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ABSTRACT

This study investigates the finite-time consensus control problem for a class of mixed-order multi-agent systems (MASs) with both stochastic noises and nonlinear dynamics. The sub-systems of the MASs under consideration are heterogenous that are described by a series of differential equations with different orders. The purpose of the addressed problem is to design a control protocol ensuring that the agents' states can achieve the desired consensus in finite time in probability 1. By using the so-called adding a power integrator technique in combination with Lyapunov stability theory, the required distributed consensus control protocol is developed and the corresponding settling time is estimated. Finally, a simulation example is given to demonstrate the correctness and usefulness of the proposed theoretical results.

1. Introduction

In recent years, along with the fast development of network communications (Zou et al., 2020), multi-agent systems (MASs) have been stirring considerable research interest due to the broad practical applications in various fields, ranging from autonomous vehicles to sensor networks (Chen et al., 2015; Ge & Han, 2016, 2017; Li et al., 2017; Ma et al., 2017a; Oh et al., 2015; Tariverdi et al., 2021; Wang & Han, 2018; Yousuf et al., 2020; Zhang et al., 2011; Zou, Wang, Dong et al., 2020; Zou, Wang, Hu et al., 2020). MASs consist of a multitude of agents (sub-systems) that can interact with neighbours for the purpose of achieving the common goals collectively. It should be mentioned that the consensus control problem, which aims to seek a control law/protocol that enables the agents' states to reach certain common values, is one of the most fundamental yet active research topics in the study of MASs (Ma et al., 2017b). Many other tasks (e.g. Ma, Wang, Han et al., 2017) can be equivalently converted to the consensus control issue, and therefore, the consensus control of MASs have been extensively investigated and a huge amount of results have been reported in literature, see Wang and Wang (2020), Xu and Wu (2021), Herzallah (2021) and Liu et al. (2020) for some recent publications.

Among the aforementioned works, most algorithms have been exploited to reach the consensus in the asymptotic mean. In other words, the required consensus might be achieved when the time approaches infinity rather than in a finite time interval (Zhu et al., 2014; Zou et al., 2019). It is widely known that the convergence rate is critical which is utilized to evaluate the speed of attaining the consensus, as in many practical systems, faster convergence speed indicates better performance. Consequently, the finite-time consensus control issue for MASs with the purpose of reaching consensus in a limited/required time interval has started to gain research interest. So far, much effort has been devoted to the investigation on the finite-time consensus control, resulting in many research fruits available in the literature, see e.g. Li et al. (2011), Wang et al. (2016), Lu et al. (2017) and Li et al. (2019) and the references therein.

Note that to date, almost all of the investigation regarding the finite-time consensus control of MASs have been concerned with the homogeneous case where the considered MASs are comprised of sub-systems of identical dynamics. In engineering practice, however, quite a few types of MASs are consisting of agents with different parameters, dynamics and/or structures. In such cases, the existing results on homogeneous MASs cannot be employed directly to deal with the heterogeneous ones. This gives rise to the study toward the consensus control problem for heterogeneous MASs (Shi et al., 2020). For instance, for a class of heterogeneous linear MASs, the

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consensus control problem has been solved in Wieland and Allgower (2009) where the sub-systems are of the same structure but with different parameters. Not only the parameters but also the structures can be different in a heterogeneous MAS. A quintessential example that should be mentioned is the multi-vehicle systems composed of unmanned ground vehicles (UGVs) and unmanned aerial vehicles (UAVs), which can be found wide utilizations in various areas like patrol, search and rescue (Luo et al., 2016). Notice that agents of UGVs are usually modelled by second-order differential equations, whereas those of UAVs are modelled by fourth-order ones.

Unfortunately, the aforementioned literature concerning heterogenous MASs are mainly focus on agents with relatively low-order dynamics (Du et al., 2020; Zheng & Wang, 2012), and therefore, the corresponding algorithms would be unapplicable for the high-order MASs (Du et al., 2017; Li et al., 2019; Sun et al., 2015; You et al., 2019). Limited research has been carried out but the obtained results can be applied only for relatively simple dynamics, see e.g. Zhou et al. (2015). As for the more complex cases such as general nonlinear high-order MASs, the corresponding research has been far from adequate which still remains challenging. On the other hand, in real-world applications, all the systems are inherently nonlinear (Liu, H. et al., 2020; Liu, Ma et al., 2020; Ma, Fang et al., 2020; Ma, Wang et al., 2020; Zhang et al., 2020) and subject to stochastic disturbances (Hu & Feng, 2010; Li & Zhang, 2010; Wen et al., 2012; Zhao & Jia, 2015). Accordingly, it is of vital importance to take into account nonlinearities and stochasticity when handling the consensus problem of MASs, which gives us the main motivation of conducting the current research.

In response to the above dissuasions, this paper tackles the finite-time consensus control problem for a class of mixed-order stochastic nonlinear MASs. The main difficulties of the addressed problem can be identified as follows: (i) How to develop an appropriate methodology to design a consensus control protocol ensuring the agents' states of the same order could reach some common values during a finite-time interval? (ii) In addition to mixed-order agents' dynamics (which contains high-order components), both random noise and nonlinearities are also taken into consideration in the system model, which makes the design of the consensus protocol more complicated.

The main contributions of this study can be outlined as follows: (1) The model of the heterogeneous MASs discussed in this paper is comprehensive, not only the nonlinearities but also the orders of agents can be different. (2) Stochastic noises, nonlinear terms, high-order and mixed-order dynamics are considered simultaneously that provide a comprehensive yet realistic reflection of the real-world engineering complexities. In comparison to algorithms in existing literature, the advantage of the consensus approach proposed in this paper mainly lies in its capability of dealing with finite-time consensus in probability for mixed-order dynamics and high-order dynamics in a uniform framework. (3) By resorting to adding a power integrator technique in combination with Lyapunov theory, the addressed finite-time consensus control protocol is proposed and the desired specific settling time is formulated.

The rest of this paper is organized as follows. The consensus control problem of the mixed-order heterogeneous stochastic nonlinear MAS is formulated in Section 2. The design of consensus control protocol and the analysis of the setting time are presented in Section 3. To demonstrate the usefulness of the proposed protocol, a simulation is given in Section 4. Section 5 draws our conclusion.

Notation: The notations used in this paper is quite standard except where otherwise stated. $\mathbb{R}^n$ refers to the n-dimensional Euclidean space. $|\cdot|$ denotes the absolute value. The superscript T denotes the transpose and trace(A) means the trace of matrix A. $\lambda(A)$ denotes the eigenvalue of matrix A. $\mathbf{1}_n$ refers to an n-dimensional column vector with all ones. The notation $P\{A\}$ denotes the probability of event A, while $E\{A\}$ stands for the mathematical expectation of random variable A.

