



Solution Stability and Well-Posedness for Classes of Parametric Set Optimization Problems

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Received: 4 March 2025 / Accepted: 11 September 2025
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Abstract

This paper investigates the solution stability and well-posedness for a parametric set optimization problem (PSOP), where lower and upper set order relations are induced by an improvement set. We provide new sufficient conditions for the outer-continuity, outer-openness and inner-openness of the solution mapping of (PSOP). By utilizing the property of cone-continuity, we derive sufficient conditions ensuring the Levitin-Polyak well-posedness for (PSOP) and the Hadamard well-posedness for a related parametric implicit set optimization problem (ISOP). Numerical examples are also given to illustrate the main results.

Keywords Set optimization · Improvement set · Solution Stability · Levitin-Polyak well-posedness · Hadamard well-posedness

Mathematics Subject Classification 49J53 · 49K40 · 90C31

1 Introduction

In the past few years, the set-valued optimization problem has become a vibrant and growing branch of applied mathematics due to its practical application in diverse fields, such as engineering, control theory, welfare economies, game theory and many others; for details, see [4, 23, 26] and the references therein. The vector criterion and the set criterion are two well known criteria for finding the optimal solution to set-valued optimization problems. In the vector criterion, optimal solutions of a set-valued optimization problem are defined as the efficient points of the union of all images of the set-valued objective mapping. In set-valued optimization (known also as set optimization), a solution concept is defined by a comparison among all images of the set-valued objective mapping using set order relations. Due to its extensive

Dedicated to Professor Q. H. Ansari on his 65th birthday.

Communicated by Jen-Chih Yao.

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use in practical situations, set optimization though a relatively new direction in the field of optimization has attracted a lot of attention from researchers, and we refer the interested reader to [7, 8, 25, 26, 32, 33, 45, 46] for various important and interesting results.

The stability analysis of solutions is one of the very interesting and meaningful topics in optimization and related fields. In the literatures, we can find a number of papers that studied the stability of solutions for set-valued optimization problems, vector variational inequalities and vector equilibrium problems [2, 19–21, 24, 29, 32–34, 36, 37]. For instance, Khanh and Luc [24] proposed two concepts of superior open limit and inferior open limit and used them to examine out-continuity and inner-continuity of set-valued mapping for parametric variational relation problems. Later, Hung [17] studied the outer-continuity and the outer-openness of solutions for parametric generalized vector mixed quasivariational inequality problems. Recently, under free-disposal set mappings, Li et al. [28] obtained some sufficient conditions of the outer-continuity, outer-openness and inner-openness for a parametric unified quasivariational inequality.

The well-posedness of an optimization problem investigates the behavior of the variables when the value of the corresponding objective function is near to the optimal value. It is closely related to the analysis of solution stability of optimization problems. The study of the well-posedness of optimization problems starts with scalar optimization problems, and it is divided into two main categories: one known as Hadamard well-posedness and the other as Tikhonov well-posedness. Hadamard well-posedness was initiated by Hadamard [13] in 1902 to study a mathematical model related to physical phenomena. It requires the existence and uniqueness of the optimal solution of the problem together with the continuous dependence of the optimal solution set on the data. Meanwhile, Tikhonov well-posedness [41] was proposed in 1966 to find optimal solutions of unconstrained optimization problems. Levitin-Polyak (LP, for short) well-posedness was originally proposed by Levitin and Polyak [27], and it is an extension of Tikhonov well-posedness. In Tikhonov well-posedness, each minimizing sequence is required to stay in the feasible set of the problem. Whereas in LP well-posedness, minimizing sequence can be outside of the feasible set, but the distance between it and the feasible set has to be approximately zero. In the literature, the concept of LP well-posedness has further been extended for variational inequalities [16, 40, 42] and for mathematical programs with equilibrium constraints [1].

The study of well-posedness for set optimization problems was initiated by Zhang et al. [44], where they introduced three kinds of well-posedness and obtained sufficient and necessary conditions that guarantee the well-posedness for set optimization problems involving the lower set less order relations. Later, Long et al. [30] introduced two types of pointwise well-posedness for set optimization problems and investigated the relationships among them and well-posedness in [44]. Vui et al. [43] presented the sufficient and necessary conditions for set optimization problems to be LP well-posed, and established the links between stability and LP well-posedness of the underlying problem. Khoshkhabar-amiranloo [25] generalized the well-posedness in [44] and obtained a characterization in terms of Berge-upper semicontinuity and closedness of an approximate solution map. Recently, Duy [8] presented various notions of LP well-

posedness for set optimization problems and investigated the closeness of efficient solution sets for a set optimization problem under the Berge continuity.

According to our observation of the above literatures [8, 25, 30, 43, 44], we can find that the order relations used in set optimization problems are usually preorders induced by convex cones. However, it is worth noting that in economics [10] we may have to deal with general order relations that are not necessarily preorders. This insight has inspired many researchers to explore more general order relations for the optimality of set optimization problems that are not necessarily preorders (see e.g., [32, 33]). Moreover, many results of the well-posedness of set optimization problems rely on the Berge continuity of the objective function (see, [8, 43]), which may not be easily satisfied.

Consequently, a quite natural and interesting question is: Can we study the well-posedness for a parametric set optimization problem involving general ordering sets regardless of the Berge continuity? Additionally, how do the out-continuity, outer-openness and inner-openness properties of the solution map for a parametric set optimization problem (PSOP) manifest?

The main purpose of this paper is to answer the above questions by investigating the solution stability and well-posedness for a parametric set optimization problem (PSOP), where lower and upper set order relations are induced by an improvement set. More precisely, we provide new sufficient conditions for the outer-continuity, outer-openness and inner-openness of the solution mapping to the problem (PSOP). By using the property of cone-continuity, we derive sufficient conditions ensuring the Levitin-Polyak (*LP*) well-posedness for (PSOP) and the Hadamard well-posedness for a related parametric implicit set optimization problem (ISOP). Numerical examples are also given to illustrate the main results.

The structure of this paper is organized as follows. Sect. 2 recalls some definitions and properties of cones, improvement sets and set-valued mappings needed in the sequel. In Sect. 3, we study the outer-continuity, outer-openness and inner-openness of solutions for (PSOP). In Sect. 4, the closedness of solution sets of a set optimization problem, the *LP* well-posedness of (PSOP) and the Hadamard well-posedness of (ISOP). The last section, Sect. 5, gives a summary of the obtained results.

2 Preliminaries

Throughout this paper, unless stated otherwise, let X, Y, Z and A be real Banach spaces equipped with the norm $\|\cdot\|$. Assume that $C \subseteq Y$ is a closed, convex and pointed (i.e. $C \cap (-C) = \{0\}$) cone with nonempty interior (i.e. $\text{int}C \neq \emptyset$), where 0 denotes the apex of C . Let B_Y represent the open unit ball in Y , and $\mathcal{P}(Y)$ denote the set of all nonempty subsets of Y . For any nonempty subset A in a Banach space, its topological interior is written as $\text{int}A$.

For the above cone C , we have known the lower set less relation, upper set less relation and set less relation (see e.g., [22, 23, 26]) defined respectively by

$$\begin{aligned} A \leq_C^l B &\Leftrightarrow B \subseteq A + C; \\ A \leq_C^u B &\Leftrightarrow A \subseteq B - C; \end{aligned}$$

$$A \leqslant_C^s B \Leftrightarrow B \subseteq A + C, A \subseteq B - C,$$

where $A, B \in \mathcal{P}(Y)$.

Chicco et al. [6] defined the upper comprehensive set of a nonempty set E in Y , denoted by $\text{u-compr}(E)$ as

$$\text{u-compr}(E) := E + C.$$

A nonempty set E in Y is said to be upper comprehensive if $\text{u-compr}(E) = E$. A nonempty upper comprehensive set E in Y is said to be an improvement set if $0 \notin E$. Clearly, $C \setminus \{0\}$ and $\text{int}C$ are improvement sets in Y .

From now on, let $E \subset Y$ be an improvement set satisfying $E \subseteq C \setminus \{0\}$. In [7], for every $A, B \in \mathcal{P}(Y)$, the following lower set less order relations on $\mathcal{P}(Y)$ were defined by

$$\begin{aligned} A \leqslant_E^l B &\Leftrightarrow B \subseteq A + E; \\ A \ll_E^l B &\Leftrightarrow B \subseteq A + \text{int}E, \end{aligned}$$

where $\text{int}E \neq \emptyset$ when referring to the latter relation. In analogy to [7], Mao et al. [32] defined the upper set less order relations via improvement sets as follows:

$$\begin{aligned} A \leqslant_E^u B &\Leftrightarrow A \subseteq B - E; \\ A \ll_E^u B &\Leftrightarrow A \subseteq B - \text{int}E. \end{aligned}$$

Note that, for each $i \in \{u, l\}$, the order relations $\leqslant_E^i$ and $\ll_E^i$ are transitive but not reflexive.

