+NG_{ij})^{-1}P \end{bmatrix} < 0 \quad (45)$$

where $\Lambda_5 = -RK_1 + \lambda^* \rho_1 E_n + (\tau_M - \tau_m + 1)Q - P$, $\Lambda_3^{(i,j)}$, and $\widetilde{G}$ are defined in Corollary 1.

$$\Psi_1^{(i,j)} = \begin{bmatrix} \Lambda_1 & 0 & -RK_2 + \varepsilon W_1^T W_2 & \varepsilon W_1^T W_3 & 0 & A^T P & 0 \\ * & \Lambda_2 & 0 & -WK_2 & -SL_2 & 0 & 0 \\ * & * & -R + \varepsilon W_2^T W_2 & \varepsilon W_2^T W_3 & 0 & B^T P & 0 \\ * & * & * & -W + \varepsilon W_3^T W_3 & 0 & C^T P & 0 \\ * & * & * & * & \Lambda_3^{(i,j)} & 0 & 0 \\ * & * & * & * & * & -(1+NG_{ij})^{-1}P & PM \\ * & * & * & * & * & * & -\varepsilon E_n \end{bmatrix} < 0 \quad (37)$$

$$\Psi_2^{(i,j)} = \begin{bmatrix} \Lambda_4^{(i,j)} & 0 & -RK_2 + \varepsilon W_1^T W_2 & \varepsilon W_1^T W_3 & A^T P & 0 \\ * & -WK_1 - Q & 0 & -WK_2 & 0 & 0 \\ * & * & -R + \varepsilon W_2^T W_2 & \varepsilon W_2^T W_3 & B^T P & 0 \\ * & * & * & -W + \varepsilon W_3^T W_3 & C^T P & 0 \\ * & * & * & * & -(1+NG_{ij})^{-1}P & PM \\ * & * & * & * & * & -\varepsilon E_n \end{bmatrix} < 0 \quad (39)$$

Remark 6: Up to now, there have been considerable results on the general topic of synchronization. For example, master–slave synchronization of delayed neural networks was analyzed in [3] and [13], the synchronization problem of complex networks without delay was investigated in [26] and [42], while the delayed case was studied in [9] and [43]; synchronization of an array of delayed neural networks was also studied in [4], [6], and [22]. However, all criteria obtained in the aforementioned literature are about continuous-time systems, and the results on the corresponding discrete-time models are relatively few. In the context of discrete-time models, the master–slave synchronization was discussed in [17] without considering the parameter uncertainties, where the activation function was assumed to be of the traditional Lipschitz type. In [23]–[25], Lu and Chen investigated the synchronization problem for an array of discrete-time coupled complex networks in a systematic way and obtained a series of elegant results by using innovative manifold/graph approaches. Overall, compared to the earlier works, there are two main features of our results. First, the model studied here is in discrete-time form; second, to the best of our knowledge, this paper represents the first attempt to address both parameter uncertainties and the stochastic disturbances for the synchronization problems under a milder assumption on the activation functions.

IV. Numerical Examples

In this section, two examples will be illustrated to show the effectiveness of our results.

Example 1: Consider system (36) with $W_1 = W_2 = W_3 = [-0.1\ 0.2\ 0.3]$; $\tau_M = 4$, $\tau_m = 2$, and

$$A = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.03 \end{bmatrix} \quad B = \begin{bmatrix} 0.2 & 0 & 0.3 \\ -0.1 & -0.2 & 0 \\ 0 & -0.3 & 0.2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.1 & -0.2 & 0.3 \\ -0.2 & 0.3 & 0.1 \\ 0.4 & 0.1 & -0.1 \end{bmatrix} \quad M = \begin{bmatrix} 0.2 \\ -0.1 \\ 0.1 \end{bmatrix}$$

$$G = \begin{bmatrix} -0.3-a & 0.1+a & 0.2 \\ 0.1+a & -0.2-a & 0.1 \\ 0.2 & 0.1 & -0.3 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

$$f_1(u) = \tanh(0.6u) - 0.2\sin u$$
$$f_2(u) = \tanh(-0.4u) \quad f_3(u) = \tanh(-0.2u)$$
$$h_1(u) = \tanh(-0.4u) + 0.2\cos u$$
$$h_2(u) = \tanh(0.2u) \quad h_3(u) = \tanh(0.4u).$$

