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Optimality and solutions for conic robust multiobjective programs

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Abstract

This paper presents a robust framework for handling a *conic multiobjective* linear optimization problem, where the objective and constraint functions are involving affinely parameterized data uncertainties. More precisely, we examine optimality conditions and calculate efficient solutions of the conic robust multiobjective linear problem. We provide necessary and sufficient linear conic criteria for efficiency of the underlying conic robust multiobjective linear program. It is shown that such optimality conditions can be expressed in terms of linear matrix inequalities and second-order conic conditions for a multiobjective semidefinite program and a multiobjective second order conic program, respectively. We show how efficient solutions of the conic robust multiobjective linear problem can be found via its conic programming reformulation problems including semidefinite programming and second-order cone programming problems. Numerical examples are also provided to illustrate that the proposed conic programming reformulation schemes can be employed to find efficient solutions for concrete problems including those arisen from practical applications.

Keywords Multiobjective optimization · Robust optimization · Efficient solution · Optimality condition · Conic reformulation · Semidefinite programming

Mathematics Subject Classification 65K10 · 49K99 · 90C46 · 90C29

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1 Introduction

Conic optimization problems are constrained optimization models in which their constraints are defined via certain closed convex cones, see e.g., [4, 9, 16]. Conic optimization problems have been intensively studied by many researchers as they belong to a wide class of structured decision making optimization problems that encompass all important prominent classes of mathematical models including numerically tractable classes of *linear* programming, *second-order cone* programming and *semidefinite programming* problems [8], as well as a more general class of *convex optimization* problems because any convex optimization problem can be reformulated as a conic optimization problem [31].

In the real-world, many decision-making models not only admit *multiple objectives* (cf. [2, 3, 17, 28]) that are possible in conflict but also involve *uncertain data* (cf. [5, 7, 34]) because of, for instance, the lack of information, measurement errors or unknown future developments. So, discovering new frameworks or classes of multiobjective optimization problems, tangible approaches as well as associated methods that are capable of dealing with uncertainty data, has emerged as a crucial aim of research in multiobjective optimization. Such multiobjective optimization (called *robust multiobjective*) models are able to generate (weak) Pareto solutions, which are *immune* from uncertainty data, see e.g., [1, 12, 18–20, 24, 25, 35] and other references therein.

A recent trend in robust multiobjective optimization is to identify classes of robust multiobjective problems whose optimality conditions are numerically verifiable or their relaxation problems can be reformulated and solved by means of linear programming, second-order cone programming or semidefinite programming problems [10, 11, 13, 14, 21, 22, 26, 27]. In particular, by using an alternative theorem for a robust linear inequality system, the paper [10] provided necessary and sufficient optimality conditions for weak Pareto solutions of a robust multiobjective linear programming problem. Recently, the authors in [14] developed tractable optimality conditions as well as semidefinite reformulation schemes to identify robust (weak) efficient solutions for quadratic multiobjective problems under data uncertainty. The interested reader is referred to [26, 27] for approximate approaches to solve a subclass of robust convex polynomial multiobjective optimization problems.

This paper aims to study a broad class of conic uncertain/robust multiobjective linear programming problems defined as follows.

Conic Multiobjective Linear Optimization Programs. A conic *uncertain* multiobjective linear problem is defined by

$$\min_{x\in\mathbb{R}^n} \left\{ \left(c^1(u^1)^\top x + \beta^1(u^1), \ldots, c^p(u^p)^\top x + \beta^p(u^p)\right) \mid A(v)x - b(v) \in -K \right\}, \tag{UC}$$

where $u^j \in U_j$, $j = 1, \ldots, p$ and $v \in V$ are *uncertain* parameters, $U_j \subset \mathbb{R}^s$, $j = 1, \ldots, p$ and $V \subset \mathbb{R}^{s_0}$ are *uncertainty* sets that are assumed to be nonempty and compact, $A : \mathbb{R}^{s_0} \to L(\mathbb{R}^n, \mathbb{R}^m)$, $b : \mathbb{R}^{s_0} \to \mathbb{R}^m$ are affine maps, $K \subset \mathbb{R}^m$ is a closed pointed (i.e., $K \cap (-K) = \{0\}$) convex cone with the nonempty interior (i.e., $\operatorname{int} K \neq \emptyset$), and $c^j : \mathbb{R}^s \to \mathbb{R}^n$, $\beta^j : \mathbb{R}^s \to \mathbb{R}$, $j = 1, \ldots, p$ are affine maps defined respectively by

$$c^j(u^j) := c_0^j + \sum_{i=1}^{s} u_i^j c_i^j, \quad \beta^j(u^j) := \beta_0^j + \sum_{i=1}^{s} u_i^j \beta_i^j \tag{1.1}$$

for $u^j := (u_1^j, \ldots, u_s^j) \in \mathbb{R}^s$ with $\beta_i^j \in \mathbb{R}$ and $c_i^j \in \mathbb{R}^n$ fixed for $j = 1, \ldots, p, i = 0, 1, \ldots, s$. Note that the notation $L(\mathbb{R}^n, \mathbb{R}^m)$ stands for the space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

To tackle the conic uncertain multiobjective problem (UC), we consider its *robust* counterpart as follows:

$$\min_{x\in\mathbb{R}^n}\left\{\Big(\max_{u^1\in U_1}\{c^1(u^1)^\top x+\beta^1(u^1)\},\ldots,\max_{u^p\in U_p}\{c^p(u^p)^\top x+\beta^p(u^p)\}\Big)\mid A(v)x-b(v)\in -K,\ \forall v\in V\right\}. \tag{RC}$$

To proceed, let us define the notions of robust (weak) efficient solutions in the sense of *minmax robustness* (cf. [1, 25]) for multiobjective optimization problems. For the sake of simplicity, we use the notation $f_j(x) := \max_{u^j\in U_j}\{c^j(u^j)^\top x + \beta^j(u^j)\}$, $j = 1,\ldots,p$ for $x\in\mathbb{R}^n$ and denote $\mathcal{F} := \{x\in\mathbb{R}^n \mid A(v)x - b(v)\in -K,\ \forall v\in V\}$ the set of all robust feasible points of problem (UC). This enables us to align notions of Pareto optimality in the robust case with those for their non-robust counterpart. We note that the direct computation of the functions f_j is inefficient in practice.

Definition 1.1 (Robust weak/efficient solutions) For the problem (UC), let $\bar{x}\in\mathcal{F}$.

(i) One says that $\bar{x}$ is a *robust weak efficient solution* of problem (UC) if it is a weak efficient solution of problem (RC); i.e., there is no other $x\in\mathcal{F}$ satisfying

$$f_j(x) < f_j(\bar{x}),\ \ j = 1,\ldots,p.$$

(ii) One says that $\bar{x}$ is a *robust efficient solution* of problem (UC) if it is an efficient solution of problem (RC); i.e., there is no other $x\in\mathcal{F}$ satisfying

$$f_j(x)\le f_j(\bar{x}),\ \ j = 1,\ldots,p\ \ \text{and}\ \ f_j(x) < f_j(\bar{x})\ \ \text{for some } j\in\{1,\ldots,p\}.$$

The model (UC) or its robust counterpart (RC) encompasses a broad class of uncertain multiobjective optimization problems including uncertain multiobjective linear programs and standard conic uncertain multiobjective linear optimization problems. In particular, if K is the nonnegative orthant of $\mathbb{R}^n$ (i.e., $K := \mathbb{R}^n_+$), V is a spectrahedron (see [36] or (2.17) below) and there are *no* affine maps β^j, $j = 1,\ldots,p$ (i.e., $\beta^j := 0$, $j = 1,\ldots,p$), then the problem (RC) collapses to a robust multiobjective optimization model studied in [13]. Moreover, if there is *no uncertainty* in the objectives (i.e., $c_i^j := 0, b_i^j := 0, j = 1,\ldots,p, i = 1,\ldots,s$), the resulting problem further reduces to a robust multiobjective model examined in [10]. The problem (UC) includes popular conic uncertain multiobjective linear programs such as uncertain *multiobjective semidefinite programming* problems and uncertain *multiobjective second-order cone programming* problems, which we will examine in the forthcoming sections. It also covers other uncertain multiobjective linear programs such as those discussed in [22] by appropriately specifying the data of K, A, b and U_j, $j = 1,\ldots,p$.

To the best of our knowledge, a study of optimality conditions and conic reformulations for finding robust (weak) efficient solutions of the conic uncertain multiobjective linear program of type (UC) has not been available in the literature. Such an investigation would be complicated due to the challenges arisen in handling data uncertainties of the objectives and conic constraints. Furthermore, the obtained optimality conditions and the relaxation/reformulation schemes for solving the underlying problem would not be numerically verified by virtue of the general structures of the uncertainty sets. To this end, we assume throughout the paper that the uncertainty set V is a polytope given by $V := \text{conv}\,\{\bar{v}^1,\ldots,\bar{v}^q\}$ with $\bar{v}^l\in\mathbb{R}^{s_0}$ for $l = 1,\ldots,q$ (see e.g., [5]), and the uncertainty sets U_j, $j = 1,\ldots,p$ are *cone-based sets* (see e.g., [6]) given by

$$U_j := \{u^j := (u^j_1,\ldots,u^j_s)\in\mathbb{R}^s \mid C_j u^j - d^j\in -K_j\} \tag{1.2}$$

with $C_j \in L(\mathbb{R}^s, \mathbb{R}^{m_j})$, $d^j \in \mathbb{R}^{m_j}$ and $K_j \subset \mathbb{R}^{m_j}$, $j = 1, \ldots, p$, where K_j, $j = 1, \ldots, p$ are closed pointed convex cones with $\operatorname{int} K_j \neq \emptyset$.

The main purposes of this work are to examine robust optimality conditions and compute robust (weak) efficient solutions for the uncertain conic multiobjective linear problem (UC). More exactly, we provide necessary and sufficient criteria in terms of linear *conic conditions* for robust (weak) efficiency of the uncertain conic multiobjective linear problem (UC). We establish that these optimality conditions can be displayed by way of linear matrix inequalities and second-order cone conditions for subclasses of the underlying conic multiobjective problems such as the class of multiobjective semidefinite programming problems and the class of multiobjective second-order cone programming problems. With the help of conic optimality conditions, we also show how robust weak/efficient solutions of the uncertain conic multiobjective linear problem (UC) can be located by solving conic programming reformulation problems including semidefinite programming and second-order cone programming problems.

To show the verifiability and efficacy of our approach, we give numerical examples that illustrate how the proposed conic programming reformulation schemes can be employed to identify robust (weak) efficient solutions for (concrete) uncertain multiobjective problems inspired by practical applications. The simulation results demonstrate that the proposed conic uncertain multiobjective problem is capable of modeling practical problems under data uncertainties. Moreover, the corresponding conic reformulation schemes are able to generate multiple robust Pareto solutions for such models. As a result, the proposed conic uncertain/robust multiobjective models and associated conic reformulation schemes not only empower the decision-maker to more readily identify preferred (optimal) trade-off trends, but also enable the opportunity to *stabily* achieve the corresponding (optimal) trade-off values under uncertainty of the inputs for actual problems.