Remark 2.1 Consider a nonempty subset A and the above cone C of Y . The element $a \in A$ is called a minimal point of A with respect to C if $(A - a) \cap (-C) = \{0\}$. Corollary 3.8 from [31] ensures that if A is compact and nonempty, then the set of all minimal points of A , denoted by $\text{Min}(A)$, is nonempty.

Let $F : X \rightrightarrows Y$ be a set-valued mapping. The domain and graph of F are defined respectively as

$$\text{dom}F = \{x \in X : F(x) \neq \emptyset\}, \quad \text{gr}F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Let M be a nonempty subset of X , and assume throughout that $M \subseteq \text{dom}F$. We consider the following set optimization problem (SOP):

$$\min\{F(x) : x \in M\}. \quad (\text{SOP})$$

Definition 2.1 [26] For each $i \in \{u, l\}$, $\bar{x} \in M$ is said to be

- (i) an E - i -minimal solution of (SOP) if $x \in M$ and $F(x) \leqslant_E^i F(\bar{x})$ implies $F(\bar{x}) \leqslant_E^i F(x)$;
- (ii) a weak E - i -minimal solution of (SOP) if $x \in M$ and $F(x) \ll_E^i F(\bar{x})$ implies $F(\bar{x}) \ll_E^i F(x)$.

In what follows, we denote the sets of all E - i -minimal and weak E - i -minimal solutions of (SOP) by $E_i(F, M)$ and $W_i(F, M)$, respectively, and assume that $E_i(F, M) \neq \emptyset$ for each $i \in \{u, l\}$.

Lemma 2.1 [32] Let $x_0 \in M$. If $F(x_0)$ is compact, then $x_0 \in E_i(F, M)$ if and only if there is no $x \in M$ satisfying $F(x) \leq_E^i F(x_0)$ for $i \in \{u, l\}$.

The following result demonstrates that the assertion in Lemma 2.1 also holds for the weak E - i -minimal solutions with $i \in \{u, l\}$.

Proposition 2.2 Let $x_0 \in M$. If $F(x_0)$ is compact, then $x_0 \in W_i(F, M)$ if and only if there is no $x \in M$ satisfying $F(x) \ll_E^i F(x_0)$ for $i \in \{u, l\}$.

Proof We only need to prove for the case of $i = l$ as it can be done similarly for the case $i = u$. Clearly, if there is no $x \in M$ satisfying $F(x) \ll_E^l F(x_0)$, then $x_0 \in W_l(F, M)$ by definition. Now, we prove the “only if” part. Assume on the contrary that there exists $x \in M$ satisfying $F(x) \ll_E^l F(x_0)$. Then $F(x_0) \ll_E^l F(x)$ because $x_0 \in W_l(F, M)$. Using the above two relations we get

$$F(x_0) \subseteq F(x) + \text{int}E \subseteq F(x_0) + \text{int}E + \text{int}E \subseteq F(x_0) + C \setminus \{0\}, \quad (2.1)$$

where we should note that $\text{int}E \subseteq \text{int}C$ due to our assumption that $E \subset C \setminus \{0\}$. Therefore, $\text{int}E + \text{int}E \subset \text{int}C + \text{int}C \subset \text{int}C \subset C \setminus \{0\}$. Since $F(x_0)$ is a compact and nonempty set, we get by Remark 2.1 that $\text{Min}(F(x_0)) \neq \emptyset$. Let $z_0 \in \text{Min}(F(x_0))$. This means that

$$(F(x_0) - z_0) \cap (-C) = \{0\}. \quad (2.2)$$

As $\text{Min}(F(x_0)) \subset F(x_0)$, according to the equation (2.1), there exist $u_0 \in F(x_0)$ and $c_0 \in C \setminus \{0\}$ such that $z_0 = u_0 + c_0$. Then, $0 \neq -c_0 = u_0 - z_0 \in (F(x_0) - z_0) \cap (-C)$, which contradicts (2.2) and so the proof is complete. $\square$

Let us now recall some basic definitions and properties of multifunction/set-valued mappings.

Definition 2.2 [3, 5] Let $F : X \rightrightarrows Y$ be a set-valued mapping and C be the above cone.

- (i) F is said to be Berge-upper semicontinuous (B-u.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighbourhood U of x_0 in X such that $F(x) \subseteq V$ for all $x \in U$.
- (ii) F is said to be Berge-lower semicontinuous (B-l.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \cap V \neq \emptyset$, there exists a neighbourhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.
- (iii) F is said to be B-u.s.c. (resp. B-l.s.c.) on X , if it is B-u.s.c. (resp. B-l.s.c.) at x for all $x \in X$; F is Berge continuous on X if it is both B-u.s.c and B-l.s.c. on X .

- (iv) F is said to be Hausdorff C -upper semicontinuous (H- C -u.s.c.) at $x_0 \in X$, if for any neighbourhood V of 0_Y , there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x) \subseteq F(x_0) + V + C, \quad \forall x \in U(x_0).$$

- (v) F is said to be Hausdorff C -lower semicontinuous (H- C -l.s.c.) at $x_0 \in X$, if for any neighbourhood V of 0_Y , there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0) \subseteq F(x) + V + C, \quad \forall x \in U(x_0).$$

- (vi) F is said to be C -lower semicontinuous (C -l.s.c.) at $x_0 \in X$, if for any $y \in F(x_0)$ and any neighbourhood V of 0_Y , there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x) \cap (y + V - C) \neq \emptyset, \quad \forall x \in U(x_0).$$

- (vii) F is said to be C -upper semicontinuous (C -u.s.c.) at $x_0 \in X$, if for any neighbourhood V of $F(x_0)$, there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x) \subseteq V + C, \quad \forall x \in U(x_0).$$

- (viii) F is called compact at $x_0 \in X$ if for every sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq \text{gr } F$ with $x_n \rightarrow x_0$, there exists a convergent subsequence $\{y_{n_k}\}$ such that

$$y_{n_k} \rightarrow y_0 \quad \text{for some } y_0 \in F(x_0).$$

Lemma 2.3 [11] Let $F : X \rightrightarrows Y$ be a set-valued mapping, $x \in X$ and C be the above cone.

- (i) If F is C -u.s.c at x , then F is H- C -u.s.c. at x .
- (ii) If F is compact and H- C -u.s.c at x , then F is C -u.s.c. at x .
- (iii) If F is H- C -l.s.c. at x , then F is C -l.s.c. at x .
- (iv) If F is compact and C -l.s.c. at x , then F is H- C -l.s.c. at x .
- (v) If F is B-l.s.c. at x , then F is H- C -l.s.c. at x .

Note that the B-u.s.c. and B-l.s.c. properties of $F : X \rightrightarrows Y$ at a point can be characterized via the sequential forms as follows.

Lemma 2.4 Consider the set-valued mapping $F : X \rightrightarrows Y$ between the Banach spaces X and Y .

- (i) [11, Proposition 2.5.6] F is B-l.s.c. at $x_0 \in X$ if and only if for any sequence $\{x_n\} \subset X$ with $x_n \rightarrow x_0$ and for any $y_0 \in F(x_0)$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and elements $y_n \in F(x_{n_k})$ for all n such that $y_n \rightarrow y_0$.
- (ii) [11, Proposition 2.5.9] Let $x_0 \in X$ be such that $F(x_0)$ is compact. Then, F is B-u.s.c. at x_0 if and only if for any sequence $\{x_n\} \subset X$ with $x_n \rightarrow x_0$ and $y_n \in F(x_n)$ for all n , there exist $y_0 \in F(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightarrow y_0$.

Definition 2.3 [23] Let D be a nonempty convex subset of X , $F : D \rightrightarrows Y$ be a set-valued mapping and C be the above cone.

