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On K-stability of One-nodal Prime Fano Threefolds of Genus 12

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To Professor Yuri Prokhorov on the occasion of his 60th birthday

Abstract. We show that general one-nodal prime Fano threefolds of genus 12 are K-polystable.

1. Introduction

1.1. Singular Fano threefolds of genus 12

Let X be a Fano threefold with terminal Gorenstein singularities. By [7, 12], $X \hookrightarrow \mathcal{X}$ has a smoothing and $\mathcal{X}_t$ for $t \neq 0$ is a smooth Fano threefold with Picard rank $\rho(X)$ and anticanonical degree $(-K_X)^3$. Unless mentioned otherwise, a prime Fano threefold of genus 12 will refer to a terminal Gorenstein Fano threefold with Picard rank 1 and anticanonical degree 22. Recent advances in the theory of K-stability show that there is a projective moduli space $\mathrm{M}^{\mathrm{Kps}}_{3,22}$ whose closed points over $\mathbb{C}$ parameterize K-polystable Fano threefolds of anticanonical degree 22 that admit a smoothing (see [16] as a reference on the general theory of K-moduli).

Let X be a prime Fano threefold of genus 12, then X is $\mathbb{Q}$-factorial precisely when X is smooth [11]. Smooth prime Fano threefolds of genus 12 form a 6-dimensional family, which contains both K-polystable and strictly K-semistable members [2, Section 7.1]. A precise description of which smooth prime Fano threefolds of genus 12 are K-polystable or semistable is still conjectural. Denote by $\overline{M}$ the (non-empty 6-dimensional) component of $\mathrm{M}^{\mathrm{Kps}}_{3,22}$ parametrizing those K-polystable Fano threefolds of anticanonical degree 22 with a smoothing to a prime Fano threefold of genus 12.

Prokhorov classifies prime Fano threefolds of genus 12 with one node and shows that they form four 5-dimensional families [14]. The goal of this note is to show

Theorem 1.1. *A general one-nodal prime Fano threefold of genus* 12 *is K-polystable. There are four boundary divisors of* $\overline{M}$ *parametrising K-polystable degenerations of one-nodal prime Fano threefolds of genus* 12.

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We now describe the geometry of the four families of one-nodal prime Fano threefolds of genus 12 briefly.

Theorem 1.2. [14] *Let X be a prime Fano threefold of genus* 12*, and assume that* $\operatorname{rk}\operatorname{Cl}(X) = 2$ *(or equivalently that* $\operatorname{Sing}(X)$ *consists of precisely one ordinary double point). Then X is the midpoint of a Sarkisov link*

$$\begin{array}{ccccccc} & & X_1 & \overset{\chi}{\dashrightarrow} & X_2 & & \\ & \swarrow^{f_1} & & \searrow^{\pi_1} \quad \swarrow^{\pi_2} & & \searrow^{f_2} & \\ Z_1 & & & X & & & Z_2 \end{array}$$

where π_1 and π_2 are small $\mathbb{Q}$-factorializations, χ is a flop, and f_1 and f_2 are K-negative extremal contractions described as follows:

(I) *$Z_1 = \mathbb{P}^3$ and $Z_2 = \mathbb{P}^3$, f_1 and f_2 are the blowups of curves $\Gamma_1 \subset Z_1$ and $\Gamma_2 \subset Z_2$ respectively. Both Γ_1 and Γ_2 are rational quintic curves that do not lie on quadric surfaces.*

(II) *$Z_1 = Q \subset \mathbb{P}^4$ and $Z_2 = \mathbb{P}^2$, f_1 is the blowup of a rational quintic curve $\Gamma_1 \subset Q$ that does not lie on a hyperplane section of Q, and f_2 is a conic bundle with discriminant δ of degree* 3.

(III) *$Z_1 = V_5$ a quintic del Pezzo threefold, $Z_2 = \mathbb{P}^1$, f_1 is the blowup of a rational quartic curve $\Gamma \subset V_5$ and f_2 is a del Pezzo fibration of degree* 6.

(IV) *$Z_1 = \mathbb{P}^2$ and $Z_2 = \mathbb{P}^1$, $f_1 \colon \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \to \mathbb{P}^2$, where $\mathscr{E}$ is a stable rank* 2 *vector bundle and f_2 is a del Pezzo fibration of degree* 5.

Remark 1.3. These four families appear in [8, Table 2] under references (12na), (12nb), (12nc) and (12nd). The blowup of a general member X of Family (I) (resp. (II), (III), (IV)) at its node is a weak Fano threefold $\widehat{X}$ whose anticanonical model admits a smoothing in Family MM_{2-12} (resp. MM_{2-13}, MM_{2-14}, MM_{3-5}) in the classification of Fano threefolds [10].

Theorem 1.4. [2, Proposition 5.66] *There is a K-stable member of Family* (IV).

In this note, we prove

Theorem 1.5. *There exist K-stable members of Families* (I) *and* (II). *There is a K-polystable member of Family* (III).

2. Preliminary results on explicit K-stability of Fano threefolds

All varieties considered are defined over $\mathbb{C}$. Let X be a Fano variety with at most Kawamata log terminal singularities of dimension $n \geq 2$, and let G be a reductive subgroup in $\mathrm{Aut}(X)$. Let Ξ be a divisor over X, that is Ξ is a prime divisor on a normal variety $\widetilde{X}$ with a birational morphism $\varphi\colon \widetilde{X} \to X$. Define $\beta(\Xi) = A_X(\Xi) - S_X(\Xi)$, where $A_X(\Xi) = 1 + \mathrm{ord}_{\Xi}(K_{\widetilde{X}/X})$ is the log discrepancy of Ξ and

$$S_X(\Xi) = \frac{1}{(-K_X)^n} \int_0^{\tau(\Xi)} \mathrm{vol}(\varphi^*(-K_X) - u\Xi)\, du$$

for $\tau(\Xi) = \sup\{u \in \mathbb{R}_{>0} \mid \varphi^*(-K_X) - u\Xi \text{ is big }\}$.

Theorem 2.1. [17, Corollary 4.14] *Suppose that $\beta(\Xi) > 0$ for every G-invariant prime divisor Ξ over X. Then X is K-polystable.*

Recall the definition of the number $\alpha_{G,Z}(X)$, where $Z \subset X$ is a G-invariant subvariety:

$$\alpha_{G,Z}(X) = \sup\left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is log canonical at general point of } Z \text{ for any} \\ \text{effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}.$$

Then $\alpha_G(X) \leq \alpha_{G,Z}(X)$.

Lemma 2.2. [2, 1.44] *Let $f\colon \widetilde{X} \to X$ be an arbitrary G-equivariant birational morphism, let Ξ be a G-invariant prime divisor in X such that $Z \subseteq f(\Xi)$, then we have*

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \frac{n+1}{n} \alpha_{G,Z}(X).$$

In particular, in dimension 3, the existence of a G-invariant divisor Ξ over X with $\beta(\Xi) < 0$ and $Z \subset c_X(\Xi)$ implies that $\alpha_{G,Z}(X) < 3/4$, so that Z is contained in $\mathrm{Nklt}(X, B_X)$ for some $B_X \sim_{\mathbb{Q}} -\lambda K_X$ and rational number $\lambda < 3/4$.

The next theorem is an application of the general inductive argument developed by Abban and Zhuang to bound the ratio $\frac{A_X(\Xi)}{S_X(\Xi)}$ [1] to the case of smooth Fano threefolds.

Theorem 2.3. [2, Corollary 1.110] *Let X be a smooth Fano threefold, let Y be an irreducible normal surface in the threefold X, let Z be an irreducible curve in Y, and Ξ a prime divisor over X with $C_X(\Xi) = Z$. Then*

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min\left\{ \frac{1}{S_X(Y)}, \frac{1}{S(W^Y_{\bullet,\bullet}; Z)} \right\}$$

and

$$S(W^Y_{\bullet,\bullet};Z) = \frac{3}{(-K_X)^3}\int_0^\tau (P(u)^2\cdot Y)\cdot \mathrm{ord}_Z(N(u)|_Y)\,du + \frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty \mathrm{vol}(P(u)|_Y - vZ)\,dv\,du$$

where $P(u)$ is the positive part of the Zariski decomposition of the divisor $-K_X - uY$, and $N(u)$ is its negative part.

Remark 2.4. Here, $W^Y_{\bullet,\bullet}$ is a $\mathbb{N}^2$ linear series defined as the refinement of the anticanonical ring

$$V^X_\bullet = \bigoplus_{m\in\mathbb{N}} H^0(X,-mK_X)$$

by the divisor Y. We refer to [1, Section 2] or [2, Section 1.7] for the definition of $W^Y_{\bullet,\bullet}$ and of the associated invariant $S(W^Y_{\bullet,\bullet};Z)$. We take the expression in Theorem 2.3 as a definition of $S(W^Y_{\bullet,\bullet};Z)$. Note that an expression for $S(W^Y_{\bullet,\bullet};Z)$ can be computed in the more general context of $\mathbb{Q}$-factorial Mori Dream spaces [2, Theorem 1.106].

We recall a few results on nonklt centres of pairs (X,B_X) where $X\sim -\lambda K_X$ for $\lambda\in\mathbb{Q}$ when X admits morphisms to projective spaces.

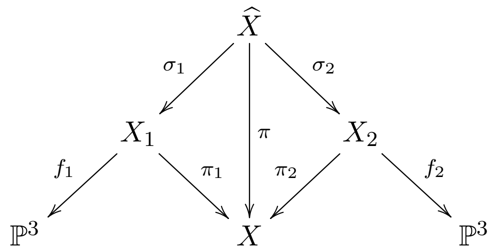
Lemma 2.5. [2, Corollary A.10] *Suppose $X=\mathbb{P}^3$ and $B_X\sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda < 3/4$. Let Z be the union of one-dimensional components of* $\mathrm{Nklt}(X,B_X)$. *Then $\mathcal{O}_{\mathbb{P}^3}(1)\cdot Z\le 1$. In particular, if $Z\ne 0$, then Z is a line.*

Lemma 2.6. [2, Corollary A.12] *Suppose that X is a smooth Fano threefold, $B_X\sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda<1$, and there exists a surjective morphism with connected fibers $\phi\colon X\to\mathbb{P}^1$. Set $H=\phi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Let Z be the union of one-dimensional components of* $\mathrm{Nklt}(X,\lambda B_X)$. *Then $H\cdot Z\le 1$.*

Lemma 2.7. [2, Corollary A.13] *Suppose that $-K_X$ is nef and big, $B_X\sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda<1$, and there exists a surjective morphism with connected fibers $\phi\colon X\to\mathbb{P}^2$. Set $H=\phi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Let Z be the union of one-dimensional components of* $\mathrm{Nklt}(X,\lambda B_X)$. *Then $H\cdot Z\le 2$.*

3. Family (I)

Let X be a one-nodal prime Fano threefold of genus 12 that belongs to Family (I) of Theorem 1.2, then X is the midpoint of a Sarkisov link associated to a Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ which is a degeneration of the cubo-cubic transformation [3]. We describe the associated birational geometry briefly, see [3, 5, 14] and [8] for proofs and precise statements.



Denote by $H_i = \sigma_i^*\big(f_i^*\mathcal{O}_{\mathbb{P}^3}(1)\big)$ for $i = 1, 2$, and by $H = \pi^*(-K_X)$ the pullbacks to $\widehat{X}$ (or to any of the models) of the ample generators of $\mathrm{Pic}(\mathbb{P}^3)$ and of $\mathrm{Pic}(X)$. Given a curve $\Gamma \subset \mathbb{P}^3$, we (sloppily) denote by $|nH_1 - \Gamma|$ the linear system $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n) \otimes \mathcal{I}_\Gamma)$ of surfaces of degree n on which Γ lies. The morphism f_1 is the blowup of a smooth rational quintic curve $\Gamma_1 \subset \mathbb{P}^3$ that does not lie on a quadric ($|2H_1 - \Gamma_1| = \emptyset$), and there is a unique quadrisecant line L_1 to Γ_1. The curve Γ_1 lies on a cubic surface, $|3H_1 - \Gamma_1|$ has dimension 4 and $\mathrm{Bs}\,|3H_1 - \Gamma_1| = \Gamma_1 \cup L_1$. The birational map associated to $|3H_1 - \Gamma_1| = |H_2|$ is precisely the Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ induced by the Sarkisov link above. The threefold X_1 is weak Fano,

$$-K_{X_1} \sim H \sim 4H_1 - E_1,$$

where $E_1 = \mathrm{Exc}\, f_1$, so that the proper transform of L_1 (still denoted L_1) is the unique flopping curve on X_1. The map π_1 contracts L_1 to a node $\{x_0\} = \mathrm{Sing}(X) \in X$.