The rest of the paper is organized as follows. In Section 2, we first establish conic conditions for robust (weak) efficiency of the uncertain conic multiobjective linear problem (UC). We then derive corresponding results for the class of multiobjective semidefinite programs and for the class of multiobjective second-order cone programming problems. Section 3 presents the conic reformulations and there we develop schemes as to how to calculate robust (weak) efficient solutions for the problem (UC) via its conic reformulations. In Section 4, we present numerical examples including those inspired by practical applications. Section 5 concludes the obtained results with an outlook on conic uncertain/robust multiobjective optimization problems.

2 Conic optimality conditions

Let us provide some notations and definitions, which will be used throughout this paper. The notation $\mathbb{R}^n$ signifies the Euclidean space whose norm is denoted by $\|\cdot\|$ for each $n \in \mathbb{N} := \{1, 2, \ldots\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y \rangle := x^\top y$ for all $x, y \in \mathbb{R}^n$. For each $j \in \{1, \ldots, n\}$, e^n_j is the unit vector in $\mathbb{R}^n$ whose jth element is one and the other elements are all zero. We denote by 0 the origin of a space and we also use 0_n to denote the origin of $\mathbb{R}^n$ for more clarification. We denote by $\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$, while $\mathbb{R}^1_+ := \mathbb{R}_+ = [0, +\infty)$. For a nonempty set $\Omega \subset \mathbb{R}^n$, $\operatorname{conv} \Omega$ denotes the convex hull of Ω and $\operatorname{int} \Omega$ stands for the interior of Ω. As usual, the notation $L(W, Z)$ stands for the space of all linear transformations between finite dimensional spaces W and Z. The dual cone of a cone $K \subset \mathbb{R}^m$ is given by

$$K^* := \{y \in \mathbb{R}^m \mid y^\top k \geq 0 \text{ for all } k \in K\}.$$

An $(n \times n)$ real matrix A is symmetric if $A^\top = A$, where $A^\top$ is the transpose of A. The set of all symmetric $(n \times n)$ real matrices is denoted by S^n. A matrix $B \in S^n$ is said to be positive semidefinite, denoted by $B \succeq 0$, whenever $x^\top Bx \geq 0$ for all $x \in \mathbb{R}^n$. If $x^\top Bx > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$, then B is called positive definite, denoted by $B \succ 0$. The notation S^n_+ stands for the set of all positive semidefinite matrices in S^n. The trace of an $(n \times n)$ real matrix A is defined by $\text{Tr}(A) = \sum_{j=1}^n a_{jj}$, where a_{jj} is the entry in the jth row and jth column of A for $j = 1, \ldots, n$. Note that the space S^n can be treated as a Euclidean space equipped with the Frobenius inner product $\langle A, B\rangle := \text{Tr}(AB)$ for $A, B \in S^n$ (see e.g., [5, Page 150]). Given $v := (v_1, \ldots, v_n)$, the notation $\text{diag}(v)$ or $\text{diag}(v_1, \ldots, v_n)$ denotes a diagonal matrix with entries $v_1, \ldots, v_n$ along the diagonal and zeros elsewhere. Similarly, $\text{diag}(A_1, \ldots, A_n)$ denotes the block diagonal matrix with submatrices $A_1, \ldots, A_n$ along the diagonal and zero submatrices elsewhere.

For a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, the *adjoint* linear transformation, denoted by $A^\top$, is the map $A^\top : \mathbb{R}^m \to \mathbb{R}^n$ satisfying

$$\langle Ax, y\rangle = \langle x, A^\top y\rangle \text{ for all } x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

We are now in a position to present necessary/sufficient optimality criteria, which are exhibited via linear conic conditions for robust (weak) efficient solutions of problem (UC).

Theorem 2.1 *For the problem (UC), let $\bar{x} \in \mathcal{F}$.*

(i) **(Necessary conic conditions)** *Assume that the strict constraint qualification holds, i.e., there exists $x^0 \in \mathbb{R}^n$ such that*

$$A(v)x^0 - b(v) \in -\text{int}\, K, \ \forall v \in V. \tag{2.3}$$

Let $\bar{x}$ be a robust weak efficient solution of (UC). Then, there exist $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+ \backslash \{0\}$, $\alpha^i_j \in \mathbb{R}, i = 1, \ldots, s, j = 1, \ldots, p$ and $\lambda^l \in K^*, l = 1, \ldots, q$ such that

$$\sum_{j=1}^p (\alpha_j c_0^j + \sum_{i=1}^s \alpha^i_j c^j_i) + \sum_{l=1}^q (\bar{A}^l)^\top \lambda^l = 0, \tag{2.4}$$

$$\sum_{j=1}^p (\alpha_j \beta_0^j + \sum_{i=1}^s \alpha^i_j \beta^j_i) - \sum_{l=1}^q (\lambda^l)^\top \bar{b}^l - \sum_{j=1}^p \alpha_j f_j(\bar{x}) \geq 0, \tag{2.5}$$

$$C_j(\alpha^1_j, \ldots, \alpha^s_j) - \alpha_j d^j \in -K_j, \ j = 1, \ldots, p, \tag{2.6}$$

where $\bar{A}^l := A(\bar{v}^l)$ and $\bar{b}^l := b(\bar{v}^l)$ for $l = 1, \ldots, q$.

(ii) **(Sufficient conditions for robust weak efficiency)** *Assume that there exist $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+ \backslash \{0\}$, $\alpha^i_j \in \mathbb{R}, i = 1, \ldots, s, j = 1, \ldots, p$ and $\lambda^l \in K^*, l = 1, \ldots, q$ satisfying (2.4), (2.5) and (2.6). Then, $\bar{x}$ is a robust weak efficient solution of (UC).*

(iii) **(Sufficient conditions for robust efficiency)** *Assume that there exist $(\alpha_1, \ldots, \alpha_p) \in \text{int}\mathbb{R}^p_+$, $\alpha^i_j \in \mathbb{R}, i = 1, \ldots, s, j = 1, \ldots, p$ and $\lambda^l \in K^*, l = 1, \ldots, q$ satisfying (2.4), (2.5) and (2.6). Then, $\bar{x}$ is a robust efficient solution of (UC).*

Proof Note that $V := \text{conv}\,\{\bar{v}^1, \ldots, \bar{v}^q\}$ and we denote $\bar{A}^l := A(\bar{v}^l)$ and $\bar{b}^l := b(\bar{v}^l)$, where $\bar{v}^l \in \mathbb{R}^{s_0}, l = 1, \ldots, q$ are fixed. For each $v \in V$, there exist $\gamma_l \geq 0$, $\sum_{l=1}^q \gamma_l = 1$ such that

$v = \sum_{l=1}^{q} \gamma_l \bar{v}^l$. Since A and b are affine maps, it holds that

$$A(v)x - b(v) = \sum_{l=1}^{q} \gamma_l A(\bar{v}^l)x - \sum_{l=1}^{q} \gamma_l b(\bar{v}^l) = \sum_{l=1}^{q} \gamma_l \big(\bar{A}^l x - \bar{b}^l\big).$$

Moreover, by the convexity of K, we see that $A(v)x - b(v) \in -K$ for all $v \in V$ if and only if $\bar{A}^l x - \bar{b}^l \in -K$ for all $l = 1, \ldots, q$, and so the problem (RC) is equivalent to the following problem

$$\min_{x \in \mathbb{R}^n} \left\{ \big(f_1(x), \ldots, f_p(x)\big) \mid \bar{A}^l x - \bar{b}^l \in -K,\ l = 1, \ldots, q \right\}, \tag{AP}$$

where $f_j(x) := \max_{u^j \in U_j} \{c^j(u^j)^\top x + \beta^j(u^j)\}$, $j = 1, \ldots, p$ for $x \in \mathbb{R}^n$ as above.

(i) Assume that the strict constraint qualification in (2.3) holds, and that $\bar{x}$ is a robust weak efficient solution of (UC). Letting

$$\Omega := \{(r_1, \ldots, r_p, y^1, \ldots, y^q) \in \mathbb{R}^{p+qm} \mid \exists x \in \mathbb{R}^n,\ f_j(x) - f_j(\bar{x}) < r_j,\ j = 1, \ldots, p,\ y^l + \bar{b}^l \in \bar{A}^l x + K,\ l = 1, \ldots, q\},$$

we see that $\Omega \neq \emptyset$ due to $(f_1(x^0) - f_1(\bar{x}) + \epsilon, \ldots, f_p(x^0) - f_p(\bar{x}) + \epsilon, 0_{qm}) \in \Omega$ for any $\epsilon > 0$. Observe further that Ω is a convex set and, as $\bar{x}$ is a robust weak efficient solution of (UC), it follows that $(0_p, 0_{qm}) \notin \Omega$. Using a separation theorem (see e.g., [30, Theorem 2.5]), we find $0 \neq (\alpha, \lambda) \in \mathbb{R}^p \times \mathbb{R}^{qm}$ such that

$$\inf \left\{ \alpha^\top r + \lambda^\top y \mid (r, y) \in \Omega \right\} \geq 0, \tag{2.7}$$

where $r := (r_1, \ldots, r_p) \in \mathbb{R}^p$ and $y := (y^1, \ldots, y^q) \in \mathbb{R}^{qm}$. Observe by (2.7) that $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+$ and $\lambda := (\lambda^1, \ldots, \lambda^q)$, $\lambda^l \in K^*, l = 1, \ldots, q$.
Let $\epsilon > 0$. Since $(f_1(x) - f_1(\bar{x}) + \epsilon, \ldots, f_p(x) - f_p(\bar{x}) + \epsilon, \bar{A}^1 x - \bar{b}^1, \ldots, \bar{A}^q x - \bar{b}^q) \in \Omega$ for each $x \in \mathbb{R}^n$, we get by (2.7) that

$$\sum_{j=1}^{p} \alpha_j \big(f_j(x) - f_j(\bar{x}) + \epsilon\big) + \sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x - \bar{b}^l) \geq 0 \tag{2.8}$$

for all $x \in \mathbb{R}^n$. If $\alpha = 0$, there exists $l_0 \in \{1, \ldots, q\}$ such that $\lambda^{l_0} \in K^* \backslash \{0\}$. Therefore, by (2.3), we assert (cf. [3, Lemma 3.21]) that $(\lambda^{l_0})^\top (\bar{A}^{l_0} x^0 - \bar{b}^{l_0}) < 0$, which ensures that $\sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x^0 - \bar{b}^l) < 0$ due to the fact that $\lambda^l \in K^*$ for all $l = 1, \ldots, q$. This contradicts (2.8) and so $\alpha \neq 0$.
As $\epsilon > 0$ was arbitrarily chosen, we conclude from (2.8) that

$$\sum_{j=1}^{p} \alpha_j f_j(x) + \sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x - \bar{b}^l) \geq \sum_{j=1}^{p} \alpha_j f_j(\bar{x}) \ \text{ for all } \ x \in \mathbb{R}^n,$$

which entails that

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{p} \alpha_j \max_{u^j \in U_j} \{c^j(u^j)^\top x + \beta^j(u^j)\} + \sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x - \bar{b}^l) \right\} \geq \sum_{j=1}^{p} \alpha_j f_j(\bar{x}). \tag{2.9}$$

