

ON K -STABILITY OF ONE-NODAL PRIME FANO THREEFOLDS OF GENUS 12

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To Professor Yuri Prokhorov on the occasion of his 60th birthday

ABSTRACT. We show that general one-nodal prime Fano threefolds of genus 12 are K -polystable.

1. INTRODUCTION

1.1. Singular Fano threefolds of genus 12. Let X be a Fano threefold with terminal Gorenstein singularities. By [12, 7], $X \hookrightarrow \mathcal{X}$ has a smoothing and $\mathcal{X}_t$ for $t \neq 0$ is a smooth Fano threefold with Picard rank $\rho(X)$ and anticanonical degree $(-K_X)^3$. Unless mentioned otherwise, a prime Fano threefold of genus 12 will refer to a terminal Gorenstein Fano threefold with Picard rank 1 and anticanonical degree 22. Recent advances in the theory of K -stability show that there is a projective moduli space $M_{3,22}^{\text{Kps}}$ whose closed points over $\mathbb{C}$ parameterize K -polystable Fano threefolds of anticanonical degree 22 that admit a smoothing (see [16] as a reference on the general theory of K -moduli).

Let X be a prime Fano threefold of genus 12, then X is $\mathbb{Q}$ -factorial precisely when X is smooth [11]. Smooth prime Fano threefolds of genus 12 form a 6-dimensional family, which contains both K -polystable and strictly K -semistable members [2, Section 7.1]. A precise description of which smooth prime Fano threefolds of genus 12 are K -polystable or semistable is still conjectural. Denote by $\overline{M}$ the (non-empty 6-dimensional) component of $M_{3,22}^{\text{Kps}}$ parametrizing those K -polystable Fano threefolds of anticanonical degree 22 with a smoothing to a prime Fano threefold of genus 12. Prokhorov classifies prime Fano threefolds of genus 12 with one node and shows that they form four 5-dimensional families [14]. The goal of this note is to show:

Theorem 1.1. *A general one-nodal prime Fano threefold of genus 12 is K -polystable. There are four boundary divisors of $\overline{M}$ parametrising K -polystable degenerations of one-nodal prime Fano threefolds of genus 12.*

We now describe the geometry of the four families of one-nodal prime Fano threefolds of genus 12 briefly.

Theorem 1.2. [14] *Let X be a prime Fano threefold of genus 12, and assume that $\text{rk Cl}(X) = 2$ (or equivalently that $\text{Sing}(X)$ consists of precisely one ordinary double point). Then X is the midpoint of a Sarkisov link*

$$\begin{array}{ccccc}
 & X_1 & \overset{\chi}{\dashrightarrow} & X_2 & \\
 f_1 \swarrow & & \pi_1 \searrow & \pi_2 \swarrow & \searrow f_2 \\
 Z_1 & & X & & Z_2
 \end{array}$$

where π_1 and π_2 are small $\mathbb{Q}$ -factorializations, χ is a flop, and f_1 and f_2 are K -negative extremal contractions described as follows

- (I) $Z_1 = \mathbb{P}^3$ and $Z_2 = \mathbb{P}^3$, f_1 and f_2 are the blowups of curves $\Gamma_1 \subset Z_1$ and $\Gamma_2 \subset Z_2$ respectively. Both Γ_1 and Γ_2 are rational quintic curves that do not lie on quadric surfaces.

- (II) $Z_1 = Q \subset \mathbb{P}^4$ and $Z_2 = \mathbb{P}^2$, f_1 is the blowup of a rational quintic curve $\Gamma_1 \subset Q$ that does not lie on a hyperplane section of Q , and f_2 is a conic bundle with discriminant δ of degree 3.
- (III) $Z_1 = V_5$ a quintic del Pezzo threefold, $Z_2 = \mathbb{P}^1$, f_1 is the blowup of a rational quartic curve $\Gamma \subset V_5$ and f_2 is a del Pezzo fibration of degree 6.
- (IV) $Z_1 = \mathbb{P}^2$ and $Z_2 = \mathbb{P}^1$, $f_1: \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}) \rightarrow \mathbb{P}^2$, where $\mathcal{E}$ is a stable rank 2 vector bundle and f_2 is a del Pezzo fibration of degree 5.

Remark 1.3. These four families appear in [8, Table 2] under references (12na),(12nb),(12nc) and (12nd). The blowup of a general member X of Family (I) (resp. (II), resp. (III), resp. (IV)) at its node is a weak Fano threefold $\hat{X}$ whose anticanonical model admits a smoothing in Family MM_{2-12} (resp. MM_{2-13} , resp. MM_{2-14} , resp. MM_{3-5}) in the classification of Fano threefolds [10].

Theorem 1.4. [2, Proposition 5.66] *There is a K -stable member of Family (IV).*

In this note, we prove:

Theorem 1.5. *There exist K -stable members of Families (I) and (II). There is a K -polystable member of Family (III).*

Acknowledgements. We thank Ivan Cheltsov, Yuchen Liu, Antoine Pinardin, Junyan Zhao and the anonymous referee for valuable discussions and comments. Anne-Sophie Kaloghiros was supported by EPSRC grant EP/V056689/1, Elena Denisova is a PhD student at the University of Edinburgh supported by the School of Mathematics Studentship (funded via EPSRC DTP).

2. PRELIMINARY RESULTS ON EXPLICIT K -STABILITY OF FANO THREEFOLDS

All varieties considered are defined over $\mathbb{C}$. Let X be a Fano variety with at most Kawamata log terminal singularities of dimension $n \geq 2$, and let G be a reductive subgroup in $\text{Aut}(X)$. Let Ξ be a divisor over X , that is Ξ is a prime divisor on a normal variety $\tilde{X}$ with a birational morphism $\varphi: \tilde{X} \rightarrow X$. Define $\beta(\Xi) = A_X(\Xi) - S_X(\Xi)$, where $A_X(\Xi) = 1 + \text{ord}_{\Xi}(K_{\tilde{X}/X})$ is the log discrepancy of Ξ and

$$S_X(\Xi) = \frac{1}{(-K_X)^n} \int_0^{\tau(\Xi)} \text{vol}(\varphi^*(-K_X) - u\Xi) du$$

for $\tau(\Xi) = \sup\{u \in \mathbb{R}_{>0} \mid \varphi^*(-K_X) - u\Xi \text{ is big}\}$.

Theorem 2.1. [17, Corollary 4.14] *Suppose that $\beta(\Xi) > 0$ for every G -invariant prime divisor Ξ over X . Then X is K -polystable.*

Recall the definition of the number $\alpha_{G,Z}(X)$, where $Z \subset X$ is a G -invariant subvariety:

$$\alpha_{G,Z}(X) = \sup \left\{ \lambda \in \mathbb{Q} \mid \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is log canonical at general point of } Z \text{ for any,} \\ \text{effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}.$$

Then $\alpha_G(X) \leq \alpha_{G,Z}(X)$.

Lemma 2.2. [2, 1.44] *Let $f: \tilde{X} \rightarrow X$ be an arbitrary G -equivariant birational morphism, let Ξ be a G -invariant prime divisor in X such that $Z \subseteq f(\Xi)$, then we have*

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \frac{n+1}{n} \alpha_{G,Z}(X).$$

In particular, in dimension 3, the existence of a G -invariant divisor Ξ over X with $\beta(\Xi) < 0$ and $Z \subset c_X(\Xi)$ implies that $\alpha_{G,Z}(X) < \frac{3}{4}$, so that Z is contained in $\text{Nklt}(X, B_X)$ for some $B_X \sim_{\mathbb{Q}} -\lambda K_X$ and rational number $\lambda < \frac{3}{4}$.

The next theorem is an application of the general inductive argument developed by Abban and Zhuang to bound the ratio $\frac{A_X(\Xi)}{S_X(\Xi)}$ [1] to the case of smooth Fano threefolds.

Theorem 2.3. [2, Corollary 1.110] *Let X be a smooth Fano threefold, let Y be an irreducible normal surface in the threefold X , let Z be an irreducible curve in Y , and Ξ a prime divisor over X with $C_X(\Xi) = Z$. Then*

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min \left\{ \frac{1}{S_X(Y)}, \frac{1}{S(W_{\bullet,\bullet}^Y; Z)} \right\}$$

and

$$\begin{aligned} S(W_{\bullet,\bullet}^Y; Z) = & \frac{3}{(-K_X)^3} \int_0^\tau (P(u)^2 \cdot Y) \cdot \text{ord}_Z(N(u)|_Y) du + \\ & + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_Y - vZ) dv du \end{aligned}$$

where $P(u)$ is the positive part of the Zariski decomposition of the divisor $-K_X - uY$, and $N(u)$ is its negative part.

Remark 2.4. Here, $W_{\bullet,\bullet}^Y$ is a $\mathbb{N}^2$ linear series defined as the refinement of the anticanonical ring

$$V_{\bullet}^X = \bigoplus_{m \in \mathbb{N}} H^0(X, -mK_X)$$

by the divisor Y . We refer to [1, Section 2] or [2, Section 1.7] for the definition of $W_{\bullet,\bullet}^Y$ and of the associated invariant $S(W_{\bullet,\bullet}^Y; Z)$. We take the expression in Theorem 2.3 as a definition of $S(W_{\bullet,\bullet}^Y; Z)$. Note that an expression for $S(W_{\bullet,\bullet}^Y; Z)$ can be computed in the more general context of $\mathbb{Q}$ -factorial Mori Dream spaces [2, Theorem 1.106].

We recall a few results on nonklt centres of pairs (X, B_X) where $X \sim -\lambda K_X$ for $\lambda \in \mathbb{Q}$ when X admits morphisms to projective spaces.

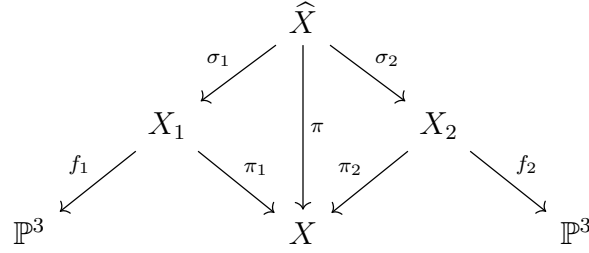
Lemma 2.5 ([2, Corollary A.10]). *Suppose $X = \mathbb{P}^3$ and $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda < \frac{3}{4}$. Let Z be the union of one-dimensional components of $\text{Nklt}(X, B_X)$. Then $\mathcal{O}_{\mathbb{P}^3}(1) \cdot Z \leq 1$. In particular, if $Z \neq 0$, then Z is a line.*

Lemma 2.6 ([2, Corollary A.12]). *Suppose that X is a smooth Fano threefold, $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda < 1$, and there exists a surjective morphism with connected fibers $\phi: X \rightarrow \mathbb{P}^1$. Set $H = \phi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Let Z be the union of one-dimensional components of $\text{Nklt}(X, \lambda B_X)$. Then $H \cdot Z \leq 1$.*

Lemma 2.7 ([2, Corollary A.13]). *Suppose that $-K_X$ is nef and big, $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda < 1$, and there exists a surjective morphism with connected fibers $\phi: X \rightarrow \mathbb{P}^2$. Set $H = \phi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Let Z be the union of one-dimensional components of $\text{Nklt}(X, \lambda B_X)$. Then $H \cdot Z \leq 2$.*

3. FAMILY I

Let X be a one-nodal prime Fano threefold of genus 12 that belongs to Family I of Theorem 1.2, then X is the midpoint of a Sarkisov link associated to a Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ which



is a degeneration of the cubo-cubic transformation [3]. We describe the associated birational geometry briefly, see [3, 5, 14] and [8] for proofs and precise statements.

Denote by $H_i = \sigma_i^*(f_i^*\mathcal{O}_{\mathbb{P}^3}(1))$ for $i = 1, 2$, and by $H = \pi^*(-K_X)$ the pullbacks to $\hat{X}$ (or to any of the models) of the ample generators of $\text{Pic}(\mathbb{P}^3)$ and of $\text{Pic}(X)$. Given a curve $\Gamma \subset \mathbb{P}^3$, we (sloppily) denote by $|nH_1 - \Gamma|$ the linear system $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n) \otimes \mathcal{I}_\Gamma)$ of surfaces of degree n on which Γ lies. The morphism f_1 is the blowup of a smooth rational quintic curve $\Gamma_1 \subset \mathbb{P}^3$ that does not lie on a quadric ($|2H_1 - \Gamma_1| = \emptyset$), and there is a unique quadrisecant line L_1 to Γ_1 . The curve Γ_1 lies on a cubic surface, $|3H_1 - \Gamma_1|$ has dimension 4 and $\text{Bs } |3H_1 - \Gamma_1| = \Gamma_1 \cup L_1$. The birational map associated to $|3H_1 - \Gamma_1| = |H_2|$ is precisely the Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ induced by the Sarkisov link above. The threefold X_1 is weak Fano,

$$-K_{X_1} \sim H \sim 4H_1 - E_1,$$

where $E_1 = \text{Exc } f_1$, so that the proper transform of L_1 (still denoted L_1) is the unique flopping curve on X_1 . The map π_1 contracts L_1 to a node $\{x_0\} = \text{Sing}(X) \in X$.