(i) We say that F is C -convex on D if for each $x_1, x_2 \in D$ and $\lambda \in [0, 1]$, we have

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^s \lambda F(x_1) + (1 - \lambda)F(x_2).$$

(ii) We say that F is strictly naturally C -quasiconvex on D if for each $x_1, x_2 \in D$ and $\lambda \in (0, 1)$, there is an $\alpha \in [0, 1]$ such that

$$F(\lambda x_1 + (1 - \lambda)x_2) \ll_C^s \alpha F(x_1) + (1 - \alpha)F(x_2).$$

For a set-valued mapping $F : X \rightrightarrows Y$ between Banach spaces X and Y , the superior limit, superior open limit and inferior open limit of F (see e.g., [24, 39]) are defined respectively by

$$\limsup_{x \rightarrow x_0} F(x) := \{y_0 \in Y \mid \exists x_n \rightarrow x_0, \exists y_n \in F(x_n), \forall n, y_n \rightarrow y_0\},$$

$\limsupo_{x \rightarrow x_0} F(x) := \{y \in Y \mid \text{There exist an open neighbourhood } U \text{ of } y \text{ and a sequence } \{x_n\} \subseteq X \text{ converging to } x_0 \text{ and } x_n \neq x_0 \text{ such that } U \subseteq F(x_n) \text{ for all } n\},$

$\liminfo_{x \rightarrow x_0} F(x) := \{y \in Y \mid \text{There exist open neighbourhoods } U \text{ of } x_0 \text{ and } V \text{ of } y \text{ such that } V \subseteq F(x) \text{ for all } x \in U \text{ with } x \neq x_0\}.$

Definition 2.4 [24, 39] Let $F : X \rightrightarrows Y$ be a set-valued mapping between Banach spaces.

(i) F is called outer-continuous at $x_0 \in X$ if

$$\limsup_{x \rightarrow x_0} F(x) \subseteq F(x_0).$$

F is termed outer-continuous on X if it is outer-continuous at every $x \in X$.

(ii) F is called outer-open at $x_0 \in X$ if

$$\limsupo_{x \rightarrow x_0} F(x) \subseteq F(x_0).$$

F is outer-open on X if this holds for all $x \in X$.

(iii) F is called inner-open at $x_0 \in X$ if

$$F(x_0) \subseteq \liminfo_{x \rightarrow x_0} F(x).$$

F is inner-open on X if it is inner-open at every $x \in X$.

The next lemmas give some properties of the superior limit, superior open limit and inferior open limit which are useful in the sequel.

Lemma 2.5 [24] For a set-valued mapping $F : X \rightrightarrows Y$ and $x_0 \in X$, the following relations hold.

- (i) $\liminf_{x \rightarrow x_0} F(x) \subseteq \limsup_{x \rightarrow x_0} F(x) \subseteq \limsup_{x \rightarrow x_0} F(x)$;
- (ii) $\liminf_{x \rightarrow x_0} F(x) = \left[\limsup_{x \rightarrow x_0} (F(x))^{Co} \right]^{Co}$, where $(F(x))^{Co} = Y \setminus F(x)$ for $x \in X$.

Lemma 2.6 [15] Let A and B be two nonempty subsets of Y . If $0 < \varphi < \delta$, B is convex and $A + \delta B_Y \subseteq B + \varphi B_Y$, then $A \subseteq \text{int} B$.

Proposition 2.7 Let M be a convex set and assume that $F(\cdot)$ is strictly naturally C -quasiconvex on M , where C is a cone as above. If $F(\cdot)$ is compact for each $x \in M$, then $E_i(F, M) = W_i(F, M)$ for $i \in \{u, l\}$.

Proof Let $i \in \{u, l\}$. By the definition of strictly naturally C -quasiconvex and the convexity of M , we can easily obtain the inclusion $W_i(F, M) \subset E_i(F, M)$. Since $E_i(F, M) \subset W_i(F, M)$ is obvious, the proof is complete. $\square$

3 Outer-Continuity and Openness for the Solution Mapping

In this section, we analyze the outer-continuity, outer-openness and inner-openness properties of the solution mapping for a parametric set optimization problem.

Let $F : X \rightrightarrows Y$ and $M : \Lambda \rightrightarrows X$ be two set-valued mappings with nonempty values. Consider the following parametric set optimization problem (PSOP): For each parameter $\lambda \in \Lambda$, the associated set optimization problem is defined by

$$\min\{F(x) : x \in M(\lambda)\}. \quad (\text{SOP}_\lambda)$$

As above, we denote the sets of all E - i -minimal and weak E - i -minimal solutions of (SOP_λ) by $E_i(F, M(\lambda))$ and $W_i(F, M(\lambda))$, respectively, and assume that $E_i(F, M(\lambda)) \neq \emptyset$ for $i \in \{u, l\}$.

The first main result of this section provides sufficient conditions that guarantee the outer-continuity for the weak E - i -minimal solution map of (PSOP), where $i \in \{u, l\}$.

Theorem 3.1 Let $\lambda_0 \in \Lambda$ and assume that the following conditions hold:

- (i) $M(\cdot)$ is outer-continuous at λ_0 and B -l.s.c. at λ_0 ;
- (ii) $F(\cdot)$ is Berge continuous on $M(\lambda_0)$ and has nonempty compact values on X .

Then, $W_i(F, M(\cdot))$ is outer-continuous at λ_0 for $i \in \{u, l\}$.

Proof We only give a proof for the case $i = l$ as it can be done similarly for the case $i = u$. Namely, we will justify that

$$\limsup_{\lambda \rightarrow \lambda_0} W_l(F, M(\lambda)) \subset W_l(F, M(\lambda_0)). \quad (3.1)$$

Let $x_0 \in \limsup_{\lambda \rightarrow \lambda_0} W_l(F, M(\lambda))$. This implies that there exists a sequence $\{\lambda_\alpha\}$ converging to λ_0 with $x_\alpha \in W_l(F, M(\lambda_\alpha))$ such that $x_\alpha \rightarrow x_0$. Since $M(\cdot)$ is outer-continuous at λ_0 , we have $x_0 \in M(\lambda_0)$. Note that $F(x_0)$ is compact by (ii).

We show that $x_0 \in W_l(F, M(\lambda_0))$. Suppose to the contrary that $x_0 \notin W_l(F, M(\lambda_0))$. Then, by Proposition 2.2, there exists $z_0 \in M(\lambda_0)$ such that $F(z_0) \ll_E^l F(x_0)$, i.e.,

$$F(x_0) \subseteq F(z_0) + \text{int}E. \quad (3.2)$$

Since $M(\cdot)$ is B -l.s.c. at λ_0 , we conclude from Lemma 2.4(i) that there exists $\{z_\alpha\}$ with $z_\alpha \in M(\lambda_\alpha)$ such that $z_\alpha \rightarrow z_0$. We now claim that there exists $\alpha_0 \in \mathbb{N}$ such that

$$F(x_\alpha) \subseteq F(z_\alpha) + \text{int}E, \quad \forall \alpha \geq \alpha_0. \quad (3.3)$$

Assume on the contrary that (3.3) is not true. Then, there exist subsequences $\{x_{\alpha_k}\}$ of $\{x_\alpha\}$ and $\{z_{\alpha_k}\}$ of $\{z_\alpha\}$ such that $F(x_{\alpha_k}) \not\subseteq F(z_{\alpha_k}) + \text{int}E$ for all k . Without loss of generality, we assume that for all $\alpha \in \mathbb{N}$,

$$F(x_\alpha) \not\subseteq F(z_\alpha) + \text{int}E.$$

Therefore, for each α , one can find $v_\alpha \in F(x_\alpha)$ such that

$$v_\alpha \notin F(z_\alpha) + \text{int}E. \quad (3.4)$$

Note that $F(\cdot)$ is B -u.s.c. at x_0 and $F(x_0)$ is compact. Lemma 2.4(ii) yields $v_0 \in F(x_0)$ and a subsequence $\{v_{\alpha_k}\}$ with $v_{\alpha_k} \rightarrow v_0$. Without loss of generality, assume $v_\alpha \rightarrow v_0$. From (3.2), there exists $u_0 \in F(z_0)$ satisfying

$$v_0 - u_0 \in \text{int}E. \quad (3.5)$$

Moreover, $F(\cdot)$ is B -l.s.c. at z_0 , we get by Lemma 2.4(i) that there exist elements $u_\alpha \in F(z_\alpha)$ such that $u_\alpha \rightarrow u_0$. This together with (3.5) and $v_\alpha \rightarrow v_0$ implies that there exists $\alpha_1 \in \mathbb{N}$ such that

$$v_\alpha - u_\alpha \in \text{int}E, \quad \forall \alpha \geq \alpha_1,$$

which contradicts (3.4). So, (3.3) must be true.

However, it follows from (3.3) and Proposition 2.2 that $x_\alpha \notin W_l(F, M(\lambda_\alpha))$ for all $\alpha \geq \alpha_0$. This is a contradiction and so, $x_0 \in W_l(F, M(\lambda_0))$. Consequently, (3.1) holds and the proof is complete. $\square$

The following example illustrates the outer-continuity of the solution map in Theorem 3.1.