By denoting $U := \Pi_{j=1}^{p} U_j$, (2.9) reduces further to the following one

$$\inf_{x\in\mathbb{R}^n} \max_{(u^1,\dots,u^p)\in U} \left\{ \sum_{j=1}^{p} \alpha_j \Big((c_0^j)^\top x + \sum_{i=1}^{s} u_i^j (c_i^j)^\top x + \beta_0^j + \sum_{i=1}^{s} u_i^j \beta_i^j \Big) + \sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x - \bar{b}^l) \right\}$$
$$\geq \sum_{j=1}^{p} \alpha_j f_j(\bar{x}), \tag{2.10}$$

where $u^j := (u_1^j, \dots, u_s^j)$, $j = 1, \dots, p$. Consider a function $F : \mathbb{R}^n \times \mathbb{R}^{ps} \to \mathbb{R}$ given by $F(x,u) := \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top x + \sum_{i=1}^{s} u_i^j (c_i^j)^\top x + \beta_0^j + \sum_{i=1}^{s} u_i^j \beta_i^j \big) + \sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x - \bar{b}^l)$ for $x \in \mathbb{R}^n$ and $u := (u^1, \dots, u^p) \in \mathbb{R}^{ps}$. Since F is an affine function in variable x and in variable u, (2.10) and a minimax theorem (cf. [33, Theorem 4.2]) entail that

$$\max_{u\in U} \inf_{x\in\mathbb{R}^n} F(x,u) = \inf_{x\in\mathbb{R}^n} \max_{u\in U} F(x,u) \geq \sum_{j=1}^{p} \alpha_j f_j(\bar{x}).$$

This shows that there exists $\bar{u} := (\bar{u}^1, \dots, \bar{u}^p)$, where $\bar{u}^j := (\bar{u}_1^j, \dots, \bar{u}_s^j) \in U_j$, $j = 1, \dots, p$, such that

$$\inf_{x\in\mathbb{R}^n} F(x,\bar{u}) \geq \sum_{j=1}^{p} \alpha_j f_j(\bar{x}). \tag{2.11}$$

Letting $\alpha_j^i := \alpha_j \bar{u}_i^j$, $j = 1, \dots, p$, $i = 1, \dots, s$, we claim that

$$C_j(\alpha_j^1, \dots, \alpha_j^s) - \alpha_j d^j \in -K_j, \;\; j = 1, \dots, p. \tag{2.12}$$

To see this, consider any $j \in \{1, \dots, p\}$. If $\alpha_j = 0$, then $\alpha_j^i = 0$ for all $i = 1, \dots, s$ and thus (2.12) holds trivially. Otherwise, we have $\alpha_j > 0$. Then, by $\bar{u}^j \in U_j$, $j = 1, \dots, p$, it holds that

$$\alpha_j \left(C_j \left(\frac{\alpha_j^1}{\alpha_j}, \dots, \frac{\alpha_j^s}{\alpha_j} \right) - d^j \right) = \alpha_j (C_j \bar{u}^j - d^j) \in -K_j, \tag{2.13}$$

which shows that (2.12) holds as well. Now, we derive from (2.11) that

$$\sum_{j=1}^{p} \alpha_j \left((c_0^j)^\top x + \sum_{i=1}^{s} \bar{u}_i^j (c_i^j)^\top x + \beta_0^j + \sum_{i=1}^{s} \bar{u}_i^j \beta_i^j \right) + \sum_{l=1}^{q} (\lambda^l)^\top (\bar{A}^l x - \bar{b}^l) - \sum_{j=1}^{p} \alpha_j f_r(\bar{x}) \geq 0$$

for all $x \in \mathbb{R}^n$. This is equivalent to the following conditions

$$\sum_{j=1}^{p} \left(\alpha_j c_0^j + \sum_{i=1}^{s} \alpha_j^i c_i^j \right) + \sum_{l=1}^{q} (\bar{A}^l)^\top \lambda^l = 0,$$
$$\sum_{j=1}^{p} \left(\alpha_j \beta_0^j + \sum_{i=1}^{s} \alpha_j^i \beta_i^j \right) - \sum_{l=1}^{q} (\lambda^l)^\top \bar{b}^l - \sum_{j=1}^{p} \alpha_j f_j(\bar{x}) \geq 0,$$

which show that (2.4) and (2.5) are valid. So (i) holds.

(ii) Let $(\alpha_1,\ldots,\alpha_p)\in\mathbb{R}^p_+\setminus\{0\}$, $\alpha^i_j\in\mathbb{R}, i=1,\ldots,s, j=1,\ldots,p$ and $\lambda^l\in K^*, l=1,\ldots,q$ be such that (2.4), (2.5) and (2.6) hold.
Consider any $j\in\{1,\ldots,p\}$. By the compactness of U_j, we claim by (2.6) that if $\alpha_j=0$, then $\alpha^i_j=0$ for all $i=1,\ldots,s$. Assume on the contrary that $\alpha_j=0$ but there exists $i_0\in\{1,\ldots,s\}$ with $\alpha^{i_0}_j\neq 0$. In this case, we get by (2.6) that $C_j(\alpha^1_j,\ldots,\alpha^s_j)\in -K_j$. Take any $\bar u^j:=(\bar u^j_1,\ldots,\bar u^j_s)\in U_j$. By definition, $C_j\bar u^j-d^j\in -K_j$ and thus

$$C_j\big(\bar u^j+t(\alpha^1_j,\ldots,\alpha^s_j)\big)-d^j=(C_j\bar u^j-d^j)+tC_j(\alpha^1_j,\ldots,\alpha^s_j)\in -K_j \text{ for all } t>0.$$

This means that $\bar u^j+t(\alpha^1_j,\ldots,\alpha^s_j)\in U_j$ for all $t>0$, which contradicts the fact that $(\alpha^1_j,\ldots,\alpha^s_j)\neq 0_s$ and U_j is bounded. Consequently, our claim is valid.
Let us take $\hat u^j:=(\hat u^j_1,\ldots,\hat u^j_s)\in U_j$ and define $\tilde u^j:=(\tilde u^j_1,\ldots,\tilde u^j_s)$ with

$$\tilde u^j_i:=\begin{cases}\hat u^j_i & \text{if } \alpha_j=0,\\ \frac{\alpha^i_j}{\alpha_j} & \text{if } \alpha_j\neq 0,\end{cases}\quad i=1,\ldots,s.$$

Note by (2.6) that $C_j\big(\frac{\alpha^1_j}{\alpha_j},\ldots,\frac{\alpha^s_j}{\alpha_j}\big)-d^j\in -K_j$ whenever $\alpha_j\neq 0$, and so $\tilde u^j\in U_j$. Then, for any $x\in\mathbb{R}^n$, we have

$$\Big(\alpha_jc^j_0+\sum_{i=1}^s\alpha^i_jc^j_i\Big)^\top x=\alpha_j\Big(c^j_0+\sum_{i=1}^s\tilde u^j_ic^j_i\Big)^\top x=\alpha_jc^j\Big(\tilde u^j\Big)^\top x,\ \ j=1,\ldots,p, \tag{2.14}$$

where we remind that if $\alpha_j=0$, then $\alpha^i_j=0$ for all $i=1,\ldots,s$ as proved above. Similarly, we obtain that $\alpha_j\beta^j_0+\sum_{i=1}^s\alpha^i_j\beta^j_i=\alpha_j(\beta^j_0+\sum_{i=1}^s\tilde u^j_i\beta^j_i)=\alpha_j\beta^j(\tilde u^j)$ for $j=1,\ldots,p$. So, we get by (2.4) and (2.5) that

$$\sum_{j=1}^p\alpha_jc^j(\tilde u^j)^\top x+\sum_{j=1}^p\alpha_j\beta^j(\tilde u^j)+\sum_{l=1}^q(\lambda^l)^\top(\bar A^lx-\bar b^l)\geq\sum_{j=1}^p\alpha_jf_j(\bar x)\ \text{ for all }\ x\in\mathbb{R}^n \tag{2.15}$$

Now, assume that $\hat x\in\mathbb{R}^n$ is robust feasible for the problem (UC). Then, $\bar A^l\hat x-\bar b^l\in -K$ for $l=1,\ldots,q$, which guarantee that $\sum_{l=1}^q(\lambda^l)^\top(\bar A^lx-\bar b^l)\leq 0$. Evaluating (2.15) at $\hat x$, we arrive at $\sum_{j=1}^p\alpha_j\big(c^j(\tilde u^j)^\top\hat x+\beta^j(\tilde u^j)\big)\geq\sum_{j=1}^p\alpha_jf_j(\bar x)$ and thus,

$$\sum_{j=1}^p\alpha_jf_j(\hat x)\geq\sum_{j=1}^p\alpha_jf_j(\bar x) \tag{2.16}$$

due to the fact that $\alpha_j\geq 0$ and $f_j(\hat x)\geq c^j(\tilde u^j)^\top\hat x+\beta^j(\tilde u^j)$ for all $j=1,\ldots,p$. Keeping in mind that $(\alpha_1,\ldots,\alpha_p)\in\mathbb{R}^p_+\setminus\{0\}$, (2.16) ensures that there is no other $x\in\mathcal{F}$ with

$$f_j(x)<f_j(\bar x) \text{ for all } j=1,\ldots,p.$$

Consequently, $\bar x$ is a robust weak efficient solution of (UC).

(iii) Let $(\alpha_1,\ldots,\alpha_p)\in\operatorname{int}\mathbb{R}^p_+$, $\lambda^l\in K^*, l=1,\ldots,q$ and $\alpha^i_j\in\mathbb{R}, i=1,\ldots,s, j=1,\ldots,p$ be such that (2.4), (2.5) and (2.6) hold. Similarly, following the same argument of (ii), we come to the assertion in (2.16). This together with $(\alpha_1, \ldots, \alpha_p) \in \mathrm{int}\mathbb{R}^p_+$ entails that $\bar{x}$ is a robust efficient solution of problem (UC), which completes the proof. □

Multiobjective Semidefinite Programming Problems. Let us now consider a particular case of the problem (UC), which is defined by

$$\min_{x\in\mathbb{R}^n} \left\{ \left(c^1(u^1)^\top x + \beta^1(u^1), \ldots, c^p(u^p)^\top x + \beta^p(u^p)\right) \mid B(v) - \sum_{i=1}^n x_i A_i(v) \succeq 0 \right\}, \tag{USP}$$

where uncertain parameters u^j, $j = 1, \ldots, p$ and v, the uncertainty set $V := \mathrm{conv}\,\{\bar{v}^1, \ldots, \bar{v}^q\}$ with $\bar{v}^l \in \mathbb{R}^{s_0}$ for $l = 1, \ldots, q$ are defined as above, the maps $c^j : \mathbb{R}^s \to \mathbb{R}^n, \beta^j : \mathbb{R}^s \to \mathbb{R}$, $j = 1, \ldots, p$ are declared in (1.1), $A_i : \mathbb{R}^{s_0} \to S^k, i = 1, \ldots, n, B : \mathbb{R}^{s_0} \to S^k$ are affine maps, and the uncertainty sets U_j, $j = 1, \ldots, p$ are *spectrahedra* (see e.g., [36]) given by

$$U_j := \left\{ u^j := (u^j_1, \ldots, u^j_s) \in \mathbb{R}^s \mid D^j + \sum_{i=1}^s u^j_i C^j_i \succeq 0 \right\} \tag{2.17}$$

with given matrices $D^j \in S^{k_j}, C^j_i \in S^{k_j}, i = 1, \ldots, s$. Note that the spectrahedral sets in (2.17) encompass almost commonly used uncertainty sets in robust optimization including ball, box, cylinder and ellipsoid uncertainty data.