Let $\pi: \hat{X} \rightarrow X$ be the blowup of x_0 , and σ_1 the induced map to X_1 . Note that X_1 and X_2 are the two small resolutions of the node x_0 , $\chi: X_1 \dashrightarrow X_2$ is the associated Atiyah flop and $L_1 = \sigma_1(F)$, where $F = \text{Exc } \pi$. Then, $\hat{X}$ is a weak Fano threefold of $\rho = 3$ and we have [8]:

$$-K_{\hat{X}} \sim H - F \sim H_1 + H_2$$

and from

$$H \sim 4H_1 - E_1 \sim 4H_2 - E_2$$

we deduce

$$H \sim 2(H_1 + H_2) - \frac{E_1 + E_2}{2} \quad \text{and} \quad H_1 + H_2 \sim \frac{E_1 + E_2}{2} + F.$$

For future reference, let T_1 be a cubic surface containing Γ_1 , and denote by T its proper transform on $\hat{X}$. Since $\text{Bs } |3H_1 - \Gamma_1| = \Gamma_1 \cup L_1$, L_1 lies on T_1 and:

$$T \sim 3\sigma_1^*(f_1^*\mathcal{O}_{\mathbb{P}^3}(1)) - E_1 - F \sim 3H_1 - E_1 - F \sim H_2,$$

so that

$$-K_{\hat{X}} - uT \sim H_1 + H_2 - uH_2 \in \mathbb{Z}_{\geq 0}[H_1] + \mathbb{Z}_{\geq 0}[H_2] \subset \text{Nef}(\hat{X})$$

is nef for $0 \leq u \leq 1$. For $u > 1$, $-K_{\hat{X}} - uT$ is no longer nef. If C is the proper transform on $\hat{X}$ of a minimal rational curve contracted by f_1 , then $H_1 \cdot C = 0$ and $H_2 \cdot C > 0$, so that

$$-K_{\hat{X}} - uT \sim H_1 \cdot C - (u - 1)H_2 \cdot C < 0.$$

We may write for $u \geq 1$

$$\begin{aligned} -K_{\hat{X}} - uT &\sim uH_1 - (u - 1)(H_1 + H_2) \sim uH_1 - (u - 1)(4H_1 - E_1 - F) \\ &\sim (4 - 3u)H_1 + (u - 1)(E_1 + F) \end{aligned}$$

showing that the pseudo-effective threshold is $u = \frac{4}{3}$, and that $-K_{\hat{X}} - uT$ admits a Zariski decomposition with nef positive part $P(u) = (4 - 3u)H_1$ and negative part $N(u) = (u - 1)(E_1 + F)$.

3.1. Construction of a member with $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action. We now consider a special member of Family (I). Let $C_{(a,b)}$ be the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by

$$[x : y] \rightarrow [x^5 : ax^4y + bx^2y^3 : bx^3y^2 + axy^4 : y^5] \text{ for } a, b \in \mathbb{C}^*;$$

then $C_{(a,b)}$ is a rational quintic curve that does not lie on a quadric surface for $|a| \neq |b|$. The curve $C_{(a,b)}$ is invariant under the action of $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{P}^3$ defined by:

$$\begin{aligned} \tau_1 : [x_0 : x_1 : x_2 : x_3] &\rightarrow [x_3 : x_2 : x_1 : x_0], \\ \tau_2 : [x_0 : x_1 : x_2 : x_3] &\rightarrow [x_0 : -x_1 : x_2 : -x_3]. \end{aligned}$$

In fact, the action of τ_1 (resp. τ_2) on $C_{(a,b)}$ is induced by that of the involution of $\mathbb{P}^1$ given by $[x : y] \leftrightarrow [y : x]$ (resp. $[x : y] \leftrightarrow [x : -y]$). We consider the element of Family (I) obtained by taking the curve $\Gamma_1 = C_{(1,-4)}$.

Since Γ_1 is G -invariant, L_1 is also G -invariant and X_1 and X are endowed with a G -action.

Claim 3.1. *The group $\text{Aut}(X)$ is finite.*

Proof. The curve Γ_1 is not contained in a hypersurface of $\mathbb{P}^3$, the stabilizer of Γ in $\text{Aut}(\mathbb{P}^3)$ is $\text{Aut}(\mathbb{P}^3; \Gamma_1) \simeq \text{Aut}(\Gamma_1) \simeq \text{Aut}(\mathbb{P}^1)$. By construction of X , $\text{Aut}(X)$ is a subgroup of the group $\text{Aut}(\mathbb{P}^3; \Gamma_1) \simeq \text{Aut}(\mathbb{P}^1)$ that preserves the four points of intersection $\Gamma_1 \cap L_1$, so it is a finite group. $\square$

We will apply Theorem 2.1 to prove that X is K-stable. To do so, we first describe possible centres of G -invariant divisors over X . In what follows, Ξ always denotes a G -invariant prime divisor over X .

Claim 3.2. *If the centre of Ξ on X is 0-dimensional, it is the singular point $c_X(\Xi) = \{x_0\}$.*

Proof. There is no point of $\mathbb{P}^3$ fixed by the action of G . $\square$

Claim 3.3. *If the centre of Ξ on $\mathbb{P}^3$ is a line L , then*

$$L = L_{[\lambda:\mu]} = \begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0. \end{cases}.$$

All G -invariant lines lie on the quadric $Q = \{x_1x_0 - x_2x_3 = 0\}$. Any two distinct G -invariant lines are disjoint. A G -invariant line $L \neq L_1$ is either disjoint from Γ_1 or meets Γ_1 in precisely two points.

Proof. Let $L \subset \mathbb{P}^3$ be a G -invariant line, and consider any two distinct hyperplanes $H_1 = \{f_1 = 0\}$ and $H_2 = \{f_2 = 0\}$ containing L , so that $L = H_1 \cap H_2 = \{f_1 = f_2 = 0\}$. Then, $L = \text{Bs } \mathcal{H}$ is the base locus of the pencil $\mathcal{H} = \{uf_1 + vf_2 = 0; [u : v] \in \mathbb{P}^1\}$.

The line $L = \text{Bs } \mathcal{H}$ is G -invariant precisely when G fixes $\mathcal{H}$, or equivalently when both τ_1 and τ_2 induce involutions on $\mathcal{H}$, and on its base $\mathbb{P}^1$. Up to reparametrizing the pencil $\mathcal{H}$ we may assume that $[u : v] = [1 : 0]$ is a τ_2 -invariant hyperplane, that is, the linear form $f_1(x_0, \dots, x_3)$ is one of

$$\lambda x_0 + \mu x_2 \text{ or } \lambda x_3 + \mu x_1 \text{ for } [\lambda : \mu] \in \mathbb{P}^1,$$

and $H_1 = \{\lambda x_0 + \mu x_2 = 0\}$ or $H_1 = \{\lambda x_3 + \mu x_1 = 0\}$. The condition that $\mathcal{H}$ is G -invariant is then that $\tau_1 \cdot H_1$ is a fibre of the pencil, so that (noting that H_1 is not fixed by τ_1)

$$L = L_{[\lambda:\mu]} = H_1 \cap \tau_1 \cdot H_1 = \begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0. \end{cases}$$

which gives the desired expression.

Check that $L_{[\lambda:\mu]} \subset Q$ for all $[\lambda:\mu] \in \mathbb{P}^1$, that $L_{[\lambda:\mu]} \cap L_{[\lambda':\mu']} = \emptyset$ for $[\lambda:\mu] \neq [\lambda':\mu']$, and that $L_{[\lambda:\mu]} \cap \Gamma_1 = \emptyset$ unless $[\lambda:\mu] \in \{[0:1], [3:1], [-5:1]\}$ and $L_{[\lambda:\mu]} \cap \Gamma_1$ consists of 2 points, or $[\lambda:\mu] = [1:1]$ and $L_{[1:1]} = L_1$ is the unique quadrisecant to Γ_1 . $\square$

Remark 3.1. Given that the Sarkisov link of which X is a midpoint is G -equivariant, E_2 and $\Gamma_2 = \Gamma^+$ are also invariant under the induced G -action. Since the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is induced by $|H_2| = |3H_1 - \Gamma_1|$, the fibres of $E_2 \rightarrow \Gamma_2$ are the transforms of trisecant lines of Γ_1 . Since none of these are G -invariant, the action of G on Γ_2 does not fix Γ_2 pointwise either.

Claim 3.4. *Let $H_{[\lambda:\mu]}$ be a general hyperplane containing $L_{[\lambda:\mu]}$.*

Then $H_{[\lambda:\mu]} \cap \Gamma_1 = \{b_1, \dots, b_5\}$ and $H_{[\lambda:\mu]} \cap L_1 = \{b_0\}$, where $b_1, \dots, b_5$ (resp. $b_0, \dots, b_5$) consists of 5 (resp. 6) points in general position.

Proof. Fix $[\lambda:\mu] \in \mathbb{P}^1$, and let $\mathcal{H}$ be the pencil of hyperplanes containing $L_{[\lambda:\mu]}$. We compute that the general fibre of $\mathcal{H}$ intersects $\Gamma_1 \cup L_1$ in 6 distinct points. Assume that for some fibre H of $\mathcal{H}$, 3 of the 5 points of $H \cap \Gamma_1$ lie on a line (resp. the 6 points $H \cap (\Gamma_1 \cup L_1)$ lie on a conic). Then, this line (resp. conic) is contracted by the Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ to a point lying on Γ_2 . If the points of intersection of a general hyperplane containing $L_{[\lambda:\mu]}$ are not in general position, then we define a dominant rational map $\mathbb{P}^1 \dashrightarrow \Gamma_2$ from the base of $\mathcal{H}$ to Γ_2 , leading to a contradiction. $\square$

We now turn to the proof that no G -invariant prime divisor Ξ over X with $\beta(\Xi) \leq 0$ has 1-dimensional centre $Z = c_{\mathbb{P}^3}(\Xi)$. If Z is 1-dimensional, then either $Z = \Gamma_1$, or by Lemma 2.2, Z is the union of 1-dimensional components of $\text{Nklt}(\mathbb{P}^3, B)$ for some $B \sim \mathcal{O}_{\mathbb{P}^3}(4\lambda)$ with $\lambda \in \mathbb{Q}$, $\lambda < 3/4$. Then, by Lemma 2.5, Z can only be a line.

Lemma 3.2. *If $Z = c_{\mathbb{P}^3}(\Xi)$ is a G -invariant line distinct from L_1 , $\beta(\Xi) > 0$.*

Proof. We will use Lemma 2.5 to find a lower bound for $\beta(\Xi)$. To this effect, we find an irreducible normal surface $S \subset \widehat{X}$ containing Z and use the inequality

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min \left\{ \frac{1}{S_X(S)}, \frac{1}{S(W_{\bullet, \bullet}^S; Z)} \right\}.$$

Let $S_1 \subset X_1$ and $S \subset \widehat{X}$ be the pullbacks to X_1 and $\widehat{X}$ of a general hyperplane containing Z . By Claim 3.4, $S_1 \subset X_1$ is a del Pezzo surface of degree 4 and S is a cubic surface. Recall that E_1 and E_2 are the f_1 and f_2 exceptional divisors, and that F is the π -exceptional divisor. All these are G -invariant, and E_2 is covered by the (proper transforms of) trisecant lines of Γ_1 .

We first compute $S_X(S)$; on $\widehat{X}$, we have the following intersection numbers:

$$\begin{aligned} S^3 &= 1, & S^2 \cdot E_1 &= 0, & S \cdot E_1^2 &= -5, & E_1^3 &= -18, \\ S^2 \cdot F &= 0, & S \cdot F^2 &= -1, & F^3 &= 2, \\ S \cdot E_1 \cdot F &= 0, & E_1 \cdot F^2 &= -4, & E_1^2 \cdot F &= 0. \end{aligned}$$

and relations [8]

$$S \sim H_1 \sim 3H_2 - E_2 - F \quad H \sim 4H_1 - E_1 \sim 4H_2 - E_2 \quad H - F \sim H_1 + H_2.$$

Define, for $u \geq 0$, the divisor

$$\begin{aligned} D_u &= \pi^*(-K_X) - uS \sim (1-u)H + u(H-S) \sim (1-u)H + u(H_2 + F) = \\ &= (1-u)H + u\left(\frac{H + E_2}{4} + F\right), \end{aligned}$$

then D_u is pseudo-effective for $u \leq 4/3$, and has a Zariski decomposition with nef positive part

$$D_u = P(u) + N(u), \text{ where } \begin{cases} P(u) = (1 - \frac{3u}{4})H, \\ N(u) = u(\frac{E_2}{4} + F). \end{cases}$$

and we compute:

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^\tau \text{vol}(\pi^*(-K_X) - uS) du = \frac{1}{22} \int_0^{4/3} \frac{11(4-3u)^3}{32} du = \frac{1}{3}.$$

We now show that $S(W_{\bullet,\bullet}^S; Z) < 1$ in order to apply Theorem 2.3.