Let $\pi\colon \widehat{X} \to X$ be the blowup of x_0, and σ_1 the induced map to X_1. Note that X_1 and X_2 are the two small resolutions of the node x_0, $\chi\colon X_1 \dashrightarrow X_2$ is the associated Atiyah flop and $L_1 = \sigma_1(F)$, where $F = \mathrm{Exc}\,\pi$. Then, $\widehat{X}$ is a weak Fano threefold of $\rho = 3$ and we have [8]:

$$-K_{\widehat{X}} \sim H - F \sim H_1 + H_2$$

and from

$$H \sim 4H_1 - E_1 \sim 4H_2 - E_2$$

we deduce

$$H \sim 2(H_1 + H_2) - \frac{E_1 + E_2}{2} \quad \text{and} \quad H_1 + H_2 \sim \frac{E_1 + E_2}{2} + F.$$

For future reference, let T_1 be a cubic surface containing Γ_1, and denote by T its proper transform on $\widehat{X}$. Since $\mathrm{Bs}\,|3H_1 - \Gamma_1| = \Gamma_1 \cup L_1$, L_1 lies on T_1 and

$$T \sim 3\sigma_1^*(f_1^*\mathcal{O}_{\mathbb{P}^3}(1)) - E_1 - F \sim 3H_1 - E_1 - F \sim H_2,$$

so that

$$-K_{\widehat{X}} - uT \sim H_1 + H_2 - uH_2 \in \mathbb{Z}_{\geq 0}[H_1] + \mathbb{Z}_{\geq 0}[H_2] \subset \mathrm{Nef}(\widehat{X})$$

is nef for $0 \leq u \leq 1$. For $u > 1$, $-K_{\widehat{X}} - uT$ is no longer nef. If C is the proper transform on $\widehat{X}$ of a minimal rational curve contracted by f_1, then $H_1 \cdot C = 0$ and $H_2 \cdot C > 0$, so that

$$-K_{\widehat{X}} - uT \sim H_1 \cdot C - (u-1)H_2 \cdot C < 0.$$

We may write for $u \geq 1$,

$$\begin{aligned} -K_{\widehat{X}} - uT &\sim uH_1 - (u-1)(H_1 + H_2) \sim uH_1 - (u-1)(4H_1 - E_1 - F) \\ &\sim (4-3u)H_1 + (u-1)(E_1 + F) \end{aligned}$$

showing that the pseudo-effective threshold is $u = 4/3$, and that $-K_{\widehat{X}} - uT$ admits a Zariski decomposition with nef positive part $P(u) = (4-3u)H_1$ and negative part $N(u) = (u-1)(E_1 + F)$.

3.1. Construction of a member with $\mathbb{Z}_2 \times \mathbb{Z}_2$-action

We now consider a special member of Family (I). Let $C_{(a,b)}$ be the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by

$$[x:y] \to [x^5 : ax^4y + bx^2y^3 : bx^3y^2 + axy^4 : y^5] \quad \text{for } a, b \in \mathbb{C}^*;$$

then $C_{(a,b)}$ is a rational quintic curve that does not lie on a quadric surface for $|a| \neq |b|$. The curve $C_{(a,b)}$ is invariant under the action of $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{P}^3$ defined by

$$\begin{aligned} \tau_1 &: [x_0 : x_1 : x_2 : x_3] \to [x_3 : x_2 : x_1 : x_0], \\ \tau_2 &: [x_0 : x_1 : x_2 : x_3] \to [x_0 : -x_1 : x_2 : -x_3]. \end{aligned}$$

In fact, the action of τ_1 (resp. τ_2) on $C_{(a,b)}$ is induced by that of the involution of $\mathbb{P}^1$ given by $[x:y] \leftrightarrow [y:x]$ (resp. $[x:y] \leftrightarrow [x:-y]$). We consider the element of Family (I) obtained by taking the curve $\Gamma_1 = C_{(1,-4)}$.

Since Γ_1 is G-invariant, L_1 is also G-invariant and X_1 and X are endowed with a G-action.

Claim 3.1. The group $\mathrm{Aut}(X)$ is finite.

Proof. The curve Γ_1 is not contained in a hypersurface of $\mathbb{P}^3$, the stabilizer of Γ in $\mathrm{Aut}(\mathbb{P}^3)$ is $\mathrm{Aut}(\mathbb{P}^3; \Gamma_1) \simeq \mathrm{Aut}(\Gamma_1) \simeq \mathrm{Aut}(\mathbb{P}^1)$. By construction of X, $\mathrm{Aut}(X)$ is a subgroup of the group $\mathrm{Aut}(\mathbb{P}^3, \Gamma_1) \simeq \mathrm{Aut}(\mathbb{P}^1)$ that preserves the four points of intersection $\Gamma_1 \cap L_1$, so it is a finite group. □

We will apply Theorem 2.1 to prove that X is K-stable. To do so, we first describe possible centres of G-invariant divisors over X. In what follows, Ξ always denotes a G-invariant prime divisor over X.

Claim 3.2. If the centre of Ξ on X is 0-dimensional, it is the singular point $c_X(\Xi) = \{x_0\}$.

Proof. There is no point of $\mathbb{P}^3$ fixed by the action of G. □

Claim 3.3. If the centre of Ξ on $\mathbb{P}^3$ is a line L, then

$$L = L_{[\lambda:\mu]} = \begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0. \end{cases}$$

All G-invariant lines lie on the quadric $Q = \{x_1x_0 - x_2x_3 = 0\}$. Any two distinct G-invariant lines are disjoint. A G-invariant line $L \neq L_1$ is either disjoint from Γ_1 or meets Γ_1 in precisely two points.

Proof. Let $L \subset \mathbb{P}^3$ be a G-invariant line, and consider any two distinct hyperplanes $H_1 = \{f_1 = 0\}$ and $H_2 = \{f_2 = 0\}$ containing L, so that $L = H_1 \cap H_2 = \{f_1 = f_2 = 0\}$. Then, $L = \operatorname{Bs} \mathscr{H}$ is the base locus of the pencil $\mathscr{H} = \{uf_1 + vf_2 = 0; [u : v] \in \mathbb{P}^1\}$.

The line $L = \operatorname{Bs} \mathscr{H}$ is G-invariant precisely when G fixes $\mathscr{H}$, or equivalently when both τ_1 and τ_2 induce involutions on $\mathscr{H}$, and on its base $\mathbb{P}^1$. Up to reparametrizing the pencil $\mathscr{H}$ we may assume that $[u : v] = [1 : 0]$ is a τ_2-invariant hyperplane, that is, the linear form $f_1(x_0, \ldots, x_3)$ is one of

$$\lambda x_0 + \mu x_2 \quad \text{or} \quad \lambda x_3 + \mu x_1 \quad \text{for } [\lambda : \mu] \in \mathbb{P}^1,$$

and $H_1 = \{\lambda x_0 + \mu x_2 = 0\}$ or $H_1 = \{\lambda x_3 + \mu x_1 = 0\}$. The condition that $\mathscr{H}$ is G-invariant is then that $\tau_1 \cdot H_1$ is a fibre of the pencil, so that (noting that H_1 is not fixed by τ_1)

$$L = L_{[\lambda:\mu]} = H_1 \cap \tau_1 \cdot H_1 = \begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0 \end{cases}$$

which gives the desired expression.

Check that $L_{[\lambda:\mu]} \subset Q$ for all $[\lambda : \mu] \in \mathbb{P}^1$, that $L_{[\lambda:\mu]} \cap L_{[\lambda':\mu']} = \emptyset$ for $[\lambda : \mu] \neq [\lambda' : \mu']$, and that $L_{[\lambda:\mu]} \cap \Gamma_1 = \emptyset$ unless $[\lambda : \mu] \in \{[0 : 1], [3 : 1], [-5 : 1]\}$ and $L_{[\lambda:\mu]} \cap \Gamma_1$ consists of 2 points, or $[\lambda : \mu] = [1 : 1]$ and $L_{[1:1]} = L_1$ is the unique quadrisecant to Γ_1. □

Remark 3.4. Given that the Sarkisov link of which X is a midpoint is G-equivariant, E_2 and $\Gamma_2 = \Gamma^+$ are also invariant under the induced G-action. Since the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is induced by $|H_2| = |3H_1 - \Gamma_1|$, the fibres of $E_2 \to \Gamma_2$ are the transforms of trisecant lines of Γ_1. Since none of these are G-invariant, the action of G on Γ_2 does not fix Γ_2 pointwise either.

Claim 3.5. Let $H_{[\lambda:\mu]}$ be a general hyperplane containing $L_{[\lambda:\mu]}$. Then $H_{[\lambda:\mu]} \cap \Gamma_1 = \{b_1, \ldots, b_5\}$ and $H_{[\lambda:\mu]} \cap L_1 = \{b_0\}$, where $b_1, \ldots, b_5$ (resp. $b_0, \ldots, b_5$) consists of 5 (resp. 6) points in general position.

Proof. Fix $[\lambda : \mu] \in \mathbb{P}^1$, and let $\mathscr{H}$ be the pencil of hyperplanes containing $L_{[\lambda:\mu]}$. We compute that the general fibre of $\mathscr{H}$ intersects $\Gamma_1 \cup L_1$ in 6 distinct points. Assume that for some fibre H of $\mathscr{H}$, 3 of the 5 points of $H \cap \Gamma_1$ lie on a line (resp. the 6 points $H \cap (\Gamma_1 \cup L_1)$ lie on a conic). Then, this line (resp. conic) is contracted by the Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ to a point lying on Γ_2. If the points of intersection of a general hyperplane containing $L_{[\lambda:\mu]}$ are not in general position, then we define a dominant rational map $\mathbb{P}^1 \dashrightarrow \Gamma_2$ from the base of $\mathscr{H}$ to Γ_2, leading to a contradiction. □

We now turn to the proof that no G-invariant prime divisor Ξ over X with $\beta(\Xi) \leq 0$ has 1-dimensional centre $Z = c_{\mathbb{P}^3}(\Xi)$. If Z is 1-dimensional, then either $Z = \Gamma_1$, or by Lemma 2.2, Z is the union of 1-dimensional components of $\mathrm{Nklt}(\mathbb{P}^3, B)$ for some $B \sim \mathcal{O}_{\mathbb{P}^3}(4\lambda)$ with $\lambda \in \mathbb{Q}$, $\lambda < 3/4$. Then, by Lemma 2.5, Z can only be a line.

Lemma 3.6. *If $Z = c_{\mathbb{P}^3}(\Xi)$ is a G-invariant line distinct from L_1, $\beta(\Xi) > 0$.*

Proof. We will use Lemma 2.5 to find a lower bound for $\beta(\Xi)$. To this effect, we find an irreducible normal surface $S \subset \widehat{X}$ containing Z and use the inequality

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet}; Z)}\right\}.$$

Let $S_1 \subset X_1$ and $S \subset \widehat{X}$ be the pullbacks to X_1 and $\widehat{X}$ of a general hyperplane containing Z. By Claim 3.5, $S_1 \subset X_1$ is a del Pezzo surface of degree 4 and S is a cubic surface. Recall that E_1 and E_2 are the f_1 and f_2 exceptional divisors, and that F is the π-exceptional divisor. All these are G-invariant, and E_2 is covered by the (proper transforms of) trisecant lines of Γ_1.

We first compute $S_X(S)$; on $\widehat{X}$, we have the following intersection numbers:

$$\begin{aligned}
&S^3 = 1, \quad && S^2 \cdot E_1 = 0, \quad && S \cdot E_1^2 = -5, \quad && E_1^3 = -18,\\
& && S^2 \cdot F = 0, && S \cdot F^2 = -1, && F^3 = 2,\\
& && S \cdot E_1 \cdot F = 0, && E_1 \cdot F^2 = -4, && E_1^2 \cdot F = 0,
\end{aligned}$$

and relations [8]

$$S \sim H_1 \sim 3H_2 - E_2 - F, \quad H \sim 4H_1 - E_1 \sim 4H_2 - E_2, \quad H - F \sim H_1 + H_2.$$

Define, for $u \geq 0$, the divisor

$$\begin{aligned}
D_u &= \pi^*(-K_X) - uS \sim (1-u)H + u(H - S) \sim (1-u)H + u(H_2 + F)\\
&= (1-u)H + u\left(\frac{H + E_2}{4} + F\right),
\end{aligned}$$

then D_u is pseudo-effective for $u \leq 4/3$, and has a Zariski decomposition with nef positive part

$$D_u = P(u) + N(u), \quad \text{where } P(u) = \left(1 - \frac{3u}{4}\right) H \text{ and } N(u) = u\left(\frac{E_2}{4} + F\right)$$

and we compute

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^\tau \operatorname{vol}(\pi^*(-K_X) - uS)\, du = \frac{1}{22} \int_0^{4/3} \frac{11(4-3u)^3}{32}\, du = \frac{1}{3}.$$

We now show that $S(W^S_{\bullet,\bullet}; Z) < 1$ in order to apply Theorem 2.3.

The surface S is a cubic surface obtained by blowing up a general hyperplane $\mathbb{P}^2$ at 6 points $\{b_0, \ldots, b_5\}$ in general position. Let ℓ be the pullback of the generator of $\operatorname{Pic}(\mathbb{P}^2)$, and $e_0, \ldots, e_5$ the exceptional curves. The Mori cone $\overline{\mathrm{NE}}(S)$ is generated by $e_0, \ldots, e_5$, by the proper transforms $l_{i,j} = \ell - e_i - e_j$ of lines through two of the blownup points, and by the proper transforms $q_i = 2l - \sum e_j + e_i$ of the conics through any 5 of the blownup points $\{b_0, \ldots, b_5\}$.