The robust counterpart of problem (USP) can be captured as follows:

$$\min_{x\in\mathbb{R}^n} \left\{ \left(\max_{u^1\in U_1} \{c^1(u^1)^\top x + \beta^1(u^1)\}, \ldots, \max_{u^p\in U_p} \{c^p(u^p)^\top x + \beta^p(u^p)\} \right) \mid \right. \tag{RSP}$$
$$\left. B(v) - \sum_{i=1}^n x_i A_i(v) \succeq 0, \ \forall v \in V \right\}.$$

We are now ready to derive linear matrix inequality (LMI) conditions for robust (weak) efficiency of (USP).

Corollary 2.2 (LMI optimality conditions) *For the problem (USP), let $\bar{x} \in \{x \in \mathbb{R}^n \mid B(v) - \sum_{i=1}^n x_i A_i(v) \succeq 0, \ \forall v \in V\}$. We have the following assertions.*

(i) *Let $x^0 \in \mathbb{R}^n$ be such that*

$$B(v) - \sum_{i=1}^n x^0_i A_i(v) \succ 0, \ \forall v \in V. \tag{2.18}$$

Assume that $\bar{x}$ is a robust weak efficient solution of problem (USP). Then, we can find $\alpha^i_j \in \mathbb{R}, i = 1, \ldots, s, j = 1, \ldots, p$, $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+\backslash\{0\}$ and $\lambda^l \in S^k_+, l = 1, \ldots, q$ such that

$$\sum_{j=1}^{p}\left(\alpha_j c_0^j+\sum_{i=1}^{s}\alpha_j^i c_i^j\right)+\sum_{l=1}^{q}\left(\mathrm{Tr}(\bar{A}_1^l\lambda^l),\ldots,\mathrm{Tr}(\bar{A}_n^l\lambda^l)\right)=0, \tag{2.19}$$

$$\sum_{j=1}^{p}\left(\alpha_j\beta_0^j+\sum_{i=1}^{s}\alpha_j^i\beta_i^j\right)-\sum_{l=1}^{q}\mathrm{Tr}(\bar{B}^l\lambda^l)-\sum_{j=1}^{p}\alpha_j f_j(\bar{x})\geq 0, \tag{2.20}$$

$$\alpha_j D^j+\sum_{i=1}^{s}\alpha_j^i C_i^j\succeq 0,\ \ j=1,\ldots,p, \tag{2.21}$$

where $f_j(\bar{x}) := \max_{u^j\in U_j}\{c^j(u^j)^\top\bar{x}+\beta^j(u^j)\}$, $j=1,\ldots,p$ *and* $\bar{A}_i^l := A_i(\bar{v}^l)$, $\bar{B}^l := B(\bar{v}^l), l=1,\ldots,q, i=1,\ldots,n$.

(ii) *Assume that there exist* $\alpha_j^i\in\mathbb{R}, i=1,\ldots,s,\ j=1,\ldots,p,\ (\alpha_1,\ldots,\alpha_p)\in\mathbb{R}_+^p\backslash\{0\}$ *and* $\lambda^l\in S_+^k, l=1,\ldots,q$ *satisfying (2.19), (2.20) and (2.21). Then, we assert that* $\bar{x}$ *is a robust weak efficient solution of (USP).*

(iii) *Assume that there exist* $\alpha_j^i\in\mathbb{R}, i=1,\ldots,s,\ j=1,\ldots,p,\ (\alpha_1,\ldots,\alpha_p)\in\mathrm{int}\mathbb{R}_+^p$ *and* $\lambda^l\in S_+^k, l=1,\ldots,q$ *satisfying (2.19), (2.20) and (2.21). Then,* $\bar{x}$ *is a robust efficient solution of (USP).*

Proof Consider a map $\mathcal{A}:\mathbb{R}^{s_0}\to L(\mathbb{R}^n,S^k)$ defined as follows: For each $v\in\mathbb{R}^{s_0}$, one has a linear transformation $\mathcal{A}(v):\mathbb{R}^n\to S^k$ given by $\mathcal{A}(v)x:=\sum_{i=1}^n x_iA_i(v)$ for $x\in\mathbb{R}^n$, where $A_i(v)\in S^k, i=1,\ldots,n$. The maps $A_i, i=1,\ldots,n$ are affine, so is the map $\mathcal{A}$. Moreover, for each $v\in V$, it holds that $B(v)-\sum_{i=1}^n x_iA_i(v)\succeq 0$ if and only if $\mathcal{A}(v)x-B(v)\in -S_+^k$. We also consider linear transformations $\mathcal{C}_j:\mathbb{R}^s\to S^{k_j}, j=1,\ldots,p$ defined by $\mathcal{C}_ju^j:=-\sum_{i=1}^s u_i^jC_i^j$ for $u^j\in\mathbb{R}^s$, where $C_i^j\in S^{k_j}, j=1,\ldots,p, i=1,\ldots,s$. Then, the problem (USP) can be rewritten as the following one

$$\min_{x\in\mathbb{R}^n}\left\{\left(c^1(u^1)^\top x+\beta^1(u^1),\ldots,c^p(u^p)^\top x+\beta^p(u^p)\right)\mid \mathcal{A}(v)x-B(v)\in -S_+^k\right\}, \tag{UAP}$$

where the uncertainty sets $U_j, j=1,\ldots,p$ are given by $U_j:=\{u^j:=(u_1^j,\ldots,u_s^j)\in\mathbb{R}^s\mid \mathcal{C}_ju^j-D^j\in -S_+^{k_j}\}$. This problem lands in the form of problem (UC) with $K:=S_+^k$ and $K_j:=S_+^{k_j}, j=1,\ldots,p$. Moreover, the condition (2.18) means that $\mathcal{A}(v)x^0-B(v)\in -\mathrm{int}\,K$ for all $v\in V$. We now invoke Theorem 2.1 to assert that there exist $(\alpha_1,\ldots,\alpha_p)\in\mathbb{R}_+^p\backslash\{0\}$, $\alpha_j^i\in\mathbb{R}, j=1,\ldots,p, i=1,\ldots,s$ and $\lambda^l\in K^*=S_+^k, l=1,\ldots,q$ such that

$$\sum_{j=1}^{p}\left(\alpha_j c_0^j+\sum_{i=1}^{s}\alpha_j^i c_i^j\right)+\sum_{l=1}^{q}(\bar{\mathcal{A}}^l)^\top\lambda^l=0,$$
$$\sum_{j=1}^{p}\left(\alpha_j\beta_0^j+\sum_{i=1}^{s}\alpha_j^i\beta_i^j\right)-\sum_{l=1}^{q}\langle\lambda^l,\bar{B}^l\rangle-\sum_{j=1}^{p}\alpha_j f_j(\bar{x})\geq 0,$$
$$\mathcal{C}_j(\alpha_j^1,\ldots,\alpha_j^s)-\alpha_jD^j\in -K_j,\ \ j=1,\ldots,p,$$

where $\bar{\mathcal{A}}^l:=\mathcal{A}(\bar{v}^l)$, $\bar{B}^l:=B(\bar{v}^l), l=1,\ldots,q$.

Denote $\bar{A}_i^l:=A_i(\bar{v}^l), l=1,\ldots,q, i=1,\ldots,n$. We note that since $\bar{\mathcal{A}}^l, l=1,\ldots,q$ are linear transformations defined by $\bar{\mathcal{A}}^lx:=\mathcal{A}(\bar{v}^l)x=\sum_{i=1}^n x_i\bar{A}_i^l$ for $x\in\mathbb{R}^n$, the corresponding adjoint operators $(\bar{\mathcal{A}}^l)^\top, l=1,\ldots,q$ are computed by $(\bar{\mathcal{A}}^l)^\top\lambda=\left(\mathrm{Tr}(\bar{A}_1^l\lambda),\ldots,\mathrm{Tr}(\bar{A}_n^l\lambda)\right)$ for $\lambda\in S^k$. Moreover, it holds that $\langle\lambda^l,\bar{B}^l\rangle=\mathrm{Tr}(\bar{B}^l\lambda^l)$ for

$l = 1, \ldots, q$ and that

$$\alpha_j D^j + \sum_{i=1}^{s} \alpha_j^i C_i^j \succeq 0 \Leftrightarrow \mathcal{C}_j(\alpha_j^1, \ldots, \alpha_j^s) - \alpha_j D^j \in -K_j$$

for all $j = 1, \ldots, p$. Consequently, we arrive at the desired conclusions. □

Multiobjective Second-Order Cone Programming Problems. We consider another special case of the problem (UC), which is defined by

$$\min_{x \in \mathbb{R}^n} \left\{ \left(c^1(u^1)^\top x + \beta^1(u^1), \ldots, c^p(u^p)^\top x + \beta^p(u^p) \right) \mid A(v)x - b(v) \in -\mathcal{L}_m \right\}, \quad \text{(UOP)}$$

where uncertain parameters u^j, $j = 1, \ldots, p$ and v, the uncertainty set $V := \text{conv}\{\bar{v}^1, \ldots, \bar{v}^q\}$ with $\bar{v}^l \in \mathbb{R}^{s_0}$ for $l = 1, \ldots, q$, the maps $A : \mathbb{R}^{s_0} \to L(\mathbb{R}^n, \mathbb{R}^m)$, $b : \mathbb{R}^{s_0} \to \mathbb{R}^m$ are defined as above, the maps $c^j : \mathbb{R}^s \to \mathbb{R}^n$, $\beta^j : \mathbb{R}^s \to \mathbb{R}$, $j = 1, \ldots, p$ are given in (1.1), and $\mathcal{L}_m := \left\{ (y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_1 \geq ||(y_2, \ldots, y_m)|| \right\}$ is the second-order cone or Lorentz cone in $\mathbb{R}^m (m \geq 2)$, while the uncertainty sets U_j, $j = 1, \ldots, p$ are *ellipsoids* given by

$$U_j := \left\{ u^j \in \mathbb{R}^s \mid (u^j)^\top M^j u^j \leq 1 \right\} \tag{2.22}$$

with $M^j \in S^s$ satisfying $M^j \succ 0$. For each $j \in \{1, \ldots, p\}$, we let E^j be an $(m_j \times s)$ matrix, which is a decomposition factor of M^j, i.e.,

$$M^j = (E^j)^\top E^j. \tag{2.23}$$

The robust counterpart of problem (UOP) is given by

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u^1 \in U_1} \{c^1(u^1)^\top x + \beta^1(u^1)\}, \ldots, \max_{u^p \in U_p} \{c^p(u^p)^\top x + \beta^p(u^p)\} \right) \mid A(v)x - b(v) \in -\mathcal{L}_m, \ \forall v \in V \right\}. \quad \text{(ROP)}$$

In this case, we obtain second-order conic (SOC) conditions for robust (weak) efficiency of (UOP) as follows.