Example 3.1 Let $X = \Lambda = \mathbb{R}$ and $Y = \mathbb{R}^2$. We consider $C = \mathbb{R}_+^2$ and $E = \mathbb{R}_+^2 + \{(2, 0)\}$ and define a set-valued map $M : \Lambda \rightrightarrows X$ as

$$M(\lambda_0) = \{x \in \mathbb{R} : 0 \leq x \leq 2 + |\lambda_0|\} \text{ for } \lambda_0 \in \mathbb{R}.$$

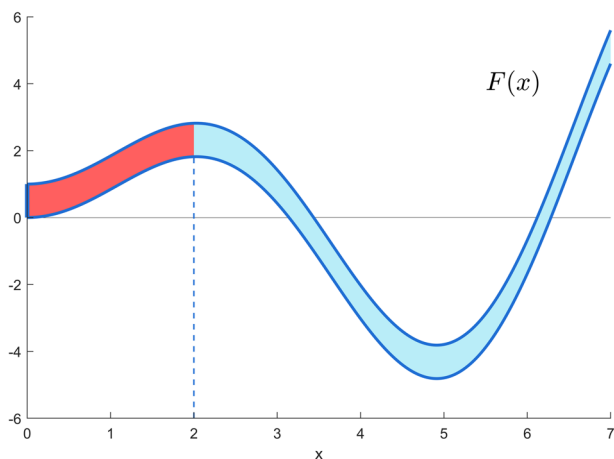


Fig. 1 The graph of F

Let $F : X \rightrightarrows Y$ be a set-valued map defined by

$$F(x) = [(x, x \sin x), (x, 1 + x \sin x)] \text{ for } x \in \mathbb{R}.$$

The graph of F is shown in Figure 1.

We can verify that the conditions (i) and (ii) of Theorem 3.1 are fulfilled. By direct computation, we obtain that $W_l(F, M(\lambda_0)) = [0, 2]$ for a given $\lambda_0 \in \Lambda$. So $W_l(F, M(\cdot))$ is outer-continuous at any $\lambda_0 \in \Lambda$.

The next result provides sufficient conditions that guarantee the outer-openness for the weak E - i -minimal solution map of (PSOP), where $i \in \{u, l\}$.

Corollary 3.2 *Let $\lambda_0 \in \Lambda$. Assume that the assumptions of Theorem 3.1 are satisfied. Then, $W_i(F, M(\cdot))$ is outer-open at λ_0 for $i \in \{u, l\}$.*

Proof Let $\lambda_0 \in \Lambda$ and $i \in \{u, l\}$. It follows from Lemma 2.5(i) and the proof of Theorem 3.1 that

$$\limsup_{\lambda \rightarrow \lambda_0} W_i(F, M(\lambda)) \subseteq \limsup_{\lambda \rightarrow \lambda_0} W_i(F, M(\lambda)) \subseteq W_i(F, M(\lambda_0)).$$

Therefore, $W_i(F, M(\cdot))$ is outer-open at λ_0 . This completes the proof. $\square$

The following theorem provides sufficient conditions that guarantee the inner-openness for the E - i -minimal solution map of (PSOP), where $i \in \{u, l\}$.

Theorem 3.3 *Let $\lambda_0 \in \Lambda$ and assume that E is a closed improvement set. Suppose that the following conditions hold.*

(i) $M(\cdot)$ is inner-open at λ_0 and B -u.s.c. with compact values at λ_0 ;

(ii) $F(\cdot)$ is Berge continuous on $M(\lambda_0)$ and has nonempty compact values on X .

Then, $E_i(F, M(\cdot))$ is inner-open at λ_0 for $i \in \{u, l\}$.

Proof We only give a proof for the case $i = l$ as it can be done similarly for the case $i = u$. Suppose by contradiction that $E_l(F, M(\cdot))$ is not inner-open at λ_0 . Then, we can find $x_0 \in E_l(F, M(\lambda_0))$ such that $x_0 \notin \liminf_{\lambda \rightarrow \lambda_0} E_l(F, M(\lambda))$. It follows from Lemma 2.5(ii) that

$$\liminf_{\lambda \rightarrow \lambda_0} E_l(F, M(\lambda)) = \left[\limsup_{\lambda \rightarrow \lambda_0} (E_l(F, M(\lambda)))^{Co} \right]^{Co}.$$

This implies that $x_0 \in \limsup_{\lambda \rightarrow \lambda_0} (E_l(F, M(\lambda)))^{Co}$. Then, there exist a sequence $\{\lambda_\alpha\}$ with $\lambda_\alpha \rightarrow \lambda_0$ and a sequence $\{x_\alpha\}$ with $x_\alpha \in (E_l(F, M(\lambda_\alpha)))^{Co}$ such that $x_\alpha \rightarrow x_0$. Since $M(\cdot)$ is inner-open at λ_0 and $x_0 \in M(\lambda_0)$, we have

$$x_0 \in \liminf_{\lambda \rightarrow \lambda_0} M(\lambda).$$

Thus, by the definition of the inferior open limit, we find neighbourhoods U of λ_0 and V of x_0 such that $V \subseteq M(\lambda)$ for all $\lambda \in U$, $\lambda \neq \lambda_0$. As $\lambda_\alpha \rightarrow \lambda_0$ and $x_\alpha \in (E_l(F, M(\lambda_\alpha)))^{Co}$, there exist subsequences $\{\lambda_\beta\}$ of $\{\lambda_\alpha\}$ and $\{x_\beta\}$ of $\{x_\alpha\}$ with $x_\beta \notin E_l(F, M(\lambda_\beta))$. By the compactness of $F(x_\beta)$, from Lemma 2.1, we can find $z_\beta \in M(\lambda_\beta)$ such that $F(z_\beta) \leq_E^l F(x_\beta)$, i.e.,

$$F(x_\beta) \subseteq F(z_\beta) + E \text{ for all } \beta. \quad (3.6)$$

Note that $M(\cdot)$ is B -u.s.c. at λ_0 and $M(\lambda_0)$ is compact. Then, by $\lambda_\beta \rightarrow \lambda_0$ and $z_\beta \in M(\lambda_\beta)$, we get by Lemma 2.4(ii) that there exist $z_0 \in M(\lambda_0)$ and a subsequence $\{z_{\beta_k}\}$ of $\{z_\beta\}$ such that $z_{\beta_k} \rightarrow z_0$. There is no loss of generality in assuming that

$$z_\beta \rightarrow z_0.$$

Take any $v_0 \in F(x_0)$ and note that $x_\beta \rightarrow x_0$. Since $F(\cdot)$ is B -l.s.c. at x_0 , we derive from Lemma 2.4(i) that there exist $v_\beta \in F(x_\beta)$ for all β such that

$$v_\beta \rightarrow v_0.$$

Now, take a sequence $\{u_\beta\} \subset Y$ with $u_\beta \in F(z_\beta)$. Note that $F(\cdot)$ is B -u.s.c. at z_0 and $F(z_0)$ is compact. Since $z_\beta \rightarrow z_0$, we get by Lemma 2.4(ii) that there exist $u_0 \in F(z_0)$ and a subsequence $\{u_{\beta_k}\}$ of $\{u_\beta\}$ such that $u_{\beta_k} \rightarrow u_0$. Without loss of generality, we may assume that $u_\beta \rightarrow u_0$. Hence, by (3.6),

$$v_\beta - u_\beta \in E \text{ for all } \beta.$$

Passing to the limit and noting the closeness of E , we obtain $v_0 - u_0 \in E$. This implies that $v_0 \in F(z_0) + E$. Since v_0 was arbitrarily chosen, we conclude that $F(x_0) \subseteq F(z_0) + E$, i.e., $F(z_0) \leq_E^l F(x_0)$. However, this contradicts $x_0 \in E_l(F, M(\lambda_0))$ by virtue of Lemma 2.1 under the compactness of $F(x_0)$. Consequently, $E_l(F, M(\cdot))$ is inner-open at λ_0 , which completes the proof. $\square$

Remark 3.1 The inner continuity, the outer continuity, the inner openness and the outer openness of the solution map for a class of (generalized) variational inequality problems have been investigated in the literature, e.g., [17, 18, 24, 28]. Recently, in [29], authors investigated the stability of solution sets for set optimization problems via improvement sets. Some sufficient conditions for the upper (lower) semicontinuity and compactness of E -minimal solution mappings have been given in the above-mentioned papers for a parametric set optimization under suitable conditions. Our contribution for this section is to provide sufficient conditions for guaranteeing the outer-continuity, outer-openness and inner-openness of the solution mappings of (PSOP) via improvement sets.

4 Closedness and Well-posedness for Solutions

In this section, we examine the closedness and the Levitin-Polyak well-posedness of the solution set of a set optimization problem and the Hadamard well-posedness of the solution map of a parametric implicit set optimization problem.