2. Preliminaries and problem formulation

In this paper, we use an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{H})$ to describe the interaction among N_2 agents. Denote, respectively, $\mathcal{V} = \{1, \ldots, N_2\}$, $\mathcal{E}$ and $\mathcal{H} = [h_{ij}]$, as the set of N_2 agents, the set of edges, and the adjacency matrix of $\mathcal{G}$. If there is an edge between agent i and agent j, it means the two agents can communicate with each other. In this case, $h_{ij} = h_{ji} > 0$. Specially, we set $h_{ii} = 0$. The matrix $L = [l_{ij}]$ represents the Laplacian matrix, where $l_{ii} = \sum_{j=1}^{N_2} h_{ij}$ and $l_{ij} = -h_{ij}, i \neq j$. If a path can be found between any two nodes, then the graph is connected. Here, we suppose $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{H})$ is connected.

Suppose that the addressed heterogeneous MAS is composed of agents whose dynamics are described by kth-order ($k = 2, \ldots, n$) differential equations. The amount of all agents is N_2. Agents with nth-order dynamics are labelled as $i = 1, \ldots, N_n$, and agents with mth-order ($2 \leq m < n$) dynamics are labelled as $i = N_{m+1} + 1, \ldots, N_m$, where $N_n \leq N_{n-1} \leq \cdots \leq N_2$. The model of

agent i is described as follows:

$$\begin{cases} \dot{x}_{i,j}(t) = x_{i,j+1}(t) & j = 1, \ldots, n_i - 1 \\ \dot{x}_{i,n_i}(t) = f_i(x_{i,n_i}(t)) + u_i(t) + g_i(x_{i,n_i}(t))\eta(t) \end{cases} \quad (1)$$

where $x_{i,1}(t), \ldots, x_{i,n_i}(t) \in \mathbb{R}$ are the entries of state of agent i; $u_i(t) \in \mathbb{R}$ is the control input; n_i is the order of agent i; $\eta(t)$ is a scalar Gaussian noise with $E(\eta(t)) = 0$ and $E(\eta^2(t)) = 1$; $f_i : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R} \to \mathbb{R}$ are continuously differentiable nonlinear functions, satisfying $f_i(0) = 0$ and $g_i(0) = 0$.

In the following, we apply the abbreviated notations $x_{i,1}, \ldots, x_{i,n_i}$ and u_i for $x_{i,1}(t), \ldots, x_{i,n_i}(t)$ and $u_i(t)$. Then, we can rewrite equations in (1) as follows Zhao and Jia (2015):

$$\begin{cases} dx_{i,j} = x_{i,j+1}dt & j = 1, \ldots, n_i - 1, \\ dx_{i,n_i} = (f_i(x_{i,n_i}) + u_i)dt + g_i(x_{i,n_i})dW \end{cases} \quad (2)$$

where $W \in \mathbb{R}$ denotes the standard Wiener process.

Remark 2.1: It is worth mentioning that MASs composed of different orders of agents are quite common. The system model considered in this paper can represent both multi-agent systems with different order dynamics and multi-agent systems with the same order dynamics. Here are two examples to help readers understand the system model proposed above. For instance, when $n = 4$, $N_3 - N_4 = 0$, the MAS can be described as a combination of N_4 fourth-order agents

$$\begin{cases} dx_{i,1} = x_{i,2}dt \\ dx_{i,2} = x_{i,3}dt \\ dx_{i,3} = x_{i,4}dt \\ dx_{i,4} = (f_i(x_{i,4}) + u_i)dt + g_i(x_{i,4})dW, \\ i = 1, \ldots, N_4, \end{cases} \quad (3)$$

and $N_2 - N_4$ second-order agents

$$\begin{cases} dx_{i,1} = x_{i,2}dt \\ dx_{i,2} = (f_i(x_{i,2}) + u_i)dt + g_i(x_{i,2})dW \\ i = N_4 + 1, \ldots, N_2 \end{cases} \quad (4)$$

When $n = k(k \geq 2)$, $N_k = N_{k-1} = \cdots = N_2$, the MAS is only composed of kth-order agents:

$$\begin{cases} dx_{i,1} = x_{i,2}dt \\ \vdots \\ dx_{i,k} = (f_i(x_{i,k}) + u_i)dt + g_i(x_{i,k}))dW \\ i = 1, \ldots, N_k \end{cases} \quad (5)$$

The following assumption and definition are needed in our subsequent development.

Assumption 2.1: There exist constants $p_{i1} \geq 0$ and $p_{i2} \geq 0$ such that $|f_i(x_1) - f_i(x_2)| \leq p_{i1}|x_1 - x_2|$ and $|g_i(x_1) - g_i(x_2)| \leq p_{i2}|x_1 - x_2|$.

Definition 2.1: The mixed-order MAS (2) is said to achieve finite-time consensus in probability 1 if the following holds:

$$P\{x_{i,1} = x_{j,1}\} = 1, \quad \forall t \geq T, \forall i \neq j, i, j = 1, \ldots, N_2$$

$$P\{x_{i,2} = x_{j,2}\} = 1, \quad \forall t \geq T, \forall i \neq j, i, j = 1, \ldots, N_2$$

$$\vdots$$

$$P\{x_{i,n} = x_{j,n}\} = 1, \quad \forall t \geq T, \forall i \neq j, i, j = 1, \ldots, N_n,$$

where T is the settling time.

3. Main results

Before the establishment of main results, the following lemmas are firstly introduced.

Lemma 3.1 (Qian & Wei, 2001): For $a > 0$, $b > 0$ and $\gamma(m, n) > 0$,

$$|m|^a|n|^b \leq \frac{a\gamma(m,n)|m|^{a+b}}{a+b} + \frac{b\gamma(m,n)^{-\frac{a}{b}}|n|^{a+b}}{a+b}. \quad (6)$$

Lemma 3.2 (Hardy et al., 1952): For given scalars $\xi_1, \xi_2, \ldots, \xi_m > 0$, the following inequalities hold.

$$\sum_{i=1}^m \xi_i^k \geq m^{1-k}\left(\sum_{i=1}^m \xi_i\right)^k \quad \text{if } 1 < k < \infty \quad (7)$$

$$\sum_{i=1}^m \xi_i^k \geq \left(\sum_{i=1}^m \xi_i\right)^k \quad \text{if } 0 < k \leq 1 \quad (8)$$

Lemma 3.3 (Zhao & Jia, 2015): For scalars $\xi_1 \in \mathbb{R}, \xi_2 \in \mathbb{R}$ and $m = \frac{m_1}{m_2}$, where m_1 and m_2 are positive odd integers,

$$\left|\xi_1^m - \xi_2^m\right| \leq 2^{1-m}|\xi_1 - \xi_2|^m. \quad (9)$$

Consider the following system:

$$dx(t) = f(t, x(t))dt + g(t, x(t))dw(t), x(0) = x_0 \in \mathbb{R}^n,$$

$$t \geq 0, \quad (10)$$

where $x \in \mathbb{R}^n$ is the system state, $w(t)$ is an m-dimensional standard Wiener process (Brown motion) defined on a complete probability space (Ω, F, P) with the augmented filtration $\{F_t\}_{t \geq 0}$ generated by $\{w_t\}_{t \geq 0}$. In (10), $f(,)$ and $g(,)$ are continuous functions of appropriate dimensions with $f(t, 0) = 0$ and $g(t, 0) = 0$ for all $t \geq 0$.