The surface S is a cubic surface obtained by blowing up a general hyperplane $\mathbb{P}^2$ at 6 points $\{b_0, \dots, b_5\}$ in general position. Let ℓ be the pullback of the generator of $\text{Pic}(\mathbb{P}^2)$, and $e_0, \dots, e_5$ the exceptional curves. The Mori cone $\overline{\text{NE}}(S)$ is generated by $e_0, \dots, e_5$, by the proper transforms $l_{i,j} = \ell - e_i - e_j$ of lines through two of the blownup points, and by the proper transforms $q_i = 2\ell - \sum e_j + e_i$ of the conics through any 5 of the blownup points $\{b_0, \dots, b_5\}$.

We want to evaluate

$$\begin{aligned} S(W_{\bullet,\bullet}^S; Z) &= \frac{3}{(-K_X)^3} \int_0^\tau (P(u)^2 \cdot S) \cdot \text{ord}_Z(N(u)|_S) du + \\ &\quad + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_S - vZ) dv du. \end{aligned}$$

Since there are no G -fixed points, $Z \subset S$ is one of ℓ or the lines $l_{i,j}$. Recall that $E_2 \sim 8H_1 - 3E_1 - 4F$, and restricting to S gives $E_2|_S \sim 8\ell - 3(e_1 + \dots + e_5) - 4e_0$. From the description of $\overline{\text{NE}}(S)$,

$$\text{ord}_Z(E_2|_S) \leq 2$$

and

$$\text{ord}_Z(N(u)|_S) \leq \begin{cases} 2 \cdot \frac{1}{4} = \frac{1}{2} & \text{if } Z \subset E_2, \\ 0 & \text{otherwise.} \end{cases}$$

The first term of the expression $S(W_{\bullet,\bullet}^S; Z)$ is bounded by

$$\frac{3}{(-K_X)^3} \int_0^\tau (P(u)^2 \cdot S) \cdot \text{ord}_Z(N(u)|_S) du \leq \frac{3}{22} \int_0^{4/3} \frac{11(4-3u)^2}{16} \cdot \frac{1}{2} du = \frac{1}{3}.$$

Case 1: $Z \cap \Gamma_1 = \emptyset$. In this case, $Z \sim \ell$, and the Zariski decomposition of

$$(-\pi^*(K_X) - uS)|_S - vZ = P(u, v) + N(u, v);$$

for $u \in [0, \frac{4}{3}]$ is given by

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3(4-3u)}{8}, \\ \frac{8v-3(4-3u)}{4} q_0 & \text{for } \frac{3(4-3u)}{8} \leq v \leq \frac{4-3u}{2}. \end{cases}$$

and we compute:

$$P(u, v)^2 = \begin{cases} v^2 - 8v + 6uv + \frac{99u^2}{16} - \frac{33u}{2} + 11 & \text{for } 0 \leq v \leq \frac{3(4-3u)}{8}, \\ \frac{5(2v-(4-3u))^2}{4} & \text{for } \frac{3(4-3u)}{8} \leq v \leq \frac{4-3u}{2}. \end{cases}$$

This yields:

$$S(W_{\bullet,\bullet}^S; Z) \leq \frac{1}{3} + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_S - vZ) dv du \leq$$

$$\begin{aligned} &\leq \frac{1}{3} + \frac{3}{22} \int_0^{4/3} \left(\int_0^{\frac{3(4-3u)}{8}} v^2 - 8v + 6uv + \frac{99u^2}{16} - \frac{33u}{2} + 11dv + \right. \\ &\quad \left. + \int_{\frac{3(4-3u)}{8}}^{\frac{4-3u}{2}} \frac{5(2v - (4-3u))^2}{4} dv \right) du \leq \frac{1}{3} + \frac{53}{132} = \frac{97}{132} < 1, \end{aligned}$$

which is what we wanted.

Case 2: $Z \cap \Gamma_1 \neq \emptyset$. As Z is one of the bisecant lines of Γ_1 , $Z \sim l_{i,j} = \ell - e_i - e_j$ for some $1 \leq i < j \leq 5$. We may assume that $Z \sim l_{1,2}$. Write the Zariski decomposition of $(-\pi^*(K_X) - uS)|_S - vZ$ for $0 \leq u \leq \frac{4}{3}$ we have:

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3(4-3u)}{4}, \\ \frac{4v-3(4-3u)}{4}(e_1 + e_2) & \text{for } \frac{3(4-3u)}{4} \leq v \leq \frac{4-3u}{2}, \\ \frac{4v-3(4-3u)}{4}(e_1 + e_2) + \frac{2v-3(4-3u)}{2}(\ell_{34} + \ell_{35} + \ell_{45}) & \text{for } \frac{4-3u}{2} \leq v \leq \frac{5(4-3u)}{8}. \end{cases}$$

This time, we compute:

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &\leq \frac{1}{3} + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_S - vZ) dv du \leq \\ &\leq \frac{1}{3} + \frac{3}{22} \int_0^{4/3} \left(\int_0^{\frac{4-3u}{4}} -v^2 - 4v + 3uv + \frac{99u^2}{16} - \frac{33u}{2} + 11dv + \right. \\ &\quad \left. + \int_{\frac{4-3u}{4}}^{\frac{4-3u}{2}} v^2 - 8v + 6uv + \frac{117u^2}{16} - \frac{39u}{2} + 13dv + \right. \\ &\quad \left. + \int_{\frac{4-3u}{2}}^{\frac{5(4-3u)}{8}} \frac{(8v - 5(4-3u))^2}{16} dv \right) du \leq \frac{1}{3} + \frac{23}{44} = \frac{113}{132} < 1. \end{aligned}$$

and this finishes the proof in this case. $\square$

Lemma 3.3. *Let Ξ be a prime divisor over X with $c_{\mathbb{P}^3}(\Xi) = L_1$ then $\beta(\Xi) > 0$.*

Proof. By [8], there are precisely 4 lines through $x_0 \in X$. Since G fixes x_0 and G sends lines to lines, G sends a line through x_0 to a line through x_0 . If $L \ni x_0$ is a line and $\widehat{L}$ is its proper transform on $\widehat{X}$, $-K_{\widehat{X}} \cdot \widehat{L} = 0$ and $\widehat{L}$ is a flopping curve. Let $\omega: \widetilde{X} \rightarrow \widehat{X}$ be the blowup of the proper transforms of the 4 lines through $x_0 \in X$, and denote by $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$ its $(G$ -invariant) exceptional divisor. Denote by $\widetilde{F} = \omega^*F$, $\widetilde{E}_1 = \omega^*E_1 - \Lambda$ the proper transforms of F and E_1 on $\widetilde{X}$. On $\widetilde{X}$, we have the intersection numbers:

$$\begin{aligned} \Lambda^3 &= 8, & \Lambda^2 \cdot \omega^*F &= -4, & \Lambda \cdot \omega^*(F)^2 &= 0, & \Lambda \cdot \omega^*(F) \cdot \omega^*(E_1) &= 0, \\ \Lambda^2 \cdot \omega^*(E_1) &= 4, & \Lambda \cdot \omega^*(E_1)^2 &= 0, & \omega^*F^3 &= 2, & \omega^*E_1^3 &= -18. \end{aligned}$$

We first show that the Zariski decomposition of $\omega^*\pi^*(-K_X) - u\widetilde{F}$ exists and writes $P(u) + N(u)$, where $P(u)$ is nef and

$$N(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ (u-1)\Lambda & \text{for } 1 \leq u \leq 3. \end{cases}$$

We have $A_X(\tilde{F}) = 2$ and compute

$$(3.1.1) \quad S_X(\tilde{F}) = \frac{1}{(-K_X)^3} \int_0^\tau \text{vol}(\omega^* \pi^*(-K_X) - u\tilde{F}) du =$$

$$(3.1.2) \quad = \frac{1}{22} \left(\int_0^1 22 - 2u^3 du + \int_1^3 2(u-3)(u^2 - 3u - 3) du \right) = \frac{83}{44}.$$

So that $\beta(\tilde{F}) > 0$.

Now we assume that the centre of Ξ over $\tilde{X}$ is one-dimensional, so that $Z = c_{\tilde{X}}(\Xi) \subset \tilde{F}$ is an irreducible curve. The surface $\tilde{F}$ is the blowup of $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ at 4 points, so it is a del Pezzo surface of degree 4. Let ℓ_1 and ℓ_2 be the pullbacks to $\tilde{F}$ of the two rulings and e_1, e_2, e_3, e_4 the exceptional divisors. Then $\bar{\text{NE}}(\tilde{F})$ is generated by the proper transforms of rulings through one of the blowup points ($\ell_{1,i} = \ell_1 - e_i$ or $\ell_{2,i} = \ell_2 - e_i$) and by the proper transforms of $(1, 1)$ curves on F through 3 blowup points $\ell_{i,j,k} = \ell_1 + \ell_2 - e_i - e_j - e_k$. We use Theorem 2.3 to find a lower bound for $\beta(\Xi)$. We have

$$S(W_{\bullet, \bullet}^{\tilde{F}}; Z) = \frac{3}{(-K_X)^3} \int_0^3 (P(u)^2 \cdot \tilde{F}) \cdot \text{ord}_Z(N(u)|_{\tilde{F}}) du + \\ + \frac{3}{(-K_X)^3} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du.$$

Since $Z \notin \{e_1, e_2, e_3, e_4\} = \Lambda_{\tilde{F}}$, $\text{ord}_Z(N(u)|_{\tilde{F}}) = 0$. By construction, we may write

$$Z \sim \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \sum_{\substack{i \in \{1,2,3,4\}, \\ j \in \{1,2\}}} \alpha_{ij} \ell_{i(j)} + \alpha_{123} \ell_{123} + \alpha_{124} \ell_{124} + \alpha_{134} \ell_{134} + \alpha_{234} \ell_{234}.$$

Since there is no G -fixed point in $\mathbb{P}^3$ on either side of the link, one of α_{123} , α_{124} , α_{134} , or α_{234} is greater than 1. Without loss of generality we assume that $\alpha_{123} \geq 1$; by convexity of volume we get the inequality

$$S(W_{\bullet, \bullet}^{\tilde{F}}; Z) = \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du \leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\ell_{123}) dv du.$$

so it is enough to show that the last integral is less than 1 to conclude.

We now assume $Z \sim \ell_{123}$, and denote by $P(u, v)$ and $N(u, v)$ the positive and negative parts of the Zariski decomposition of $(\omega^* \pi^*(-K_X) - u\tilde{F})|_{\tilde{F}} - vZ$. Then

- if $u \in [0, 1]$ then for $0 \leq v \leq u$ $N(u, v) = v(e_1 + e_2 + e_3)$ so that $P(u, v)^2 = 2(u - v)^2$.
- if $u \in [1, 2]$ then:

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u - 1, \\ (-u + v + 1)(e_1 + e_2 + e_3) & \text{for } u - 1 \leq v \leq 1, \\ (-u + v + 1)(e_1 + e_2 + e_3) + (v - 1)(\ell_{1,4} + \ell_{2,4}) & \text{for } 1 \leq v \leq \frac{u+1}{2}. \end{cases}$$

- if $u \in [2, 3]$ then:

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 1, \\ (v - 1)(\ell_{1,4} + \ell_{2,4}) & \text{for } 1 \leq v \leq u - 1, \\ (-u + v + 1)(e_1 + e_2 + e_3) + (v - 1)(\ell_{1,4} + \ell_{2,4}) & \text{for } u - 1 \leq v \leq \frac{u+1}{2}. \end{cases}$$

We have

$$\begin{aligned}
S(W_{\bullet, \bullet}^{\tilde{F}}; Z) &\leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\ell_{123}) dv du = \\
&= \frac{3}{22} \left(\int_0^1 \int_0^u 2(u-v)^2 dv du + \right. \\
&\quad + \int_1^2 \left(\int_0^{u-1} (-2u^2 + 2uv - v^2 + 8u - 6v - 4) dv + \int_{u-1}^1 (u^2 - 4uv + 2v^2 + 2u - 1) dv + \right. \\
&\quad \left. \left. + \int_1^{\frac{u+1}{2}} (1+u-2v)^2 dv \right) du + \right. \\
&\quad + \int_2^3 \left(\int_0^1 (-2u^2 + 2uv - v^2 + 8u - 6v - 4) dv + \int_1^{u-1} (-2u^2 + 2uv + v^2 + 8u - 10v - 2) dv + \right. \\
&\quad \left. \left. + \int_{u-1}^{\frac{u+1}{2}} (1+u-2v)^2 dv \right) du \right) = \frac{29}{44} < 1.
\end{aligned}$$

We see that $S_X(\tilde{F}) < 2$ and $S(W_{\bullet, \bullet}^{\tilde{F}}; Z) < 1$ thus

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min \left\{ \frac{2}{S_X(\tilde{F})}, \frac{1}{S(W_{\bullet, \bullet}^{\tilde{F}}; Z)} \right\} > 1.$$

and $\beta(\Xi) = A_X(\Xi) - S_X(\Xi) > 0$. □

Finally, we exclude the case where $Z \subset \Gamma_1$.