4.1 Levitin-Polyak Well-posedness of Set Optimization Problems

In this subsection, we examine the closedness and the Levitin-Polyak (LP) well-posedness of the solution set of the set optimization problem (SOP).

The following theorem presents the closedness of the sets of weak E - i -minimal solutions and E - i -minimal solutions of (SOP) for $i \in \{u, l\}$.

Theorem 4.1 For the set optimization problem (SOP), let M be a convex and closed subset of $\text{dom} F$ and E be a convex improvement set. Suppose that $F(\cdot)$ is C -convex and has compact values on M , where C is a cone defined in Section 2. Then, the following statements hold.

- (a) $W_l(F, M)$ is closed if $F(\cdot)$ is Hausdorff C -continuous on M .
- (b) $W_u(F, M)$ is closed if $-F(\cdot)$ is Hausdorff C -continuous on M .

Proof We only give a proof for (a) as it can be done similarly for (b). Let $F(\cdot)$ be Hausdorff C -continuous on M and we will show that $W_l(F, M)$ is closed.

Let $\{u_n\} \subseteq W_l(F, M)$ be such that $u_n \rightarrow u_0$. Observe first that $u_0 \in M$ due to the closedness of M . We will justify that $u_0 \in W_l(F, M)$. Assume on the contrary that $u_0 \notin W_l(F, M)$. Then, by Proposition 2.2, there exists $x_0 \in M$ such that $F(x_0) \ll_E^l F(u_0)$, i.e.

$$F(u_0) \subseteq F(x_0) + \text{int} E.$$

Due to the compactness of $F(u_0)$ and the fact that $F(x_0) + \text{int}E = \bigcup_{z \in F(x_0)} (z + \text{int}E)$, we can find $\{z_1, z_2, \dots, z_k\} \subseteq F(x_0)$ such that

$$F(u_0) \subseteq \bigcup_{i=1}^k \{z_i + \text{int}E\} = \{z_1, z_2, \dots, z_k\} + \text{int}E.$$

The compactness of $F(u_0)$ ensures further that there exists $\delta > 0$ such that

$$F(u_0) + 3\delta B_Y \subseteq \{z_1, z_2, \dots, z_k\} + \text{int}E. \quad (4.1)$$

Since $F(\cdot)$ is H - C - $u.s.c.$ at u_0 , for the neighbourhood δB_Y of 0_Y , there exists a neighbourhood $U(u_0)$ of u_0 such that

$$F(u) \subseteq F(u_0) + \delta B_Y + C, \quad \forall u \in U(u_0).$$

As $u_n \rightarrow u_0$, there exists $n_1 \in \mathbb{N}$ such that $u_n \in U(u_0)$ for any $n \geq n_1$ and that

$$F(u_n) \subseteq F(u_0) + \delta B_Y + C. \quad (4.2)$$

As $F(\cdot)$ is H - C - $l.s.c.$ at x_0 , for the above set δB_Y , there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0) \subseteq F(x') + \delta B_Y + C, \quad \forall x' \in U(x_0). \quad (4.3)$$

It follows from (4.1) and (4.2) that

$$\begin{aligned} F(u_n) + 2\delta B_Y &\subseteq F(u_0) + 3\delta B_Y + C \subseteq \{z_1, z_2, \dots, z_k\} + \text{int}E + C \\ &\subseteq \{z_1, z_2, \dots, z_k\} + \text{int}E, \quad \forall n \geq n_1, \end{aligned} \quad (4.4)$$

where we should recall that $\text{int}E + C \subset \text{int}E$ (cf. [12, Proposition 2.4]).

For $n \geq n_1$, take any $\vartheta \in F(u_n) + 2\delta B_Y$. By (4.4), there are $i_0 \in \{1, 2, \dots, k\}$ and $e \in \text{int}E$ such that

$$\vartheta = z_{i_0} + e. \quad (4.5)$$

Moreover, by $z_{i_0} \in F(x_0)$, we get by (4.3) that for each $x' \in U(x_0)$ there exist $t \in F(x')$, $b_0 \in B_Y$ and $p \in C$ such that $z_{i_0} = t + \delta b_0 + p$. Hence, it follows from (4.5) that

$$\vartheta = t + \delta b_0 + p + e \in F(x') + \delta B_Y + C + \text{int}E, \quad \forall n \geq n_1.$$

Because of the arbitrariness of $\vartheta \in F(u_n) + 2\delta B_Y$, one has

$$\begin{aligned} F(u_n) + 2\delta B_Y &\subseteq F(x') + \delta B_Y + C + \text{int}E \\ &\subseteq F(x') + \delta B_Y + E, \quad \forall n \geq n_1, \forall x' \in U(x_0). \end{aligned}$$

Sine $F(\cdot)$ is C -convex and E is a convex improvement set, we conclude that $F(x') + E$ is convex (cf. [33, Lemma 4.3]) for any $x' \in U(x_0)$. This fact together with Lemma 2.6 implies that

$$F(u_n) \subseteq \text{int}(F(x') + E) \subseteq F(x') + \text{int}E, \quad \forall n \geq n_1, \forall x' \in U(x_0).$$

So we get that $F(x') \ll_E^l F(u_n)$, $\forall n \geq n_1, \forall x' \in U(x_0)$. However, by Proposition 2.2, this is a contradiction to the fact that $u_n \in W_l(F, M)$ for $n \geq n_1$. Therefore, $u_0 \in W_l(F, M)$ and $W_l(F, M)$ is closed. The proof is complete. $\square$

Corollary 4.2 For the set optimization problem (SOP), let all assumptions in Theorem 4.1 hold. Suppose that $F(\cdot)$ is strictly naturally C -quasiconvex. Then, $E_i(F, M)$ is closed if $F(\cdot)$ is Hausdorff C -continuous on M , where $i \in \{u, l\}$.

Proof Let $i \in \{u, l\}$. Since $F(\cdot)$ is strictly naturally C -quasiconvex, we get $E_i(F, M) = W_i(F, M)$ thanks to Proposition 2.7. So we obtain by Theorem 4.1 that $E_i(F, M)$ is closed if $F(\cdot)$ is Hausdorff C -continuous on M . $\square$

Remark 4.1 Theorem 4.1 can be considered as an improvement of Theorem 4.4 in [8], where the author used the Berge semicontinuity of F to obtain the closedness of solutions of a set optimization problem. It is worth mentioning here that the Hausdorff C -continuity is weaker than the Berge continuity for a set-valued mapping.

To proceed, we present the concepts of generalized Levitin-Polyak (LP)-minimizing sequence and LP well-posedness for the set optimization problem (SOP) (see also in [8]).

Definition 4.1 Consider the set optimization problem (SOP) and a cone C defined in Section 2.

(i) A sequence $\{x_n\} \subseteq \text{dom} F$ is called a *generalized LP minimizing sequence* with respect to $e \in \text{int} C$ for (SOP) if there exist a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \downarrow 0$ and a sequence $\{u_n\} \subseteq E_i(F, M)$ such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq_E^l F(u_n) + \varepsilon_n e$ for all $n \in \mathbb{N}$, where $i \in \{u, l\}$ and $d(x_n, M) := \inf \{\|x_n - a\| : a \in M\}$.

(ii) The problem (SOP) is called *generalized LP well-posed* if for each generalized LP minimizing sequence $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to an element in $E_i(F, M)$, where $i \in \{u, l\}$.

We are now ready to provide sufficient conditions of generalized LP well-posedness for the problem (SOP). For the sake of convenience, we use l -(SOP) (resp. u -(SOP)) to refer to the set optimization problem (SOP) with respect to the lower (resp. upper) set less order relation.

Theorem 4.3 For the set optimization problem (SOP), let M be a convex and compact subset of $\text{dom} F$ and E be a convex and closed improvement set. Suppose that $F(\cdot)$ is C -convex, strictly naturally C -quasiconvex and has compact values on M , where C is a cone defined in Section 2. Then, the following statements hold.

- (a) The problem l -(SOP) is generalized LP well-posed if $F(\cdot)$ is Hausdorff C -continuous on M .
- (b) The problem u -(SOP) is generalized LP well-posed if $-F(\cdot)$ is Hausdorff C -continuous on M .

Proof We only give a proof for (a) as it can be done similarly for (b). Let $\{x_n\}$ be a generalized LP -minimizing sequence with respect to $e \in \text{int} C$ for the problem l -(SOP). By definition, there exist a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \downarrow 0$ and a sequence

$\{u_n\} \subseteq E_l(F, M)$ such that $d(x_n, M) \leq \varepsilon_n$, and $F(x_n) \leq_E^l F(u_n) + \varepsilon_n e$, i.e.

$$F(u_n) + \varepsilon_n e \subseteq F(x_n) + E, \quad \forall n \in \mathbb{N}. \quad (4.6)$$

Since M is compact and $E_l(F, M)$ is closed by virtue of Corollary 4.2, $E_l(F, M)$ is compact. By the compactness of $E_l(F, M)$, we may assume without loss of generality that $\{u_n\}$ converges to some $u' \in E_l(F, M)$.