The differential operator of Lyapunov function V in regard to (10) is defined as:

$$\mathcal{L}V(x) = \frac{\partial V(x)}{\partial x}f(t,x) + \frac{1}{2}\,\text{trace}\left\{g^{\mathsf{T}}(t,x)\frac{\partial^2 V(x)}{\partial x^2}g(t,x)\right\}. \tag{11}$$

Definition 3.1 (Yin & Khoo, 2015): The trivial solution of (10) is said to be finite-time stable in probability if the following two requirements are met simultaneously:

(1) System (10) admits a solution (either in the strong sense or in the weak sense) for any initial data $x_0 \in \mathbb{R}^n$, denoted by $x(t;x_0)$. Moreover, for every initial value $x_0 \in \mathbb{R}^n\backslash\{0\}$, the first hitting time $\tau_{x_0} = \inf\{t : x(t;x_0) = 0\} = \inf\{t : |x(t;x_0)| = 0\}$, called stochastic settling time, is finite almost surely, that is, $P\{\tau_{x_0} < \infty\} = 1$;

(2) For every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that $P(|x(t;x_0)| < r$ for all $t \geq 0) \geq 1 - \varepsilon$ whenever $|x_0| < \delta$.

Next, we introduce a lemma regarding the finite-time stability in probability for the nonlinear stochastic system (10).

Lemma 3.4 (Yin et al., 2015; Yin & Khoo, 2015): *Suppose that system* (10) *has a solution for each initial value $x_0 \in \mathbb{R}^n$, if there is a $C_2^0(\mathbb{R}^n)$ function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that:*

(1) $h_1(x) \leq V(x) \leq h_2(x)$, $x \in \mathbb{R}^n$ *for some K_∞ class functions $h_1(x)$ and $h_2(x)$;*

(2) $\mathcal{L}V(x) \leq -aV(x)^b$, $x \in \mathbb{R}^n\backslash\{0\}$ *for some real numbers $a > 0$, $0 < b < 1$,*

then the origin of (10) *is finite-time stochastically stable and the stochastic settling time τ_{x_0} can be estimated by $E\{\tau_{x_0}\} \leq \frac{V(x_0)^{1-b}}{a(1-b)}$.*

Lemma 3.5 (Olfati-Saber & Murray, 2004): *Laplacian matrix $L \in \mathbb{R}^{N_2 \times N_2}$ is a semi-positive matrix and its eigenvalues are greater than or equal to 0. Define $\lambda_1(L), \lambda_2(L), \dots, \lambda_{N_2}(L)$ to be the eigenvalues of L sorted from small to large. The eigenvector corresponding to eigenvalue 0 is $\mathbf{1}_{N_2}$. When the graph corresponding to the Laplacian matrix L is a connected graph, the second smallest eigenvalue $\lambda_2(L)$ is greater than 0. If $\mathbf{1}_{N_2}^{\mathsf{T}}x_1 = 0$, then $\lambda_2(L)x_1^{\mathsf{T}}x_1 \leq x_1^{\mathsf{T}}Lx_1$.*

3.1. Design of the finite-time consensus protocol

In this section, we shall design the finite-time consensus protocols for the considered mixed-order MASs. To this end, we first present three lemmas that play vital roles in the establishment of our main results. To begin with,

define

$$\xi_{i,1} = \sum_{j=1}^{N_2} a_{ij}(x_{i,1} - x_{j,1}), \quad i = 1, \dots, N_2,$$

$$x_{i,2}^* = -\beta_1 \xi_{i,1}^{q_2}, \quad \xi_{i,2} = (x_{i,2})^{\frac{1}{q_2}} - (x_{i,2}^*)^{\frac{1}{q_2}},$$
$$i = 1, \dots, N_2,$$

$$\vdots$$

$$x_{i,n}^* = -\beta_{n-1}\xi_{i,n-1}^{q_n}, \quad \xi_{i,n} = (x_{i,n})^{\frac{1}{q_n}} - (x_{i,n}^*)^{\frac{1}{q_n}},$$
$$i = 1, \dots, N_n,$$

where $q_{j+1} = q_j + \alpha$, $\alpha = -\frac{p_1}{p_2} \in (-\frac{1}{n}, 0)$, $q_1 = 1$, $p_1 > 0$ is a known even integer, $p_2 > 0$ is a known odd integer and $\beta_j > 0$, $j = 1, \dots, n$ are suitable constants.

Lemma 3.6 (Li et al., 2019): *Consider MAS* (2). *For $2 \leq k \leq n_i$, there exists a positive scalar $\hat{\mu}_k$ such that*

$$\frac{\mathrm{d}\left(-(x_{i,k}^*)^{\frac{1}{q_k}}\right)}{\mathrm{d}t}$$

$$\leq \hat{\mu}_k\left(\sum_{m=1}^{k}|\xi_{i,m}|^{q_2} + \sum_{j=1}^{N_2}\left(|\xi_{j,1}|^{q_2} + |\xi_{j,2}|^{q_2}\right)\right). \tag{12}$$

In this paper, the proposed distributed control protocol is of the following form:

$$u_i = -\xi_{i,n_i}^{q_{n_i}-2}(p_{i,1}|\xi_{i,n_i}|^{1-q_2}(\theta_{n_i,1}|\xi_{i,n_i-1}|^{1+q_2}$$
$$+ \theta_{n_i,2}|\xi_{i,n_i}|^{1+q_2}) + p_{i,2}^2|\xi_{i,n_i}|^{1-q_2}(\theta_{n_i,3}|\xi_{i,n_i-1}|^{1+q_2}$$
$$+ \theta_{n_i,4}|\xi_{i,n_i}|^{1+q_2})) - \beta_{n_i}\xi_{i,n_i}^{q_{n_i}+1} \tag{13}$$

where $\theta_{n_i,1}, \theta_{n_i,2}, \theta_{n_i,3}, \theta_{n_i,4}$ and β_{n_i} are positive parameters to be determined later.

Defining two functions V_1 and V_2 by

$$V_1 = \frac{1}{2}x_1^{\mathsf{T}}Lx_1 = \frac{1}{4}\sum_{i=1}^{N_2}\sum_{j=1}^{N_2}a_{ij}\left(x_{i,1} - x_{j,1}\right)^2 \tag{14}$$

where $x_1 = [x_{1,1}, \dots, x_{N_2,1}]^{\mathsf{T}}$, and

$$V_2 = V_1 + \sum_{i=1}^{N_2}W_{i,2} \tag{15}$$

where

$$W_{i,2} = \int_{x_{i,2}^*}^{x_{i,2}}\left(s^{\frac{1}{q_2}} - (x_{i,2}^*)^{\frac{1}{q_2}}\right)^{2-q_2}\mathrm{d}s, \tag{16}$$

we present the following lemma which is pivotal in the design of control parameters.