Lemma 3.4. *If the center $Z = c_{X_1}(\Xi)$ is one-dimensional and is contained in E_1 , $\beta(\Xi) > 0$.*

Proof. Assume that $Z \subset E_1$, then since there is no G -fixed point on $\mathbb{P}^3$, $\phi_1(Z) = c_{\mathbb{P}^3}(\Xi)$ is the curve Γ_1 . Denote by $Y_1 \rightarrow \mathbb{P}^3$ the blowup of the line L_1 and by $Y_1 \rightarrow \mathbb{P}^1$ the morphism induced by the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ away from L_1 . Let $\hat{X}^+ \rightarrow Y_1$ be the blowup of the proper transform of Γ_1 , then $\hat{X}^+ \dashrightarrow \hat{X}$ is a flop, and there is a morphism $\tilde{X} \rightarrow \hat{X}$. Denote by η the composition $\tilde{X} \rightarrow \hat{X} \rightarrow Y_1 \rightarrow \mathbb{P}^1$ and by $\tilde{Z}$ the centre $c_{\tilde{X}}(\Xi)$. If T is a general fiber of η , $T \cdot \tilde{Z} \geq 5$, hence, by Lemma 2.6, $\beta(\Xi) > 0$. □

Lemma 3.5. *There is no G -invariant prime divisor Ξ over X with centre a prime divisor $D_X = c_X(\Xi)$ such that $\beta(\Xi) \leq 0$.*

Proof. By [2, Corollary 1.44], for any divisor Ξ over X , if $\alpha_{G,Z}(X) > 3/4$, where $Z = c_X(\Xi)$, then $\beta(\Xi) > 0$. Assume now that there is a divisor Ξ over X with $\beta(\Xi) \leq 0$ and $c_X(\Xi) = D_X$ a divisor, so that $\alpha_{G,D_X}(X) \leq \frac{3}{4}$. First assume that $\alpha_{G,D_X}(X) < \frac{3}{4}$, then D_X is the G -orbit of a minimal log canonical centre of a suitable pair $(X, \frac{3}{4}\mathcal{D})$ for $\mathcal{D} \subset |-K_X|_{\mathbb{Q}}$ a G -invariant linear system. By [2, Theorem 1.52], D_X is a G -invariant irreducible normal surface with

$$-K_X \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$$

for Δ_X an effective $\mathbb{Q}$ -divisor and a rational number $\lambda > \frac{4}{3}$.

We show that there is no such divisor D_X . Recall that $\pi: X_i \rightarrow X$ for $i = 1, 2$ are small $\mathbb{Q}$ -factorialisations so that

$$-K_{X_i} \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$$

where we still denote by D_X, Δ_X the pullbacks of these divisors to X_i . We have $\overline{\text{Eff}}(X_i) = \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[E_2]$, and $E_2 = 8H_1 - 3E_1$. If $D_X = E_1$, then $\Delta_X \sim 4H_1 - (1 + \lambda)E_1$, but this

is impossible as $(1 + \lambda) > 3/2$. If $D_X \neq E_1$, the image of D_X by f_1 is a G -invariant irreducible surface of degree $d \in \mathbb{N}$ on $\mathbb{P}^3$, and since

$$f_1(4H_1) \sim \lambda f_1(D_X) + f_1(\Delta_X)$$

we have $4 \geq \lambda d$, and since $\lambda > \frac{4}{3}$, $d \leq 2$. Since there is no G -invariant hyperplane of $\mathbb{P}^3$, $d = 2$ and $f_1(D_X)$ is a G -invariant quadric. Since Γ_1 doesn't lie on a quadric, $D_X \sim 2H_1$ and

$$\Delta_X \sim (4 - 2\lambda)H_1 - E_1 \sim xH_1 - E_1$$

for some $x < 4/3$, which is impossible for an effective divisor.

Now assume that $\alpha_{G,D_X}(X) = \frac{3}{4}$, then since π_1 is small, $\alpha_{G,D_X}(X_1) = \frac{3}{4}$ by [2, Lemma 1.47], and $\beta(\Xi) = A_X(D_X) - S_X(D_X) = A_{X_1}(D_X) - S_{X_1}(D_X)$. Since X_1 is smooth, as in the proof of [2, Theorem 1.51], assuming that $\beta(\Xi) = 0$ would imply $X_1 \simeq \mathbb{P}^3$, a contradiction. We conclude that $\beta(\Xi) > 0$ for all G -invariant prime divisors Ξ with $c_X(\Xi) = D_X$ a prime divisor on X . $\square$

We now have all the elements to prove

Main Theorem (I). *The threefold X is K -polystable.*

Proof. Assume that X is not K -polystable, then there is a G -invariant prime divisor Ξ over X such that $\beta(\Xi) < 0$. Lemma 3.5 shows that the centre of Ξ on X is not a surface. If the centre of Ξ on $\mathbb{P}^3$ is a curve other than Γ_1 , by Lemma 2.5, this curve is a line. Lemma 3.2 shows that this line cannot be a G -invariant line that is not the unique quadriseccant of Γ_1 , while Lemma 3.3 excludes the quadriseccant line L_1 . Lemma 3.4 shows that the centre of Ξ on $\mathbb{P}^3$ is not Γ_1 . As there is no G -fixed point on $\mathbb{P}^3$, if $c_X(\Xi)$ is 0-dimensional, it is the singular point $x_0 \in X$, and its centre on $\mathbb{P}^3$ is L_1 , so that this case is also excluded by Lemma 3.3. $\square$

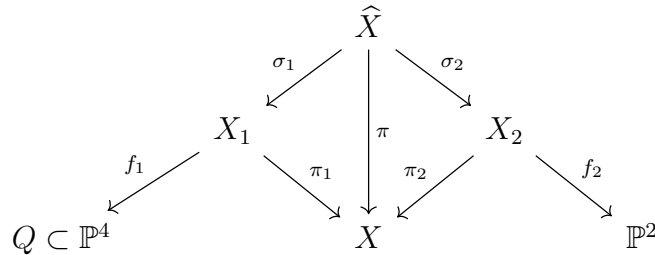
Since $\text{Aut}(X)$ is finite, X is K -stable, and by openness of K -stability [4], this implies:

Corollary 3.6. *A general one-nodal prime Fano threefold of genus 12 in Family I is K -stable.*

Remark 3.7. Liu and Zhao have constructed a K -semistable degeneration of one-nodal prime Fano threefolds in Family I, in which the curve Γ_1 is taken to lie on a quadric (this corresponds to $C_{a,b}$ with $|a| = |b|$ above). The resulting prime Fano threefold of genus 12 has (non-isolated) canonical singularities [9].

4. FAMILY II

Let X be a one-nodal prime Fano threefold of genus 12 that belongs to Family II of Theorem 1.2, then X is the midpoint of a Sarkisov link associated to a rational map $Q \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$; we describe the associated birational geometry briefly, see [6, 14] and [8] for proofs and precise statements.



Denote by $H_1 = \sigma_1^*(f_1^*\mathcal{O}_Q(1))$, by $H_2 = \sigma_2^*(f_2^*\mathcal{O}_{\mathbb{P}^2}(1))$, and by $H = \pi^*(-K_X)$ the pullbacks to $\widehat{X}$ (or to any of the models) of the ample generators of $\text{Pic}(Q)$, $\text{Pic}(\mathbb{P}^2)$ and $\text{Pic}(X)$ respectively. The morphism f_1 is the blowup of a smooth rational quintic curve $\Gamma_1 \subset Q \subset \mathbb{P}^4$ that does not lie on a hyperplane section of Q ($|H_1 - \Gamma_1| = \emptyset$), and there is a unique trisecant line L_1 to Γ_1 . The

curve Γ_1 lies on a section of $|2H_1|$ on Q , that is on a del Pezzo surface of degree 4, and the linear system $|2H_1 - \Gamma_1|$ has dimension 3 and $\text{Bs } |2H_1 - \Gamma_1| = \Gamma_1 \cup L_1$. The rational map associated to $|2H_1 - \Gamma_1| = |H_2|$ is precisely the $Q \dashrightarrow \mathbb{P}^2$ induced by the Sarkisov link above. The threefold X_1 is weak Fano,

$$-K_{X_1} \sim H \sim 3H_1 - E_1$$

where $E_1 = \text{Exc } f_1$, so that the proper transform of L_1 (still denoted L_1) is the unique flopping curve on X_1 . The map π_1 contracts L_1 to a node $\{x_0\} = \text{Sing}(X) \in X$.

Let $\pi: \widehat{X} \rightarrow X$ be the blowup of x_0 , and σ_1 the induced map to X_1 , note that $L_1 = \sigma_1(F)$, where $F = \text{Exc } \pi$. The threefolds X_1 and X_2 are the two small resolutions of the node x_0 and χ is the induced birational map between these (the Atiyah flop associated to $x_0 \in X$). Then, $\widehat{X}$ is a weak Fano threefold ($-K_{\widehat{X}}$ is nef and big) with $\rho = 3$ and we have [8]:

$$-K_{\widehat{X}} \sim H - F \sim H_1 + H_2$$

and from $H \sim 3H_1 - E_1 \sim H_1 + H_2$, we deduce

$$H_2 \sim 2H_1 - E_1.$$

The map f_2 is a conic bundle (a Mori fibre space with one-dimensional fibres) and its discriminant curve $\delta = -f_{2*}(K_{X_2/\mathbb{P}^2})^2 = -f_{2*}(H - 3H_2)^2$ has degree 12.

Denoting by $\mathcal{E} = \phi_{2*}H$, then $X_2 \subset \mathbb{P}(\mathcal{E})$ is a section of $2H - 3H_2$, where, abusing notation, we denote by H the tautological class of $\mathbb{P}(\mathcal{E})$ and by H_2 the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$.

Since $H_2 \cdot L_1 = 1$, L_1 maps to a line $\ell \subset \mathbb{P}^2$. Let $R = f_2^{-1}\ell$ be its preimage on X_2 , and by abuse of notation, also denote by $R = \sigma_2^*(f_2^{-1}\ell)$ its proper transform on $\widehat{X}$. By construction, R is the unique section of $|2H_1 - E_1 - 2F| = |H_2 - F|$.

4.1. Construction of a member with $\mathbb{Z}_2 \rtimes \mathbb{Z}_3$ -action. Let $Q \subset \mathbb{P}^4$ be the smooth quadric threefold

$$Q = \{2x_2^2 = x_1x_3 - x_0x_4\}$$

and let Γ_1 be the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$ given by

$$[x : y] \rightarrow [x^5 : 2x^3y^2 + y^5 : x^4y + xy^4 : 2x^2y^3 + x^5 : y^5];$$

Γ_1 lies on Q but on no hyperplane section of Q .

Let ω be a primitive cube root of unity, and define an action of $G := \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ on $\mathbb{P}^4$ by the action of its generators:

$$\begin{aligned} \tau : [x_0 : x_1 : x_2 : x_3 : x_4] &\rightarrow [x_4 : x_3 : x_2 : x_1 : x_0], \\ \sigma : [x_0 : x_1 : x_2 : x_3 : x_4] &\rightarrow [x_0 : \omega^2 x_1 : \omega x_2 : x_3 : \omega^2 x_4]. \end{aligned}$$

and observe that Γ_1 is G -invariant, and that

$$L_1 = \{x_0 + x_3 = x_4 + x_1 = x_2 = 0\}$$

is G -invariant and trisecant to Γ_1 (the intersection $L_1 \cap \Gamma_1$ consists of the image of the points $[1 : -\omega^i]$ for $i = 0, 1, 2$). The threefolds X, X_1 and X_2 are equipped with a G -action and the Sarkisov link above is G -equivariant. For instance, a G -invariant basis of $|H_2|$ is

$$\begin{cases} S_1 = \{x_0^2 - x_3^2 + 2(x_1 - x_4) = 0\}, \\ S_2 = \{x_1^2 - x_4^2 + 2(x_0 - x_3) = 0\}, \\ S_3 = \{(x_0 - x_2)^2 + (x_4 - x_2)^2 + x_0x_1 + 2x_0x_4 + x_3x_4 = 2x_2^2 + (x_1 - x_2)^2 + (x_3 - x_2)^2\}. \end{cases}$$

and the only section of $|H_2|$ that is singular along L_1 is $f_1(\sigma_1 R)$ (which we still call R by abuse of notation). We have:

$$R = \{x_0x_1 + 2x_0x_4 + x_3x_4 = 2x_2^2\}.$$

The discriminant curve of f_2 is the smooth plane cubic

$$\delta = \{2y_0^3 + 6y_0^2y_1 + 5y_0y_1^2 + y_0y_1y_2 + 3y_1^3 + 5y_1^2y_2 + 6y_1y_2^2 + 2y_2^3 = 0\} \subset \mathbb{P}^2.$$

Claim 4.1. *The group $\text{Aut}(X)$ is finite.*

Proof. Since $\text{Aut}(X)$ is a subgroup of $\text{Aut}(X_1) = \text{Aut}(Q, \Gamma_1)$, and since Γ_1 does not lie on a hyperplane section of Q , $\text{Aut}(Q, \Gamma_1) = \text{Aut}(X_1)$ is a subgroup of $\text{Aut}(\Gamma_1) = \text{Aut}(\mathbb{P}^1)$ by [15, Lemma 2.1]. Consequently, $\text{Aut}(X)$ is a subgroup of $\text{Aut}(\mathbb{P}^1)$ preserving the three points of intersection $\Gamma_1 \cap L_1$, therefore it is finite. $\square$

The intersection numbers associated to the Sarkisov link are

$$\begin{aligned} H_1^3 &= 2, & H_1^2 \cdot E_1 &= 0, & H_1 \cdot E_1^2 &= -5, & E_1^3 &= -13, \\ H_1^2 \cdot F &= 0, & H_1 \cdot F^2 &= -1, & F^3 &= 2, \\ E_1 \cdot F \cdot H_1 &= 0, & E_1 \cdot F^2 &= -3, & E_1^2 \cdot F &= 0, \\ H_2^3 &= 0, & H_2^2 \cdot H &= 2, & H_2 \cdot H^2 &= 12 - \deg \delta. \end{aligned}$$

We will apply Theorem 2.1 to prove that X is K-stable. To do so, we first describe possible centres of G -invariant divisors over X . In what follows, Ξ always denotes a G -invariant prime divisor over X .