From $d(x_n, M) \leq \varepsilon_n$, for each n , there exists $\bar{x}_n \in M$ such that

$$\|x_n - \bar{x}_n\| \leq d(x_n, M) + \frac{1}{n} \leq \varepsilon_n + \frac{1}{n}.$$

By the compactness of M , we may assume by passing to a subsequence if necessary that $\{\bar{x}_n\}$ converges to some $x' \in M$. It is easy to see that

$$\|x_n - x'\| \leq \|x_n - \bar{x}_n\| + \|\bar{x}_n - x'\| \leq \varepsilon_n + \frac{1}{n} + \|\bar{x}_n - x'\|.$$

As $\varepsilon_n \downarrow 0$, $\frac{1}{n} \rightarrow 0$, and $\|\bar{x}_n - x'\| \rightarrow 0$, we obtain that $x_n \rightarrow x' \in M$.

We will show that $x' \in E_l(F, M)$. To see this, we assume on the contrary that $x' \notin E_l(F, M)$. Then, by the compactness of $F(x')$, we claim by Lemma 2.1 that there exists $z' \in M$ such that $F(z') \leq_E^l F(x')$, i.e., $F(x') \subseteq F(z') + E$.

Now, we justify that $F(u') \subseteq F(z') + E$. Assume by contradiction that there exists $v \in F(u')$ such that $v \notin F(z') + E$. Note that $F(z') + E$ is closed as $F(z')$ is compact and E is closed. Hence, there exists $\delta > 0$ such that

$$(v + 2\delta B_Y) \cap (F(z') + E) = \emptyset. \quad (4.7)$$

Let B_1 and B_2 be zero neighbourhoods in Y such that

$$B_1 + B_2 \subseteq \delta B_Y. \quad (4.8)$$

Since $F(\cdot)$ is H-C-u.s.c. at $x' \in M$, for the above B_1 , there exists a neighbourhood $U(x')$ of x' such that

$$F(x) \subseteq F(x') + B_1 + C, \quad \forall x \in U(x').$$

As $x_n \rightarrow x'$, there exists $n_1 \in \mathbb{N}$ such that $x_n \in U(x')$ for all $n \geq n_1$. So, we have

$$F(x_n) \subseteq F(x') + B_1 + C, \quad \forall n \geq n_1. \quad (4.9)$$

As $F(\cdot)$ is H-C-l.s.c. at $u' \in M$, for the above B_2 , there exists a neighbourhood $U(u')$ of u' such that

$$F(u') \subseteq F(u) + B_2 + C, \quad \forall u \in U(u').$$

By $u_n \rightarrow u'$, there exists $n_2 \in \mathbb{N}$ such that $u_n \in U(u')$ for all $n \geq n_2$. Therefore,

$$F(u') \subseteq F(u_n) + B_2 + C, \quad \forall n \geq n_2. \quad (4.10)$$

Let $n_0 = \max\{n_1, n_2\}$. We conclude from (4.6) and (4.8)-(4.10) that, for any $n \geq n_0$,

$$\begin{aligned} F(u') + \varepsilon_n e &\subseteq F(u_n) + B_2 + C + \varepsilon_n e \\ &\subseteq F(x_n) + E + B_2 + C \\ &\subseteq F(x') + B_1 + C + E + B_2 + C \\ &\subseteq F(x') + \delta B_Y + E. \\ &\subseteq F(z') + E + \delta B_Y + E. \\ &\subseteq F(z') + C + \delta B_Y + E = F(z') + \delta B_Y + E, \end{aligned}$$

where we should note that $E \subseteq C \setminus \{0\}$ and $E + C = E$. Therefore, it holds that

$$v + \varepsilon_n e \in F(z') + \delta B_Y + E, \quad \forall n \geq n_0.$$

Then, for each $n \geq n_0$, we can find $b_n \in B_Y$ such that

$$v + \varepsilon_n e - \delta b_n \in F(z') + E. \quad (4.11)$$

As $\varepsilon_n \downarrow 0$, we can take $\bar{n} > n_0$ such that $\varepsilon_n < \frac{\delta}{\|e\|}$ for all $n \geq \bar{n}$. Now, for $n \geq \bar{n}$, it holds that $\|\varepsilon_n e - \delta b_n\| \leq \|\varepsilon_n e\| + \|\delta b_n\| < 2\delta$, i.e., $\varepsilon_n e - \delta b_n \in 2\delta B_Y$. This and the relation in (4.11) show that

$$v + \varepsilon_n e - \delta b_n \in (v + 2\delta B_Y) \cap (F(z') + E), \quad \forall n \geq \bar{n},$$

which clearly contradicts (4.7). Consequently, $F(u') \subseteq F(z') + E$, which means that $F(z') \leq_E^l F(u')$. Then, by the compactness of $F(u')$, we claim by Lemma 2.1 that $u' \notin E_l(F, M)$. This contradicts $u' \in E_l(F, M)$ as shown above. Therefore, $x' \in E_l(F, M)$ holds, and the proof is complete. $\square$

Remark 4.2 By virtue of the Hausdorff C -continuity, the above results present sufficient conditions of generalized LP well-posedness of problem (SOP), where the lower/upper set order relations are induced by an improvement set rather than a cone. Obviously, the assumptions of Theorem 4.3 are weaker than the Berge continuous assumption of [8, Theorem 4.5] and [43, Theorem 3.2], and so Theorem 4.3 can be considered as an improvement of [8, Theorem 4.5] and [43, Theorem 3.2].

The following example illustrates how one can verify the sufficient conditions for the generalized LP well-posedness of a concrete set optimization problem given in Theorem 4.3.

Example 4.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [0, 1]$, $C = \mathbb{R}_+^2$ and $E = \mathbb{R}_+^2 + \{(\frac{1}{2}, 0)\}$. We define a set-valued map $F : X \rightrightarrows Y$ as follows:

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } x = 0, \\ [x, 1] \times [-x, 0], & \text{if } x \in (0, 1], \\ [0, 1] \times [0, 0], & \text{if } x < 0, \\ [1, 1] \times [-1, 0], & \text{if } x > 1. \end{cases}$$

One can verify that $F(\cdot)$ is Hausdorff C -continuous and compact-valued on M . Clearly, E is convex and closed improvement set and M is a convex and closed compact subset of $\text{dom} F$. Moreover, $F(\cdot)$ is C -convex, strictly naturally C -quasiconvex, i.e., all assumptions of Theorem 4.3 (a) are satisfied.

By calculating, we have $E_l(F, M) = [0, 1]$. It can be seen that l -(SOP) is generalized LP well-posed. Indeed, taking $x_n = -\frac{1}{2n}$, $u_n = \frac{1}{2n}$ and $\varepsilon_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, we see that $\{x_n\}$ is a generalized LP -minimizing sequence for (SOP) and $-\frac{1}{2n} \rightarrow 0 \in E_l(F, M)$. Therefore, Theorem 4.3 (a) is applicable.

However, $F(\cdot)$ is not B -l.s.c., and so $F(\cdot)$ is not Berge continuous. In fact, let $x_0 = 0$ and $V = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-1)^2 < \frac{1}{4}\}$ with $F(x_0) \cap V \neq \emptyset$. Then, for any neighbourhood U of x_0 , there exists $\tilde{x} \in U$ such that $F(\tilde{x}) \cap V = \emptyset$. Therefore, Theorem 4.5 in [8] is not applicable for this setting.

4.2 Hadamard Well-posedness of Parametric Implicit Set Optimization Problem

Let $D \subseteq Z$ be an improvement set with nonempty interior, $A \subseteq X$ be a nonempty subset, and T be a nonempty compact subset of a real Hausdorff topological space. Define

- $\mathcal{O}$ as the space of all vector-valued maps from $A \times T$ to Z ;
- $\mathcal{L}$ as the space of all set-valued maps from A to Y .