Lemma 3.7: *Consider MAS* (2) *under protocol* (13) *and functions V_1 and V_2. With appropriately selected parameters β_1, β_2, $\theta_{2,1}$, $\theta_{2,2}$, $\theta_{2,3}$ and $\theta_{2,4}$, the following inequality holds*

$$\mathcal{L}V_2 \leq -(n-1)\sum_{i=1}^{N_2}|\xi_{i,1}|^{1+q_2} - (n-1)\sum_{i=1}^{N_2}|\xi_{i,2}|^{1+q_2}$$
$$+ \sum_{i=1}^{N_3}\xi_{i,2}^{2-q_2}(x_{i,3} - x_{i,3}^*). \tag{17}$$

Proof: Differentiating V_1 along the trajectory of MAS (2) with (13), we arrive at

$$\frac{dV_1}{dt} = \sum_{i=1}^{N_2}\xi_{i,1}\dot{x}_{i,1} = \sum_{i=1}^{N_2}\xi_{i,1}x_{i,2}^* + \sum_{i=1}^{N_2}\xi_{i,1}(x_{i,2} - x_{i,2}^*). \tag{18}$$

Set $\beta_1 = n$, then

$$\frac{dV_1}{dt} = -n\sum_{i=1}^{N_2}\xi_{i,1}^{1+q_2} + \sum_{i=1}^{N_2}\xi_{i,1}(x_{i,2} - x_{i,2}^*). \tag{19}$$

It is easy to have from (11) that

$$\mathcal{L}V_2 = \frac{dV_1}{dt} + \sum_{i=1}^{N_3}\frac{dW_{i,2}}{dt} + \sum_{i=N_3+1}^{N_2}\mathcal{L}W_{i,2}$$
$$= \frac{dV_1}{dt} + \sum_{i=1}^{N_3}\frac{dW_{i,2}}{dt} + \sum_{i=N_3+1}^{N_2}\frac{\partial W_{i,2}}{\partial x_{i,2}}u_i$$
$$+ \sum_{i=N_3+1}^{N_2}\frac{\partial W_{i,2}}{\partial x_{i,2}}f_i(x_{i,2}) + \sum_{i=N_3+1}^{N_2}\frac{\partial W_{i,2}}{\partial \xi_{i,1}}\frac{d\xi_{i,1}}{dt}$$
$$+ \frac{1}{2}\sum_{i=N_3+1}^{N_2}\frac{\partial^2 W_{i,2}}{\partial x_{i,2}^2}g_i^2(x_{i,2}). \tag{20}$$

It is inferred from Lemmas 3.1 and 3.3 that

$$\sum_{i=1}^{N_2}\xi_{i,1}(x_{i,2} - x_{i,2}^*) \leq \frac{1}{2}\sum_{i=1}^{N_2}|\xi_{i,1}|^{1+q_2} + \hat{c}_2\sum_{i=1}^{N_2}|\xi_{i,2}|^{1+q_2} \tag{21}$$

where $\hat{c}_2 > 0$.

Furthermore, we have

$$\int_{x_{i,2}^*}^{x_{i,2}}\left(s^{\frac{1}{q_2}} - (x_{i,2}^*)^{\frac{1}{q_2}}\right)^{1-q_2}ds \leq 2^{1-q_2}|\xi_{i,2}|. \tag{22}$$

By defining $d_i = \sum_{i=1}^{N_2}a_{ij}$, we obtain that

$$\frac{d\left(-(x_{i,2}^*)^{\frac{1}{q_2}}\right)}{dt} \leq \beta_1^{\frac{1}{q_2}}\left(d_i|x_{i,2}| + \sum_{j=1}^{N_2}a_{ij}|x_{j,2}|\right). \tag{23}$$

It follows from Zhou et al. (2015) that

$$|x_{i,2}||\xi_{i,2}| \leq \left(\beta_1\frac{1}{1+q_2}\gamma_1^{-q_2} + 2^{1-q_2}\right)|\xi_{i,2}|^{1+q_2}$$
$$+ \beta_1\frac{q_2}{1+q_2}\gamma_1|\xi_{i,1}|^{1+q_2} \tag{24}$$

and

$$|x_{j,2}||\xi_{i,2}| \leq \beta_1\frac{q_2}{1+q_2}\gamma_1|\xi_{j,1}|^{1+q_2}$$
$$+ 2^{1-q_2}\frac{q_2}{1+q_2}\gamma_2|\xi_{j,2}|^{1+q_2}$$
$$+ \left(2^{1-q_2}\frac{1}{1+q_2}\gamma_2^{-q_2} + \beta_1\frac{1}{1+q_2}\gamma_1^{-q_2}\right)|$$
$$\times \xi_{i,2}|^{1+q_2} \tag{25}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$.

Consequently, based on (23)–(25) in combination with Lemma 3.1, we derive that

$$\sum_{i=1}^{N_2}(2-q_2)\frac{d\left(-(x_{i2}^*)^{\frac{1}{q_2}}\right)}{dt}\int_{x_{i,2}^*}^{x_{i,2}}\left(s^{\frac{1}{q_2}} - (x_{i,2}^*)^{\frac{1}{q_2}}\right)^{1-q_2}ds$$
$$\leq \sum_{i=1}^{N_2}(2-q_2)2^{1-q_2}\beta_1^{\frac{1}{q_2}}\left(d_i|x_{i,2}| + \sum_{j=1}^{N_2}a_{ij}|x_{j,2}|\right)|\xi_{i,2}|$$
$$\leq \frac{1}{2}\sum_{i=1}^{N_2}|\xi_{i,1}|^{1+q_2} + \tilde{c}_2\sum_{i=1}^{N_2}|\xi_{i,2}|^{1+q_2} \tag{26}$$

where $\tilde{c}_2 > 0$.

On the other hand, we know from (26) that

$$\sum_{i=1}^{N_3}\frac{dW_{i,2}}{dt} + \sum_{i=N_3+1}^{N_2}\frac{\partial W_{i,2}}{\partial \xi_{i,1}}\frac{d\xi_{i,1}}{dt}$$
$$= \sum_{i=1}^{N_3}\xi_{i2}^{2-q_2}x_{i3} + \sum_{i=1}^{N_2}(2-q_2)\frac{d\left(-(x_{i2}^*)^{\frac{1}{q_2}}\right)}{dt}$$
$$\times \int_{x_{i2}^*}^{x_{i2}}\left(s^{\frac{1}{q_2}} - (x_{i2}^*)^{\frac{1}{q_2}}\right)^{1-q_2}ds$$
$$\leq \frac{1}{2}\sum_{i=1}^{N_2}|\xi_{i,1}|^{1+q_2} + \tilde{c}_2\sum_{i=1}^{N_2}|\xi_{i,2}|^{1+q_2} + \sum_{i=1}^{N_3}\xi_{i,2}^{2-q_2}x_{i,3}. \tag{27}$$

Then, the following is true

$$\sum_{i=N_3+1}^{N_2}\frac{\partial W_{i,2}}{\partial x_{i,2}} = \sum_{i=N_3+1}^{N_2}\xi_{i,2}^{2-q_2}. \tag{28}$$