Claim 4.2. *If the centre of Ξ on X is 0-dimensional, it is the singular point $c_X(\Xi) = \{x_0\}$.*

Proof. There is no point of $Q \subset \mathbb{P}^4$ fixed by the action of G . $\square$

We now consider the case when the centre $Z = c_Q(\Xi)$ on Q is one-dimensional. First, we assume that Z lies on a (smooth) section S of the linear system $|H_2| = |2H_1 - E_1|$. As an intersection of two quadrics in $\mathbb{P}^4$, S is a del Pezzo of degree 4, and $p: S \rightarrow \mathbb{P}^2$ is the blowup of five points $p_1, \dots, p_5$ in general position. Let ℓ be the pullback of a line on $\mathbb{P}^2$, and $e_1, \dots, e_5$ the p -exceptional curves. Then the Mori cone $\overline{\text{NE}}(S)$ is generated by $\ell, e_1, \dots, e_5, \ell_{i,j}$ for $1 \leq i < j \leq 5$ and q where $\ell_{i,j}$ is the proper transform of the line through p_i and p_j and q that of the conic through $p_1, \dots, p_5$. For a smooth curve $C \subset S$, if $C \sim k\ell + \sum m_i e_i$, then

$$\deg C = -K_S \cdot C = H_1 \cdot C = 3k - \sum m_i \text{ and } p_a(C) = \frac{(k-1)(k-2)}{2} - \sum \frac{m_i(m_i-1)}{2}$$

so that without loss of generality, we may assume that $\Gamma_1 = 2\ell - e_1$ and $L_1 = q$.

Lemma 4.1. *If $Z = c_Q(\Xi)$ is a G -invariant irreducible curve lying on $S \in |H_2|$, and if $Z \not\subset \Gamma_1 \cup L_1$, then $\beta(\Xi) > 0$.*

Proof. We use Theorem 2.3 to bound $\beta(\Xi)$ below. Let $D_u = H - uS$ on $\hat{X}$ for $u \geq 0$, and write its Zariski decomposition $D_u = P(u) + N(u)$, where for $0 \leq u \leq \frac{3}{2}$, $P(u)$ is nef and

$$P(u) = H - uS - u\left(\frac{E_1}{3} + F\right) = \left(1 - \frac{2}{3}u\right)(3H_1 - E_1) \text{ and } N(u) = u\left(\frac{E_1}{3} + F\right),$$

which gives:

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^\tau \text{vol}(\pi^*(-K_X) - uS) du = \frac{1}{22} \int_0^{3/2} \frac{22(3-2u)^3}{27} du = \frac{3}{8} < 1.$$

Note that since $Z \not\subset (E_1 \cup F)$, $\text{ord}_Z(N(u)|_S) = 0$.

We now consider $(H - uS)|_S - vZ$ on S and denote by $P(u, v) + N(u, v)$ its Zariski decomposition for $0 \leq u \leq \frac{3}{2}$. We have

$$Z \sim \alpha \ell + \sum_{i=1}^5 \alpha_i e_i + \sum_{1 \leq i < j \leq 5} \alpha_{ij} \ell_{ij} + \beta q.$$

Since $Z \not\subset F$, $Z \neq q$ and at least one of the coefficients $\alpha, \alpha_i, \alpha_{i,j}$ is ≥ 1 . If $\mathbf{l}$ is the corresponding curve, since $Z \geq \mathbf{l}$, by convexity of volume:

$$S(W_{\bullet, \bullet}^S; Z) = \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du \leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\mathbf{l}) dv du.$$

so it is enough to show that the last integral is less than 1 when $Z = \mathbf{l}$, for each possible $\mathbf{l}$.

Case 1. $Z \sim \ell$. For $0 \leq u \leq \frac{3}{2}$, and $0 \leq v \leq \frac{3-2u}{3}$, $N(u, v) = vq$, and we compute

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &\leq \frac{3}{22} \int_0^{3/2} \int_0^\infty \text{vol}(P(u)|_S - v\ell) dv du = \\ &= \frac{3}{22} \int_0^{3/2} \int_0^{1-\frac{2u}{3}} \frac{(2u+3v-3)(6u+5v-9)}{3} dv du = \frac{3}{16} < 1. \end{aligned}$$

Case 2. $Z \sim e_1$. For $0 \leq u \leq \frac{3}{2}$, we have:

$$N(u, v) = \begin{cases} vq & \text{for } 0 \leq v \leq \frac{2(3-2u)}{3}, \\ vq + (v-2 + \frac{4u}{3})(\ell_{12} + \ell_{13} + \ell_{14} + \ell_{15}) & \text{for } \frac{2(3-2u)}{3} \leq v \leq \frac{5(3-2u)}{6}. \end{cases}$$

We obtain

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &\leq \frac{3}{22} \int_0^{3/2} \int_0^\infty \text{vol}(P(u)|_S - ve_1) dv du = \\ &= \frac{3}{22} \int_0^{3/2} \left(\int_0^{\frac{2(3-2u)}{3}} \frac{(2u-3)(6u+4v-9)}{3} dv + \int_{\frac{2(3-2u)}{3}}^{\frac{5(3-2u)}{6}} \frac{5(3-2u)}{6} \frac{(10u+6v-15)^2}{9} dv \right) du = \\ &= \frac{182}{352} < 1. \end{aligned}$$

Case 3. $Z \sim e_2$ (or e_i , $i \neq 1$). For $0 \leq u \leq \frac{3}{2}$, we have:

$$N(u, v) = \begin{cases} vq & \text{for } 0 \leq v \leq \frac{3-2u}{3}, \\ vq + (v-1 + \frac{2u}{3})(\ell_{23} + \ell_{24} + \ell_{25}) & \text{for } \frac{3-2u}{3} \leq v \leq \frac{2(3-2u)}{3}. \end{cases}$$

In addition

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &\leq \frac{3}{22} \int_0^{3/2} \int_0^\infty \text{vol}(P(u)|_S - ve_2) dv du = \\ &= \frac{3}{22} \int_0^{3/2} \left(\int_0^{\frac{3-2u}{3}} (2u-3)(2u+2v-3) dv + \int_{\frac{3-2u}{3}}^{\frac{2(3-2u)}{3}} \frac{(4u+3v-6)^2}{3} dv \right) du = \\ &= \frac{63}{176} < 1. \end{aligned}$$

Case 4. $Z \sim \ell_{12}$ (or ℓ_{1j}). For $0 \leq u \leq \frac{3}{2}$, we have:

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3-2u}{3}, \\ (v-1 + \frac{2u}{3})(\ell_{34} + \ell_{35} + \ell_{45}) & \text{for } \frac{3-2u}{3} \leq v \leq \frac{2(3-2u)}{3}. \end{cases}$$

In addition

$$\begin{aligned}
S(W_{\bullet,\bullet}^S; Z) &\leq \frac{3}{22} \int_0^{3/2} \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) dv du = \\
&= \frac{3}{22} \int_0^{3/2} \left(\int_0^{\frac{3-2u}{3}} 4u^2 + \frac{8}{3}uv - 12u - 9 - 4v - v^2 dv + \int_{\frac{3-2u}{3}}^{\frac{2(3-2u)}{3}} \frac{2(4u+3v-6)(2u+v-3)}{3} dv \right) du = \\
&= \frac{75}{176} < 1.
\end{aligned}$$

Case 5. $Z \sim \ell_{23}$ (or ℓ_{ij} , $i \neq 1$). For $0 \leq u \leq \frac{3}{2}$, we have:

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3-2u}{3}, \\ (v-1 + \frac{2u}{3})\ell_{45} & \text{for } \frac{3-2u}{3} \leq v \leq \frac{2(3-2u)}{3}, \\ (v-1 + \frac{2u}{3})\ell_{45} - (v-2 + \frac{4u}{3})(\ell_{14} + \ell_{15}) & \text{for } \frac{2(3-2u)}{3} \leq v \leq 3-2u. \end{cases}$$

In addition

$$\begin{aligned}
S(W_{\bullet,\bullet}^S; Z) &\leq \frac{3}{22} \int_0^{3/2} \int_0^\infty \text{vol}(P(u)|_S - v\ell_{23}) dv du = \\
&= \frac{3}{22} \int_0^{3/2} \left(\int_0^{\frac{3-2u}{3}} 4u^2 + \frac{4}{3}uv - 12u + 9 - 2v - v^2 dv + \right. \\
&\quad \left. + \int_{\frac{3-2u}{3}}^{\frac{2(3-2u)}{3}} \frac{2(2u-3)(10u+6v-15)}{3} dv + \int_{\frac{2(3-2u)}{3}}^{3-2u} 2(2u+v-3)^2 dv \right) du = \\
&= \frac{111}{176} < 1.
\end{aligned}$$

This finishes the proof, as in all cases we have $\min \left\{ \frac{1}{s_X(S)}, \frac{1}{s(W_{\bullet,\bullet}^S; Z)} \right\} > 1$ □

Lemma 4.2. *If $Z = c_Q(\Xi)$ is a line other than L_1 , $\beta(\Xi) > 0$.*

Proof. Since there is no G -fixed point on Q , $Z \cap \Gamma_1$ is empty or consists of two points. In the second case, $H_2 \cdot Z = 0$, so that Z lies on a section $S \in |H_2|$ and $\beta(\Xi) > 0$ by Lemma 4.1.

We now assume that Z is disjoint from Γ_1 and denote by $S^Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ the general hyperplane section of Q containing Z , and by S its proper transform on $\widehat{X}$. The intersection $S^Q \cap (\Gamma_1 \cup L_1) = \{p_1, \dots, p_6\}$ consists of six points, and these points are in general position because any line through 3 of the points (respectively conic through 6 of the points) would be contracted by π_1 , the anticanonical map of X_1 , but the only flopping curve on X_1 is L_1 . As S is the blowup of S^Q at $\{p_1, \dots, p_6\}$, S a del Pezzo surface of degree 2. Denote by ℓ_1, ℓ_2 the pullbacks of the two rulings of $S^Q = \mathbb{P}^1 \times \mathbb{P}^1$, and by $e_1, \dots, e_6$ the exceptional divisors. The Mori cone $\overline{\text{NE}}(S)$ is generated by $\ell_1, \ell_2, e_1, \dots, e_6$, and by the classes of

- the proper transforms $\ell_{i(1)}$ and $\ell_{i(2)}$ of rulings through the points p_i for $1 \leq i \leq 6$,
- the proper transforms $\ell_{i,j,k}$ for $1 \leq i < j < k \leq 6$ of irreducible conics through 3 of the blowup points ($\ell_{i,j,k} = \ell_1 + \ell_2 - e_i - e_j - e_k$),
- the proper transforms $\kappa_{j(1)}$ and $\kappa_{j(2)}$ of rational cubic curves through 5 of the p_i s (where $\kappa_{j(1)} = 2\ell_1 + \ell_2 - \sum e_i + e_j$) for $1 \leq j \leq 6$,
- and the proper transforms q_j of elliptic quartic curves through $p_1, \dots, p_6$, which have multiplicity 2 at p_j for $1 \leq j \leq 6$ ($q_j = 2\ell_1 + 2\ell_2 - \sum e_i - e_j$).

The Zariski decomposition of $\pi^*(-K_X) - uS$ writes $P(u) + N(u)$ where $P(u)$ is nef, and for $0 \leq u \leq 1$, $N(u) = uF$. We have

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^\tau \text{vol}(\pi^*(-K_X) - uS) du = \frac{1}{22} \int_0^1 (1-u)(u^2 - 17u + 22) du = \frac{3}{8} < 1.$$

Since Z is disjoint from Γ_1 , without loss of generality we may assume that $Z \sim \ell_1$.