Obviously, these activation functions satisfy Assumption 1 with

$$k_1^- = -0.2 \quad k_1^+ = 0.8$$
$$k_2^- = -0.4 \quad k_2^+ = 0$$
$$k_3^- = -0.2 \quad k_3^+ = 0$$
$$l_1^- = -0.2 \quad l_1^+ = 0.6$$
$$l_2^- = 0 \quad l_2^+ = 0.2$$
$$l_3^- = 0 \quad l_3^+ = 0.4$$

and, therefore, it follows from the definitions of K_i and L_i $(i = 1, 2)$ that

$$K_1 = \mathrm{diag}\{-0.16, 0, 0, \} \quad K_2 = \mathrm{diag}\{-0.3, 0.2, 0.1\}$$
$$L_1 = \mathrm{diag}\{-0.12, 0, 0, \} \quad L_2 = \mathrm{diag}\{0.2, -0.1, -0, 2\}.$$

Let the scalar a vary from -0.1; by using the Matlab LMI Toolbox, LMI (37) can be solved with the feasible solutions until a $= 0.6175$. By taking $a = 0$, we have the solutions as follows:

$$P = \begin{bmatrix} 263.6908 & 45.1525 & -25.7545 \\ 45.1525 & 211.2515 & 24.1494 \\ -25.7545 & 24.1494 & 145.5831 \end{bmatrix}$$

$$Q = \begin{bmatrix} 60.5222 & 11.9773 & -3.7023 \\ 11.9773 & 46.6390 & 3.3782 \\ -3.7023 & 3.3782 & 30.0475 \end{bmatrix}$$

$$R = \begin{bmatrix} 118.3256 & 0 & 0 \\ 0 & 157.8252 & 0 \\ 0 & 0 & 198.5518 \end{bmatrix}$$

$$W = \begin{bmatrix} 100.5035 & 0 & 0 \\ 0 & 181.2791 & 0 \\ 0 & 0 & 193.0692 \end{bmatrix}$$

$$S = \mathrm{diag}\{62.7657, 118.0793, 102.5650\} \quad \varepsilon = 134.2146.$$

According to Corollary 1, the array of coupled uncertain discrete-time delayed neural networks (36) can achieve globally robust synchronization when $a \in [-0.1, 0.6175]$.

Example 2: Consider system (43) with $\rho_1 = 0.1$ and $\rho_2 = 0.15$, $\tau_M = 5$ and $\tau_m = 3$, and

$$A = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix} \quad B = \begin{bmatrix} 0.2 & -0.1 \\ 0.3 & -0.2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.3 & 0.1 \\ -0.3 & 0.2 \end{bmatrix}$$

$$G = \begin{bmatrix} -0.3-b & 0.1 & 0.2+b \\ 0.1 & -0.2 & 0.1 \\ 0.2+b & 0.1 & -0.3-b \end{bmatrix}$$

$$D = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0 & -0.16 \end{bmatrix} \quad K_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.3 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 0 & 0 \\ 0 & -0.12 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

Let the scalar b vary from -0.2; by using the Matlab LMI Toolbox, LMIs (44) and (45) can be solved with the feasible solutions until $b = 0.0893$. When $b = 0$, we have the feasible solutions given as follows:

$$P = \begin{bmatrix} 14.5037 & 0.0413 \\ 0.0413 & 15.6396 \end{bmatrix} \quad Q = \begin{bmatrix} 3.5933 & 0.0036 \\ 0.0036 & 3.9363 \end{bmatrix}$$

$$R = \begin{bmatrix} 30.2621 & 0 \\ 0 & 6.1613 \end{bmatrix}$$

$$W = \begin{bmatrix} 17.7166 & 0 \\ 0 & 3.5723 \end{bmatrix} \quad S = \begin{bmatrix} 6.5919 & 0 \\ 0 & 0.9187 \end{bmatrix}$$

$$\lambda^* = 15.8564.$$

According to Corollary 4, the array (43) is globally synchronized in the mean square when $b \in [-0.2, 0.0893]$.

V. Conclusion

In this paper, we have investigated the robust synchronization problem for an array of coupled stochastic discrete-time neural networks with time-varying delay. We have established easy-to-verify conditions under which the addressed neural networks are synchronized. By using the Kronecker product as an effective tool, an LMI approach has been developed to derive several sufficient criteria ensuring the coupled delayed neural networks to be globally, robustly, exponentially synchronized in the mean square. The LMI-based conditions obtained are dependent not only on the lower bound but also on the upper bound of the time-varying delay, and can be solved efficiently via the Matlab LMI Toolbox. Two numerical examples have been given to demonstrate the usefulness of the proposed synchronization scheme.