According to Assumption 2.1 and $f_i(0) = 0$, it is readily inferred that $|f_i(x_{i,2})| \leq p_{i,1}|x_{i,2}|$. Moreover, we acquire

$$\sum_{i=N_3+1}^{N_2} \frac{\partial W_{i,2}}{\partial x_{i,2}} f_i(x_{i,2})$$

$$\leq \sum_{i=N_3+1}^{N_2} p_{i,1}|\xi_{i,2}|^{1-q_2}|\xi_{i,2}| \left(|\xi_{i,2}| + \beta_1^{\frac{1}{q_2}}|\xi_{i,1}| \right)^{q_2}$$

$$\leq \sum_{i=N_3+1}^{N_2} p_{i,1}|\xi_{i,2}|^{1-q_2} \left(|\xi_{i,2}|^{1+q_2} + \beta_1|\xi_{i,1}|^{q_2}|\xi_{i,2}| \right)$$

$$\leq \sum_{i=N_3+1}^{N_2} p_{i,1}|\xi_{i,2}|^{1-q_2} \left(\beta_1 \frac{q_2}{1+q_2}\gamma|\xi_{i,1}|^{1+q_2} \right.$$

$$\left. + \left(\beta_1 \frac{1}{1+q_2}\gamma^{-q_2} + 1 \right)|\xi_{i,2}|^{1+q_2} \right)$$

$$\leq \sum_{i=N_3+1}^{N_2} p_{i,1}|\xi_{i,2}|^{1-q_2} \left(\theta_{2,1}|\xi_{i,1}|^{1+q_2} + \theta_{2,2}|\xi_{i,2}|^{1+q_2} \right)$$

$$(29)$$

where $\theta_{2,1} \geq \beta_1 \frac{q_2}{1+q_2}\gamma$ and $\theta_{2,2} \geq \beta_1 \frac{1}{1+q_2}\gamma^{-q_2} + 1$ with γ an arbitrary positive constant.

Next, it can be obtained that

$$\sum_{i=N_3+1}^{N_2} \frac{\partial^2 W_{i,2}}{\partial x_{i,2}^2} = \sum_{i=N_3+1}^{N_2} \frac{(2-q_2)}{q_2}\xi_{i,2}^{1-q_2}x_{i,2}^{\frac{1}{q_2}-1}. \quad (30)$$

From Assumption 2.1 and $g_i(0) = 0$, we have $|g_i(x_{i,2})| \leq p_{i,2}|x_{i,2}|$. Using Lemma 3.2 again, it is obtained that

$$\sum_{i=N_3+1}^{N_2} \frac{1}{2}\frac{\partial^2 W_{i,2}}{\partial x_{i,2}^2}g_i^2(x_{i,2})$$

$$\leq \sum_{i=N_3+1}^{N_2} \frac{2-q_2}{2q_2}p_{i,2}^2\xi_{i,2}^{1-q_2} \left| \xi_{i2} + \beta_1^{\frac{1}{q_2}}\xi_{i,1} \right|^{1+q_2}$$

$$\leq \sum_{i=N_3+1}^{N_2} p_{i,2}^2|\xi_{i,2}|^{1-q_2} \left(\theta_{2,3}|\xi_{i,1}|^{1+q_2} + \theta_{2,4}|\xi_{i,2}|^{1+q_2} \right)$$

$$(31)$$

where $\theta_{2,3} \geq \frac{2-q_2}{2q_2}2^{q_2}\beta_1^{\frac{1+q_2}{q_2}}$ and $\theta_{2,4} \geq \frac{2-q_2}{2q_2}2^{q_2}$.

Selecting $\beta_2 = n - 1 + \tilde{c}_2 + \hat{c}_2$, and taking into account (13), (20), (21), (27), (29) and (31), it is seen that inequality (17) holds. The proof is complete now. ∎

By following a similar line, by defining a function V_k as

$$V_k = V_1 + \sum_{j=2}^{k}\sum_{i=1}^{N_j} W_{i,j}, \quad 3 \leq k \leq n - 1 \quad (32)$$

where

$$W_{i,j} = \int_{x_{i,j}^*}^{x_{i,j}} (s^{\frac{1}{q_j}} - (x_{i,j}^*)^{\frac{1}{q_j}})^{2-q_j}ds, \quad (33)$$

we give the following lemma.

Lemma 3.8: *Consider MAS* (2) *under protocol* (13) *with function* V_k. *By selecting appropriate values of* β_k, $\theta_{k,1}$, $\theta_{k,2}$, $\theta_{k,3}$ *and* $\theta_{k,4}$, *the following inequality holds*

$$\mathcal{L}V_k \leq -(n-k+1)\sum_{i=1}^{N_2}|\xi_{i,1}|^{1+q_2}$$

$$- (n-k+1)\sum_{j=2}^{k}\sum_{i=1}^{N_j}|\xi_{i,j}|^{1+q_2}$$

$$+ \sum_{i=1}^{N_{k+1}}\xi_{i,k}^{2-q_k}(x_{i,k+1} - x_{i,k+1}^*). \quad (34)$$

Proof: The proof is performed by induction.

Initial Step: For $k = 2$, it can be known directly from Lemma 3.7 that inequality (34) holds.

Inductive Step: Given that at step $k-1$, $3 < k \leq n - 1$, the following inequality is true:

$$\mathcal{L}V_{k-1} \leq -(n-k+2)\sum_{i=1}^{N_2}|\xi_{i,1}|^{1+q_2}$$

$$- (n-k+2)\sum_{j=2}^{k-1}\sum_{i=1}^{N_j}|\xi_{i,j}|^{1+q_2}$$

$$+ \sum_{i=1}^{N_k}\xi_{i,k-1}^{2-q_{k-1}}(x_{i,k} - x_{i,k}^*). \quad (35)$$

Then it remains to show that inequality (34) still holds at step k.

In fact, one can obtain that

$$\mathcal{L}V_k = \mathcal{L}V_{k-1} + \sum_{i=1}^{N_{k+1}}\frac{dW_{i,k}}{dt} + \sum_{i=N_{k+1}+1}^{N_k}\mathcal{L}W_{i,k}$$

$$= \mathcal{L}V_{k-1} + \sum_{i=1}^{N_{k+1}}\frac{dW_{i,k}}{dt} + \sum_{i=N_{k+1}+1}^{N_k}\frac{\partial W_{i,k}}{\partial x_{i,k}}u_i$$

$$+ \sum_{i=N_{k+1}+1}^{N_k}\frac{\partial W_{i,k}}{\partial x_{i,k}}f_i(x_{i,k}) + \sum_{i=N_{k+1}+1}^{N_k}\frac{\partial W_{i,k}}{\partial \xi_{i,k-1}}\frac{d\xi_{i,k-1}}{dt}$$

$$+ \frac{1}{2}\sum_{i=N_{k+1}+1}^{N_k}\frac{\partial^2 W_{i,k}}{\partial x_{i,k}^2}g_i^2(x_{i,k}). \quad (36)$$

Similar to (21) and (22), we have

$$|\xi_{i,k-1}^{2-q_{k-1}}(x_{i,k} - x_{i,k}^*)| \leq \frac{1}{2}|\xi_{i,k-1}|^{1+q_2} + \hat{c}_k|\xi_{i,k}|^{1+q_2} \quad (37)$$

and

$$\int_{x_{i,k}^*}^{x_{i,k}} \left(s^{\frac{1}{q_k}} - (x_{i,k}^*)^{\frac{1}{q_k}} \right)^{1-q_k} ds \leq 2^{1-q_k} |\xi_{i,k}| \qquad (38)$$

where $\hat{c}_k > 0$.