Using the same notation as before, for $0 \leq u \leq \frac{5}{7}$, we have

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{4-3u}{2}, \\ (2v - 4 + 3u)\kappa_{6(2)} & \text{if } \frac{4-3u}{2} \leq v \leq \frac{5-4u}{2}, \\ (2v - 5 + 4u) \sum \kappa_{i(2)} + (1-u)\kappa_{6(2)} & \text{if } \frac{5-4u}{2} \leq v \leq \frac{7-5u}{3}. \end{cases}$$

For $\frac{5}{7} \leq u \leq 1$, we have

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{4-3u}{2}, \\ (2v - 4 + 3u)\kappa_{6(2)} & \text{if } \frac{4-3u}{2} \leq v \leq \frac{5-4u}{2}, \\ (2v - 5 + 4u) \sum \kappa_{i(2)} + (1-u)\kappa_{6(2)} & \text{if } \frac{5-4u}{2} \leq v \leq \frac{11-9u}{4}. \end{cases}$$

Since $Z \not\subset F$, $\text{ord}_Z(N(u)|_S) = 0$ and

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &= \frac{3}{22} \int_0^1 \int_0^\infty \text{vol}(P(u)|_S - vZ) dv du = \\ &= \frac{3}{22} \int_0^{5/7} \left(\int_0^{\frac{4-2u}{2}} u^2 + 2uv - 12u - 6v + 13dv + \int_{\frac{4-3u}{2}}^{\frac{5-4u}{2}} 10u^2 + 14uv - 36u + 4v^2 - 22v + 29dv + \right. \\ &\quad \left. + \int_{\frac{5-4u}{2}}^{\frac{7-5u}{3}} 2(5u + 3v - 7)(9u + 4v - 11)dv \right) du + \\ &+ \frac{3}{22} \int_{5/7}^1 \left(\int_0^{\frac{4-2u}{2}} u^2 + 2uv - 12u - 6v + 13dv + \int_{\frac{4-3u}{2}}^{\frac{5-4u}{2}} 10u^2 + 14uv - 36u + 4v^2 - 22v + 29dv + \right. \\ &\quad \left. + \int_{\frac{5-4u}{2}}^{\frac{11-9u}{4}} 2(5u + 3v - 7)(9u + 4v - 11)dv \right) du = \frac{18969}{1108811} < 1. \end{aligned}$$

As above, this completes proof that $\beta(\Xi) > 0$. □

Lemma 4.3. *If $c_Q(\Xi) = L_1$, then $\beta(\Xi) > 0$.*

Proof. By [8], there are precisely 3 lines through $x_0 \in X$, and by construction, the set of lines through $x_0 \in X$ is G -invariant. If $L \ni x_0$ is a line and $\widehat{L}$ is its proper transform on $\widehat{X}$, $-K_{\widehat{X}} \cdot \widehat{L} = 0$ and $\widehat{L}$ is a flopping curve. Let $\omega: \widetilde{X} \rightarrow \widehat{X}$ be the blowup of the proper transforms of the 3 lines through $x_0 \in X$, and denote by $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ its (G -invariant) exceptional divisor. Denote by $\widetilde{F} = \omega^*F$, $\widetilde{E}_1 = \omega^*E_1 - \Lambda$ the proper transforms of F and E_1 on $\widetilde{X}$, and by $\widetilde{R} = \omega^*R - \Lambda$, the proper transform of the unique section of $|2H_1 - E_1 - 2F| = |H_2 - F|$. On $\widetilde{X}$, we have the intersection numbers:

$$\begin{aligned} \Lambda^3 &= 6, & \Lambda^2 \cdot \omega^*F &= -3, & \Lambda \cdot \omega^*(F)^2 &= 0, & \Lambda \cdot \omega^*(F) \cdot \omega^*(E_1) &= 0, \\ \Lambda^2 \cdot \omega^*(E_1) &= 3, & \Lambda \cdot \omega^*(E_1)^2 &= 0, & \omega^*F^3 &= 2, & \omega^*E_1^3 &= -13. \end{aligned}$$

We first show that $\beta(F) > 0$. The Zariski decomposition of $\omega^*\pi^*(-K_X) - u\tilde{F}$ can be written $P(u) + N(u)$, where $P(u)$ is nef and

$$N(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ (u-1)\Lambda & \text{for } 1 \leq u \leq 2, \\ (u-1)\Lambda + (u-2)\tilde{R} & \text{for } 2 \leq u \leq 3. \end{cases}$$

We have $A_X(\tilde{F}) = 2$ and

$$\begin{aligned} S_X(\tilde{F}) &= \frac{1}{(-K_X)^3} \int_0^\tau \text{vol}(\omega^*\pi^*(-K_X) - u\tilde{F}) du \\ &= \frac{1}{22} \left(\int_0^1 22 - 2u^3 du + \int_1^2 (u+1)(u^2 - 10u + 19) du + \int_2^3 3(u-3)(2u-7) du \right) = \frac{161}{88}. \end{aligned}$$

So that $\beta(F) = \frac{15}{88} > 0$.

Now assume that Ξ is not F and denote by Z the centre of Ξ on $\tilde{X}$. By construction, $Z = c_{\tilde{X}}(\Xi) \subset \tilde{F}$ is a curve, and $\tilde{F}$ is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in three points in general position, so it is a del Pezzo surface of degree 5. We denote by ℓ_1, ℓ_2 the proper transforms of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, and by e_1, e_2, e_3 the (-1) -curves. The extremal rays of the Mori cone $\bar{NE}(\tilde{F})$ are the (-1) -curves e_1, e_2, e_3 , the proper transforms $\ell_{i(1)}$ and $\ell_{i(2)}$ of rulings through the blowup points for $1 \leq i \leq 3$, and the proper transform of the conic through the three blowup points $\ell_{123} = \ell_1 + \ell_2 - e_1 - e_2 - e_3$.

We will estimate $\beta(\Xi)$ by considering the flag $Z \subset \tilde{F} \subset \tilde{X}$; we write

$$\begin{aligned} S(W_{\bullet, \bullet}^{\tilde{F}}; Z) &= \frac{3}{(-K_X)^3} \int_0^3 (P(u)^2 \cdot \tilde{F}) \cdot \text{ord}_Z(N(u)|_{\tilde{F}}) du + \\ &\quad + \frac{3}{(-K_X)^3} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du. \end{aligned}$$

Since $c_Q(\Xi)$ is one-dimensional, $Z \not\subset \Lambda|_{\tilde{F}}$, and $\text{ord}_Z(N(u)|_{\tilde{F}}) = 0$ unless $Z = \tilde{R}|_{\tilde{F}}$.

We first assume that $Z \neq \tilde{R}|_{\tilde{F}}$. There are positive integers α_i, α_{ij} and α_{123} so that

$$Z \sim \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \sum \alpha_{ij} \ell_{i(j)} + \alpha_{123} \ell_{123}.$$

Since $Z \not\subset \Lambda|_{\tilde{F}}$, α_{ij} and α_{123} are not all simultaneously 0. Let $\mathbf{l}$ denote one of the (-1) curves other than e_1, e_2, e_3 such that $Z \geq \mathbf{l}$, then by convexity of volume:

$$S(W_{\bullet, \bullet}^S; Z) = \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du \leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\mathbf{l}) dv du.$$

so it is enough to show that the last integral is less than 1 when $Z = \mathbf{l}$.

Case 1. Assume that $Z \sim \ell_{123}$, and let $P(u, v)$ and $N(u, v)$ be the positive and negative parts of the Zariski decomposition of $(\omega^*\pi^*(-K_X) - u\tilde{F})|_{\tilde{F}} - vZ$. Then, for $0 \leq u \leq 1$, $N(u, v) = v(e_1 + e_2 + e_3)$ for $0 \leq v \leq u$; for $1 \leq u \leq 2$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u-1, \\ (v-u+1)(e_1 + e_2 + e_3) & \text{for } u-1 \leq v \leq u. \end{cases}$$

and for $2 \leq u \leq 3$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 1, \\ (v-1)(e_1 + e_2 + e_3) & \text{for } 1 \leq v \leq 4-u. \end{cases}$$

Putting things together, we get:

$$\begin{aligned}
S(W_{\bullet, \bullet}^{\tilde{F}}; Z) &\leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\ell_{123}) dv du = \\
&= \frac{3}{22} \left(\int_0^1 \int_0^u 2(u-v)^2 dv du + \right. \\
&\quad + \int_1^2 \left(\int_0^{u-1} -u^2 + 2uv + 6u - v^2 - 6v - 3 dv + \int_{u-1}^u 2(u-v)^2 dv \right) du + \\
&\quad \left. + \int_2^3 \left(\int_0^1 2uv - v^2 - 4u - 6v + 13 dv + \int_1^{4-u} 2(v-2)(v+u-4) dv \right) du \right) = \frac{29}{44} < 1.
\end{aligned}$$

and $\beta(\Xi) > 0$.

Case 2. Now assume that $Z \sim \ell_{1(2)}$ (or any $\ell_{i(j)}$). The positive and negative parts of the Zariski decomposition of $(\omega^* \pi^*(-K_X) - u\tilde{F})|_{\tilde{F}} - vZ$ are as follows.

For $0 \leq u \leq 1$, $N(u, v) = ve_1$ for $0 \leq v \leq u$; for $1 \leq u \leq 2$

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u-1, \\ (v-u+1)e_1 & \text{for } u-1 \leq v \leq 1, \\ (v-u+1)e_1 + (v-1)(\ell_{2(1)} + \ell_{3(1)}) & \text{for } 1 \leq v \leq u. \end{cases}$$

In addition for $2 \leq u \leq 3$

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 1, \\ (v-3+u)(\ell_{2(1)} + \ell_{3(1)}) & \text{for } 3-u \leq v \leq 1, \\ (v-3+u)(\ell_{2(1)} + \ell_{3(1)}) + (v-1)e_1 & \text{for } 1 \leq v \leq 4-u. \end{cases}$$

We have

$$\begin{aligned}
S(W_{\bullet, \bullet}^{\tilde{F}}; Z) &\leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\ell_{1(2)}) dv du = \\
&= \frac{3}{22} \left(\int_0^1 \int_0^u 2u(u-v) dv du + \right. \\
&\quad + \int_1^2 \left(\int_0^{u-1} -u^2 - v^2 + 6u - 2v - 3 dv + \int_{u-1}^1 -2uv + 4u - 2dv + \int_1^u 2(2-v)(u-v) dv \right) du + \\
&\quad + \int_2^3 \left(\int_0^{3-u} -v^2 - 4u - 2v + 13 dv + \int_{3-u}^1 2u^2 + 4uv + v^2 - 16u - 14v + 31 dv + \right. \\
&\quad \left. \left. + \int_1^{4-u} 2(u+v-4)^2 dv \right) du \right) = \frac{59}{88} < 1.
\end{aligned}$$

This finishes the proof that $\beta(\Xi) > 0$ when $Z \neq \tilde{R}|_{\tilde{F}}$.

Assume that $Z = \tilde{R}|_{\tilde{F}}$, so that $\text{ord}_Z(N(u)|_{\tilde{F}}) = 1$ when $2 \leq u \leq 3$. We have

$$\frac{3}{(-K_X)^3} \int_2^3 (P(u)^2 \cdot \tilde{F}) \cdot \text{ord}_Z(N(u)|_{\tilde{F}}) du = \frac{9}{22}.$$

As before, denote by $P(u, v)$ and $N(u, v)$ the positive and negative parts of the Zariski decomposition of $\omega^* \pi^*(-K_X - u\tilde{F})|_{\tilde{F}} - vZ$. When $0 \leq u \leq 1$, $N(u, v) = v(e_1 + e_2 + e_3)$ for $0 \leq v \leq u/2$,

when $1 \leq u \leq 2$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u - 1, \\ (v - u + 1)(e_1 + e_2 + e_3) & \text{for } u - 1 \leq v \leq u/2, \end{cases}$$

and finally, when $2 \leq u \leq 3$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 3 - u, \\ (v - 3 + u)(\ell_{1(1)} + \ell_{2(1)} + \ell_{3(1)}) & \text{for } 3 - u \leq v \leq 2 - u/2. \end{cases}$$

We have

$$\begin{aligned} S(W_{\bullet, \bullet}^{\tilde{F}}; Z) &= \frac{9}{22} + \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du = \\ &= \frac{9}{22} + \frac{3}{22} \left(\int_0^1 \int_0^{u/2} 2(u-v)(u-2v) dv du + \right. \\ &\quad + \int_1^2 \left(\int_0^{u-1} -u^2 + v^2 + 6u - 6v - 3 dv + \int_{u-1}^{u/2} 2(u-v)(u-2v) dv \right) du + \\ &\quad \left. + \int_2^3 \left(\int_0^{3-u} 2uv + v^2 - 4u - 10v + 13 dv + \int_{3-u}^{2-u/2} (u+2v-4)(3u+2v-10) dv \right) du \right) = \frac{3}{4} < 1. \end{aligned}$$

We see that $S_X(\tilde{F}) < 2$ and $S(W_{\bullet, \bullet}^{\tilde{F}}; Z) < 1$, so that $\beta(\Xi) > 0$. $\square$

Now we need to consider G -invariant prime divisors Ξ whose centre on Q lies on Γ_1 .

Lemma 4.4. *If $Z = c_{\tilde{X}}(\Xi) \subset E_1$, then $\beta(\Xi) > 0$.*

Proof. Assume that $Z \subset E_1$, then since there is no G -fixed point on $Q \subset \mathbb{P}^4$, $f_1(Z) = c_Q(\Xi)$ is the curve Γ_1 . Denote by $Q_1 \rightarrow Q$ the blowup of the line L_1 and by $Q_1 \rightarrow \mathbb{P}^2$ the morphism induced by the projection $Q \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ away from L_1 . Let $\hat{X}^+ \rightarrow Q_1$ be the blowup of the proper transform of Γ_1 , then $\hat{X}^+ \dashrightarrow \hat{X}$ is a flop, and there is a morphism $\tilde{X} \rightarrow \hat{X}$. Denote by η the composition $\tilde{X} \rightarrow \hat{X} \rightarrow Q_1 \rightarrow \mathbb{P}^2$.