We consider a *parametric implicit set optimization problem* (ISOP) under perturbations of both the objective and constraint maps on the parameter space $P := \mathcal{L} \times \mathcal{O}$ as follows: For each $p := (F, h) \in P$ (regarded as a parameter), one has an implicit set optimization problem

$$\min\{F(x) : x \in K(p)\}, \quad (\text{ISOP}_p)$$

where

$$K(p) = \{x \in A : h(x, t) \notin \text{int} D, \forall t \in T\}.$$

The “min” is understood with respect to $\leq_E^i$ or $\ll_E^i$ and so we have the corresponding parametric implicit set optimization i -(ISOP _{p}) problems for $i \in \{u, l\}$. We also denote the set of all E - i -minimal and weak E - i -minimal solutions of i -(ISOP _{p}) by $E_i(p)$ and $W_i(p)$ respectively for $i \in \{u, l\}$.

Remark 4.3 For a given $p := (F, h) \in P$, let $K(p)$ be nonempty and compact. Then we have the following assertions:

(i) If F is l.s.c. and compact on A , then the set $E_u(p)$ is nonempty (see, [32, Lemma 2.5]). Note that $E_u(p) \subset W_u(p)$ and so $W_u(p)$ is nonempty.

(ii) If F is C-u.s.c. on A , then the set $E_l(p)$ is nonempty (see, [15, Lemma 2.3]), and so $W_l(p)$ is also nonempty.

Definition 4.2 [33] Let $F_n : X \rightrightarrows Y$ and $F : X \rightrightarrows Y$, A be a nonempty subset of X with $A \subseteq \text{dom} F_n \cap \text{dom} F$ for $n \in \mathbb{N}$ and C be a cone defined in Section 2. We say that

(i) $F_n \xrightarrow{HCl} F$ on A if for any neighbourhood V of 0_Y , there exists $n_0 \in \mathbb{N}$ such that

$$F(x) \subseteq F_n(x) + V - C, \quad \forall x \in A, \quad \forall n \geq n_0.$$

(ii) $F_n \xrightarrow{HCu} F$ on A if for any neighbourhood V of 0_Y , there exists $n_0 \in \mathbb{N}$ such that

$$F_n(x) \subseteq F(x) + V - C, \quad \forall x \in A, \quad \forall n \geq n_0.$$

We say that $F_n \xrightarrow{HC} F$ (also called, Hausdorff C -convergence) on A if $F_n \xrightarrow{HCl} F$ and $F_n \xrightarrow{HCu} F$ on A .

The following concepts of generalized Hadamard well-posedness were given in [9] for a type of set optimization problems.

Definition 4.3 Let $p \in P$ be a given parameter and let $i \in \{u, l\}$. The problem i -(ISOP) is said to be generalized Hadamard well-posed for the efficiency (resp. weak efficiency) at p if the following statements hold:

- (i) $E_i(p) \neq \emptyset$ (resp. $W_i(p) \neq \emptyset$);
- (ii) Let $p_n \rightarrow p$, where $p_n \in P$ for all $n \in \mathbb{N}$. Then, every sequence $\{x_n\}$ with $x_n \in E_i(p_n)$ (resp. $x_n \in W_i(p_n)$), $n \in \mathbb{N}$, admits a subsequence converging to some $\bar{x} \in E_i(p)$ (resp. $\bar{x} \in W_i(p)$).

We now provide sufficient conditions for the problem u -(ISOP) to be generalized Hadamard well-posed for the weak efficiency at a reference parameter.

Theorem 4.4 Let E be a convex improvement set, C be a cone defined in Section 2. For a given $p := (F, h) \in P$, let F be compact-valued on A , where A is a convex set. For any $p_n := (F_n, h_n)$ and $n \in \mathbb{N}$, assume that the following conditions hold:

- (i) $K(\cdot)$ is Berge continuous and compact at p ;
- (ii) $-F(\cdot)$ is H-C-u.s.c. on A and $F(\cdot)$ is l.s.c. on A ;
- (iii) $F_n \xrightarrow{HC} F$ on A ;
- (iv) $F_n(\cdot)$ is $-C$ -convex on A .

Then, the parametric implicit set optimization problem u -(ISOP) is generalized Hadamard well-posed for the weak efficiency at p .

To prove this theorem, we need the following technical lemma.

Lemma 4.5 Let E be a convex improvement set, $F_n : X \rightrightarrows Y$ and $F : X \rightrightarrows Y$, A be a convex subset of X with $A \subseteq \text{dom} F_n \cap \text{dom} F$ for $n \in \mathbb{N}$ and C be a cone defined in Section 2. Let $\{x_n\} \subseteq A$ with $x_n \rightarrow x \in A$ and $\{y_n\} \subseteq A$ with $y_n \rightarrow y \in A$. Suppose that the following conditions hold:

- (i) $F(y)$ is compact and $F(y) \ll_E^u F(x)$;
- (ii) $-F(\cdot)$ is H-C-u.s.c. at y and $-F(\cdot)$ is C-l.s.c. at x ;
- (iii) $F_n \xrightarrow{HC} F$ on A ;
- (iv) $F_n(\cdot)$ is $-C$ -convex on A .

Then, there exists $n_0 \in \mathbb{N}$ such that

$$F_n(y_n) \ll_E^u F_n(x_n), \quad \forall n \geq n_0.$$

Proof The proof of this lemma is done by using similar arguments as in the proof of Theorem 4.4 of [33].

Proof of Theorem 4.4. Since $K(p)$ is compact and F is l.s.c and compact on A , we assert by Remark 4.3(i) that $W_u(p)$ is nonempty. Let $p_n \rightarrow p$ and let $\{x_n\}$ be a sequence such that $x_n \in W_u(p_n)$ for all n . Since $K(\cdot)$ is compact and B-u.s.c. at p , we get by Lemma 2.4 (ii) that the sequence $\{x_n\}$ admits a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow \bar{x} \in K(p)$. Now, we claim that $\bar{x} \in W_u(p)$. Suppose by contrary that $\bar{x}$ is not a weak E - u -minimal solution of u -(ISOP) $_p$. Then, there exists $x' \in K(p)$ such that $F(x') \ll_E^u F(\bar{x})$, i.e.

$$F(x') \subseteq F(\bar{x}) - \text{int} E.$$

It follows from $K(\cdot)$ is B-l.s.c. at p and Lemma 2.4 (i) that there exists a sequence $\{\hat{x}_{n_k}\}$ with $\hat{x}_{n_k} \in K(p_{n_k})$ such that $\hat{x}_{n_k} \rightarrow x'$. We conclude from Lemma 4.5 that there exists $n_0 \in \mathbb{N}$ such that

$$F_{n_k}(\hat{x}_{n_k}) \subseteq F_{n_k}(x_{n_k}) - \text{int} E, \quad \forall k \geq k_0.$$

i.e., $F_{n_k}(\hat{x}_{n_k}) \ll_E^u F_{n_k}(x_{n_k})$. This is a contradiction as $x_n \in W_u(p_n)$ for all n . So, the problem u -(ISOP) is generalized Hadamard well-posed for the weak efficiency, which completes the proof. $\square$

Similarly, we have sufficient conditions ensuring the generalized Hadamard well-posed for the weak efficiency of l -(ISOP), and its proof is analogous to that of the above theorem and is therefore omitted.

Proposition 4.6 Let E be a convex improvement set, C be a cone defined in Section 2. For a given $p := (F, h) \in P$, let F be compact-valued on A , where A is a convex set. For any $p_n := (F_n, h_n)$ and $n \in \mathbb{N}$, assume that the following conditions hold:

- (i) $K(\cdot)$ is Berge continuous and compact at p ;

- (ii) $F(\cdot)$ is H-C-u.s.c. on A and $F(\cdot)$ is l.s.c. on A ;
- (iii) $-F_n \xrightarrow{H(-C)} -F$ on A ;
- (iv) $F_n(\cdot)$ be C -convex on A .

Then, the problem l -(ISOP) is generalized Hadamard well-posed for the weak efficiency at p .

Let us now provide sufficient conditions for the problem i -(ISOP), $i \in \{u, l\}$, to be generalized Hadamard well-posed for the efficiency at a reference parameter.

Theorem 4.7 Let E be a convex improvement set, C be a cone defined in Section 2 and $n \in \mathbb{N}$. For a given $p = (F, h) \in P$, let F be compact-valued on A , where A is a convex set. For any $p_n := (F_n, h_n)$ and $n \in \mathbb{N}$, consider the following conditions:

- (i) $K(\cdot)$ is Berge continuous and compact at p ;
- (ii) $F_n \xrightarrow{HC} F$ on A and $F_n(\cdot)$ is $-C$ -convex on A ;
- (iii) $F_n \xrightarrow{H(-C)} F$ on A and $F_n(\cdot)$ is C -convex on A ;
- (iv) $-F(\cdot)$ is H-C-u.s.c. on A and $F(\cdot)$ is l.s.c. on A ;
- (v) $F(\cdot)$ is H-C-u.s.c. on A and $F(\cdot)$ is l.s.c. on A ;
- (vi) $F(\cdot)$ is strictly naturally C -quasiconvex on A .