Corollary 2.3 (SOC optimality conditions) *For the problem (UOP), let* $\bar{x} \in \{x \in \mathbb{R}^n \mid A(v)x - b(v) \in -\mathcal{L}_m, \ \forall v \in V\}$.

(i) *Let* $x^0 \in \mathbb{R}^n$ *be such that*

$$A(v)x^0 - b(v) \in -\mathcal{L}_m^0, \ \forall v \in V, \tag{2.24}$$

where $\mathcal{L}_m^0 := \left\{ (y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_1 > ||(y_2, \ldots, y_m)|| \right\}$. *Assume that* $\bar{x}$ *is a robust weak efficient solution of (UOP). Then, there exist* $\alpha_j^i \in \mathbb{R}$, $i = 1, \ldots, s$, $j = 1, \ldots, p$, $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p \backslash \{0\}$ *and* $\lambda^l := (\lambda_1^l, \ldots, \lambda_m^l) \in \mathbb{R}^m$, $\lambda_1^l \geq ||(\lambda_2^l, \ldots, \lambda_m^l)||$, $l = 1, \ldots, q$ *such that*

$$\sum_{j=1}^{p}\left(\alpha_j c_0^j + \sum_{i=1}^{s}\alpha_j^i c_i^j\right) + \sum_{l=1}^{q}(\bar{A}^l)^\top \lambda^l = 0, \tag{2.25}$$

$$\sum_{j=1}^{p}\left(\alpha_j \beta_0^j + \sum_{i=1}^{s}\alpha_j^i \beta_i^j\right) - \sum_{l=1}^{q}(\lambda^l)^\top \bar{b}^l - \sum_{j=1}^{p}\alpha_j f_j(\bar{x}) \geq 0, \tag{2.26}$$

$$||E^j(\alpha_j^1, \ldots, \alpha_j^s)|| \leq \alpha_j, \ \ j = 1, \ldots, p, \tag{2.27}$$

where $f_j(\bar{x}) := \max_{u^j \in U_j} \{c^j(u^j)^\top \bar{x} + \beta^j(u^j)\}, j = 1, \ldots, p$ *and* $\bar{A}^l := A(\bar{v}^l), \bar{b}^l := b(\bar{v}^l), l = 1, \ldots, q$.

(ii) *Assume that there exist* $\alpha_j^i \in \mathbb{R}, i = 1, \ldots, s, j = 1, \ldots, p$, $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+\backslash\{0\}$ *and* $\lambda^l := (\lambda_1^l, \ldots, \lambda_m^l) \in \mathbb{R}^m, \lambda_1^l \geq ||(\lambda_2^l, \ldots, \lambda_m^l)||, l = 1, \ldots, q$ *satisfying (2.25), (2.26) and (2.27). Then,* $\bar{x}$ *is a robust weak efficient solution of (UOP).*

(iii) *Assume that there exist* $\alpha_j^i \in \mathbb{R}, i = 1, \ldots, s, j = 1, \ldots, p$, $(\alpha_1, \ldots, \alpha_p) \in \text{int}\mathbb{R}^p_+$ *and* $\lambda^l := (\lambda_1^l, \ldots, \lambda_m^l) \in \mathbb{R}^m, \lambda_1^l \geq ||(\lambda_2^l, \ldots, \lambda_m^l)||, l = 1, \ldots, q$ *satisfying (2.25), (2.26) and (2.27). Then,* $\bar{x}$ *is a robust efficient solution of (UOP).*

Proof Consider any $j \in \{1, \ldots, p\}$. Let $C_j : \mathbb{R}^s \to \mathbb{R} \times \mathbb{R}^{m_j}$ be given by $C_j u^j := (0, -E^j u^j), u^j \in \mathbb{R}^s$, and denote $d^j := (1, \underbrace{0, \ldots, 0}_{m_j}) \in \mathbb{R} \times \mathbb{R}^{m_j}$. Since $M^j = (E^j)^\top E^j$, we see that $u^\top M^j u \leq 1$ is equivalent to $||E^j u|| \leq 1$ for each $u \in \mathbb{R}^s$, and therefore the ellipsoid U_j in (2.22) can be rewritten as the following cone-based set

$$C_j u^j - d^j \in -K_j,$$

where $K_j := \{(k, y) \in \mathbb{R} \times \mathbb{R}^{m_j} \mid k \geq ||y||\}$. Now, the problem (UOP) lands in the form of (UC) with $K := \mathcal{L}_m$, and the condition (2.24) means that $A(v)x^0 - b(v) \in -\text{int}\, K$ for all $v \in V$. We employ Theorem 2.1 to assert that there exist $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+\backslash\{0\}$, $\alpha_j^i \in \mathbb{R}, j = 1, \ldots, p, i = 1, \ldots, s$ and $\lambda^l \in K^* = \mathcal{L}_m, l = 1, \ldots, q$ such that

$$\sum_{j=1}^{p}\left(\alpha_j c_0^j + \sum_{i=1}^{s}\alpha_j^i c_i^j\right) + \sum_{l=1}^{q}(\bar{A}^l)^\top \lambda^l = 0,$$

$$\sum_{j=1}^{p}\left(\alpha_j \beta_0^j + \sum_{i=1}^{s}\alpha_j^i \beta_i^j\right) - \sum_{l=1}^{q}(\lambda^l)^\top \bar{b}^l - \sum_{j=1}^{p}\alpha_j f_j(\bar{x}) \geq 0,$$

$$C_j(\alpha_j^1, \ldots, \alpha_j^s) - \alpha_j d^j \in -K_j, \ \ j = 1, \ldots, p,$$

where $f_j(\bar{x}) := \max_{u^j \in U_j} \{c^j(u^j)^\top \bar{x} + \beta^j(u^j)\}, j = 1, \ldots, p$ and $\bar{A}^l := A(\bar{v}^l), \bar{b}^l := b(\bar{v}^l), l = 1, \ldots, q$. Note in this setting that

$$||E^j(\alpha_j^1, \ldots, \alpha_j^s)|| \leq \alpha_j \ \Leftrightarrow \ C_j(\alpha_j^1, \ldots, \alpha_j^s) - \alpha_j d^j \in -K_j$$

for all $j = 1, \ldots, p$, and so the proof is completed by using Theorem 2.1. □

Remark 2.4 It is worth mentioning that if the cone K in the constraint of problem (UC) is the nonnegative orthant (i.e., $K := \mathbb{R}^m_+$), and the cones $K_j, j = 1, \ldots, p$ in (1.2) are polyhedral cones (i.e., $K_j := \{y \in \mathbb{R}^{m_j} \mid \widetilde{M}_j y \geq 0\}$ for given matrices $\widetilde{M}_j$), then the *conic* optimality conditions obtained in Theorem 2.1 such as $\lambda^l \in K^*, l = 1, \ldots, q$ and (2.6) reduce to classical *linear* conditions.

3 Robust efficient solutions with conic reformulations

This section is devoted to showing how robust weak/efficient solutions of the uncertain conic multiobjective linear programming problem (UC) can be calculated via its (scalar) conic programming problems. To this end, we examine *robust* scalarized optimization models for (UC) as follows.

Conic Programming Reformulations. For $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+ \backslash \{0\}$, one considers a *robust* scalarized problem of the problem (UC) given by

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{p} \alpha_j \max_{u^j \in U_j} \{c^j(u^j)^\top x + \beta^j(u^j)\} \mid A(v)x - b(v) \in -K, \ \forall v \in V \right\}, \qquad (\mathrm{R}_\alpha)$$

where $c^j(u^j) := c_0^j + \sum_{i=1}^s u_i^j c_i^j$ and $\beta^j(u^j) := \beta_0^j + \sum_{i=1}^s u_i^j \beta_i^j$, $j = 1, \ldots, p$ are given in (1.1), U_j, $j = 1, \ldots, p$ are given in (1.2), $V := \mathrm{conv}\{\bar{v}^1, \ldots, \bar{v}^q\}$ is given as before, $A : \mathbb{R}^{s_0} \to L(\mathbb{R}^n, \mathbb{R}^m)$, $b : \mathbb{R}^{s_0} \to \mathbb{R}^m$ and K are given in the definition of (UC).

Let us define a *conic programming* reformulation problem for (R_α) given by

$$\inf_{(y, z^1, \ldots, z^p)} \left\{ \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top y + \beta_0^j + (d^j)^\top z^j\big) \mid \bar{A}^l y - \bar{b}^l \in -K, \ l = 1, \ldots, q, \right. \qquad (\mathrm{R}^*_\alpha)$$
$$\left. \big((c_1^j)^\top y + \beta_1^j, \ldots, (c_s^j)^\top y + \beta_s^j\big) - C_j^\top z^j = 0, \, y \in \mathbb{R}^n, z^j \in K_j^*, j = 1, \ldots, p \right\},$$

where $\bar{A}^l := A(\bar{v}^l)$ and $\bar{b}^l := b(\bar{v}^l)$ for $l = 1, \ldots, q$.

In the following theorem, we present relationships of solutions between the conic uncertain multiobjective linear programming problem (UC) and a (scalar) conic programming problem (R^*_α). This shows how to find *robust weak efficient* and *robust efficient* solutions of the conic uncertain multiobjective linear programming problem (UC) by using the (scalar) conic programming reformulation problem (R^*_α).

Theorem 3.1 (Efficiency with conic reformulations) *For the problem (UC), let $\hat{u}^j \in \mathbb{R}^s$, $j = 1, \ldots, p$ be such that*

$$C_j \hat{u}^j - d^j \in -\mathrm{int} K_j. \qquad (3.1)$$

Then, we have the following assertions.