We want to evaluate

$$\begin{aligned} S(W^S_{\bullet,\bullet}; Z) = {} & \frac{3}{(-K_X)^3} \int_0^\tau (P(u)^2 \cdot S) \cdot \operatorname{ord}_Z(N(u)|_S)\, du \\ & + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \operatorname{vol}(P(u)|_S - vZ)\, dv\, du. \end{aligned}$$

Since there are no G-fixed points, $Z \subset S$ is one of ℓ or the lines $l_{i,j}$. Recall that $E_2 \sim 8H_1 - 3E_1 - 4F$, and restricting to S gives $E_2|_S \sim 8\ell - 3(e_1 + \cdots + e_5) - 4e_0$. From the description of $\overline{\mathrm{NE}}(S)$,

$$\operatorname{ord}_Z(E_2|_S) \leq 2 \quad \text{and} \quad \operatorname{ord}_Z(N(u)|_S) \leq \begin{cases} 2 \cdot \frac{1}{4} = \frac{1}{2} & \text{if } Z \subset E_2, \\ 0 & \text{otherwise.} \end{cases}$$

The first term of the expression $S(W^S_{\bullet,\bullet}; Z)$ is bounded by

$$\frac{3}{(-K_X)^3} \int_0^\tau (P(u)^2 \cdot S) \cdot \operatorname{ord}_Z(N(u)|_S)\, du \leq \frac{3}{22} \int_0^{4/3} \frac{11(4-3u)^2}{16} \cdot \frac{1}{2}\, du = \frac{1}{3}.$$

Case 1*:* $Z \cap \Gamma_1 = \emptyset$. In this case, $Z \sim \ell$, and the Zariski decomposition of

$$(-\pi^*(K_X) - uS)|_S - vZ = P(u,v) + N(u,v);$$

for $u \in [0, 4/3]$ is given by

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3(4-3u)}{8}, \\ \frac{8v - 3(4-3u)}{4} q_0 & \text{for } \frac{3(4-3u)}{8} \leq v \leq \frac{4-3u}{2}, \end{cases}$$

and we compute

$$P(u,v)^2 = \begin{cases} v^2 - 8v + 6uv + \frac{99u^2}{16} - \frac{33u}{2} + 11 & \text{for } 0 \le v \le \frac{3(4-3u)}{8}, \\ \frac{5(2v-(4-3u))^2}{4} & \text{for } \frac{3(4-3u)}{8} \le v \le \frac{4-3u}{2}. \end{cases}$$

This yields

$$\begin{aligned} S(W^S_{\bullet,\bullet}; Z) &\le \frac{1}{3} + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \mathrm{vol}(P(u)|_S - vZ)\, dv\, du \\ &\le \frac{1}{3} + \frac{3}{22} \int_0^{4/3} \Bigg(\int_0^{\frac{3(4-3u)}{8}} v^2 - 8v + 6uv + \frac{99u^2}{16} - \frac{33u}{2} + 11\, dv \\ &\qquad + \int_{\frac{3(4-3u)}{8}}^{\frac{4-3u}{2}} \frac{5(2v-(4-3u))^2}{4}\, dv \Bigg) du \\ &\le \frac{1}{3} + \frac{53}{132} = \frac{97}{132} < 1, \end{aligned}$$

which is what we wanted.

Case 2: $Z \cap \Gamma_1 \neq \emptyset$. As Z is one of the bisecant lines of Γ_1, $Z \sim l_{i,j} = \ell - e_i - e_j$ for some $1 \le i < j \le 5$. We may assume that $Z \sim l_{1,2}$. Write the Zariski decomposition of $(-\pi^*(K_X) - uS)|_S - vZ$ for $0 \le u \le 4/3$ we have

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \le v \le \frac{3(4-3u)}{4}, \\ \frac{4v-3(4-3u)}{4}(e_1+e_2) & \text{for } \frac{3(4-3u)}{4} \le v \le \frac{4-3u}{2}, \\ \frac{4v-3(4-3u)}{4}(e_1+e_2) + \frac{2v-3(4-3u)}{2}(\ell_{34}+\ell_{35}+\ell_{45}) & \text{for } \frac{4-3u}{2} \le v \le \frac{5(4-3u)}{8}. \end{cases}$$

This time, we compute

$$\begin{aligned} S(W^S_{\bullet,\bullet}; Z) &\le \frac{1}{3} + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \mathrm{vol}(P(u)|_S - vZ)\, dv\, du \\ &\le \frac{1}{3} + \frac{3}{22} \int_0^{4/3} \Bigg(\int_0^{\frac{4-3u}{4}} -v^2 - 4v + 3uv + \frac{99u^2}{16} - \frac{33u}{2} + 11\, dv \\ &\qquad + \int_{\frac{4-3u}{4}}^{\frac{4-3u}{2}} v^2 - 8v + 6uv + \frac{117u^2}{16} - \frac{39u}{2} + 13\, dv \\ &\qquad + \int_{\frac{4-3u}{2}}^{\frac{5(4-3u)}{8}} \frac{(8v-5(4-3u))^2}{16}\, dv \Bigg) du \\ &\le \frac{1}{3} + \frac{23}{44} = \frac{113}{132} < 1, \end{aligned}$$

and this finishes the proof in this case. ∎

Lemma 3.7. *Let Ξ be a prime divisor over X with $c_{\mathbb{P}^3}(\Xi) = L_1$ then $\beta(\Xi) > 0$.*

Proof. By [8], there are precisely 4 lines through $x_0 \in X$. Since G fixes x_0 and G sends lines to lines, G sends a line through x_0 to a line through x_0. If $L \ni x_0$ is a line and $\widehat{L}$ is its proper transform on $\widehat{X}$, $-K_{\widehat{X}} \cdot \widehat{L} = 0$ and $\widehat{L}$ is a flopping curve. Let $\omega\colon \widetilde{X} \to \widehat{X}$ be the blowup of the proper transforms of the 4 lines through $x_0 \in X$, and denote by $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$ its (G-invariant) exceptional divisor. Denote by $\widetilde{F} = \omega^*F$, $\widetilde{E}_1 = \omega^*E_1 - \Lambda$ the proper transforms of F and E_1 on $\widetilde{X}$. On $\widetilde{X}$, we have the intersection numbers:

$$\begin{aligned}
&\Lambda^3 = 8, && \Lambda^2 \cdot \omega^*F = -4, && \Lambda \cdot \omega^*(F)^2 = 0, && \Lambda \cdot \omega^*(F) \cdot \omega^*(E_1) = 0,\\
&\Lambda^2 \cdot \omega^*(E_1) = 4, && \Lambda \cdot \omega^*(E_1)^2 = 0, && \omega^*F^3 = 2, && \omega^*E_1^3 = -18.
\end{aligned}$$

We first show that the Zariski decomposition of $\omega^*\pi^*(-K_X) - u\widetilde{F}$ exists and writes $P(u) + N(u)$, where $P(u)$ is nef and

$$N(u) = \begin{cases} 0 & \text{for } 0 \le u \le 1,\\ (u-1)\Lambda & \text{for } 1 \le u \le 3. \end{cases}$$

We have $A_X(\widetilde{F}) = 2$ and compute

$$\begin{aligned}
S_X(\widetilde{F}) &= \frac{1}{(-K_X)^3} \int_0^\tau \operatorname{vol}(\omega^*\pi^*(-K_X) - u\widetilde{F})\, du\\
&= \frac{1}{22}\left(\int_0^1 22 - 2u^3\, du + \int_1^3 2(u-3)(u^2-3u-3)\, du\right) = \frac{83}{44}.
\end{aligned}$$

So that $\beta(\widetilde{F}) > 0$.

Now we assume that the centre of Ξ over $\widetilde{X}$ is one-dimensional, so that $Z = c_{\widehat{X}}(\Xi) \subset \widetilde{F}$ is an irreducible curve. The surface $\widetilde{F}$ is the blowup of $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ at 4 points, so it is a del Pezzo surface of degree 4. Let ℓ_1 and ℓ_2 be the pullbacks to $\widetilde{F}$ of the two rulings and e_1, e_2, e_3, e_4 the exceptional divisors. Then $\overline{\mathrm{NE}}(\widetilde{F})$ is generated by the proper transforms of rulings through one of the blownup points ($\ell_{1,i} = \ell_1 - e_i$ or $\ell_{2,i} = \ell_2 - e_i$) and by the proper transforms of $(1,1)$ curves on F through 3 blownup points $\ell_{i,j,k} = \ell_1 + \ell_2 - e_i - e_j - e_k$. We use Theorem 2.3 to find a lower bound for $\beta(\Xi)$. We have

$$\begin{aligned}
S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) = {}& \frac{3}{(-K_X)^3} \int_0^3 (P(u)^2 \cdot \widetilde{F}) \cdot \operatorname{ord}_Z(N(u)|_{\widetilde{F}})\, du\\
&+ \frac{3}{(-K_X)^3} \int_0^3 \int_0^\infty \operatorname{vol}(P(u)|_{\widetilde{F}} - vZ)\, dv\, du.
\end{aligned}$$

Since $Z \notin \{e_1, e_2, e_3, e_4\} = \Lambda_{\widetilde{F}}$, $\operatorname{ord}_Z(N(u)|_{\widetilde{F}}) = 0$. By construction, we may write

$$\begin{aligned}
Z \sim \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_3 e_4 + \sum_{\substack{i \in \{1,2,3,4\}\\ j \in \{1,2\}}} \alpha_{ij} \ell_{i(j)}\\
+ \alpha_{123}\ell_{123} + \alpha_{124}\ell_{124} + \alpha_{134}\ell_{134} + \alpha_{234}\ell_{234}.
\end{aligned}$$

Since there is no G-fixed point in $\mathbb{P}^3$ on either side of the link, one of α_{123}, α_{124}, α_{134}, or α_{234} is greater than 1. Without loss of generality we assume that $\alpha_{123} \geq 1$; by convexity of volume we get the inequality

$$S(W^{\widetilde{F}}_{\bullet,\bullet};Z) = \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - vZ)\,dv\,du \leq \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - v\ell_{123})\,dv\,du,$$

so it is enough to show that the last integral is less than 1 to conclude.

We now assume $Z \sim \ell_{123}$, and denote by $P(u,v)$ and $N(u,v)$ the positive and negative parts of the Zariski decomposition of $(\omega^*\pi^*(-K_X) - u\widetilde{F})|_{\widetilde{F}} - vZ$. Then

- if $u \in [0,1]$ then for $0 \leq v \leq u$, $N(u,v) = v(e_1+e_2+e_3)$ so that $P(u,v)^2 = 2(u-v)^2$.
- if $u \in [1,2]$ then

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u-1, \\ (-u+v+1)(e_1+e_2+e_3) & \text{for } u-1 \leq v \leq 1, \\ (-u+v+1)(e_1+e_2+e_3) + (v-1)(\ell_{1,4}+\ell_{2,4}) & \text{for } 1 \leq v \leq \frac{u+1}{2}. \end{cases}$$

- if $u \in [2,3]$ then

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 1, \\ (v-1)(\ell_{1,4}+\ell_{2,4}) & \text{for } 1 \leq v \leq u-1, \\ (-u+v+1)(e_1+e_2+e_3) + (v-1)(\ell_{1,4}+\ell_{2,4}) & \text{for } u-1 \leq v \leq \frac{u+1}{2}. \end{cases}$$

We have

$$\begin{aligned}
&S(W^{\widetilde{F}}_{\bullet,\bullet};Z) \\
&\leq \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - v\ell_{123})\,dv\,du \\
&= \frac{3}{22}\bigg(\int_0^1\int_0^u 2(u-v)^2\,dv\,du + \int_1^2\bigg(\int_0^{u-1}(-2u^2+2uv-v^2+8u-6v-4)\,dv \\
&\qquad + \int_{u-1}^1 (u^2-4uv+2v^2+2u-1)\,dv + \int_1^{\frac{u+1}{2}}(1+u-2v)^2\,dv\bigg)\,du \\
&\qquad + \int_2^3\bigg(\int_0^1(-2u^2+2uv-v^2+8u-6v-4)\,dv \\
&\qquad\quad + \int_1^{u-1}(-2u^2+2uv+v^2+8u-10v-2)\,dv + \int_{u-1}^{\frac{u+1}{2}}(1+u-2v)^2\,dv\bigg)\,du\bigg) \\
&= \frac{29}{44} < 1.
\end{aligned}$$

We see that $S_X(\widetilde{F}) < 2$ and $S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) < 1$ thus

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min\left\{\frac{2}{S_X(\widetilde{F})}, \frac{1}{S(W^{\widetilde{F}}_{\bullet,\bullet}; Z)}\right\} > 1,$$

and $\beta(\Xi) = A_X(\Xi) - S_X(\Xi) > 0$. □

Finally, we exclude the case where $Z \subset \Gamma_1$.