If T is a general fiber of η , $T \cdot \tilde{Z} \geq 5$, hence, by Lemma 2.7, $\beta(\Xi) > 0$. $\square$

Lemma 4.5. *If Ξ is a G -invariant prime divisor over X with centre a prime divisor $D_X = c_X(\Xi)$ such that $\beta(\Xi) < 0$, then $D_X \in |H_2|$.*

Proof. The centre $c_X(\Xi) = D_X$ is the G -orbit of a minimal log canonical centre of a suitable pair $(X, \frac{3}{4}\mathcal{D})$ for $\mathcal{D} \subset |-K_X|_{\mathbb{Q}}$ a G -invariant linear system, so that D_X is a G -invariant irreducible normal surface with

$$-K_X \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$$

for some effective $\mathbb{Q}$ -divisor Δ_X and rational number $\lambda > \frac{4}{3}$ (see proof of [2, Theorem 1.52]). We show that then, D_X is linearly equivalent to H_2 (here since $X_1 \rightarrow X$ is a small map, we also denote $c_{X_1}(\Xi)$ by D_X).

Recall that $\overline{\text{Eff}}(X_1) = \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[H_2]$, and $H_2 \sim 2H_1 - E_1$. If $D_X = E_1$, then

$$\Delta \sim 3H_1 - (1 + \lambda)E_1 \sim \frac{3}{2}(2H_1 - E_1) + \left(\frac{3}{2} - (1 + \lambda)\right)E_1$$

and since $\lambda > \frac{1}{2}$, this is impossible.

Now assume that $D_X \neq E_1$, so that $f_1(D_X)$ is a G -invariant surface on Q , and let d be its degree. Since

$$3H_1 \sim \lambda f_1(D_X) + f_1(\Delta_X),$$

$3 \geq \lambda d$ and $d = 1$ or $d = 2$. As there is no G -invariant hyperplane section, $d = 2$ and

$$\Delta \sim (3 - 2\lambda)H_1 + (\lambda m_1 - 1)E_1$$

where m_1 is the multiplicity of $f_1(D_X)$ along Γ_1 . Since

$$\Delta_X \sim \frac{3 - 2\lambda}{2}(2H_1 - E_1) + \left(\frac{3 - 2\lambda}{2} + \lambda m_1 - 1\right)E_1$$

we see that $m_1 \geq 1$ and $D_X \in |H_2|$. $\square$

Lemma 4.6. *Let $Z = c_{\tilde{X}}(\Xi)$ be an irreducible curve that is not contained in E_1 . Then, $\beta(\Xi) > 0$ unless $c_Q(\Xi)$ is a line.*

Proof. By Lemma 4.5, a G -invariant surface containing Z is either F or the G -invariant element of $|H_2|$. We have seen that for such Z , $\beta(\Xi) > 0$. If $Z \not\subset H_2$, as in the proof of Lemma 4.4, there is a surjective morphism $\tilde{X} \rightarrow \mathbb{P}^2$ and $H_2 \cdot Z \leq 2$. Since L_1 is in the base locus of H_2 , this implies that $H_1 \cdot Z \leq 1$. $\square$

Main Theorem (II). *X is K -polystable.*

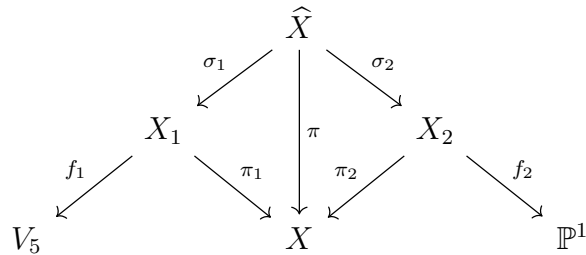
Proof. Assume that X is not K -polystable, and denote by Ξ a G -invariant prime divisor over X with $\beta(\Xi) \leq 0$. If $c_X(\Xi)$ is 0-dimensional, it is $\{x_0\}$, and $c_Q(\Xi) = L_1$, so that $\beta(\Xi) > 0$ by Lemma 4.3. If $c_Q(\Xi)$ is a curve and lies on a section S of $|H_2|$, then $\beta(\Xi) > 0$ by Lemma 4.1. If $c_Q(\Xi)$ is a line, then $\beta(\Xi) > 0$ by Lemma 4.2 and Lemma 4.3. If $c_{\tilde{X}}(\Xi)$ is a curve lying on E_1 , then $\beta(\Xi) > 0$ by Lemma 4.4, and if $c_{\tilde{X}}(\Xi)$ is a curve not lying on E_1 and such that $c_Q(\Xi)$ is not a line, then $\beta(\Xi) > 0$ by Lemma 4.6. This exhausts the cases where $c_X(\Xi)$ is 1-dimensional. Assume now that $c_X(\Xi)$ is a prime divisor. Then, by Lemma 4.5, $\beta(\Xi) > 0$ unless $c_X(\Xi) \in |H_2|$. We have seen that $\beta(S) > 0$ for $S \in |H_2|$ in the proof of Lemma 4.1, and this concludes the proof. $\square$

As in the case of Family I, since $\text{Aut}(X)$ is finite, X is K -stable and this implies by openness of K -stability [4]:

Corollary 4.7. *A general one-nodal prime Fano threefold of genus 12 in Family II is K -stable.*

5. FAMILY III

Let X be a one-nodal prime Fano threefold of genus 12 that belongs to Family III of Theorem 1.2 is the midpoint of a Sarkisov link associated to a rational map $V_5 \dashrightarrow \mathbb{P}^1$; we describe the associated birational geometry briefly, see [6, 14] and [8] for precise statements.



Denote by $H_1 = \sigma_1^*(f_1^*\mathcal{O}_{V_5}(1))$ and $H_2 = \sigma_2^*(f_2^*\mathcal{O}_{\mathbb{P}^1}(1))$, and by $H = \pi^*(-K_X)$ the pullbacks to $\widehat{X}$ (or to any of the models) of the ample generators of $\text{Pic}(V_5)$, $\text{Pic}(\mathbb{P}^1)$ and $\text{Pic}(X)$ respectively. The morphism f_1 is the blowup of a smooth rational quartic curve $\Gamma_1 \subset V_5 \subset \mathbb{P}^6$, and there is a unique bisecant line L_1 to Γ_1 . The linear system $|H_1 - \Gamma_1|$ has dimension 2, $\text{Bs } |H_1 - \Gamma_1| = \Gamma_1 \cup L$,

and the rational map associated to $|H_1 - \Gamma_1| = |H_2|$ is precisely $V_5 \dashrightarrow \mathbb{P}^1$ induced by the Sarkisov link above. The threefold X_1 is weak Fano, and

$$-K_{X_1} \sim H = 2H_1 - E_1$$

where $E_1 = \text{Exc } f_1$, so that the proper transform of L_1 (still denoted L_1) is the unique flopping curve on X_1 . The map π_1 contracts L_1 to a node $\{x_0\} = \text{Sing}(X) \in X$. Let $\pi: \widehat{X} \rightarrow X$ be the blowup of x_0 , and σ_1 the induced map to X_1 ; χ is the Atiyah flop associated to $x_0 \in X$ and $L_1 = \sigma_1(F)$, where $F = \text{Exc } \pi$. Then, $\widehat{X}$ is a weak Fano threefold of $\rho = 3$ and we have [8]:

$$-K_{\widehat{X}} = H - F \sim H_1 + H_2$$

and from $H \sim 2H_1 - E_1 \sim H_1 + H_2$, we deduce

$$H_2 \sim H_1 - E_1.$$

The map f_2 is a del Pezzo fibration (a Mori fibre space with two-dimensional fibres) of degree $H^2 \cdot H_2 = 6$. For later reference, the intersection numbers on $\widehat{X}$ are:

$$\begin{array}{llll} H_1^3 = 5, & H_1^2 \cdot E_1 = 0, & H_1 \cdot E_1^2 = -4, & E_1^3 = -6, \\ H_1^2 \cdot F = 0, & H_1 \cdot F^2 = -1, & F^3 = 2, & \\ E_1 \cdot F \cdot H_1 = 0, & E_1 \cdot F^2 = -2, & E_1^2 \cdot F = 0. & \end{array}$$

5.1. Construction of a member with $\mathbb{G}_m \rtimes \mathbb{Z}_2$ -action. Recall from [2, Section 5.8] that the quintic threefold $V_5 \subset \mathbb{P}^6$ can be defined scheme theoretically by

$$\begin{cases} x_4x_5 - x_0x_2 + x_1^2 = 0, \\ x_4x_6 - x_1x_3 + x_2^2 = 0, \\ x_4^2 - x_0x_3 + x_1x_2 = 0, \\ x_1x_4 - x_0x_6 - x_2x_5 = 0, \\ x_2x_4 - x_3x_5 - x_1x_6 = 0. \end{cases}$$

and is endowed with an action of $G = \mathbb{G}_m \rtimes \mathbb{Z}_2$ defined by the involution

$$\tau: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_3 : x_2 : x_1 : x_0 : x_4 : x_6 : x_5],$$

and by the automorphisms λ_s

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [s^3x_0 : s^5x_1 : s^7x_2 : s^9x_3 : s^6x_4 : s^4x_5 : s^8x_6].$$

Consider the curve $\Gamma_1 \subset V_5$ defined by the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$ given by

$$[x : y] \mapsto [0 : ix^3y : ixy^3 : 0 : -x^2y^2 : -x^4 : -y^4],$$

where $i^2 = -1$, then Γ_1 is a G -invariant rational curve of degree 4. The line $L_1 = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$ is the unique bisecant line to Γ_1 and it is also G -invariant. Note that Γ_1 lies on $\{x_0 = x_3 = 0\} \cap V_5$, and the pencil of hyperplanes containing Γ_1 is the restriction of

$$\mathcal{H} = \{H_{[\lambda:\mu]} = \{\lambda x_0 + \mu x_3 = 0\}; [\lambda : \mu] \in \mathbb{P}^1\}$$

to V_5 . Denote by $S_{[\lambda:\mu]} = H_{[\lambda:\mu]} \cap V_5$, and note that for any hyperplane in the pencil, $L_1 \cup \Gamma_1 \subset S_{[\lambda:\mu]}$. The midpoint X of the Sarkisov link above is endowed with a G -action. Finally, denote by $S = \{x_4 = 0\} \cap V_5$ the only G -invariant hyperplane section of V_5 , and observe that S has multiplicity 2 along L , so that $\widetilde{S}_{[\lambda:\mu]} = H_1 - E_1 - F$ and $\widetilde{S} = H_1 - 2F$ are the proper transforms of $S_{[\lambda:\mu]}$ and S on $\widehat{X}$.

Claim 5.1. *The group $\text{Aut}(X) = G$, and in particular, it is reductive.*

Proof. Since

$$G \simeq \mathbb{G}_m \rtimes \mathbb{Z}_2 \subset \text{Aut}(X) \simeq \text{Aut}(V_5; \Gamma_1) \subset \text{Aut}(V_5) = \text{PGL}_2(\mathbb{C}),$$

by [13], $\text{Aut}(X) = G$ or $\text{Aut}(X) = \text{Aut}(V_5) = \text{PGL}_2(\mathbb{C})$. The second case is impossible because Γ_1 is not $\text{Aut}(V_5)$ -invariant. $\square$

We will apply Theorem 2.1 to prove that X is K-polystable. To do so, we first describe possible centres of G -invariant divisors over X . In what follows, Ξ denotes a G -invariant prime divisor over X .

Claim 5.2. *If the centre of Ξ on X is 0-dimensional, it is the singular point $c_X(\Xi) = \{x_0\}$.*

Proof. There is no point of $V_5 \subset \mathbb{P}^6$ fixed by the action of G . $\square$

We now consider those G -invariant prime divisors over X which have one-dimensional centre $Z = c_{V_5}(\Xi)$ on V_5 . By [2, Corollary 5.39], the G -invariant curves on V_5 are precisely the line L_1 , the conic C defined parametrically by $[x : y] \mapsto [x^2 : 0 : 0 : y^2 : xy : 0 : 0]$, the twisted cubic defined parametrically by $[x : y] \mapsto [x^3 : x^2y : xy^2 : y^3 : 0 : 0 : 0]$ and a family of sextic curves C_γ for $\gamma \in \mathbb{C}^*$ in each of the hyperplane sections $\{x_4 = 0\} \cap V_5$ and $\{\lambda x_0 + \mu x_3 = 0\} \cap V_5$.