(i) *Assume that the strict constraint qualification (2.3) holds and that $\bar{x}$ is a robust weak efficient solution of (UC). Then, there exist $\alpha \in \mathbb{R}^p_+ \backslash \{0\}$ and $\bar{z}^j \in \mathbb{R}^{m_j}$, $j = 1, \ldots, p$ such that $(\bar{x}, \bar{z}^1, \ldots, \bar{z}^p)$ is a solution of problem (R^*_α).*

(ii) **(Robust weak efficient solution)** *Assume that the problem (R_α) possesses a solution for $\alpha \in \mathbb{R}^p_+ \backslash \{0\}$ and let $(\bar{w}, \bar{z}^1, \ldots, \bar{z}^p)$ be a solution of (R^*_α). Then, it holds that $\bar{w}$ is a robust weak efficient solution of (UC).*

(iii) **(Robust efficient solution)** *Assume that the problem (R_α) possesses a solution for $\alpha \in \mathrm{int}\mathbb{R}^p_+$ and let $(\bar{w}, \bar{z}^1, \ldots, \bar{z}^p)$ be a solution of (R^*_α). Then, it holds that $\bar{w}$ is a robust efficient solution of (UC).*

Proof Let $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+ \backslash \{0\}$ and suppose that the robust weighted sum optimization problem (R_α) possesses an optimal solution, say $\bar{x}$. We denote by val(R_α) the optimal value of (R_α). We claim that there exist $\bar{z}^j \in K_j^*$, $j = 1, \ldots, p$ such that $(\bar{x}, \bar{z}^1, \ldots, \bar{z}^p)$ is an optimal solution of (R^*_α) satisfying

$$\text{val}(\text{R}_\alpha) = \text{val}(\text{R}_\alpha^*) = \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top \bar{x} + \beta_0^j + (d^j)^\top \bar{z}^j\big), \tag{3.2}$$

where val(R_α^*) denotes the optimal value of (R_α^*). Indeed, since $\bar{x}$ is optimal for (R_α), we see that $\bar{A}^l \bar{x} - \bar{b}^l \in -K,\ l = 1, \ldots, q$ and

$$\text{val}(\text{R}_\alpha) = \sum_{j=1}^{p} \alpha_j f_j(\bar{x}), \tag{3.3}$$

where $\bar{A}^l := A(\bar{v}^l), \bar{b}^l := b(\bar{v}^l)$ and $f_j(\bar{x}) := \max_{u^j \in U_j} \{c^j(u^j)^\top \bar{x} + \beta^j(u^j)\}$ for $j = 1, \ldots, p$. Considering any $j \in \{1, \ldots, p\}$, we derive from $f_j(\bar{x}) = \max_{u^j \in U_j} \{c^j(u^j)^\top \bar{x} + \beta^j(u^j)\}$ that

$$\min_{u^j \in \mathbb{R}^s} \{(-\bar{t}^j)^\top u^j \mid C_j u^j - d^j \in -K_j\} = \bar{t}_0^j - f_j(\bar{x}),$$

where $\bar{t}^j := (\bar{t}_1^j, \ldots, \bar{t}_s^j)$ and $\bar{t}_i^j := (c_i^j)^\top \bar{x} + \beta_i^j, i = 0, 1, \ldots, s$. Under the strict condition in (3.1), we invoke a strong duality in conic programming (see e.g., [5, Theorem A.2.1]) to assert that

$$\bar{t}_0^j - f_j(\bar{x}) = \max_{z^j \in \mathbb{R}^{m_j}} \{(-d^j)^\top z^j \mid C_j^\top z^j - \bar{t}^j = 0, z^j \in K_j^*\},$$

and so there exists $\bar{z}^j \in K_j^*$ such that

$$\begin{aligned} &(c_0^j)^\top \bar{x} + \beta_0^j + (d^j)^\top \bar{z}^j = f_j(\bar{x}), \\ &\big((c_1^j)^\top \bar{x} + \beta_1^j, \ldots, (c_s^j)^\top \bar{x} + \beta_s^j\big) - C_j^\top \bar{z}^j = 0. \end{aligned} \tag{3.4}$$

Hence, it holds that $(\bar{x}, \bar{z}^1, \ldots, \bar{z}^p)$ is a feasible point of problem (R_α^*), and so we get by (3.4) that

$$\text{val}(\text{R}_\alpha^*) \leq \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top \bar{x} + \beta_0^j + (d^j)^\top \bar{z}^j\big) = \sum_{j=1}^{p} \alpha_j f_j(\bar{x}) = \text{val}(\text{R}_\alpha), \tag{3.5}$$

where the last equality is valid due to (3.3).

To prove val(R_α) $\leq$ val(R_α^*), assume that $(w, z^1, \ldots, z^p)$ is a feasible point of (R_α^*). Then, we have $w \in \mathbb{R}^n, z^j \in K_j^*, j = 1, \ldots, p$ and

$$\bar{A}^l w - \bar{b}^l \in -K,\ l = 1, \ldots, q, \tag{3.6}$$

$$\big((c_1^j)^\top w + \beta_1^j, \ldots, (c_s^j)^\top w + \beta_s^j\big) - C_j^\top z^j = 0. \tag{3.7}$$

As $V := \text{conv}\,\{\bar{v}^1, \ldots, \bar{v}^q\}$, A and b are affine maps, K is a convex cone, arguing as in the proof of Theorem 2.1, we get by (3.6) that $A(v)w - b(v) \in -K$ for all $v \in V$. This means that w is feasible for the problem (R_α), and so

$$\text{val}(\text{R}_\alpha) \leq \sum_{j=1}^{p} \alpha_j f_j(w), \tag{3.8}$$

where $f_j(w) := \max_{u^j \in U_j} \{c^j(u^j)^\top w + \beta^j(u^j)\}$, $j = 1, \ldots, p$. Consider any $j \in \{1, \ldots, p\}$ and any $u^j \in U_j$. The later relation means that $C_j u^j - d^j \in -K$. Therefore, we assert that $(C_j u^j)^\top z^j \leq (d^j)^\top z^j$ due to $z^j \in K_j^*$. This, together with (3.7), entails that

$$c^j(u^j)^\top w + \beta^j(u^j) = (c_0^j)^\top w + \beta_0^j + (u^j)^\top (C_j^\top z^j) \leq (c_0^j)^\top w + \beta_0^j + (d^j)^\top z^j.$$

Since u^j was arbitrarily chosen in U_j, we conclude that

$$f_j(w) = \max_{u^j \in U_j} \left\{ c^j(u^j)^\top w + \beta^j(u^j) \right\} \leq (c_0^j)^\top w + \beta_0^j + (d^j)^\top z^j.$$

Now, noting that $\alpha_j \geq 0$, $j = 1, \ldots, p$ and taking (3.8) into account, we arrive at

$$\text{val}(\text{R}_\alpha) \leq \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top w + \beta_0^j + (d^j)^\top z^j \big),$$

which guarantees that $\text{val}(\text{R}_\alpha) \leq \text{val}(\text{R}_\alpha^*)$ as $(w, z^1, \ldots, z^p)$ was an arbitrary feasible point of problem (R_α^*).

Invoking now (3.5), we obtain that

$$\text{val}(\text{R}_\alpha) = \text{val}(\text{R}_\alpha^*) = \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top \bar{x} + \beta_0^j + (d^j)^\top \bar{z}^j \big),$$

which also confirms that $(\bar{x}, \bar{z}^1, \ldots, \bar{z}^p)$ is optimal for the problem (R_α^*). Thus, so our claim in (3.2) holds.

(i) Assume that the problem (UC) admits a robust weak efficient solution $\bar{x}$. Under the strict constraint qualification (2.3), we employ Theorem 2.1(i) to find $\alpha_j^i \in \mathbb{R}$, $i = 1, \ldots, s$, $j = 1, \ldots, p$, $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p \backslash \{0\}$ and $\lambda^l \in K^*$, $l = 1, \ldots, q$ such that

$$\sum_{j=1}^{p} \left(\alpha_j c_0^j + \sum_{i=1}^{s} \alpha_j^i c_i^j \right) + \sum_{l=1}^{q} (\bar{A}^l)^\top \lambda^l = 0, \tag{3.9}$$

$$\sum_{j=1}^{p} \left(\alpha_j \beta_0^j + \sum_{i=1}^{s} \alpha_j^i \beta_i^j \right) - \sum_{l=1}^{q} (\lambda^l)^\top \bar{b}^l - \sum_{j=1}^{p} \alpha_j f_j(\bar{x}) \geq 0, \tag{3.10}$$

$$C_j \left(\alpha_j^1, \ldots, \alpha_j^s \right) - \alpha_j d^j \in -K_j, \; j = 1, \ldots, p, \tag{3.11}$$

where $\bar{A}^l := A(\bar{v}^l)$, $\bar{b}^l := b(\bar{v}^l)$, $l = 1, \ldots, q$ and $f_j(\bar{x}) := \max_{u^j \in U_j} \{c^j(u^j)^\top \bar{x} + \beta^j(u^j)\}$, $j = 1, \ldots, p$. Recall here that $\mathcal{F} := \{x \in \mathbb{R}^n \mid A(v)x - b(v) \in -K, \; \forall v \in V\}$ is the robust feasible set of problem (UC) and so $\mathcal{F}$ is also the feasible set of problem (R_α). We can derive from (3.9), (3.10) and (3.11) that

$$\sum_{j=1}^{p} \alpha_j f_j(\hat{x}) \geq \sum_{j=1}^{p} \alpha_j f_j(\bar{x})$$

for all $\hat{x} \in \mathcal{F}$, which asserts that $\bar{x}$ is optimal for the problem (R_α). So, the assertion before (3.2) tells us that there exist $\bar{z}^j \in K_j^*$, $j = 1, \ldots, p$ such that $(\bar{x}, \bar{z}^1, \ldots, \bar{z}^p)$ is an optimal solution of problem (R_α^*).

(ii) Let $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p \backslash \{0\}$ be such that the problem (R_α) possesses an optimal solution. We obtain as by (3.2) that

$$\mathrm{val}(\mathrm{R}_\alpha) = \mathrm{val}(\mathrm{R}_\alpha^*). \tag{3.12}$$

Assume that $(\bar{w}, \bar{z}^1, \ldots, \bar{z}^p)$ is optimal for the problem (R_α^*). This shows that $\bar{w} \in \mathbb{R}^n$, $\bar{z}^j \in K_j^*$, $j = 1, \ldots, p$, and

$$\mathrm{val}(\mathrm{R}_\alpha^*) = \sum_{j=1}^{p} \alpha_j \big((c_0^j)^\top \bar{w} + \beta_0^j + (d^j)^\top \bar{z}^j \big), \tag{3.13}$$

$$\bar{A}^l \bar{w} - \bar{b}^l \in -K, \ l = 1, \ldots, q, \tag{3.14}$$

$$\big((c_1^j)^\top \bar{w} + \beta_1^j, \ldots, (c_s^j)^\top \bar{w} + \beta_s^j \big) - C_j^\top \bar{z}^j = 0, \ j = 1, \ldots, p. \tag{3.15}$$

Proceeding as above, we derive from (3.14) that $\bar{w}$ is feasible for the problem (R_α), and hence $\bar{w}$ is robust feasible for the problem (UC). Denoting $f_j(x) := \max_{u^j \in U_j} \big\{ c^j(u^j)^\top x + \beta^j(u^j) \big\}$, $j = 1, \ldots, p$ for $x \in \mathbb{R}^n$, we get by (3.15) that

$$f_j(\bar{w}) \le (c_0^j)^\top \bar{w} + \beta_0^j + (d^j)^\top \bar{z}^j, \ \ j = 1, \ldots, p. \tag{3.16}$$

Therefore, we assert that $\bar{w}$ is a robust weak efficient solution of (UC). Otherwise, there exists a robust feasible point of (UC), say $\hat{x}$, such that

$$f_j(\hat{x}) < f_j(\bar{w}), \ \ j = 1, \ldots, p,$$

where we should note that $\hat{x}$ is also feasible for the problem (R_α). By taking (3.16) and (3.13) into account, we see that

$$\mathrm{val}(\mathrm{R}_\alpha) \le \sum_{j=1}^{p} \alpha_j f_j(\hat{x}) < \sum_{j=1}^{p} \alpha_j f_j(\bar{w}) \le \mathrm{val}(\mathrm{R}_\alpha^*),$$

which together with (3.12) establishes a contradiction. So, $\bar{w}$ is a robust weak efficient solution of (UC).