Lemma 3.8. *If the center $Z = c_{X_1}(\Xi)$ is one-dimensional and is contained in E_1, $\beta(\Xi) > 0$.*

Proof. Assume that $Z \subset E_1$, then since there is no G-fixed point on $\mathbb{P}^3$, $\phi_1(Z) = c_{\mathbb{P}^3}(\Xi)$ is the curve Γ_1. Denote by $Y_1 \to \mathbb{P}^3$ the blowup of the line L_1 and by $Y_1 \to \mathbb{P}^1$ the morphism induced by the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ away from L_1. Let $\widehat{X}^+ \to Y_1$ be the blowup of the proper transform of Γ_1, then $\widehat{X}^+ \dashrightarrow \widehat{X}$ is a flop, and there is a morphism $\widetilde{X} \to \widehat{X}$. Denote by η the composition $\widetilde{X} \to \widehat{X} \to Y_1 \to \mathbb{P}^1$ and by $\widetilde{Z}$ the centre $c_{\widetilde{X}}(\Xi)$. If T is a general fiber of η, $T \cdot \widetilde{Z} \geq 5$, hence, by Lemma 2.6, $\beta(\Xi) > 0$. □

Lemma 3.9. *There is no G-invariant prime divisor Ξ over X with centre a prime divisor $D_X = c_X(\Xi)$ such that $\beta(\Xi) \leq 0$.*

Proof. By [2, Corollary 1.44], for any divisor Ξ over X, if $\alpha_{G,Z}(X) > 3/4$, where $Z = c_X(\Xi)$, then $\beta(\Xi) > 0$. Assume now that there is a divisor Ξ over X with $\beta(\Xi) \leq 0$ and $c_X(\Xi) = D_X$ a divisor, so that $\alpha_{G,D_X}(X) \leq 3/4$. First assume that $\alpha_{G,D_X}(X) < 3/4$, then D_X is the G-orbit of a minimal log canonical centre of a suitable pair $\left(X, \frac{3}{4}\mathcal{D}\right)$ for $\mathcal{D} \subset |-K_X|_{\mathbb{Q}}$ a G-invariant linear system. By [2, Theorem 1.52], D_X is a G-invariant irreducible normal surface with

$$-K_X \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$$

for Δ_X an effective $\mathbb{Q}$-divisor and a rational number $\lambda > 4/3$.

We show that there is no such divisor D_X. Recall that $\pi\colon X_i \to X$ for $i = 1, 2$ are small $\mathbb{Q}$-factorialisations so that

$$-K_{X_i} \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$$

where we still denote by D_X, Δ_X the pullbacks of these divisors to X_i. We have $\overline{\mathrm{Eff}}(X_i) = \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[E_2]$, and $E_2 = 8H_1 - 3E_1$. If $D_X = E_1$, then $\Delta_X \sim 4H_1 - (1+\lambda)E_1$, but this is impossible as $(1+\lambda) > 3/2$. If $D_X \neq E_1$, the image of D_X by f_1 is a G-invariant irreducible surface of degree $d \in \mathbb{N}$ on $\mathbb{P}^3$, and since

$$f_1(4H_1) \sim \lambda f_1(D_X) + f_1(\Delta_X)$$

we have $4 \geq \lambda d$, and since $\lambda > 4/3$, $d \leq 2$. Since there is no G-invariant hyperplane of $\mathbb{P}^3$, $d = 2$ and $f_1(D_X)$ is a G-invariant quadric. Since Γ_1 doesn't lie on a quadric, $D_X \sim 2H_1$ and

$$\Delta_X \sim (4 - 2\lambda)H_1 - E_1 \sim xH_1 - E_1$$

for some $x < 4/3$, which is impossible for an effective divisor.

Now assume that $\alpha_{G,D_X}(X) = 3/4$, then since π_1 is small, $\alpha_{G,D_X}(X_1) = 3/4$ by [2, Lemma 1.47], and $\beta(\Xi) = A_X(D_X) - S_X(D_X) = A_{X_1}(D_X) - S_{X_1}(D_X)$. Since X_1 is smooth, as in the proof of [2, Theorem 1.51], assuming that $\beta(\Xi) = 0$ would imply $X_1 \simeq \mathbb{P}^3$, a contradiction. We conclude that $\beta(\Xi) > 0$ for all G-invariant prime divisors Ξ with $c_X(\Xi) = D_X$ a prime divisor on X. □

We now have all the elements to prove

Theorem 3.10 (Main Theorem (I)). *The threefold X is K-polystable.*

Proof. Assume that X is not K-polystable, then there is a G-invariant prime divisor Ξ over X such that $\beta(\Xi) < 0$. Lemma 3.9 shows that the centre of Ξ on X is not a surface. If the centre of Ξ on $\mathbb{P}^3$ is a curve other than Γ_1, by Lemma 2.5, this curve is a line. Lemma 3.6 shows that this line cannot be a G-invariant line that is not the unique quadrisecant of Γ_1, while Lemma 3.7 excludes the quadrisecant line L_1. Lemma 3.8 shows that the centre of Ξ on $\mathbb{P}^3$ is not Γ_1. As there is no G-fixed point on $\mathbb{P}^3$, if $c_X(\Xi)$ is 0-dimensional, it is the singular point $x_0 \in X$, and its centre on $\mathbb{P}^3$ is L_1, so that this case is also excluded by Lemma 3.7. □

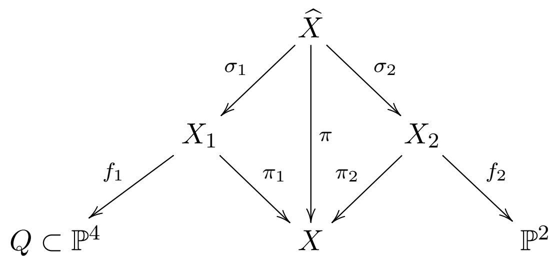
Since $\mathrm{Aut}(X)$ is finite, X is K-stable, and by openness of K-stability [4], this implies

Corollary 3.11. *A general one-nodal prime Fano threefold of genus* 12 *in Family* (I) *is K-stable.*

Remark 3.12. Liu and Zhao have constructed a K-semistable degeneration of one-nodal prime Fano threefolds in Family (I), in which the curve Γ_1 is taken to lie on a quadric (this corresponds to $C_{a,b}$ with $|a| = |b|$ above). The resulting prime Fano threefold of genus 12 has (non-isolated) canonical singularities [9].

4. Family (II)

Let X be a one-nodal prime Fano threefold of genus 12 that belongs to Family (II) of Theorem 1.2, then X is the midpoint of a Sarkisov link associated to a rational map $Q \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$; we describe the associated birational geometry briefly, see [6, 14] and [8] for proofs and precise statements.



Denote by $H_1 = \sigma_1^*\big(f_1^*\mathcal{O}_Q(1)\big)$, by $H_2 = \sigma_2^*\big(f_2^*\mathcal{O}_{\mathbb{P}^2}(1)\big)$, and by $H = \pi^*(-K_X)$ the pullbacks to $\widehat{X}$ (or to any of the models) of the ample generators of $\mathrm{Pic}(Q)$, $\mathrm{Pic}(\mathbb{P}^2)$ and $\mathrm{Pic}(X)$ respectively. The morphism f_1 is the blowup of a smooth rational quintic curve $\Gamma_1 \subset Q \subset \mathbb{P}^4$ that does not lie on a hyperplane section of Q ($|H_1 - \Gamma_1| = \emptyset$), and there is a unique trisecant line L_1 to Γ_1. The curve Γ_1 lies on a section of $|2H_1|$ on Q, that is on a del Pezzo surface of degree 4, and the linear system $|2H_1 - \Gamma_1|$ has dimension 3 and $\mathrm{Bs}\,|2H_1 - \Gamma_1| = \Gamma_1 \cup L_1$. The rational map associated to $|2H_1 - \Gamma_1| = |H_2|$ is precisely the $Q \dashrightarrow \mathbb{P}^2$ induced by the Sarkisov link above. The threefold X_1 is weak Fano,

$$-K_{X_1} \sim H \sim 3H_1 - E_1$$

where $E_1 = \mathrm{Exc}\, f_1$, so that the proper transform of L_1 (still denoted L_1) is the unique flopping curve on X_1. The map π_1 contracts L_1 to a node $\{x_0\} = \mathrm{Sing}(X) \in X$.

Let $\pi\colon \widehat{X} \to X$ be the blowup of x_0, and σ_1 the induced map to X_1, note that $L_1 = \sigma_1(F)$, where $F = \mathrm{Exc}\,\pi$. The threefolds X_1 and X_2 are the two small resolutions of the node x_0 and χ is the induced birational map between these (the Atiyah flop associated to $x_0 \in X$). Then, $\widehat{X}$ is a weak Fano threefold ($-K_{\widehat{X}}$ is nef and big) with $\rho = 3$ and we have [8]:

$$-K_{\widehat{X}} \sim H - F \sim H_1 + H_2$$

and from $H \sim 3H_1 - E_1 \sim H_1 + H_2$, we deduce

$$H_2 \sim 2H_1 - E_1.$$

The map f_2 is a conic bundle (a Mori fibre space with one-dimensional fibres) and its discriminant curve $\delta = -f_{2*}(K_{X_2/\mathbb{P}^2})^2 = -f_{2*}(H - 3H_2)^2$ has degree 12.

Denoting by $\mathcal{E} = \phi_{2*}H$, then $X_2 \subset \mathbb{P}(\mathcal{E})$ is a section of $2H - 3H_2$, where, abusing notation, we denote by H the tautological class of $\mathbb{P}(\mathcal{E})$ and by H_2 the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$).

Since $H_2 \cdot L_1 = 1$, L_1 maps to a line $\ell \subset \mathbb{P}^2$. Let $R = f_2^{-1}\ell$ be its preimage on X_2, and by abuse of notation, also denote by $R = \sigma_2^*(f_2^{-1}\ell)$ its proper transform on $\widehat{X}$. By construction, R is the unique section of $|2H_1 - E_1 - 2F| = |H_2 - F|$.

4.1. Construction of a member with $\mathbb{Z}_2 \rtimes \mathbb{Z}_3$-action

Let $Q \subset \mathbb{P}^4$ be the smooth quadric threefold

$$Q = \{2x_2^2 = x_1x_3 - x_0x_4\}$$

and let Γ_1 be the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$ given by

$$[x : y] \to [x^5 : 2x^3y^2 + y^5 : x^4y + xy^4 : 2x^2y^3 + x^5 : y^5];$$

Γ_1 lies on Q but on no hyperplane section of Q.

Let ω be a primitive cube root of unity, and define an action of $G := \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ on $\mathbb{P}^4$ by the action of its generators:

$$\begin{aligned}\tau &\colon [x_0 : x_1 : x_2 : x_3 : x_4] \to [x_4 : x_3 : x_2 : x_1 : x_0],\\ \sigma &\colon [x_0 : x_1 : x_2 : x_3 : x_4] \to [x_0 : \omega^2x_1 : \omega x_2 : x_3 : \omega^2x_4],\end{aligned}$$

and observe that Γ_1 is G-invariant, and that

$$L_1 = \{x_0 + x_3 = x_4 + x_1 = x_2 = 0\}$$

is G-invariant and trisecant to Γ_1 (the intersection $L_1 \cap \Gamma_1$ consists of the image of the points $[1 : -\omega^i]$ for $i = 0, 1, 2$). The threefolds X, X_1 and X_2 are equipped with a G-action and the Sarkisov link above is G-equivariant. For instance, a G-invariant basis of $|H_2|$ is

$$\begin{cases} S_1 = \{x_0^2 - x_3^2 + 2(x_1 - x_4) = 0\},\\ S_2 = \{x_1^2 - x_4^2 + 2(x_0 - x_3) = 0\},\\ S_3 = \{(x_0 - x_2)^2 + (x_4 - x_2)^2 + x_0x_1 + 2x_0x_4 + x_3x_4 = 2x_2^2 + (x_1 - x_2)^2 + (x_3 - x_2)^2\},\end{cases}$$

and the only section of $|H_2|$ that is singular along L_1 is $f_1(\sigma_1 R)$ (which we still call R by abuse of notation). We have

$$R = \{x_0x_1 + 2x_0x_4 + x_3x_4 = 2x_2^2\}.$$

The discriminant curve of f_2 is the smooth plane cubic

$$\delta = \{2y_0^3 + 6y_0^2y_1 + 5y_0y_1^2 + y_0y_1y_2 + 3y_1^3 + 5y_1^2y_2 + 6y_1y_2^2 + 2y_2^3 = 0\} \subset \mathbb{P}^2.$$

Claim 4.1. The group $\mathrm{Aut}(X)$ is finite.

Proof. Since $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(X_1) = \mathrm{Aut}(Q, \Gamma_1)$, and since Γ_1 does not lie on a hyperplane section of Q, $\mathrm{Aut}(Q, \Gamma_1) = \mathrm{Aut}(X_1)$ is a subgroup of $\mathrm{Aut}(\Gamma_1) = \mathrm{Aut}(\mathbb{P}^1)$ by [15, Lemma 2.1]. Consequently, $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ preserving the three points of intersection $\Gamma_1 \cap L_1$, therefore it is finite. □

The intersection numbers associated to the Sarkisov link are

$$
\begin{array}{llll}
H_1^3 = 2, & H_1^2 \cdot E_1 = 0, & H_1 \cdot E_1^2 = -5, & E_1^3 = -13, \\
& H_1^2 \cdot F = 0, & H_1 \cdot F^2 = -1, & F^3 = 2, \\
& E_1 \cdot F \cdot H_1 = 0, & E_1 \cdot F^2 = -3, & E_1^2 \cdot F = 0, \\
H_2^3 = 0, & H_2^2 \cdot H = 2, & H_2 \cdot H^2 = 12 - \deg \delta. &
\end{array}
$$

We will apply Theorem 2.1 to prove that X is K-stable. To do so, we first describe possible centres of G-invariant divisors over X. In what follows, Ξ always denotes a G-invariant prime divisor over X.

Claim 4.2. If the centre of Ξ on X is 0-dimensional, it is the singular point $c_X(\Xi) = \{x_0\}$.