Based on inequality (48) in combination with Lemmas 3.1 and 3.6, we acquire that

$$\sum_{i=1}^{N_k} (2-q_k) \frac{d\left(-(x_{i,k}^*)^{\frac{1}{q_k}}\right)}{dt} \int_{x_{i,k}^*}^{x_{i,k}} \left(s^{\frac{1}{q_k}} - (x_{i,k}^*)^{\frac{1}{q_k}} \right)^{1-q_k} ds$$

$$\leq \tilde{\mu}_k \sum_{i=1}^{N_k} \left(\sum_{m=1}^{k} |\xi_{i,m}|^{q_2} + \sum_{j=1}^{N_2} |\xi_{j,1}|^{q_2} + \sum_{j=1}^{N_2} |\xi_{j,2}|^{q_2} \right) |\xi_{i,k}|$$

$$\leq \sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2} + \sum_{m=2}^{k-2} \sum_{i=1}^{N_m} |\xi_{i,m}|^{1+q_2}$$

$$+ \frac{1}{2} \sum_{i=1}^{N_k} |\xi_{i,k-1}|^{1+q_2} + \tilde{c}_k \sum_{i=1}^{N_k} |\xi_{i,k}|^{1+q_2} \qquad (39)$$

where $\tilde{\mu}_k = (2-q_k)\hat{\mu}_k$ and $\tilde{c}_k > 0$.

It is inferred from (39) that

$$\sum_{i=1}^{N_{k+1}} \frac{dW_{i,k}}{dt} + \sum_{i=N_{k+1}+1}^{N_k} \frac{\partial W_{i,k}}{\partial \xi_{i,k-1}} \frac{d\xi_{i,k-1}}{dt}$$

$$= \sum_{i=1}^{N_{k+1}} \xi_{i,k}^{2-q_k} x_{i,k+1} + \sum_{i=1}^{N_k} (2-q_k) \frac{d\left(-(x_{i,k}^*)^{\frac{1}{q_k}}\right)}{dt}$$

$$\times \int_{x_{i,k}^*}^{x_{i,k}} \left(s^{\frac{1}{q_k}} - (x_{i,k}^*)^{\frac{1}{q_k}} \right)^{1-q_k} ds$$

$$\leq \frac{1}{2} \sum_{i=1}^{N_k} |\xi_{i,k-1}|^{1+q_2} + \tilde{c}_k \sum_{i=1}^{N_k} |\xi_{i,k}|^{1+q_2}$$

$$+ \sum_{i=1}^{N_{k+1}} \xi_{i,k}^{2-q_k} x_{i,k+1} + \sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2}$$

$$+ \sum_{i=1}^{N_2} |\xi_{i,2}|^{1+q_2} + \sum_{m=3}^{k-2} \sum_{i=1}^{N_m} |\xi_{i,m}|^{1+q_2}. \qquad (40)$$

Similar to (28)–(31), one obtains

$$\sum_{i=N_{k+1}+1}^{N_k} \frac{\partial W_{i,k}}{\partial x_{i,k}} f_i(x_{i,k})$$

$$\leq \sum_{i=N_{k+1}+1}^{N_k} p_{i,1} |\xi_{i,k}|^{1-q_2} \left(\theta_{k,1} |\xi_{i,k-1}|^{1+q_2} + \theta_{k,2} |\xi_{i,k}|^{1+q_2} \right)$$

$$\qquad (41)$$

and

$$\sum_{i=N_{k+1}+1}^{N_k} \frac{1}{2} \frac{\partial^2 W_{i,k}}{\partial x_{i,k}^2} g_i^2(x_{i,k})$$

$$\leq \sum_{i=N_{k+1}+1}^{N_k} p_{i,2}^2 |\xi_{i,k}|^{1-q_2}$$

$$\times \left(\theta_{k,3} |\xi_{i,k-1}|^{1+q_2} + \theta_{k,4} |\xi_{i,k}|^{1+q_2} \right). \qquad (42)$$

It follows from Lemma 3.1 that

$$\theta_{k,1} \geq \beta_{k-1} \frac{q_k}{1+q_2} \gamma_{k1}$$

$$\theta_{k,2} \geq \beta_{k-1} \frac{1+q_2-q_k}{1+q_2} \gamma_{k1}^{-\frac{q_k}{1+q_2-q_k}} + 1$$

$$\theta_{k,3} \geq \frac{2-q_k}{2q_k} \beta_{k-1}^{\frac{1+q_k}{q_k}} \frac{1+q_k}{1+q_2} \gamma_{k2}$$

$$\theta_{k,4} \geq \frac{2-q_k}{2q_k} \left(\beta_{k-1}^{\frac{1+q_k}{q_k}} \frac{q_2-q_k}{1+q_2} \gamma_{k2}^{-\frac{1+q_k}{q_2-q_k}} + 1 \right) \qquad (43)$$

where γ_{k1} and γ_{k2} are arbitrary positive scalars.

Selecting $\beta_k = n - k + 1 + \tilde{c}_k + \hat{c}_k$ and taking into account (13), (36), (40), (41) and (42), it can be concluded that inequality (34) holds. The proof is now complete. ■

The following theorem gives our main results of this paper.

Theorem 3.1: *The nonlinear MAS* (2) *can achieve finite-time consensus in probability under the control protocol* (13). *Moreover, the settling time T satisfies*

$$E\{T\} \leq \frac{V_n^{1-\frac{1}{1+q_2}}(0)}{\frac{1}{2} c^{-\frac{1+q_2}{2}} \left(1 - \frac{1+q_2}{2}\right)} \qquad (44)$$

where $c = \max\{\frac{1}{2\lambda_2(L)}, 2^{1-q_j}\}, j = 2, \ldots, n$.

Proof: According to Lemma 3.8, we obtain

$$\mathcal{L}V_{n-1} \leq -2 \sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2} - 2 \sum_{j=2}^{n-1} \sum_{i=1}^{N_j} |\xi_{i,j}|^{1+q_2}$$

$$+ \sum_{i=1}^{N_n} \xi_{i,n-1}^{2-q_{n-1}} (x_{i,n} - x_{i,n}^*). \qquad (45)$$

We acquire from (11) that

$$\mathcal{L}V_n = \mathcal{L}V_{n-1} + \sum_{i=1}^{N_n} \frac{\partial W_{i,n}}{\partial x_{i,n}} u_i + \sum_{i=1}^{N_n} \frac{\partial W_{i,n}}{\partial x_{i,n}} f_i(x_{i,n})$$

$$+ \sum_{i=1}^{N_n} \frac{\partial W_{i,n}}{\partial \xi_{i,n-1}} \frac{d\xi_{i,n-1}}{dt} + \frac{1}{2} \sum_{i=1}^{N_n} \frac{\partial^2 W_{i,n}}{\partial x_{i,n}^2} g_i^2(x_{i,n}).$$

$$\qquad (46)$$

Similar to (21) and (22), we have

$$|\xi_{i,n-1}^{2-q_{n-1}}(x_{i,n} - x_{i,n}^*)| \le \frac{1}{2}|\xi_{i,n-1}|^{1+q_2} + \hat{c}_n|\xi_{i,n}|^{1+q_2} \quad (47)$$

and

$$\int_{x_{i,n}^*}^{x_{i,n}} \left(s^{\frac{1}{q_n}} - (x_{i,n}^*)^{\frac{1}{q_n}}\right)^{1-q_n} ds \le 2^{1-q_n}|\xi_{i,n}| \quad (48)$$

where $\hat{c}_n > 0$.