(iii) Assume the problem (R_α) admits a solution for some $\alpha \in \mathrm{int}\mathbb{R}_+^p$, and let $(\bar{w}, \bar{z}^1, \ldots, \bar{z}^p)$ be optimal for the problem (R_α^*). Then, (3.12)–(3.16) hold true for this setting. Arguing similarly as above, we come to a conclusion that there is no other $\hat{x} \in \mathcal{F}$ with

$$f_j(\hat{x}) \le f_j(\bar{w}), \ \ j = 1, \ldots, p$$

and $f_j(\hat{x}) < f_j(\bar{w})$ for some $j \in \{1, \ldots, p\}$. So $\bar{w}$ is a robust efficient solution of (UC). □

Semidefinite Programming Reformulations. Let us now establish semidefinite programming (SDP) reformulations for finding robust (weak) efficient solutions of the uncertain multiobjective semidefinite programming problem (USP).

In this case, for each $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p \backslash \{0\}$, the *robust* weighted-sum problem of (USP) is given by

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{p} \alpha_j \max_{u^j \in U_j} \{ c^j(u^j)^\top x + \beta^j(u^j) \} \mid B(v) - \sum_{i=1}^{n} x_i A_i(v) \succeq 0, \ \forall v \in V \right\}, \tag{SR_α}$$

where $c^j(u^j) := c_0^j + \sum_{i=1}^s u_i^j c_i^j$ and $\beta^j(u^j) := \beta_0^j + \sum_{i=1}^s u_i^j \beta_i^j$, $j = 1, \ldots, p$ are given in (1.1), U_j, $j = 1, \ldots, p$ are given in (2.17), $A_i : \mathbb{R}^{s_0} \to S^k$, $i = 1, \ldots, n$, $B : \mathbb{R}^{s_0} \to S^k$ and $V := \operatorname{conv}\{\bar{v}^1, \ldots, \bar{v}^q\}$ are given as in the definition of (USP). An SDP reformulation problem for (SR_α) reads as follows:

$$\inf_{(y, Z^1, \ldots, Z^p)} \Big\{ \sum_{j=1}^p \alpha_j \big((c_0^j)^\top y + \beta_0^j + \operatorname{Tr}(D^j Z^j) \big) \tag{SR_α^*}$$

$$\mid \bar{B}^l - \sum_{i=1}^n y_i \bar{A}_i^l \succeq 0,\ l = 1, \ldots, q,$$

$$y := (y_1, \ldots, y_n) \in \mathbb{R}^n,$$

$$(c_i^j)^\top y + \beta_i^j + \operatorname{Tr}(C_i^j Z^j) = 0,$$

$$Z^j \in S^{k_j}, Z^j \succeq 0, j = 1, \ldots, p, i = 1, \ldots, s \Big\},$$

where $\bar{A}_i^l := A_i(\bar{v}^l)$, $\bar{B}^l := B(\bar{v}^l)$, $l = 1, \ldots, q$, $i = 1, \ldots, n$.

Now, solution relationships between the uncertain multiobjective semidefinite programming problem (USP) and a corresponding SDP reformulation problem (SR_α^*) are described as in the following corollary.

Corollary 3.2 (Finding solutions via SDP reformulations)

For the uncertain multiobjective semidefinite programming problem (USP), let $\hat{u}^j := (\hat{u}_1^j, \ldots, \hat{u}_s^j) \in \mathbb{R}^s$, $j = 1, \ldots, p$ *be such that*

$$D^j + \sum_{i=1}^s \hat{u}_i^j C_i^j \succ 0. \tag{3.17}$$

Then, we have the following assertions.

(i) *Assume that the strict constraint qualification (2.18) holds and that* $\bar{x}$ *is a robust weak efficient solution of (USP). Then, there exist* $\alpha \in \mathbb{R}_+^p \backslash \{0\}$ *and* $\bar{Z}^j \in S^{k_j}$, $j = 1, \ldots, p$ *such that* $(\bar{x}, \bar{Z}^1, \ldots, \bar{Z}^p)$ *is a solution of* (SR_α^*).

(ii) **(Robust weak Pareto solution)** *Assume that the problem* (SR_α) *possesses a solution for* $\alpha \in \mathbb{R}_+^p \backslash \{0\}$ *and let* $(\bar{w}, \bar{Z}^1, \ldots, \bar{Z}^p)$ *be a solution of* (SR_α^*). *Then, it holds that* $\bar{w}$ *is a robust weak efficient solution of (USP).*

(iii) **(Robust Pareto solution)** *Assume that the problem* (SR_α) *possesses a solution for* $\alpha \in \operatorname{int}\mathbb{R}_+^p$ *and let* $(\bar{w}, \bar{Z}^1, \ldots, \bar{Z}^p)$ *be a solution of* (SR_α^*). *Then, it holds that* $\bar{w}$ *is a robust efficient solution of (USP).*

Proof Consider, as in the proof of Corollary 2.2, an affine map $\mathcal{A} : \mathbb{R}^{s_0} \to L(\mathbb{R}^n, S^k)$ defined as follows: For each $v \in \mathbb{R}^{s_0}$, one has a linear transformation $\mathcal{A}(v) : \mathbb{R}^n \to S^k$ given by $\mathcal{A}(v)x := \sum_{i=1}^n x_i A_i(v)$ for $x \in \mathbb{R}^n$, where $A_i(v) \in S^k$, $i = 1, \ldots, n$. Similarly, we consider linear transformations $\mathcal{C}_j : \mathbb{R}^s \to S^{k_j}$, $j = 1, \ldots, p$ defined by

$$\mathcal{C}_j u^j := -\sum_{i=1}^s u_i^j C_i^j \text{ for } u^j \in \mathbb{R}^s,$$

where $C_i^j \in S^{k_j}$, $j = 1, \ldots, p$, $i = 1, \ldots, s$ are given. Then, for each $x \in \mathbb{R}^n$ and each $v \in V$, $B(v) - \sum_{i=1}^n x_i A_i(v) \succeq 0$ amounts to $\mathcal{A}(v)x - B(v) \in -S_+^k$, and so the problem (USP) can be viewed in the form of problem (UC) with $K := S_+^k$ and $K_j := S_+^{k_j}$, $j = 1, \dots, p$. Consequently, the problem (SR_α) lands in the form of problem (R_α). Note that $K_j^* := S_+^{k_j}$, $j = 1, \dots, p$, $\langle D^j, Z^j \rangle = \mathrm{Tr}(D^j Z^j)$ for all $Z^j \in S^{k_j}$, $j = 1, \dots, p$ and the adjoint operators of $\mathcal{C}_j$, $j = 1, \dots, p$ are $\mathcal{C}_j^\top$, $j = 1, \dots, p$ computed by

$$\mathcal{C}_j^\top Z^j = -\big(\mathrm{Tr}(C_1^j Z^j), \dots, \mathrm{Tr}(C_s^j Z^j)\big) \text{ for } Z^j \in S^{k_j}.$$

So the problem (SR_α^*) can be viewed in the form of problem (R_α^*). Moreover, we see that for each $j \in \{1, \dots, p\}$ the condition (3.17) is nothing else but $\mathcal{C}_j \hat{u}^j - D^j \in -\mathrm{int}\, K_j$, which is in the form of (3.1). Similarly, the condition (2.18) means that $\mathcal{A}(v)x^0 - B(v) \in -\mathrm{int}\, K$ for all $v \in V$. To finish the proof, we just invoke Theorem 3.1. □

Second-Order Cone Programming Reformulations. We now derive second-order cone programming (SOCP) reformulations for calculating robust weak/efficient solutions of the uncertain multiobjective second-order cone programming problem (UOP).

In this case, for each $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, the *robust* weighted-sum problem of (UOP) is given by

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^p \alpha_j \max_{u^j \in U_j} \{c^j(u^j)^\top x + \beta^j(u^j)\} \mid A(v)x - b(v) \in -\mathcal{L}_m, \ \forall v \in V \right\}, \qquad (SO_\alpha)$$

where $c^j(u^j) := c_0^j + \sum_{i=1}^s u_i^j c_i^j$ and $\beta^j(u^j) := \beta_0^j + \sum_{i=1}^s u_i^j \beta_i^j$, $j = 1, \dots, p$ are given in (1.1), U_j, $j = 1, \dots, p$ are given in (2.22), $A : \mathbb{R}^{s_0} \to L(\mathbb{R}^n, \mathbb{R}^m)$, $b : \mathbb{R}^{s_0} \to \mathbb{R}^m$, $\mathcal{L}_m := \Big\{(y_1, \dots, y_m) \in \mathbb{R}^m \mid y_1 \geq ||(y_2, \dots, y_m)||\Big\}$ and $V := \mathrm{conv}\,\{\bar{v}^1, \dots, \bar{v}^q\}$ are given in the definition of (UOP).

An SOCP reformulation problem for (SO_α) is captured by

$$\inf_{(y, \lambda_1, \dots, \lambda_p, y^1, \dots, y^p)} \left\{ \sum_{j=1}^p \alpha_j \big((c_0^j)^\top y + \beta_0^j + \lambda_j\big) \mid \bar{A}^l y - \bar{b}^l \in -\mathcal{L}_m, \ l = 1, \dots, q, y \in \mathbb{R}^n, \right. \qquad (SO_\alpha^*)$$

$$\left. (c_i^j)^\top y + \beta_i^j + (E_i^j)^\top y^j = 0, \lambda_j \geq ||y^j||, \lambda_j \in \mathbb{R}, y^j \in \mathbb{R}^{m_j}, j = 1, \dots, p, i = 1, \dots, s \right\},$$

where E_i^j, $i = 1, \dots, s$ denote the columns of the matrix E^j given in (2.23) and $\bar{A}^l := A(\bar{v}^l)$, $\bar{b}^l := b(\bar{v}^l)$, $l = 1, \dots, q$.

The relationships between the robust weak/efficient solutions of (UOP) and a corresponding SOCP reformulation problem (SO_α^*) read as follows.