Proof. There is no point of $Q \subset \mathbb{P}^4$ fixed by the action of G. □

We now consider the case when the centre $Z = c_Q(\Xi)$ on Q is one-dimensional. First, we assume that Z lies on a (smooth) section S of the linear system $|H_2| = |2H_1 - E_1|$. As an intersection of two quadrics in $\mathbb{P}^4$, S is a del Pezzo of degree 4, and $p\colon S \to \mathbb{P}^2$ is the blowup of five points $p_1, \dots, p_5$ in general position. Let ℓ be the pullback of a line on $\mathbb{P}^2$, and $e_1, \dots, e_5$ the p-exceptional curves. Then the Mori cone $\overline{\mathrm{NE}}(S)$ is generated by $\ell, e_1, \dots, e_5, \ell_{i,j}$ for $1 \leq i < j \leq 5$ and q where $\ell_{i,j}$ is the proper transform of the line through p_i and p_j and q that of the conic through $p_1, \dots, p_5$. For a smooth curve $C \subset S$, if $C \sim k\ell + \sum m_i e_i$, then

$$\deg C = -K_S \cdot C = H_1 \cdot C = 3k - \sum m_i \quad \text{and} \quad p_a(C) = \frac{(k-1)(k-2)}{2} - \sum \frac{m_i(m_i - 1)}{2}$$

so that without loss of generality, we may assume that $\Gamma_1 = 2\ell - e_1$ and $L_1 = q$.

Lemma 4.3. *If $Z = c_Q(\Xi)$ is a G-invariant irreducible curve lying on $S \in |H_2|$, and if $Z \not\subset \Gamma_1 \cup L_1$, then $\beta(\Xi) > 0$.*

Proof. We use Theorem 2.3 to bound $\beta(\Xi)$ below. Let $D_u = H - uS$ on $\widehat{X}$ for $u \geq 0$, and write its Zariski decomposition $D_u = P(u) + N(u)$, where for $0 \leq u \leq 3/2$, $P(u)$ is nef and

$$P(u) = H - uS - u\left(\frac{E_1}{3} + F\right) = \left(1 - \frac{2}{3}u\right)(3H_1 - E_1) \quad \text{and} \quad N(u) = u\left(\frac{E_1}{3} + F\right),$$

which gives

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^\tau \mathrm{vol}(\pi^*(-K_X) - uS)\, du = \frac{1}{22} \int_0^{3/2} \frac{22(3-2u)^3}{27}\, du = \frac{3}{8} < 1.$$

Note that since $Z \not\subset (E_1 \cup F)$, $\mathrm{ord}_Z(N(u)|_S) = 0$.

We now consider $(H-uS)|_S - vZ$ on S and denote by $P(u,v)+N(u,v)$ its Zariski decomposition for $0 \leq u \leq 3/2$. We have

$$Z \sim \alpha\ell + \sum_{i=1}^{5} \alpha_i e_i + \sum_{1\leq i<j\leq 5} \alpha_{ij}\ell_{ij} + \beta q.$$

Since $Z \not\subset F$, $Z \neq q$ and at least one of the coefficients α, α_i, $\alpha_{i,j}$ is ≥ 1. If $\mathbf{l}$ is the corresponding curve, since $Z \geq \mathbf{l}$, by convexity of volume:

$$S(W^S_{\bullet,\bullet}; Z) = \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - vZ)\,dv\,du \leq \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - v\mathbf{l})\,dv\,du,$$

so it is enough to show that the last integral is less than 1 when $Z = \mathbf{l}$, for each possible $\mathbf{l}$.

Case 1*: $Z \sim \ell$.* For $0 \leq u \leq 3/2$, and $0 \leq v \leq \frac{3-2u}{3}$, $N(u,v) = vq$, and we compute

$$\begin{aligned} S(W^S_{\bullet,\bullet}; Z) &\leq \frac{3}{22}\int_0^{3/2}\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell)\,dv\,du \\ &= \frac{3}{22}\int_0^{3/2}\int_0^{1-\frac{2u}{3}} \frac{(2u+3v-3)(6u+5v-9)}{3}\,dv\,du = \frac{3}{16} < 1. \end{aligned}$$

Case 2*: $Z \sim e_1$.* For $0 \leq u \leq 3/2$, we have

$$N(u,v) = \begin{cases} vq & \text{for } 0 \leq v \leq \frac{2(3-2u)}{3}, \\ vq + \left(v - 2 + \frac{4u}{3}\right)(\ell_{12}+\ell_{13}+\ell_{14}+\ell_{15}) & \text{for } \frac{2(3-2u)}{3} \leq v \leq \frac{5(3-2u)}{6}. \end{cases}$$

We obtain

$$\begin{aligned} S(W^S_{\bullet,\bullet}; Z) &\leq \frac{3}{22}\int_0^{3/2}\int_0^\infty \mathrm{vol}(P(u)|_S - ve_1)\,dv\,du \\ &= \frac{3}{22}\int_0^{3/2}\Bigg(\int_0^{\frac{2(3-2u)}{3}} \frac{(2u-3)(6u+4v-9)}{3}\,dv \\ &\qquad + \int_{\frac{2(3-2u)}{3}}^{\frac{5(3-2u)}{6}} \frac{5(3-2u)}{6}\frac{(10u+6v-15)^2}{9}\,dv\Bigg)\,du \\ &= \frac{182}{352} < 1. \end{aligned}$$

Case 3*: $Z \sim e_2$ (or e_i, $i \neq 1$).* For $0 \leq u \leq 3/2$, we have

$$N(u,v) = \begin{cases} vq & \text{for } 0 \leq v \leq \frac{3-2u}{3}, \\ vq + \left(v - 1 + \frac{2u}{3}\right)(\ell_{23}+\ell_{24}+\ell_{25}) & \text{for } \frac{3-2u}{3} \leq v \leq \frac{2(3-2u)}{3}. \end{cases}$$

In addition,

$$S(W^S_{\bullet,\bullet}; Z)$$

$$\begin{aligned}
&\leq \frac{3}{22}\int_0^{3/2}\int_0^{\infty}\operatorname{vol}(P(u)|_S - ve_2)\,dv\,du\\
&= \frac{3}{22}\int_0^{3/2}\left(\int_0^{\frac{3-2u}{3}}(2u-3)(2u+2v-3)\,dv + \int_{\frac{3-2u}{3}}^{\frac{2(3-2u)}{3}}\frac{(4u+3v-6)^2}{3}\,dv\right)du\\
&= \frac{63}{176} < 1.
\end{aligned}$$

Case 4: $Z \sim \ell_{12}$ (or ℓ_{1j}). For $0 \leq u \leq 3/2$, we have

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3-2u}{3},\\ \left(v-1+\frac{2u}{3}\right)(\ell_{34}+\ell_{35}+\ell_{45}) & \text{for } \frac{3-2u}{3} \leq v \leq \frac{2(3-2u)}{3}. \end{cases}$$

In addition,

$$\begin{aligned}
S(W^S_{\bullet,\bullet}; Z) &\leq \frac{3}{22}\int_0^{3/2}\int_0^{\infty}\operatorname{vol}(P(u)|_S - v\ell_{12})\,dv\,du\\
&= \frac{3}{22}\int_0^{3/2}\left(\int_0^{\frac{3-2u}{3}} 4u^2 + \frac{8}{3}uv - 12u - 9 - 4v - v^2\,dv\right.\\
&\qquad\left. + \int_{\frac{3-2u}{3}}^{\frac{2(3-2u)}{3}}\frac{2(4u+3v-6)(2u+v-3)}{3}\,dv\right)du\\
&= \frac{75}{176} < 1.
\end{aligned}$$

Case 5: $Z \sim \ell_{23}$ (or ℓ_{ij}, $i \neq 1$). For $0 \leq u \leq 3/2$, we have

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{3-2u}{3},\\ \left(v-1+\frac{2u}{3}\right)\ell_{45} & \text{for } \frac{3-2u}{3} \leq v \leq \frac{2(3-2u)}{3},\\ \left(v-1+\frac{2u}{3}\right)\ell_{45} - \left(v-2+\frac{4u}{3}\right)(\ell_{14}+\ell_{15}) & \text{for } \frac{2(3-2u)}{3} \leq v \leq 3-2u. \end{cases}$$

In addition,

$$\begin{aligned}
S(W^S_{\bullet,\bullet}; Z) &\leq \frac{3}{22}\int_0^{3/2}\int_0^{\infty}\operatorname{vol}(P(u)|_S - v\ell_{23})\,dv\,du\\
&= \frac{3}{22}\int_0^{3/2}\left(\int_0^{\frac{3-2u}{3}} 4u^2 + \frac{4}{3}uv - 12u + 9 - 2v - v^2\,dv\right.\\
&\qquad\left. + \int_{\frac{3-2u}{3}}^{\frac{2(3-2u)}{3}}\frac{2(2u-3)(10u+6v-15)}{3}\,dv + \int_{\frac{2(3-2u)}{3}}^{3-2u} 2(2u+v-3)^2\,dv\right)du\\
&= \frac{111}{176} < 1.
\end{aligned}$$

This finishes the proof, as in all cases we have $\min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet}; Z)}\right\} > 1$. □

Lemma 4.4. *If $Z = c_Q(\Xi)$ is a line other than L_1, $\beta(\Xi) > 0$.*

Proof. Since there is no G-fixed point on Q, $Z \cap \Gamma_1$ is empty or consists of two points. In the second case, $H_2 \cdot Z = 0$, so that Z lies on a section $S \in |H_2|$ and $\beta(\Xi) > 0$ by Lemma 4.3.

We now assume that Z is disjoint from Γ_1 and denote by $S^Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ the general hyperplane section of Q containing Z, and by S its proper transform on $\widehat{X}$. The intersection $S^Q \cap (\Gamma_1 \cup L_1) = \{p_1, \ldots, p_6\}$ consists of six points, and these points are in general position because any line through 3 of the points (respectively conic through 6 of the points) would be contracted by π_1, the anticanonical map of X_1, but the only flopping curve on X_1 is L_1. As S is the blowup of S^Q at $\{p_1, \ldots, p_6\}$, S a del Pezzo surface of degree 2. Denote by ℓ_1, ℓ_2 the pullbacks of the two rulings of $S^Q = \mathbb{P}^1 \times \mathbb{P}^1$, and by $e_1, \ldots, e_6$ the exceptional divisors. The Mori cone $\overline{\mathrm{NE}}(S)$ is generated by $\ell_1, \ell_2, e_1, \ldots, e_6$, and by the classes of

- the proper transforms $\ell_{i(1)}$ and $\ell_{i(2)}$ of rulings through the points p_i for $1 \leq i \leq 6$,
- the proper transforms $\ell_{i,j,k}$ for $1 \leq i < j < k \leq 6$ of irreducible conics through 3 of the blownup points ($\ell_{i,j,k} = \ell_1 + \ell_2 - e_i - e_j - e_k$),
- the proper transforms $\kappa_{j(1)}$ and $\kappa_{j(2)}$ of rational cubic curves though 5 of the p_is (where $\kappa_{j(1)} = 2\ell_1 + \ell_2 - \sum e_i + e_j$) for $1 \leq j \leq 6$,
- and the proper transforms q_j of elliptic quartic curves through $p_1, \ldots, p_6$, which have multiplicity 2 at p_j for $1 \leq j \leq 6$ ($q_j = 2\ell_1 + 2\ell_2 - \sum e_i - e_j$).

The Zariski decomposition of $\pi^*(-K_X) - uS$ writes $P(u) + N(u)$ where $P(u)$ is nef, and for $0 \leq u \leq 1$, $N(u) = uF$. We have

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^\tau \mathrm{vol}(\pi^*(-K_X) - uS)\, du = \frac{1}{22} \int_0^1 (1-u)(u^2 - 17u + 22)\, du = \frac{3}{8} < 1.$$

Since Z is disjoint from Γ_1, without loss of generality we may assume that $Z \sim \ell_1$.