References

[1] S. Arik, "Stability analysis of delayed neural networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 47, no. 7, pp. 1089–1092, Jul. 2000.

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[3] J. Cao and J. Lu, "Adaptive synchronization of neural networks with or without time-varying delays," *Chaos*, vol. 16, 2006, 013133.

[4] J. Cao, P. Li, and W. W. Wang, "Global synchronization in arrays of delayed neural networks with constant and delayed coupling," *Phys. Lett. A*, vol. 353, pp. 318–325, 2006.

[5] J. L. Chen and X. H. Chen, *Special Matrices*. Tsinghua, China: Tsinghua Univ. Press, 2001.

[6] G. Chen, J. Zhou, and Z. R. Liu, "Global synchronization of coupled delayed neural networks and applications to chaotic CNN models," *Int. J. Bifur. Chaos*, vol. 14, no. 7, pp. 2229–2240, 2004.

[7] W. J. Freeman, "Spatial properties of an EEG in the olfactory bulb and olfactory cortex," *Electroencephalogr. Clin. Neurophysiol.*, vol. 44, pp. 586–605, 1978.

[8] H. Gao and T. Chen, "New results on stability of discrete-time synchronization with time-varying state delay," *IEEE Trans. Autom. Control*, vol. 52, no. 2, pp. 328–334, Feb. 2007.

[9] H. Gao, J. Lam, and G. Chen, "New criteria for synchronization stability of general complex dynamical networks with coupling delays," *Phys. Lett. A*, vol. 360, pp. 263–273, 2006.

[10] C. M. Gray, "Synchronous oscillations in neuronal systems: Mechanism and functions," *J. Comput. Neurosci.*, vol. 1, pp. 11–38, 1994.

[11] Y. He, M. Wu, and J. H. She, "Delay-dependent exponential stability of delayed neural networks with time-varying delay," *IEEE Trans. Circuits Syst. II, Exp/ Briefs*, vol. 53, no. 7, pp. 553–557, Jul. 2006.

[12] Y. He, M. Wu, and J. H. She, "An improved global asymptotic stability criterion for delayed cellular neural networks," *IEEE Trans. Neural Netw.*, vol. 17, no. 1, pp. 250–252, Jan. 2006.

[13] X. Huang and J. Cao, "Generalized synchronization for delayed chaotic neural networks: A novel coupling scheme," *Nonlinearity*, vol. 19, pp. 2797–2811, 2006.

[14] K. Kaneko, "Spatio-temporal intermittency in coupled map lattices," *Prog. Theor. Phys.*, vol. 74, pp. 1033–1044, 1985.

[15] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[16] J. Liang, J. Cao, and J. Lam, "Convergence of discrete-time recurrent neural networks with variable delay," *Int. J. Birfur. Chaos*, vol. 15, no. 2, pp. 581–595, 2005.

[17] J. Liang, Z. Wang, and X. Liu, "Exponential synchronization of stochastic delayed discrete-time complex networks," *Nonlinear Dyn.*, vol. 53, no. 1–2, pp. 153–165, 2008.

[18] Y. Liu, Z. Wang, A. Serrano, and X. Liu, "Discrete-time recurrent neural networks with time-varying delays: Exponential stability analysis," *Phys. Lett. A*, vol. 362, pp. 480–488, 2007.

[19] Y. Liu, Z. Wang, and X. Liu, "Global exponential stability of generalized recurrent neural networks with discrete and distributed delays," *Neural Netw.*, vol. 19, no. 5, pp. 667–675, 2006.

[20] Y. Liu, Z. Wang, and X. Liu, "Robust stability of discrete-time stochastic neural networks with time-varying delays," *Neurocomputing*, vol. 71, no. 4–6, pp. 823–833, 2008.

[21] H. T. Lu, "Comments on "A generalized LMI-based approach to the global asymptotic stability of delayed cellular neural networks"," *IEEE Trans. Neural Netw.*, vol. 16, no. 3, pp. 778–779, May 2005.

[22] W. L. Lu and T. P. Chen, "Synchronization of coupled connected neural networks with delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 51, no. 12, pp. 2491–2503, Dec. 2004.

[23] W. L. Lu and T. P. Chen, "Synchronization analysis of linearly coupled networks of discrete time systems," *Physica D*, vol. 198, pp. 148–168, 2004.