Next, it can be obtained that

$$\frac{\partial W_{i,n}}{\partial \xi_{i,n-1}} = (2 - q_n) \frac{\partial\left(-(x_{i,n}^*)^{\frac{1}{q_n}}\right)}{\partial \xi_{i,n-1}}$$
$$\times \int_{x_{i,n}^*}^{x_{i,n}} \left(s^{\frac{1}{q_n}} - (x_{i,n}^*)^{\frac{1}{q_n}}\right)^{1-q_n} ds. \quad (49)$$

Subsequently,

$$\sum_{i=1}^{N_n} \frac{\partial W_{i,n}}{\partial \xi_{i,n-1}} \frac{d\xi_{i,n-1}}{dt}$$
$$\le \sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2} + \sum_{m=2}^{n-2} \sum_{i=1}^{N_m} |\xi_{i,m}|^{1+q_2}$$
$$+ \frac{1}{2} \sum_{i=1}^{N_n} |\xi_{i,n-1}|^{1+q_2} + \tilde{c}_n \sum_{i=1}^{N_n} |\xi_{i,n}|^{1+q_2}. \quad (50)$$

where $\tilde{c}_n > 0$.

Similar to (28)–(31), we have that

$$\sum_{i=1}^{N_n} \frac{\partial W_i}{\partial x_{i,n}} f_i(x_{i,n})$$
$$\le \sum_{i=1}^{N_n} p_{i,1}|\xi_{i,n}|^{1-q_2} \left(\theta_{n,1}|\xi_{i,n-1}|^{1+q_2} + \theta_{n,2}|\xi_{i,n}|^{1+q_2}\right) \quad (51)$$

and

$$\sum_{i=1}^{N_n} \frac{1}{2} \frac{\partial^2 W_{i,n}}{\partial x_{i,n}^2} g_i^2(x_{i,n})$$
$$\le \sum_{i=1}^{N_n} p_{i,2}^2|\xi_{i,n}|^{1-q_2} \left(\theta_{n,3}|\xi_{i,n-1}|^{1+q_2} + \theta_{n,4}|\xi_{i,n}|^{1+q_2}\right) \quad (52)$$

where $\theta_{n,1}, \theta_{n,2}, \theta_{n,3}, \theta_{n,4}$ can be chosen according to (43).

Set $\beta_n = -(\hat{c}_n + \tilde{c}_n + 1)$. Taking into account (13) and (46)–(52), we arrive at

$$\mathcal{L}V_n \le -\left(\sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2} + \sum_{j=2}^{n} \sum_{i=1}^{N_j} |\xi_{i,j}|^{1+q_2}\right). \quad (53)$$

For $x_{i,k}^* \le x_{i,k}$, we obtain from Lemma 3.3 that

$$W_{i,k} \ge 2^{\frac{q_k-1}{q_k}} \int_{x_{i,k}^*}^{x_{i,k}} \left(s - x_{i,k}^*\right)^{\frac{2}{q_k}-1} ds \ge 0 \quad (54)$$

and

$$W_{i,k} \le |x_{i,k} - x_{i,k}^*||\xi_{i,k}|^{2-q_k} \le 2^{1-q_k}|\xi_{i,k}|^2. \quad (55)$$

Likewise, we can easily obtain (54) and (55) for $x_{i,k}^* > x_{i,k}$.

From Lemma 3.5, we have $\mathbf{1}_{N_2}^\mathsf{T} L^{\frac{1}{2}} L^{\frac{1}{2}} \mathbf{1}_{N_2} = \mathbf{1}_{N_2}^\mathsf{T} L \mathbf{1}_{N_2} = 0$. Hence, $\mathbf{1}_{N_2}^\mathsf{T} L^{\frac{1}{2}} = 0^\mathsf{T}$. According to Lemma 3.5, if $\mathbf{1}_{N_2}^\mathsf{T} L^{\frac{1}{2}} x_1 = 0$, then

$$V_1 \le \frac{1}{2\lambda_2(L)} \left(L^{\frac{1}{2}} x_1\right)^\mathsf{T} L \left(L^{\frac{1}{2}} x_1\right) = \frac{1}{2\lambda_2(L)} \sum_{i=1}^{N_2} \xi_{i,1}^2. \quad (56)$$

It is inferred from (55) and (56) that

$$V_n = V_1 + \sum_{j=2}^{n} \sum_{i=1}^{N_j} W_{i,j} \le c\left(\sum_{i=1}^{N_2} \xi_{i,1}^2 + \sum_{j=2}^{n} \sum_{i=1}^{N_j} \xi_{i,j}^2\right). \quad (57)$$

Based on Lemma 3.2, one obtains that

$$V_n^{\frac{1+q_2}{2}} \le c^{\frac{1+q_2}{2}} \left(\sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2} + \sum_{j=2}^{n} \sum_{i=1}^{N_j} |\xi_{i,j}|^{1+q_2}\right). \quad (58)$$

Moreover, from (53) and (58), we acquire that

$$\mathcal{L}V_n + \frac{1}{2} c^{-\frac{1+q_2}{2}} V_n^{\frac{1+q_2}{2}}$$
$$\le -\frac{1}{2}\left(\sum_{i=1}^{N_2} |\xi_{i,1}|^{1+q_2} + \sum_{j=2}^{n} \sum_{i=1}^{N_j} |\xi_{i,j}|^{1+q_2}\right)$$
$$\le 0. \quad (59)$$

Consequently, it can be concluded from Zhao and Jia (2015) that there exists a time T such that $E\{V_n\} = 0$, $\forall t \ge T$. According to (54), we can obtain that $E\{V_1\} = 0$ and $E\{W_{i,k}\} = 0, i = 1, \ldots, N_k, k = 1, \ldots, n, \forall t \ge T$, which

indicates that

$$P\{x_{i,1} = x_{j,1}\} = 1, \quad \forall t \geq T, \forall i \neq j, i,j = 1, \ldots, N_2$$
$$P\{x_{i,2} = x_{j,2}\} = 1, \quad \forall t \geq T, \forall i \neq j, i,j = 1, \ldots, N_2$$

$$\vdots$$

$$P\{x_{i,n} = x_{j,n}\} = 1, \quad \forall t \geq T, \forall i \neq j, i,j = 1, \ldots, N_n.$$

Based on Lemma 3.4, it can be known that the settling time T satisfies

$$E\{T\} \leq \frac{V_n^{1-\frac{1}{1+q_2}}(0)}{\frac{1}{2}c^{-\frac{1+q_2}{2}}(1 - \frac{1+q_2}{2})}.$$

The proof is now complete. ∎

Remark 3.1: So far, the finite-time consensus problem has been solved for a class of mixed-order stochastic nonlinear systems. With a recursive design method, the controller of agent i has been designed at step n_i. By assuming that the nonlinear terms satisfying Lipschitz-type conditions and with the help of the adding a power integrator technique, the consensus control problem has been solved for the case where both mixed-order and high-order dynamics are involved.