Using the same notation as before, for $0 \leq u \leq 5/7$, we have

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{4-3u}{2}, \\ (2v - 4 + 3u)\kappa_{6(2)} & \text{for } \frac{4-3u}{2} \leq v \leq \frac{5-4u}{2}, \\ (2v - 5 + 4u)\sum \kappa_{i(2)} + (1-u)\kappa_{6(2)} & \text{for } \frac{5-4u}{2} \leq v \leq \frac{7-5u}{3}. \end{cases}$$

For $5/7 \leq u \leq 1$, we have

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq \frac{4-3u}{2}, \\ (2v - 4 + 3u)\kappa_{6(2)} & \text{for } \frac{4-3u}{2} \leq v \leq \frac{5-4u}{2}, \\ (2v - 5 + 4u)\sum \kappa_{i(2)} + (1-u)\kappa_{6(2)} & \text{for } \frac{5-4u}{2} \leq v \leq \frac{11-9u}{4}. \end{cases}$$

Since $Z \not\subset F$, $\mathrm{ord}_Z(N(u)|_S) = 0$ and

$$\begin{aligned}
S(W^S_{\bullet,\bullet}; Z) &= \frac{3}{22}\int_0^1\int_0^\infty \mathrm{vol}(P(u)|_S - vZ)\,dv\,du \\
&= \frac{3}{22}\int_0^{5/7}\Bigg(\int_0^{\frac{4-2u}{2}} u^2 + 2uv - 12u - 6v + 13\,dv \\
&\qquad + \int_{\frac{4-3u}{2}}^{\frac{5-4u}{2}} 10u^2 + 14uv - 36u + 4v^2 - 22v + 29\,dv \\
&\qquad + \int_{\frac{5-4u}{2}}^{\frac{7-5u}{3}} 2(5u + 3v - 7)(9u + 4v - 11)\,dv\Bigg)\,du \\
&\quad + \frac{3}{22}\int_{5/7}^1\Bigg(\int_0^{\frac{4-2u}{2}} u^2 + 2uv - 12u - 6v + 13\,dv \\
&\qquad + \int_{\frac{4-3u}{2}}^{\frac{5-4u}{2}} 10u^2 + 14uv - 36u + 4v^2 - 22v + 29\,dv \\
&\qquad + \int_{\frac{5-4u}{2}}^{\frac{11-9u}{4}} 2(5u + 3v - 7)(9u + 4v - 11)\,dv\Bigg)\,du \\
&= \frac{18969}{1108811} < 1.
\end{aligned}$$

As above, this completes proof that $\beta(\Xi) > 0$. □

Lemma 4.5. *If $c_Q(\Xi) = L_1$, then $\beta(\Xi) > 0$.*

Proof. By [8], there are precisely 3 lines through $x_0 \in X$, and by construction, the set of lines through $x_0 \in X$ is G-invariant. If $L \ni x_0$ is a line and $\widehat{L}$ is its proper transform on $\widehat{X}$, $-K_{\widehat{X}} \cdot \widehat{L} = 0$ and $\widehat{L}$ is a flopping curve. Let $\omega\colon \widetilde{X} \to \widehat{X}$ be the blowup of the proper transforms of the 3 lines through $x_0 \in X$, and denote by $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ its (G-invariant) exceptional divisor. Denote by $\widetilde{F} = \omega^*F$, $\widetilde{E}_1 = \omega^*E_1 - \Lambda$ the proper transforms of F and E_1 on $\widetilde{X}$, and by $\widetilde{R} = \omega^*R - \Lambda$, the proper transform of the unique section of $|2H_1 - E_1 - 2F| = |H_2 - F|$. On $\widetilde{X}$, we have the intersection numbers:

$$\begin{array}{llll}
\Lambda^3 = 6, & \Lambda^2\cdot\omega^*F = -3, & \Lambda\cdot\omega^*(F)^2 = 0, & \Lambda\cdot\omega^*(F)\cdot\omega^*(E_1) = 0, \\
\Lambda^2\cdot\omega^*(E_1) = 3, & \Lambda\cdot\omega^*(E_1)^2 = 0, & \omega^*F^3 = 2, & \omega^*E_1^3 = -13.
\end{array}$$

We first show that $\beta(F) > 0$. The Zariski decomposition of $\omega^*\pi^*(-K_X) - u\widetilde{F}$ can be written $P(u) + N(u)$, where $P(u)$ is nef and

$$N(u) = \begin{cases} 0 & \text{for } 0 \le u \le 1, \\ (u-1)\Lambda & \text{for } 1 \le u \le 2, \\ (u-1)\Lambda + (u-2)\widetilde{R} & \text{for } 2 \le u \le 3. \end{cases}$$

We have $A_X(\widetilde{F}) = 2$ and

$$\begin{aligned} S_X(\widetilde{F}) &= \frac{1}{(-K_X)^3}\int_0^\tau \operatorname{vol}(\omega^*\pi^*(-K_X) - u\widetilde{F})\,du \\ &= \frac{1}{22}\left(\int_0^1 22 - 2u^3\,du + \int_1^2 (u+1)(u^2-10u+19)\,du + \int_2^3 3(u-3)(2u-7)\,du\right) \\ &= \frac{161}{88}. \end{aligned}$$

So that $\beta(F) = 15/88 > 0$.

Now assume that Ξ is not F and denote by Z the centre of Ξ on $\widetilde{X}$. By construction, $Z = c_{\widetilde{X}}(\Xi) \subset \widetilde{F}$ is a curve, and $\widetilde{F}$ is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in three points in general position, so it is a del Pezzo surface of degree 5. We denote by ℓ_1, ℓ_2 the proper transforms of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, and by e_1, e_2, e_3 the (-1)-curves. The extremal rays of the Mori cone $\overline{\mathrm{NE}}(\widetilde{F})$ are the (-1)-curves e_1, e_2, e_3, the proper transforms $\ell_{i(1)}$ and $\ell_{i(2)}$ of rulings through the blownup points for $1 \le i \le 3$, and the proper transform of the conic through the three blownup points $\ell_{123} = \ell_1 + \ell_2 - e_1 - e_2 - e_3$.

We will estimate $\beta(\Xi)$ by considering the flag $Z \subset \widetilde{F} \subset \widetilde{X}$; we write

$$\begin{aligned} S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) = {}& \frac{3}{(-K_X)^3}\int_0^3 (P(u)^2 \cdot \widetilde{F}) \cdot \operatorname{ord}_Z(N(u)|_{\widetilde{F}})\,du \\ & + \frac{3}{(-K_X)^3}\int_0^3\int_0^\infty \operatorname{vol}(P(u)|_{\widetilde{F}} - vZ)\,dv\,du. \end{aligned}$$

Since $c_Q(\Xi)$ is one-dimensional, $Z \not\subset \Lambda|_{\widetilde{F}}$, and $\operatorname{ord}_Z(N(u)|_{\widetilde{F}}) = 0$ unless $Z = \widetilde{R}|_{\widetilde{F}}$.

We first assume that $Z \neq \widetilde{R}|_{\widetilde{F}}$. There are positive integers α_i, α_{ij} and α_{123} so that

$$Z \sim \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \sum \alpha_{ij}\ell_{i(j)} + \alpha_{123}\ell_{123}.$$

Since $Z \not\subset \Lambda|_{\widetilde{F}}$, α_{ij} and α_{123} are not all simultaneously 0. Let $\mathbf{l}$ denote one of the (-1) curves other than e_1, e_2, e_3 such that $Z \ge \mathbf{l}$, then by convexity of volume:

$$S(W^S_{\bullet,\bullet}; Z) = \frac{3}{22}\int_0^3\int_0^\infty \operatorname{vol}(P(u)|_{\widetilde{F}} - vZ)\,dv\,du \le \frac{3}{22}\int_0^3\int_0^\infty \operatorname{vol}(P(u)|_{\widetilde{F}} - v\mathbf{l})\,dv\,du,$$

so it is enough to show that the last integral is less than 1 when $Z = \mathbf{l}$.

Case 1. Assume that $Z \sim \ell_{123}$, and let $P(u,v)$ and $N(u,v)$ be the positive and negative parts of the Zariski decomposition of $(\omega^*\pi^*(-K_X) - u\widetilde{F})|_{\widetilde{F}} - vZ$. Then, for $0 \le u \le 1$, $N(u,v) = v(e_1 + e_2 + e_3)$ for $0 \le v \le u$; for $1 \le u \le 2$,

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \le v \le u-1, \\ (v-u+1)(e_1+e_2+e_3) & \text{for } u-1 \le v \le u, \end{cases}$$

and for $2 \le u \le 3$,

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \le v \le 1, \\ (v-1)(e_1+e_2+e_3) & \text{for } 1 \le v \le 4-u. \end{cases}$$

Putting things together, we get

$$\begin{aligned} & S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) \\ & \le \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - v\ell_{123})\,dv\,du \\ & = \frac{3}{22}\bigg(\int_0^1\int_0^u 2(u-v)^2\,dv\,du \\ & \quad + \int_1^2\bigg(\int_0^{u-1} -u^2+2uv+6u-v^2-6v-3\,dv + \int_{u-1}^u 2(u-v)^2\,dv\bigg)\,du \\ & \quad + \int_2^3\bigg(\int_0^1 2uv-v^2-4u-6v+13\,dv + \int_1^{4-u} 2(v-2)(v+u-4)\,dv\bigg)\,du\bigg) \\ & = \frac{29}{44} < 1, \end{aligned}$$

and $\beta(\Xi) > 0$.

Case 2. Now assume that $Z \sim \ell_{1(2)}$ (or any $\ell_{i(j)}$). The positive and negative parts of the Zariski decomposition of $(\omega^*\pi^*(-K_X) - u\widetilde{F})|_{\widetilde{F}} - vZ$ are as follows.

For $0 \le u \le 1$, $N(u,v) = ve_1$ for $0 \le v \le u$; for $1 \le u \le 2$,

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \le v \le u-1, \\ (v-u+1)e_1 & \text{for } u-1 \le v \le 1, \\ (v-u+1)e_1 + (v-1)(\ell_{2(1)}+\ell_{3(1)}) & \text{for } 1 \le v \le u. \end{cases}$$

In addition for $2 \le u \le 3$,

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \le v \le 1, \\ (v-3+u)(\ell_{2(1)}+\ell_{3(1)}) & \text{for } 3-u \le v \le 1, \\ (v-3+u)(\ell_{2(1)}+\ell_{3(1)}) + (v-1)e_1 & \text{for } 1 \le v \le 4-u. \end{cases}$$

We have

$$\begin{aligned} & S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) \\ & \le \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - v\ell_{1(2)})\,dv\,du \\ & = \frac{3}{22}\bigg(\int_0^1\int_0^u 2u(u-v)\,dv\,du \end{aligned}$$

$$
\begin{aligned}
&+\int_1^2\bigg(\int_0^{u-1} -u^2-v^2+6u-2v-3\,dv+\int_{u-1}^1 -2uv+4u-2\,dv\\
&\quad+\int_1^u 2(2-v)(u-v)\,dv\bigg)\,du\\
&+\int_2^3\bigg(\int_0^{3-u} -v^2-4u-2v+13\,dv+\int_{3-u}^1 2u^2+4uv+v^2-16u-14v+31\,dv\\
&\quad+\int_1^{4-u} 2(u+v-4)^2\,dv\bigg)\,du\bigg)\\
&=\frac{59}{88}<1.
\end{aligned}
$$

This finishes the proof that $\beta(\Xi)>0$ when $Z\neq\widetilde{R}|_{\widetilde{F}}$.

Assume that $Z=\widetilde{R}|_{\widetilde{F}}$, so that $\mathrm{ord}_Z(N(u)|_{\widetilde{F}})=1$ when $2\leq u\leq 3$. We have

$$
\frac{3}{(-K_X)^3}\int_2^3 (P(u)^2\cdot\widetilde{F})\cdot\mathrm{ord}_Z(N(u)|_{\widetilde{F}})\,du=\frac{9}{22}.
$$

As before, denote by $P(u,v)$ and $N(u,v)$ the positive and negative parts of the Zariski decomposition of $\omega^*\pi^*(-K_X-u\widetilde{F})|_{\widetilde{F}}-vZ$. When $0\leq u\leq 1$, $N(u,v)=v(e_1+e_2+e_3)$ for $0\leq v\leq u/2$, when $1\leq u\leq 2$,

$$
N(u,v)=\begin{cases}0 & \text{for } 0\leq v\leq u-1,\\ (v-u+1)(e_1+e_2+e_3) & \text{for } u-1\leq v\leq u/2,\end{cases}
$$

and finally, when $2\leq u\leq 3$,

$$
N(u,v)=\begin{cases}0 & \text{for } 0\leq v\leq 3-u,\\ (v-3+u)(\ell_{1(1)}+\ell_{2(1)}+\ell_{3(1)}) & \text{for } 3-u\leq v\leq 2-u/2.\end{cases}
$$

We have

$$
\begin{aligned}
&S(W^{\widetilde{F}}_{\bullet,\bullet};Z)\\
&=\frac{9}{22}+\frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}}-vZ)\,dv\,du\\
&=\frac{9}{22}+\frac{3}{22}\bigg(\int_0^1\int_0^{u/2} 2(u-v)(u-2v)\,dv\,du\\
&\quad+\int_1^2\bigg(\int_0^{u-1} -u^2+v^2+6u-6v-3\,dv+\int_{u-1}^{u/2} 2(u-v)(u-2v)\,dv\bigg)\,du\\
&\quad+\int_2^3\bigg(\int_0^{3-u} 2uv+v^2-4u-10v+13\,dv+\int_{3-u}^{2-u/2}(u+2v-4)(3u+2v-10)\,dv\bigg)\,du\bigg)\\
&=\frac{3}{4}<1.
\end{aligned}
$$

We see that $S_X(\widetilde{F})<2$ and $S(W^{\widetilde{F}}_{\bullet,\bullet};Z)<1$, so that $\beta(\Xi)>0$. □

Now we need to consider G-invariant prime divisors Ξ whose centre on Q lies on Γ_1.

Lemma 4.6. *If $Z = c_{\widetilde{X}}(\Xi) \subset E_1$, then $\beta(\Xi) > 0$.*

Proof. Assume that $Z \subset E_1$, then since there is no G-fixed point on $Q \subset \mathbb{P}^4$, $f_1(Z) = c_Q(\Xi)$ is the curve Γ_1. Denote by $Q_1 \to Q$ the blowup of the line L_1 and by $Q_1 \to \mathbb{P}^2$ the morphism induced by the projection $Q \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ away from L_1. Let $\widehat{X}^+ \to Q_1$ be the blowup of the proper transform of Γ_1, then $\widehat{X}^+ \dashrightarrow \widehat{X}$ is a flop, and there is a morphism $\widetilde{X} \to \widehat{X}$. Denote by η the composition $\widetilde{X} \to \widehat{X} \to Q_1 \to \mathbb{P}^2$.