Lemma 5.1. *Let Ξ be a G -invariant prime divisor with centre $Z = c_{V_5}(\Xi)$ a curve. Then $Z = L_1$, $Z = \Gamma_1$ or $\beta(\Xi) > 0$.*

Proof. Assume to the contrary that $\beta(\Xi) < 0$, then by Lemma 2.2, $Z_2 = c_{X_2}(\Xi)$ is contained in $\text{Nklt}(X_2, B_{X_2})$ for some $B_{X_2} \sim_{\mathbb{Q}} -\lambda K_{X_2}$ and rational number $\lambda < \frac{3}{4}$. By Lemma 2.6, the degree $H_2 \cdot Z_2 \leq 1$, and we exclude the curves with $H_1 \cdot Z > 1$ by considering $Z_1 = c_{\Xi}(Z)$ and its intersections with Γ_1 and L_1 . If Z is a rational sextic curve contained in $\{x_4 = 0\}$ or in $\{\lambda x_0 + \mu x_3 = 0\}$, $\Gamma_1 \cap L_1 = \emptyset$, so $(H_1 - E_1) \cdot Z_1 = H_2 \cdot Z_2$ and $\Gamma_1 \cap Z$ consists of at most 2 points, so $H_2 \cdot Z_2 > 1$. Similarly, if $Z = C$ is the G -invariant conic or twisted cubic, $C \cap \Gamma_1 = C \cap L_1 = \emptyset$ and $H_2 \cdot Z_2 > 1$. The only possibilities for Z are L_1 and Γ_1 . $\square$

Lemma 5.2. *Let Ξ be a G -invariant prime divisor with centre $Z = c_{\tilde{X}}(\Xi)$ a curve lying on $\tilde{F}$, then $\beta(\Xi) > 0$.*

Proof. Consider the G -invariant blowup $\omega : \tilde{X} \rightarrow \hat{X}$ of the two flopping lines (these are the transforms of the lines through the singular point on X), and denote by $\Lambda = \Lambda_1 + \Lambda_2$ its exceptional divisor G . Let $\tilde{F} = \omega^*F$, $\tilde{H}_1 = \omega^*H_1$ and $\tilde{E}_1 = \omega^*E_1 - \Lambda$ be the proper transforms of F , H_1 and E_1 . We also have $\tilde{S} = \omega^*S - 2\Lambda$ and $\tilde{S}_{[\lambda:\mu]} = \omega^*S_{\lambda:\mu} - \Lambda$.

If we write the Zariski decomposition of $\omega^*\pi^*(-K_X) - u\tilde{F} = P(u) + N(u)$, then $P(u)$ is nef for all $0 \leq u \leq 3$ and

$$N(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ (u-1)\Lambda & \text{for } 2 \leq u \leq 3, \\ (u-1)\Lambda + (u-2)\tilde{S} & \text{for } 2 \leq u \leq 3. \end{cases}$$

We now compute

$$(5.1.1) \quad S_X(\tilde{F}) = \frac{1}{(-K_X)^3} \int_0^\tau \text{vol}(\omega^*\pi^*(-K_X) - u\tilde{F}) du =$$

$$(5.1.2) \quad = \frac{1}{22} \left(\int_0^1 22 - 2u^3 du + \int_1^2 -6u^2 + 6u + 20 du + \int_2^3 2(6-u)(u-3)^2 du \right) = \frac{39}{22}.$$

So that $\beta(\tilde{F}) = A_X(\tilde{F}) - S_X(\tilde{F}) = 2 - \frac{39}{22} = \frac{5}{22} > 0$.

We now assume that $Z = c_{\tilde{X}}(\Xi) \subset \tilde{F}$. The surface $\tilde{F}$ is the blowup of $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ at two distinct points, that is a del Pezzo surface of degree 6. Let ℓ_1 (resp. ℓ_2) be the full transform of the ruling

of class $(1, 0)$ (resp. $(0, 1)$) on $\mathbb{P}^1 \times \mathbb{P}^1$, and let e_1, e_2 be the two exceptional curves. The Mori cone $\overline{\text{NE}}(\tilde{F})$ is generated by e_1 and e_2 , and by the (-1) -curves $\ell_{i(j)} = \ell_j - e_i$.

We have:

$$S(W_{\bullet, \bullet}^{\tilde{F}}; Z) = \frac{3}{(-K_X)^3} \int_0^3 (P(u)^2 \cdot \tilde{F}) \cdot \text{ord}_Z \left(N(u)|_{\tilde{F}} \right) du + \\ + \frac{3}{(-K_X)^3} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du.$$

As there is no G -fixed point on V_5 , $c_{V_5}(\Xi)$ is not a point and $Z \notin \{e_1, e_2\}$, so that $Z \notin \Lambda|_{\tilde{F}}$. When in addition, $Z \neq \tilde{S}|_{\tilde{F}}$, $\text{ord}_Z(N(u)|_{\tilde{F}}) = 0$. Write

$$Z \sim \alpha_1 e_1 + \alpha_2 e_2 + \sum_{i,j \in \{1,2\}} \alpha_{ij} \ell_{i(j)},$$

and observe that at least one of the coefficients $\alpha_{ij} \neq 0$. By convexity of volume, if the nonzero coefficient corresponds to the curve **1**, we get:

$$S(W_{\bullet, \bullet}^{\tilde{F}}; Z) = \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du \leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - v\mathbf{1}) dv du.$$

so it is enough to show that the last integral is less than 1 to deduce a contradiction.

Case 1. Assume that $Z \neq \tilde{S}|_{\tilde{F}}$, and let $Z \sim \ell_{i(j)}$, for $i, j \in \{1, 2\}$. To fix notation, we consider $\ell_{1(2)}$. Denote by $P(u, v)$ and $N(u, v)$ the positive and negative parts of the Zariski decomposition of $(\omega^* \pi^*(-K_X) - u\tilde{F})|_{\tilde{F}} - vZ$.

- For $0 \leq u \leq 1$, $N(u, v) = ve_1$ for $0 \leq v \leq u$.
- For $1 \leq u \leq 2$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u-1, \\ (v-u+1)e_1 & \text{for } u-1 \leq v \leq 1, \\ (v-u+1)e_1 + (v-1)\ell_{2(1)} & \text{for } 1 \leq v \leq u. \end{cases}$$

- For $2 \leq u \leq 3$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 3-u, \\ (v-3+u)e_1 & \text{for } 3-u \leq v \leq 1, \\ (v-3+u)e_1 + (v-1)\ell_{2(1)} & \text{for } 1 \leq v \leq 4-u. \end{cases}$$

We have

$$S(W_{\bullet, \bullet}^{\tilde{E}_L}; Z) \leq \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{E}_L} - v\ell_{1(2)}) dv du = \\ = \frac{3}{22} \left(\int_0^1 \int_0^u 2u(u-v) dv du + \right. \\ + \int_1^2 \left(\int_0^{u-1} -v^2 + 4u - 2v - 2dv + \int_{u-1}^1 u^2 - 2uv + 2u - 1dv + \int_1^u (u-v+2)(u-v)dv \right) du + \\ + \int_2^3 \left(\int_0^{3-u} 2u^2 + 2uv - v^2 - 16u - 6v + 30dv + \int_{3-u}^1 (-3+u)(3u+4v-13)dv + \right. \\ \left. + \int_1^{4-u} (u+v-4)(3u+v-10)dv \right) du \Big) = \frac{17}{22} < 1,$$

which is what we wanted.

Case 2. Now assume that $Z = \tilde{S}|_{\tilde{F}}$ so that $\text{ord}_Z(N(u)|_{\tilde{F}}) = 1$ on $u \in [2, 3]$, and

$$\frac{3}{(-K_X)^3} \int_2^3 (P(u)^2 \cdot \tilde{F}) \cdot \text{ord}_Z(N(u)|_{\tilde{F}}) du = \frac{4}{11}.$$

As before, denote by $P(u, v)$ and $N(u, v)$ the positive and negative part of the Zariski decomposition of $(\omega^* \pi^*(-K_X) - u\tilde{F})|_{\tilde{F}} - vZ$, so that

- For $0 \leq u \leq 1$, $N(u, v) = v(e_1 + e_2)$ for $0 \leq v \leq u/3$.
- For $1 \leq u \leq 2$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{u-1}{2}, \\ (v - u + 1)(e_1 + e_2) & \text{for } \frac{u-1}{2} \leq v \leq \frac{u}{3}. \end{cases}$$

- For $2 \leq u \leq 3$,

$$N(u, v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3-u}{2}, \\ (v - 3 + u)(e_1 + e_2) & \text{for } \frac{3-u}{2} \leq v \leq 2 - \frac{2u}{3}. \end{cases}$$

We now compute

$$\begin{aligned} S(W_{\bullet, \bullet}^{\tilde{F}}; Z) &= \frac{4}{11} + \frac{3}{22} \int_0^3 \int_0^\infty \text{vol}(P(u)|_{\tilde{F}} - vZ) dv du = \\ &= \frac{4}{11} + \frac{3}{22} \left(\int_0^1 \int_0^{u/3} 2(u-2v)(u-3v) dv du + \right. \\ &\quad + \int_1^2 \left(\int_0^{(u-1)/2} -2uv + 4v^2 + 4u - 8v - 2 dv + \int_{(u-1)/2}^{u/3} 2(u-2v)(u-3v) dv \right) du + \\ &\quad \left. + \int_2^3 \left(\int_0^{(3-u)/2} 2u^2 + 6uv + 4v^2 - 16u - 24v + 30 dv + \int_{(3-u)/2}^{2-2u/3} 2(u+2v-4)(2u+3v-6) dv \right) du \right) = \\ &= \frac{25}{44} < 1. \end{aligned}$$

and this finishes the proof since

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min \left\{ \frac{2}{S_X(\tilde{F})}, \frac{1}{S(W_{\bullet, \bullet}^{\tilde{F}}; Z)} \right\} > 1.$$

□

Lemma 5.3. *Let Ξ be a G -invariant prime divisor with centre $Z = c_{X_1}(\Xi)$ a curve lying on E_1 , then $\beta(\Xi) > 0$.*

Proof. Assume to the contrary that $\beta(\Xi) < 0$, then by Lemma 2.2, $Z_2 = c_{X_2}(\Xi)$ is a one-dimensional component of (X_2, B_{X_2}) , where $B_{X_2} \sim -\lambda K_{X_2}$ for some $\lambda < 3/4$. By Lemma 2.5, $H_2 \cdot Z_2 \leq 1$. This is impossible as $Z_1 = c_{X_1}(\Xi)$ cannot be mapped to a point by ϕ_1 because there is no G -invariant point on V_5 , and $H_2 \cdot Z_1 \geq H_1 \cdot Z_1 \geq 4$. □

Remark 5.4. For the sake of completion, observe that X_1 itself is divisorially K-polystable. Indeed, for $0 \leq u \leq 1$

$$-K_{X_1} - uE_1 \sim_{\mathbb{Q}} H - uE_1 \sim_{\mathbb{Q}} 2H_1 - (1+u)E_1$$

is a mobile divisor, that is the pullback of a nef divisor on X_2 , and for $u > 1$, this divisor is not effective. We have

$$\begin{aligned} (-K_{X_1})^3 \cdot S_{X_1}(E_1) &= 22 \cdot S_{X_1}(E_1) = \int_0^1 \text{vol}(2H_1 - (1+u)E_1) du = \\ &= \int_0^1 ((1-u)(-K_{X_2} + 2uH_2))^3 du = \int_0^1 ((1-u)H + 2uH_2)^3 du \\ &= \int_0^1 (1-u)^2(22+14u) du = \frac{17}{2} \end{aligned}$$

so that $\beta(E_1) > 0$.

Lemma 5.5. *There is no G -invariant irreducible surface D_X such that $-K_X \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$ for some positive rational number $\lambda > \frac{4}{3}$ and effective $\mathbb{Q}$ -divisor Δ .*

Proof. Let D_X be such a surface, and denote by D_1, Δ_1 the proper transforms of D_X and Δ_X on X_1 . We have:

$$H \sim_{\mathbb{Q}} 2H_1 + E_1 \sim_{\mathbb{Q}} \lambda D_1 + \Delta_1.$$

Recall that the pseudo-effective cone $\overline{\text{Eff}}(X_1)$ is $\mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[H_2]$, where $H_2 \sim_{\mathbb{Q}} H_1 - E_1$. If $D_1 = E_1$, we see that

$$\Delta_1 \sim_{\mathbb{Q}} 2H_2 + (1-\lambda)E_1$$

cannot be an effective divisor. We may now assume that $D_1 \in \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2]$, that is $D_1 = xH_1 - yE_1$, for $x, y \in \mathbb{N}$ and $x \geq y$. Since $\lambda D_1 \leq -K_{X_1}$, $\lambda a \leq 2$, so that $a = 1$ and $b = 0$ or $b = 1$. As D_1 is mapped to a G -invariant surface of V_5 , $\phi_1(D_1)$ is the hyperplane section $\{x_4 = 0\} \cap V_5$, and $b = 0$. Now, $\Delta_1 \sim_{\mathbb{Q}} (2-\lambda)H_1 - E_1$, but this cannot be effective as $2-\lambda < 1$. $\square$

As in the previous two cases, we conclude:

Main Theorem (III). *X is K -polystable.*

This time X is not K -stable as $\text{Aut}(X) = \mathbb{G}_m \rtimes \mathbb{Z}_2$, but using [2, Corollary 1.16] (which still holds in the case of a nodal Fano threefold), we conclude:

Corollary 5.6. *A general one-nodal prime Fano threefold of genus 12 in Family III is K -polystable.*

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