We have the following assertions:

- (a) Under conditions (i), (ii), (iv) and (vi), the problem u -(ISOP) is generalized Hadamard well-posed for the efficiency at p .
- (b) Under conditions (i), (iii), (v) and (vi), the problem l -(ISOP) is generalized Hadamard well-posed for the efficiency at p .

Proof (a) Since $K(p)$ is compact and F is l.s.c and compact on A , we assert by Remark 4.3(i) that $E_u(p)$ is nonempty. Let $p_n \rightarrow p$ and let $\{x_n\}$ be a sequence such that $x_n \in E_u(p_n)$ for all $n \in \mathbb{N}$. Since $F(\cdot)$ is strictly naturally C -quasiconvex, it holds that $x_n \in W_u(p_n)$ for all n by Proposition 2.7. Arguing similarly as in the proof of Theorem 4.4, we can conclude that $\bar{x} \in W_u(p)$, which entails that $\bar{x} \in E_u(p)$ by noting Proposition 2.7 again. So, the problem u -(ISOP) is generalized Hadamard well-posed for the efficiency at p .

(b) Similarly, we can show that the problem l -(ISOP) is generalized Hadamard well-posed for the efficiency at p .

Remark 4.4 Note that the concepts of Hadamard well-posedness for implicit vector optimization problems were first investigated in [35]. However, to the best of our knowledge, such concepts have not been explored for the setting of (ISOP). In Theorem 4.7, by using the Hausdorff C -continuity of F and other related properties, we provide new sufficient conditions for the generalized Hadamard well-posedness for the problem (ISOP) under different kinds of set order relations.

The following example shows how we can verify the sufficient conditions for the generalized Hadamard well-posedness given in Theorem 4.7.

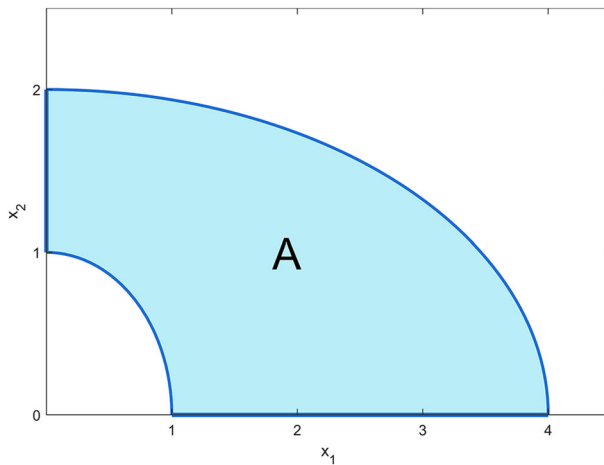


Fig. 2 The graph of A

Example 4.2 Let $X = Y = Z = \mathbb{R}^2$, $C = D = \mathbb{R}_+^2$, $E = \mathbb{R}_+^2 + \{(1, 0)\}$, $T = [0, 1]$ and

$$A = \left\{ (x_1, x_2) \in X \mid x_1^2 + x_2^2 \geq 1, \frac{x_1^2}{16} + \frac{x_2^2}{4} \leq 1, x_1 \geq 0, x_2 \geq 0 \right\}.$$

The graph of A is shown in Figure 2.

We consider $h, h_n : A \times T \rightarrow \mathbb{R}^2$, $F, F_n : A \rightrightarrows \mathbb{R}^2$, $n \in \mathbb{N}$, which are given as follows, for all $x = (x_1, x_2) \in A$, $t \in T$,

$$\begin{aligned} h(x, t) &= (x_1^2 - 2x_1 - t, x_1 + x_2 + t), \\ h_n(x, t) &= \left(x_1^2 - 2x_1 - t + \frac{1}{n}, x_1 + x_2 + t + \frac{1}{n} \right), \\ F(x) &= [(x_1, 0), (x_1, 2 + \cos \pi x_1)], \\ F_n(x) &= \left[\left(x_1 + \frac{1}{n}, 0 \right), \left(x_1 + \frac{1}{n}, 2 + \cos \pi \left(x_1 + \frac{1}{n} \right) \right) \right]. \end{aligned}$$

The graphs of F and F_n are shown in Figures 3 and 4, respectively.

Obviously, $-F(\cdot)$ is Hausdorff C -continuous and compact-valued on A and conditions (i)-(iii) of Theorem 4.7 are fulfilled. By direct computation, we obtain

$$\begin{aligned} K(p) &= \{x = (x_1, x_2) \in A \mid 0 \leq x_1 \leq 2\}, \\ K(p_n) &= \left\{ x = (x_1, x_2) \in A \mid 1 - \sqrt{\frac{n-1}{n}} \leq x_1 \leq 1 + \sqrt{\frac{n-1}{n}} \right\} \end{aligned}$$

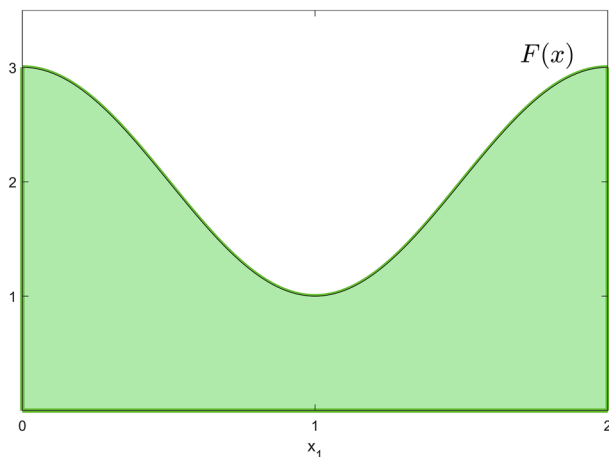


Fig. 3 The graph of F

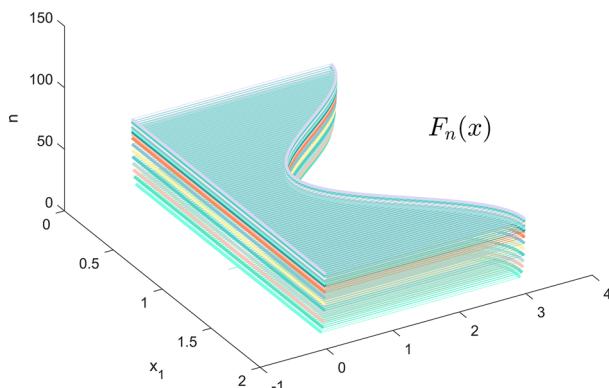


Fig. 4 The graph of F_n

and

$$E_u(p) = \left\{ (x_1, x_2) \in A \mid 0 \leq x_1 < \frac{3}{2} \right\},$$

$$E_u(p_n) = \left\{ (x_1, x_2) \in A \mid 1 - \sqrt{\frac{n-1}{n}} \leq x_1 < \frac{1}{2} + \sqrt{\frac{n-1}{n}} \right\},$$

where $p := (F, h)$ and $p_n := (F_n, h_n)$ for $n \in \mathbb{N}$. In this setting, by virtue of Definition 4.3, u -(ISOP) is generalized Hadamard well-posed for the efficiency at p , which shows that Theorem 4.7 is applicable.

5 Conclusions

In this article, we examined the stability and well-posedness for a parametric set optimization problem (PSOP) involving perturbations and the Hausdorff C -continuous set-valued mappings. More specifically, we first provided sufficient conditions for guaranteeing the outer-continuity, outer-openness and inner-openness of the solution mappings of (PSOP). By means of the improvement sets, we then investigated the closedness and the Levitin-Polyak (LP) well-posedness of E - i -minimal solution sets for a set optimization problem for $i \in \{u, l\}$. Based on the Hausdorff C -convergence and related properties, the several types of generalized Hadamard well-posedness for a parametric implicit set optimization problem (ISOP) under the set lower and upper order relations were also obtained.

It would be interesting to see how we can employ the current approach to explore the connectedness and the Hölder continuity of solutions for (PSOP) and (ISOP).

Acknowledgements The authors would like to thank the editor and reviewers for their constructive comments and valuable suggestions that helped improve the final version of the paper. The first author was partially supported by the National Natural Science Foundation of China (12271067), the Chongqing Natural Science Foundation (CSTB2024NSCQ-MSX0973), and the Science and Technology Research Key Program of Chongqing Municipal Education Commission (KJZD-K202200704). The third author was partially supported by the Mid and Early Career Academic Research Support Scheme 2024-2025 of Brunel University of London. The fifth author was supported by the Postgraduate Scientific Research and Innovation Project of Chongqing Jiaotong University (2025S0089).

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