Remark 3.2: Note that in this paper, we have proposed the algorithm to drive the agents consensus in a finite time interval. It should be mentioned that the provided design framework could not be directly used to deal with the fixed-time control that requires MAS to reach consensus in a pre-specified time interval. However, it is worth pointing out that, on basis of our obtained results, it is not difficult to extend the provided theory and techniques to deal with the fixed-time consensus study which is indeed an interesting direction of our future research work. Another research topic would be the consideration of the communication threats/attacks occurring in the data propagation among the agents (Liu et al., 2021; Ma et al., 2021).

4. Simulation results

In this section, a numerical example is given to demonstrate the effectiveness of the proposed consensus control algorithm.

Consider the following mixed-order MASs consisting of two third-order agents and three second-order agents:

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} \\ \dot{x}_{i,2} = x_{i,3} \\ \dot{x}_{i,3} = \sin(x_{i,3}) + 0.2x_{i,3} + u_i + 0.1\sin(x_{i,3})\eta \end{cases} \tag{60}$$

where $i = 1, 2$, and

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} \\ \dot{x}_{i,2} = \sin(x_{i,2}) - 0.2x_{i,2} + u_i + 0.1\sin(x_{i,2})\eta \end{cases} \tag{61}$$

where $i = 3, 4, 5$.

The communication topology of the MAS is presented in Figure 1 with Laplacian matrix L as follows:

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Choose the following initial conditions: $x_1 = [x_{11}, x_{21}, x_{31}, x_{41}, x_{51}]^T = [1, 2, 8, 3, 4]^T$, $x_2 = [x_{12}, x_{22}, x_{32}, x_{42}, x_{52}]^T = [2, 2, 5, 4, 3]^T$, $x_3 = [x_{31}, x_{32}]^T = [1, 2]^T$. According to Assumption 1, parameters $p_{i,1}$ associated with $f_i = \sin(x_{i,3}) + 0.2x_{i,3}, i = 1, 2$ can be selected as 1.2; parameters $p_{i,1}$ associated with $f_i = \sin(x_{i,2}) - 0.2x_{i,2}, i = 3, 4, 5$ can be selected as 1.2; parameters $p_{i,2}$ associated with $g_i = \sin(x_{i,3}), i = 1, 2$ can be selected as 0.1 and parameters $p_{i,2}$ associated with $g_i = 0.1\sin(x_{i,2}), i = 3, 4, 5$ can be selected as 0.1.

In addition, we select control parameters $\beta_3 = 300$, $\beta_2 = 7$, $\beta_1 = 3$, $\theta_{21} = 100$, $\theta_{22} = 1000$, $\theta_{23} = 100$, $\theta_{24} = 1000$, $\theta_{31} = 300$, $\theta_{32} = 300$, $\theta_{33} = 3000$, $\theta_{34} = 1000$, $q_2 = $
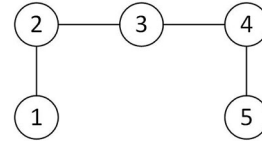

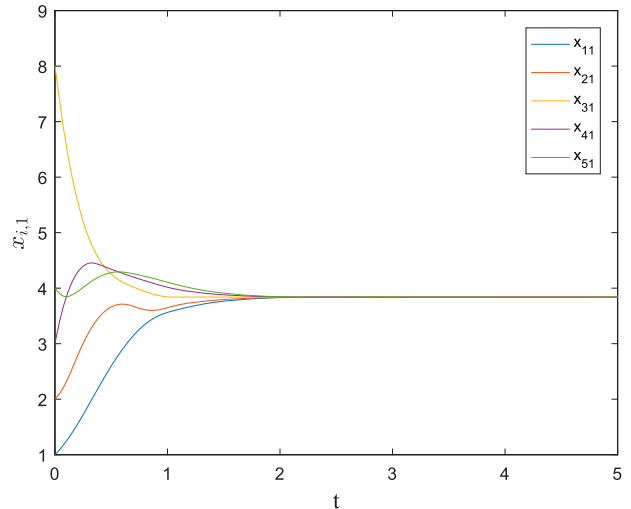**Figure 1.** The communication topology among the MAS.

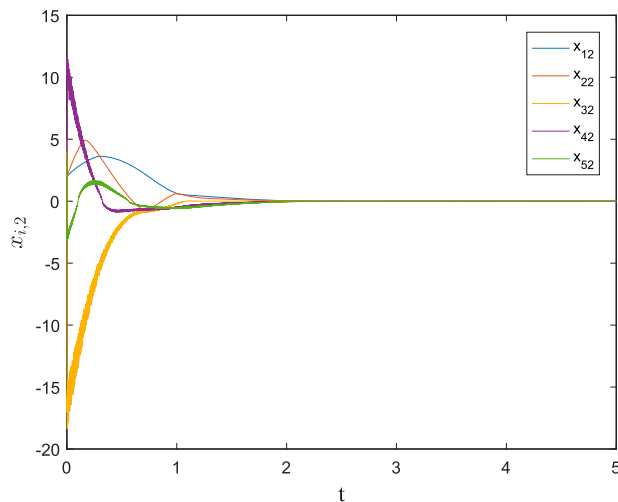**Figure 2.** Results of the first-order states of the MAS given in (60), (61) under the control law (13).

Figure 3. Results of the second-order states of the MAS given in (60), (61) under the control law (13).
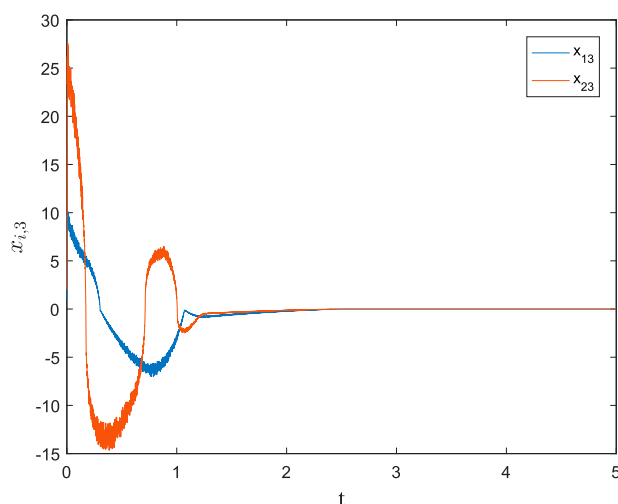

Figure 4. Results of the third-order states of the MAS given in (60), (61) under the control law (13).

$\frac{5}{7}$, $q_3 = \frac{3}{7}$ and $q_4 = \frac{1}{7}$. It can be seen from Figures 2–4, all the states of agents of different orders reach consensus within three seconds, which is indicative of the applicability of the proposed consensus protocol.

5. Conclusion

In this paper, the finite-time consensus control problem has been solved for mixed-order heterogeneous stochastic MASs with non-identical nonlinear dynamics. By assuming that nonlinear terms satisfy Lipschitz-type conditions, a novel consensus control protocol has been provided by resorting to adding a power integrator technique. Then, the finite-time consensus in probability has been proven where the boundness of the settling time has been formulated. Finally, a simulation example has

been given to verify the usefulness of the proposed control law.

Disclosure statement

No potential conflict of interest was reported by the authors.

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