If T is a general fiber of η, $T \cdot \widetilde{Z} \geq 5$, hence, by Lemma 2.7, $\beta(\Xi) > 0$. □

Lemma 4.7. *If Ξ is a G-invariant prime divisor over X with centre a prime divisor $D_X = c_X(\Xi)$ such that $\beta(\Xi) < 0$, then $D_X \in |H_2|$.*

Proof. The centre $c_X(\Xi) = D_X$ is the G-orbit of a minimal log canonical centre of a suitable pair $\left(X, \frac{3}{4}\mathcal{D}\right)$ for $\mathcal{D} \subset |-K_X|_{\mathbb{Q}}$ a G-invariant linear system, so that D_X is a G-invariant irreducible normal surface with

$$-K_X \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$$

for some effective $\mathbb{Q}$-divisor Δ_X and rational number $\lambda > 4/3$ (see proof of [2, Theorem 1.52]). We show that then, D_X is linearly equivalent to H_2 (here since $X_1 \to X$ is a small map, we also denote $c_{X_1}(\Xi)$ by D_X).

Recall that $\overline{\mathrm{Eff}}(X_1) = \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[H_2]$, and $H_2 \sim 2H_1 - E_1$. If $D_X = E_1$, then

$$\Delta \sim 3H_1 - (1+\lambda)E_1 \sim \frac{3}{2}(2H_1 - E_1) + \left(\frac{3}{2} - (1+\lambda)\right) E_1$$

and since $\lambda > 1/2$, this is impossible.

Now assume that $D_X \neq E_1$, so that $f_1(D_X)$ is a G-invariant surface on Q, and let d be its degree. Since

$$3H_1 \sim \lambda f_1(D_X) + f_1(\Delta_X),$$

$3 \geq \lambda d$ and $d = 1$ or $d = 2$. As there is no G-invariant hyperplane section, $d = 2$ and

$$\Delta \sim (3 - 2\lambda)H_1 + (\lambda m_1 - 1)E_1$$

where m_1 is the multiplicity of $f_1(D_X)$ along Γ_1. Since

$$\Delta_X \sim \frac{3-2\lambda}{2}(2H_1 - E_1) + \left(\frac{3-2\lambda}{2} + \lambda m_1 - 1\right) E_1,$$

we see that $m_1 \geq 1$ and $D_X \in |H_2|$. □

Lemma 4.8. *Let $Z = c_{\widetilde{X}}(\Xi)$ be an irreducible curve that is not contained in E_1. Then, $\beta(\Xi) > 0$ unless $c_Q(\Xi)$ is a line.*

Proof. By Lemma 4.7, a G-invariant surface containing Z is either F or the G-invariant element of $|H_2|$. We have seen that for such Z, $\beta(\Xi) > 0$. If $Z \not\subset H_2$, as in the proof of Lemma 4.6, there is a surjective morphism $\widetilde{X} \to \mathbb{P}^2$ and $H_2 \cdot Z \leq 2$. Since L_1 is in the base locus of H_2, this implies that $H_1 \cdot Z \leq 1$. □

Theorem 4.9 (Main Theorem (II))**.** *X is K-polystable.*

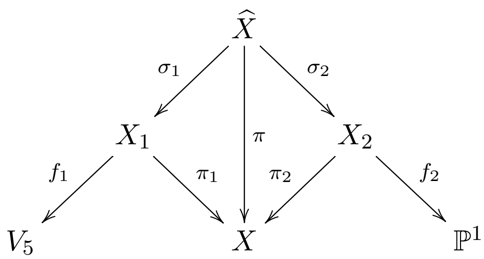
Proof. Assume that X is not K-polystable, and denote by Ξ a G-invariant prime divisor over X with $\beta(\Xi) \leq 0$. If $c_X(\Xi)$ is 0-dimensional, it is $\{x_0\}$, and $c_Q(\Xi) = L_1$, so that $\beta(\Xi) > 0$ by Lemma 4.5. If $c_Q(\Xi)$ is a curve and lies on a section S of $|H_2|$, then $\beta(\Xi) > 0$ by Lemma 4.3. If $c_Q(\Xi)$ is a line, then $\beta(\Xi) > 0$ by Lemma 4.4 and Lemma 4.5. If $c_{\widetilde{X}}(\Xi)$ is a curve lying on E_1, then $\beta(\Xi) > 0$ by Lemma 4.6, and if $c_{\widetilde{X}}(\Xi)$ is a curve not lying on E_1 and such that $c_Q(\Xi)$ is not a line, then $\beta(\Xi) > 0$ by Lemma 4.8. This exhausts the cases where $c_X(\Xi)$ is 1-dimensional. Assume now that $c_X(\Xi)$ is a prime divisor. Then, by Lemma 4.7, $\beta(\Xi) > 0$ unless $c_X(\Xi) \in |H_2|$. We have seen that $\beta(S) > 0$ for $S \in |H_2|$ in the proof of Lemma 4.3, and this concludes the proof. □

As in the case of Family (I), since $\mathrm{Aut}(X)$ is finite, X is K-stable and this implies by openness of K-stability [4]:

Corollary 4.10. *A general one-nodal prime Fano threefold of genus* 12 *in Family* (II) *is K-stable.*

5. Family (III)

Let X be a one-nodal prime Fano threefold of genus 12 that belongs to Family (III) of Theorem 1.2 is the midpoint of a Sarkisov link associated to a rational map $V_5 \dashrightarrow \mathbb{P}^1$; we describe the associated birational geometry briefly, see [6, 8, 14] for precise statements.



Denote by $H_1 = \sigma_1^*\big(f_1^*\mathcal{O}_{V_5}(1)\big)$ and $H_2 = \sigma_2^*\big(f_2^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$, and by $H = \pi^*(-K_X)$ the pullbacks to $\widehat{X}$ (or to any of the models) of the ample generators of $\mathrm{Pic}(V_5)$, $\mathrm{Pic}(\mathbb{P}^1)$ and $\mathrm{Pic}(X)$ respectively. The morphism f_1 is the blowup of a smooth rational quartic curve $\Gamma_1 \subset V_5 \subset \mathbb{P}^6$, and there is a unique bisecant line L_1 to Γ_1. The linear system $|H_1 - \Gamma_1|$ has dimension 2, $\mathrm{Bs}\,|H_1 - \Gamma_1| = \Gamma_1 \cup L$, and the rational map associated to $|H_1 - \Gamma_1| = |H_2|$

is precisely $V_5 \dashrightarrow \mathbb{P}^1$ induced by the Sarkisov link above. The threefold X_1 is weak Fano, and

$$-K_{X_1} \sim H = 2H_1 - E_1$$

where $E_1 = \operatorname{Exc} f_1$, so that the proper transform of L_1 (still denoted L_1) is the unique flopping curve on X_1. The map π_1 contracts L_1 to a node $\{x_0\} = \operatorname{Sing}(X) \in X$. Let $\pi\colon \widehat{X} \to X$ be the blowup of x_0, and σ_1 the induced map to X_1; χ is the Atiyah flop associated to $x_0 \in X$ and $L_1 = \sigma_1(F)$, where $F = \operatorname{Exc}\pi$. Then, $\widehat{X}$ is a weak Fano threefold of $\rho = 3$ and we have [8]:

$$-K_{\widehat{X}} = H - F \sim H_1 + H_2$$

and from $H \sim 2H_1 - E_1 \sim H_1 + H_2$, we deduce

$$H_2 \sim H_1 - E_1.$$

The map f_2 is a del Pezzo fibration (a Mori fibre space with two-dimensional fibres) of degree $H^2 \cdot H_2 = 6$. For later reference, the intersection numbers on $\widehat{X}$ are

$$\begin{aligned} &H_1^3 = 5, && H_1^2 \cdot E_1 = 0, && H_1 \cdot E_1^2 = -4, && E_1^3 = -6,\\ & && H_1^2 \cdot F = 0, && H_1 \cdot F^2 = -1, && F^3 = 2,\\ & && E_1 \cdot F \cdot H_1 = 0, && E_1 \cdot F^2 = -2, && E_1^2 \cdot F = 0. \end{aligned}$$

5.1. Construction of a member with $\mathbb{G}_m \rtimes \mathbb{Z}_2$-action

Recall from [2, Section 5.8] that the quintic threefold $V_5 \subset \mathbb{P}^6$ can be defined scheme theoretically by

$$\begin{cases} x_4x_5 - x_0x_2 + x_1^2 = 0,\\ x_4x_6 - x_1x_3 + x_2^2 = 0,\\ x_4^2 - x_0x_3 + x_1x_2 = 0,\\ x_1x_4 - x_0x_6 - x_2x_5 = 0,\\ x_2x_4 - x_3x_5 - x_1x_6 = 0, \end{cases}$$

and is endowed with an action of $G = \mathbb{G}_m \rtimes \mathbb{Z}_2$ defined by the involution

$$\tau\colon [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_3 : x_2 : x_1 : x_0 : x_4 : x_6 : x_5],$$

and by the automorphisms

$$\lambda_s\colon [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [s^3x_0 : s^5x_1 : s^7x_2 : s^9x_3 : s^6x_4 : s^4x_5 : s^8x_6].$$

Consider the curve $\Gamma_1 \subset V_5$ defined by the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$ given by

$$[x:y] \to [0 : ix^3y : ixy^3 : 0 : -x^2y^2 : -x^4 : -y^4],$$

where $i^2 = -1$, then Γ_1 is a G-invariant rational curve of degree 4. The line $L_1 = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$ is the unique bisecant line to Γ_1 and it is also G-invariant. Note that Γ_1 lies on $\{x_0 = x_3 = 0\} \cap V_5$, and the pencil of hyperplanes containing Γ_1 is the restriction of

$$\mathcal{H} = \left\{H_{[\lambda:\mu]} = \{\lambda x_0 + \mu x_3 = 0\}; [\lambda:\mu] \in \mathbb{P}^1\right\}$$

to V_5. Denote by $S_{[\lambda:\mu]} = H_{[\lambda:\mu]} \cap V_5$, and note that for any hyperplane in the pencil, $L_1 \cup \Gamma_1 \subset S_{[\lambda:\mu]}$. The midpoint X of the Sarkisov link above is endowed with a G-action. Finally, denote by $S = \{x_4 = 0\} \cap V_5$ the only G-invariant hyperplane section of V_5, and observe that S has multiplicity 2 along L, so that $\widetilde{S}_{[\lambda:\mu]} = H_1 - E_1 - F$ and $\widetilde{S} = H_1 - 2F$ are the proper transforms of $S_{[\lambda:\mu]}$ and S on $\widetilde{X}$.

Claim 5.1. The group $\mathrm{Aut}(X) = G$, and in particular, it is reductive.

Proof. Since

$$G \simeq \mathbb{G}_m \rtimes \mathbb{Z}_2 \subset \mathrm{Aut}(X) \simeq \mathrm{Aut}(V_5;\Gamma_1) \subset \mathrm{Aut}(V_5) = \mathrm{PGL}_2(\mathbb{C}),$$

by [13], $\mathrm{Aut}(X) = G$ or $\mathrm{Aut}(X) = \mathrm{Aut}(V_5) = \mathrm{PGL}_2(\mathbb{C})$. The second case is impossible because Γ_1 is not $\mathrm{Aut}(V_5)$-invariant. □

We will apply Theorem 2.1 to prove that X is K-polystable. To do so, we first describe possible centres of G-invariant divisors over X. In what follows, Ξ denotes a G-invariant prime divisor over X.

Claim 5.2. If the centre of Ξ on X is 0-dimensional, it is the singular point $c_X(\Xi) = \{x_0\}$.

Proof. There is no point of $V_5 \subset \mathbb{P}^6$ fixed by the action of G. □

We now consider those G-invariant prime divisors over X which have one-dimensional centre $Z = c_{V_5}(\Xi)$ on V_5. By [2, Corollary 5.39], the G-invariant curves on V_5 are precisely the line L_1, the conic C defined parametrically by $[x:y] \mapsto [x^2 : 0 : 0 : y^2 : xy : 0 : 0]$, the twisted cubic defined parametrically by $[x:y] \mapsto [x^3 : x^2y : xy^2 : y^3 : 0 : 0 : 0]$ and a family of sextic curves C_γ for $\gamma \in \mathbb{C}^*$ in each of the hyperplane sections $\{x_4 = 0\} \cap V_5$ and $\{\lambda x_0 + \mu x_3 = 0\} \cap V_5$.