[24] W. L. Lu and T. P. Chen, "A new approach to synchronization analysis of linearly coupled map lattices," *Chin. Ann. Math. B*, vol. 28, no. 2, pp. 149–160, 2007.

[25] W. L. Lu and T. P. Chen, "Global synchronization of discrete-time dynamical network with a directed graph," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 54, no. 2, pp. 136–140, Feb. 2007.

[26] J. H. Lv, X. H. Yu, and G. Chen, "Chaos synchronization of general complex dynamical network," *Physica A*, vol. 334, pp. 281–302, 2004.

[27] X. Mao, *Stochastic Differential Equations and Applications*. Chichester, U.K.: Horwood, 1997.

[28] S. Mohamad and K. Gopalsamy, "Exponential stability of continuous-time and discrete-time cellular neural networks with delays," *Appl. Math. Comput.*, vol. 135, no. 1, pp. 17–38, 2003.

[29] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 8, pp. 821–824, 1990.

[30] H. D. Qi and L. Qi, "Deriving sufficient conditions for global asymptotic stability of delayed neural networks via nonsmooth analysis," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 99–109, Jan. 2004.

[31] H. D. Qi, L. Qi, and X. Yang, "Deriving sufficient conditions for global asymptotic stability of delayed neural networks via nonsmooth analysis-II," *IEEE Trans. Neural Netw.*, vol. 16, no. 6, pp. 1701–1706, Nov. 2005.

[32] V. Singh, "A generalized LMI-Based approach to the global asymptotic stability of delayed cellular neural networks," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 223–225, Jan. 2004.

[33] V. Singh, "Global robust stability of delayed neural networks: An LMI approach," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 1, pp. 33–36, Jan. 2005.

[34] Q. Song and Z. Wang, "A delay-dependent LMI approach to dynamics analysis of discrete-time recurrent neural networks with time-varying delays," *Phys. Lett. A*, vol. 368, no. 1–2, pp. 134–145, 2007.

[35] K. Tateno, H. Hayashi, and S. Ishizuka, "Complexity of spatiotemporal activity of a neural network model which depends on the degree of synchronization," *Neural Netw.*, vol. 11, pp. 985–1003, 1998.

[36] A. A. Ukhtomsky, *Collected Works* (in Russian). Leningrad, Russia: Nauka, 1978, pp. 107–237.

[37] Z. Wang, Y. Liu, and X. Liu, "On global asymptotic stability of neural networks with discrete and distributed delays," *Phys. Lett. A*, vol. 345, no. 4–6, pp. 299–308, 2005.

[38] Z. Wang, H. Shu, J. Fang, and X. Liu, "Robust stability for stochastic Hopfield neural networks with time delays," *Nonlinear Anal.: Real World Appl.*, vol. 7, pp. 1119–1128, 2006.

[39] Z. Wang, Y. Liu, M. Li, and X. Liu, "Stability analysis for stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Trans. Neural Netw.*, vol. 17, no. 3, pp. 814–820, May 2006.

[40] Z. Wang, D. W. C. Ho, and X. Liu, "State estimation for delayed neural networks," *IEEE Trans. Neural Netw.*, vol. 16, no. 1, pp. 279–284, Jan. 2005.

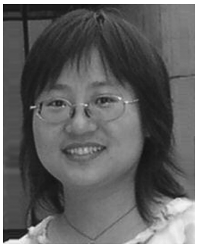
[41] J. Winson, "Patterns of hippocampal theta rhythm in the freely moving rat," *Electroencephalogr. Clin. Neurophysiol.*, vol. 36, pp. 291–301, 1974.

[42] C. W. Wu, "Synchronization in coupled arrays of chaotic oscillators with nonreciprocal coupling," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 2, pp. 294–297, Feb. 2003.

[43] C. W. Wu, "Synchronization in arrays of coupled nonlinear systems with delay and nonreciprocal time-varying coupling," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 5, pp. 282–286, May 2005.

[44] Z. G. Zeng and J. Wang, "Improved conditions for global exponential stability of recurrent neural networks with time-varying delays," *IEEE Trans. Neural Netw.*, vol. 17, no. 3, pp. 623–635, May 2006.

[45] H. Zhao, "Global asymptotic stability of Hopfield neural network involving distributed delays," *Neural Netw.*, vol. 17, pp. 47–53, 2004.



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