Lemma 5.3. *Let Ξ be a G-invariant prime divisor with centre $Z = c_{V_5}(\Xi)$ a curve. Then $Z = L_1$, $Z = \Gamma_1$ or $\beta(\Xi) > 0$.*

Proof. Assume to the contrary that $\beta(\Xi) < 0$, then by Lemma 2.2, $Z_2 = c_{X_2}(\Xi)$ is contained in $\mathrm{Nklt}(X_2, B_{X_2})$ for some $B_{X_2} \sim_{\mathbb{Q}} -\lambda K_{X_2}$ and rational number $\lambda < 3/4$. By Lemma 2.6, the degree $H_2 \cdot Z_2 \leq 1$, and we exclude the curves with $H_1 \cdot Z > 1$ by considering $Z_1 = c_{\Xi}(Z)$ and its intersections with Γ_1 and L_1. If Z is a rational sextic curve contained in $\{x_4 = 0\}$ or in $\{\lambda x_0 + \mu x_3 = 0\}$, $\Gamma_1 \cap L_1 = \emptyset$, so $(H_1 - E_1) \cdot Z_1 = H_2 \cdot Z_2$ and $\Gamma_1 \cap Z$ consists of at most 2 points, so $H_2 \cdot Z_2 > 1$. Similarly, if $Z = C$ is the G-invariant conic or twisted cubic, $C \cap \Gamma_1 = C \cap L_1 = \emptyset$ and $H_2 \cdot Z_2 > 1$. The only possibilities for Z are L_1 and Γ_1. □

Lemma 5.4. *Let Ξ be a G-invariant prime divisor with centre $Z = c_{\widetilde{X}}(\Xi)$ a curve lying on $\widetilde{F}$, then $\beta(\Xi) > 0$.*

Proof. Consider the G-invariant blowup $\omega \colon \widetilde{X} \to \widehat{X}$ of the two flopping lines (these are the transforms of the lines through the singular point on X), and denote by $\Lambda = \Lambda_1 + \Lambda_2$ its exceptional divisor G. Let $\widetilde{F} = \omega^* F$, $\widetilde{H}_1 = \omega^* H_1$ and $\widetilde{E}_1 = \omega^* E_1 - \Lambda$ be the proper transforms of F, H_1 and E_1. We also have $\widetilde{S} = \omega^* S - 2\Lambda$ and $\widetilde{S}_{[\lambda:\mu]} = \omega^* S_{\lambda:\mu} - \Lambda$.

If we write the Zariski decomposition of $\omega^*\pi^*(-K_X) - u\widetilde{F} = P(u) + N(u)$, then $P(u)$ is nef for all $0 \leq u \leq 3$ and

$$N(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ (u-1)\Lambda & \text{for } 2 \leq u \leq 3, \\ (u-1)\Lambda + (u-2)\widetilde{S} & \text{for } 2 \leq u \leq 3. \end{cases}$$

We now compute

$$\begin{aligned} S_X(\widetilde{F}) &= \frac{1}{(-K_X)^3} \int_0^\tau \mathrm{vol}(\omega^*\pi^*(-K_X) - u\widetilde{F})\, du \\ &= \frac{1}{22} \left(\int_0^1 22 - 2u^3\, du + \int_1^2 -6u^2 + 6u + 20\, du + \int_2^3 2(6-u)(u-3)^2\, du \right) = \frac{39}{22}. \end{aligned}$$

So that $\beta(\widetilde{F}) = A_X(\widetilde{F}) - S_X(\widetilde{F}) = 2 - \frac{39}{22} = \frac{5}{22} > 0$.

We now assume that $Z = c_{\widetilde{X}}(\Xi) \subset \widetilde{F}$. The surface $\widetilde{F}$ is the blowup of $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ at two distinct points, that is a del Pezzo surface of degree 6. Let ℓ_1 (resp. ℓ_2) be the full transform of the ruling of class $(1,0)$ (resp. $(0,1)$) on $\mathbb{P}^1 \times \mathbb{P}^1$, and let e_1, e_2 be the two exceptional curves. The Mori cone $\overline{\mathrm{NE}}(\widetilde{F})$ is generated by e_1 and e_2, and by the (-1)-curves $\ell_{i(j)} = \ell_j - e_i$.

We have

$$\begin{aligned} S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) &= \frac{3}{(-K_X)^3} \int_0^3 (P(u)^2 \cdot \widetilde{F}) \cdot \mathrm{ord}_Z(N(u)|_{\widetilde{F}})\, du \\ &\quad + \frac{3}{(-K_X)^3} \int_0^3 \int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - vZ)\, dv\, du. \end{aligned}$$

As there is no G-fixed point on V_5, $c_{V_5}(\Xi)$ is not a point and $Z \notin \{e_1, e_2\}$, so that $Z \not\subset \Lambda|_{\widetilde{F}}$. When in addition, $Z \neq \widetilde{S}|_{\widetilde{F}}$, $\mathrm{ord}_Z(N(u)|_{\widetilde{F}}) = 0$. Write

$$Z \sim \alpha_1 e_1 + \alpha_2 e_2 + \sum_{i,j\in\{1,2\}} \alpha_{ij}\ell_{i(j)},$$

and observe that at least one of the coefficients $\alpha_{ij} \neq 0$. By convexity of volume, if the nonzero coefficient corresponds to the curve $\mathbf{l}$, we get

$$S(W^{\widetilde{F}}_{\bullet,\bullet}; Z) = \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - vZ)\,dv\,du \leq \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}} - v\mathbf{l})\,dv\,du,$$

so it is enough to show that the last integral is less than 1 to deduce a contradiction.

Case 1. Assume that $Z \neq \widetilde{S}|_{\widetilde{F}}$, and let $Z \sim \ell_{i(j)}$, for $i, j \in \{1, 2\}$. To fix notation, we consider $\ell_{1(2)}$. Denote by $P(u, v)$ and $N(u, v)$ the positive and negative parts of the Zariski decomposition of $(\omega^*\pi^*(-K_X) - u\widetilde{F})|_{\widetilde{F}} - vZ$.

- For $0 \leq u \leq 1$, $N(u, v) = ve_1$ for $0 \leq v \leq u$.
- For $1 \leq u \leq 2$,

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq u-1,\\ (v-u+1)e_1 & \text{for } u-1 \leq v \leq 1,\\ (v-u+1)e_1 + (v-1)\ell_{2(1)} & \text{for } 1 \leq v \leq u. \end{cases}$$

- For $2 \leq u \leq 3$,

$$N(u,v) = \begin{cases} 0 & \text{for } 0 \leq v \leq 3-u,\\ (v-3+u)e_1 & \text{for } 3-u \leq v \leq 1,\\ (v-3+u)e_1 + (v-1)\ell_{2(1)} & \text{for } 1 \leq v \leq 4-u. \end{cases}$$

We have

$$\begin{aligned} &S(W^{\widetilde{E}_L}_{\bullet,\bullet}; Z)\\ &\leq \frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{E}_L} - v\ell_{1(2)})\,dv\,du\\ &= \frac{3}{22}\bigg(\int_0^1\int_0^u 2u(u-v)\,dv\,du\\ &\quad + \int_1^2\bigg(\int_0^{u-1} -v^2+4u-2v-2\,dv + \int_{u-1}^1 u^2-2uv+2u-1\,dv\\ &\qquad + \int_1^u (u-v+2)(u-v)\,dv\bigg)\,du \end{aligned}$$

$$+\int_2^3\left(\int_0^{3-u} 2u^2+2uv-v^2-16u-6v+30\,dv+\int_{3-u}^1(-3+u)(3u+4v-13)\,dv\right.$$
$$\left.+\int_1^{4-u}(u+v-4)(3u+v-10)\,dv\right)du\Bigg)$$
$$=\frac{17}{22}<1,$$

which is what we wanted.

Case 2. Now assume that $Z=\widetilde{\mathcal{S}}|_{\widetilde{F}}$ so that $\mathrm{ord}_Z(N(u)|_{\widetilde{F}})=1$ on $u\in[2,3]$, and

$$\frac{3}{(-K_X)^3}\int_2^3(P(u)^2\cdot\widetilde{F})\cdot\mathrm{ord}_Z(N(u)|_{\widetilde{F}})\,du=\frac{4}{11}.$$

As before, denote by $P(u,v)$ and $N(u,v)$ the positive and negative part of the Zariski decomposition of $(\omega^*\pi^*(-K_X)-u\widetilde{F})|_{\widetilde{F}}-vZ$, so that

- For $0\le u\le 1$, $N(u,v)=v(e_1+e_2)$ for $0\le v\le u/3$.
- For $1\le u\le 2$,

$$N(u,v)=\begin{cases}0 & \text{for } 0\le v\le \frac{u-1}{2},\\ (v-u+1)(e_1+e_2) & \text{for } \frac{u-1}{2}\le v\le \frac{u}{3}.\end{cases}$$

- For $2\le u\le 3$,

$$N(u,v)=\begin{cases}0 & \text{for } 0\le v\le \frac{3-u}{2},\\ (v-3+u)(e_1+e_2) & \text{for } \frac{3-u}{2}\le v\le 2-\frac{2u}{3}.\end{cases}$$

We now compute

$$S(W^{\widetilde{F}}_{\bullet,\bullet};Z)=\frac{4}{11}+\frac{3}{22}\int_0^3\int_0^\infty \mathrm{vol}(P(u)|_{\widetilde{F}}-vZ)\,dv\,du$$
$$=\frac{4}{11}+\frac{3}{22}\Bigg(\int_0^1\int_0^{u/3}2(u-2v)(u-3v)\,dv\,du$$
$$+\int_1^2\left(\int_0^{(u-1)/2}-2uv+4v^2+4u-8v-2\,dv\right.$$
$$\left.+\int_{(u-1)/2}^{u/3}2(u-2v)(u-3v)\,dv\right)du$$
$$+\int_2^3\left(\int_0^{(3-u)/2}2u^2+6uv+4v^2-16u-24v+30\,dv\right.$$
$$\left.+\int_{(3-u)/2}^{2-2u/3}2(u+2v-4)(2u+3v-6)\,dv\right)du\Bigg)$$
$$=\frac{25}{44}<1,$$

and this finishes the proof since

$$\frac{A_X(\Xi)}{S_X(\Xi)} \geq \min\left\{\frac{2}{S_X(\widetilde{F})}, \frac{1}{S(W^{\widetilde{F}}_{\bullet,\bullet}; Z)}\right\} > 1. \qquad \square$$

Lemma 5.5. *Let Ξ be a G-invariant prime divisor with centre $Z = c_{X_1}(\Xi)$ a curve lying on E_1, then $\beta(\Xi) > 0$.*

Proof. Assume to the contrary that $\beta(\Xi) < 0$, then by Lemma 2.2, $Z_2 = c_{X_2}(\Xi)$ is a one-dimensional component of (X_2, B_{X_2}), where $B_{X_2} \sim -\lambda K_{X_2}$ for some $\lambda < 3/4$. By Lemma 2.5, $H_2 \cdot Z_2 \leq 1$. This is impossible as $Z_1 = c_{X_1}(\Xi)$ cannot be mapped to a point by ϕ_1 because there is no G-invariant point on V_5, and $H_2 \cdot Z_1 \geq H_1 \cdot Z_1 \geq 4$. $\square$

Remark 5.6. For the sake of completion, observe that X_1 itself is divisorially K-polystable. Indeed, for $0 \leq u \leq 1$,

$$-K_{X_1} - uE_1 \sim_{\mathbb{Q}} H - uE_1 \sim_{\mathbb{Q}} 2H_1 - (1+u)E_1$$

is a mobile divisor, that is the pullback of a nef divisor on X_2, and for $u > 1$, this divisor is not effective. We have

$$\begin{aligned}
(-K_{X_1})^3 \cdot S_{X_1}(E_1) = 22 \cdot S_{X_1}(E_1) &= \int_0^1 \operatorname{vol}(2H_1 - (1+u)E_1)\, du \\
&= \int_0^1 \big((1-u)(-K_{X_2} + 2uH_2)\big)^3\, du = \int_0^1 \big((1-u)H + 2uH_2\big)^3\, du \\
&= \int_0^1 (1-u)^2(22+14u)\, du = \frac{17}{2}
\end{aligned}$$

so that $\beta(E_1) > 0$.

Lemma 5.7. *There is no G-invariant irreducible surface D_X such that $-K_X \sim_{\mathbb{Q}} \lambda D_X + \Delta_X$ for some positive rational number $\lambda > 4/3$ and effective $\mathbb{Q}$-divisor Δ.*

Proof. Let D_X be such a surface, and denote by D_1, Δ_1 the proper transforms of D_X and Δ_X on X_1. We have

$$H \sim_{\mathbb{Q}} 2H_1 + E_1 \sim_{\mathbb{Q}} \lambda D_1 + \Delta_1.$$

Recall that the pseudo-effective cone $\overline{\mathrm{Eff}}(X_1)$ is $\mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[H_2]$, where $H_2 \sim_{\mathbb{Q}} H_1 - E_1$. If $D_1 = E_1$, we see that

$$\Delta_1 \sim_{\mathbb{Q}} 2H_2 + (1-\lambda)E_1$$

cannot be an effective divisor. We may now assume that $D_1 \in \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2]$, that is $D_1 = xH_1 - yE_1$ for $x, y \in \mathbb{N}$ and $x \geq y$. Since $\lambda D_1 \leq -K_{X_1}$, $\lambda a \leq 2$, so that $a = 1$ and $b = 0$ or $b = 1$. As D_1 is mapped to a G-invariant surface of V_5, $\phi_1(D_1)$ is the hyperplane section $\{x_4 = 0\} \cap V_5$, and $b = 0$. Now, $\Delta_1 \sim_{\mathbb{Q}} (2-\lambda)H_1 - E_1$, but this cannot be effective as $2 - \lambda < 1$. $\square$

As in the previous two cases, we conclude

Theorem 5.8 (Main Theorem (III))**.** *X is K-polystable.*

This time X is not K-stable as $\mathrm{Aut}(X) = \mathbb{G}_m \rtimes \mathbb{Z}_2$, but using [2, Corollary 1.16] (which still holds in the case of a nodal Fano threefold), we conclude

Corollary 5.9. *A general one-nodal prime Fano threefold of genus* 12 *in Family* (III) *is K-polystable.*

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