Corollary 3.3 **(Solutions via SOCP reformulations)** *Consider the problem (UOP).*

(i) *Assume that the strict constraint qualification (2.24) holds and let $\bar{x}$ be a robust weak efficient solution of (UOP). Then, there exist $\alpha \in \mathbb{R}_+^p \setminus \{0\}$ and $\bar{\lambda}_j \in \mathbb{R}$, $\bar{y}^j \in \mathbb{R}^{m_j}$, $j = 1, \dots, p$ such that $(\bar{x}, \bar{\lambda}_1, \dots, \bar{\lambda}_p, \bar{y}^1, \dots, \bar{y}^p)$ is a solution of (SO_α^*).*

(ii) **(Robust weak efficient solution)** *Assume that the problem (SO_α) possesses a solution for $\alpha \in \mathbb{R}_+^p \setminus \{0\}$ and let $(\bar{w}, \bar{\lambda}_1, \dots, \bar{\lambda}_p, \bar{y}^1, \dots, \bar{y}^p)$ be a solution of (SO_α^*). Then, it holds that $\bar{w}$ is a robust weak efficient solution of (UOP).*

(iii) **(Robust efficient solution)** *Assume that the problem (SO_α) possesses a solution for $\alpha \in \mathrm{int}\mathbb{R}_+^p$ and let $(\bar{w}, \bar{\lambda}_1, \dots, \bar{\lambda}_p, \bar{y}^1, \dots, \bar{y}^p)$ be a solution of (SO_α^*). Then, it holds that $\bar{w}$ is a robust efficient solution of (UOP).*

Proof For any $j \in \{1, \ldots, p\}$, let $C_j : \mathbb{R}^s \to \mathbb{R} \times \mathbb{R}^{m_j}$ be given by $C_j u^j := (0, -E^j u^j)$ for $u^j \in \mathbb{R}^s$, where E^j is the $(m_j \times s)$ matrix satisfying $(E^j)^\top E^j = M^j$ as above, and let $d^j := (1, \underbrace{0, \ldots, 0}_{m_j}) \in \mathbb{R} \times \mathbb{R}^{m_j}$. As shown in the proof of Corollary 2.3, the ellipsoid U_j in (2.22) can be rewritten as the following cone-based set

$$C_j u^j - d^j \in -K_j,$$

where $K_j := \{(k, y) \in \mathbb{R} \times \mathbb{R}^{m_j} \mid k \geq ||y||\}$. Then, the problem (UOP) can be regarded as the problem (UC) with $K := \mathcal{L}_m$, and so the problem (SO_α) lands in the form of problem (R_α). Note that $K_j^* := K_j, j = 1, \ldots, p$, and for any $z^j := (\lambda_j, y^j) \in \mathbb{R} \times \mathbb{R}^{m_j}$ we have $(d^j)^\top z^j = \lambda_j, j = 1, \ldots, p$. In this case, the adjoint operators of $C_j, j = 1, \ldots, p$ are computed by

$$C_j^\top z^j = -(E^j)^\top y^j \text{ for } z^j := (\lambda_j, y^j) \in \mathbb{R} \times \mathbb{R}^{m_j}.$$

Thus the problem (SO_α^*) can be viewed in the form of problem (R_α^*). Moreover, the condition (2.24) means that $A(v)x^0 - b(v) \in -\text{int}\, K$ for all $v \in V$, which lands in the form of (2.3). By taking $\hat{u}^j := 0_s, j = 1, \ldots, p$, we see that

$$C_j \hat{u}^j - d^j \in -\text{int} K_j,$$

i.e., (3.1) is valid. So the proof will be completed by invoking Theorem 3.1. □

4 Solving examples with conic reformulations

In this section, we show how the proposed conic programming reformulation schemes can be employed to calculate robust (weak) efficient solutions for idealised but concrete uncertain multiobjective optimization problems involving an uncertain multiobjective optimization problem arising from practical applications.

4.1 A numerical example

Let us first consider an *uncertain* multiobjective optimization problem of the form:

$$\min_{x \in \mathbb{R}^2} \left\{ \left(h_1(x, u^1), h_2(x, u^2), h_3(x, u^3)\right) \mid 2 + v_1 x_1 + v_2 x_2 \geq \sqrt{4x_1^2 + x_2^2} \right\}, \tag{EU2}$$

where $u^j := (u_1^j, u_2^j) \in U_j, j = 1, 2, 3$ and $v := (v_1, v_2) \in V$ are *uncertain* parameters and $h_j, j = 1, 2, 3$ are functions given by

$$h_1(x, u^1) := -2u_1^1 x_1 + u_2^1 x_2 + 1 + u_1^1 - u_2^1, \; h_2(x, u^2) := u_2^2 x_1 + 3u_1^2 x_2 - 1 - u_1^2 + u_2^2,$$
$$h_3(x, u^3) := x_1 + u_2^3 x_2 + u_1^3 - u_2^3, \; x := (x_1, x_2) \in \mathbb{R}^2.$$

Here, the uncertainty sets U_j, $j = 1, 2, 3$ and V are given respectively by

$$
\begin{aligned}
U_1 &:= \{u^1 := (u_1^1, u_2^1) \in \mathbb{R}^2 \mid \frac{(u_1^1)^2}{2} + \frac{(u_2^1)^2}{3} \le 1, u_1^1 \ge 0\},\\
U_2 &:= \{u^2 := (u_1^2, u_2^2) \in \mathbb{R}^2 \mid (u_1^2)^2 + (u_2^2)^2 \le 1, u_2^2 \le 0\},\\
U_3 &:= \{u^3 := (u_1^3, u_2^3) \in \mathbb{R}^2 \mid \frac{(u_1^3)^2}{4} + \frac{(u_2^3)^2}{9} \le 1, u_1^3 \ge 0\},\\
V &:= \operatorname{conv}\{(0,0), (0,1), (1,1)\}.
\end{aligned}
$$

Consider affine maps $A_i : \mathbb{R}^2 \to S^2$, $i = 1, 2$ and $B : \mathbb{R}^2 \to S^2$ given by

$$
\begin{aligned}
B(v) &:= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, A_1(v) := \begin{pmatrix} 2 - v_1 & 0 \\ 0 & -2 - v_1 \end{pmatrix},\\
A_2(v) &:= \begin{pmatrix} -v_2 & -1 \\ -1 & -v_2 \end{pmatrix} \text{ for } v := (v_1, v_2) \in \mathbb{R}^2.
\end{aligned}
$$

Note that for $a, b, c \in \mathbb{R}$, the following equivalence holds

$$
\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0 \Leftrightarrow a + c \ge \|(a - c, 2b)\|.
$$

Then, for each $v := (v_1, v_2) \in V$ and each $x \in \mathbb{R}^2$, we see that

$$
2 + v_1 x_1 + v_2 x_2 \ge \sqrt{4x_1^2 + x_2^2} \Leftrightarrow B(v) - \sum_{i=1}^{2} x_i A_i(v) \succeq 0.
$$

Now, the problem (EU2) can be expressed in terms of problem (USP), where the uncertainty sets U_j, $j = 1, 2, 3$, are described respectively by

$$
D^1 := \begin{pmatrix} 2&0&0&0\\0&3&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, C_1^1 = C_1^3 := \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix}, C_2^1 = C_2^3 := \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix},
$$

$$
D^2 := \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, C_1^2 := \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, C_2^2 := \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1 \end{pmatrix}, D^3 := \begin{pmatrix} 4&0&0&0\\0&9&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix},
$$

and the maps $c^j : \mathbb{R}^2 \to \mathbb{R}^2, \beta^j : \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2, 3$ are given respectively by $c_0^1 := c_0^2 := c_1^3 := (0, 0)$, $c_1^1 := (-2, 0)$, $c_2^1 := c_2^3 := (0, 1)$, $c_1^2 := (0, 3)$, $c_2^2 := c_0^3 := (1, 0)$, $\beta_0^1 := \beta_1^1 := \beta_2^2 := \beta_1^3 := 1$, $\beta_0^2 := \beta_1^2 := \beta_2^1 := \beta_2^3 := -1$, $\beta_0^3 := 0$. This means that the problem (EU2) is rewritten in our multiobjective semidefinite programming model as the following one:

$$
\min_{x \in \mathbb{R}^2} \left\{ \left(c^1(u^1)^\top x + \beta^1(u^1), c^2(u^2)^\top x + \beta^2(u^2), c^3(u^3)^\top x + \beta^3(u^3) \right) \mid B(v) - \sum_{i=1}^{2} x_i A_i(v) \succeq 0 \right\}. \tag{EUP2}
$$

Let us now employ the proposed reformulation schemes in Corollary 3.2 to find a robust (weak) efficient solution of problem (EU2). Taking $\hat{u}^1 := \hat{u}^3 := (1, 0)$ and $\hat{u}^2 := (0, -\frac{1}{2})$, it holds that $D^j + \sum_{i=1}^{2} \hat{u}_i^j C_i^j \succ 0,\ j = 1, 2, 3$. This means that the condition (3.17) of Corollary 3.2 is fulfilled for this setting.

Let $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3_+ \backslash \{0\}$, and consider a robust scalarized problem of (EUP2) as follows:

$$\inf_{x \in \mathbb{R}^2} \left\{ \sum_{j=1}^{3} \alpha_j \max_{u^j \in U_j} \{c^j(u^j)^\top x + \beta^j(u^j)\} \mid B(v) - \sum_{i=1}^{2} x_i A_i(v) \succeq 0,\ \forall v \in V \right\}, \qquad (\mathrm{E}_\alpha)$$

where we should note that the problem (E_α) admits a solution as its feasible set is nonempty and compact and its objective function is a continuous function.

In this case, an SDP reformulation problem of (E_α) is described by

$$\begin{aligned} \inf_{(y, Z^1, Z^2, Z^3)} \quad & \left\{ \sum_{j=1}^{3} \alpha_j \big((c_0^j)^\top y + \beta_0^j + \mathrm{Tr}(D^j Z^j)\big) \mid \right. & (\mathrm{E}^*_\alpha) \\ & \bar{B}^l - \sum_{i=1}^{2} y_i \bar{A}_i^l \succeq 0,\ l = 1, 2, 3, \\ & (c_i^j)^\top y + \beta_i^j + \mathrm{Tr}(C_i^j Z^j) = 0, \\ & \left. Z^j \in S^4, Z^j \succeq 0, j = 1, 2, 3, i = 1, 2, y := (y_1, y_2) \in \mathbb{R}^2 \right\}, \end{aligned}$$

where $\bar{B}^1 = \bar{B}^2 = \bar{B}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and

$$\bar{A}_1^1 = \bar{A}_1^2 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \bar{A}_1^3 = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \bar{A}_2^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \bar{A}_2^2 = \bar{A}_2^3 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

We use the toolbox CVX in Matlab (see e.g., [23]) to solve the SDP problem (E^*_α) with (for example) $\alpha := (1, 1, 1) \in \mathrm{int}\mathbb{R}^3_+$ and obtain a solution of (E^*_α) as $(\bar{y}, \bar{Z}^1, \bar{Z}^2, \bar{Z}^3)$, where $\bar{y} = (0.4201, 0.6547)$. By Corollary 3.2 (iii), we assert that $\bar{y} = (0.4201, 0.6547)$ is a robust (weak) efficient solution of problem (EU2).

4.2 An example coming from practical applications

We now consider an *uncertain* multiobjective optimization problem of the form:

$$\min_{x \in \mathbb{R}^T, y \in \mathbb{R}^T, z \in \mathbb{R}^T, d \in \mathbb{R}^T, r \in \mathbb{R}^T} \big(f_1(x, y, z, u),\ f_2(x, y, z),\ f_3(x, d, r, u)\big) \qquad (\mathrm{EU3})$$

$$\text{s.t. } x^{min} \le x_t \le x^{max},\ t = 1, ..., T, \qquad (4.18)$$

$$x_{t+1} - x_t \le R^{max},\ t = 1, ..., T - 1, \qquad (4.19)$$

$$x_t - x_{t+1} \le R^{min},\ t = 1, ..., T - 1, \qquad (4.20)$$

$$d^{min} \le d_t \le d^{max},\ t = 1, ..., T, \qquad (4.21)$$

$$r^{min} \le r_t \le r^{max},\ t = 1, ..., T, \qquad (4.22)$$

$$(x_t + y_t + r_t)(x_t + y_t) \ge (z_t + d_t)^2,\ t = 1, ..., T, \qquad (4.23)$$

$$x_t + u_t W + y_t - z_t - u_{T+t} r_t + d_t \ge 0,\ t = 1, ..., T, \qquad (4.24)$$

$$0 \le \pi_t(x_t + u_t W + y_t - z_t) \le X^{max}, \ t = 1, ..., T, \tag{4.25}$$

$$y^{min} \le y_t \le y^{max}, \ t = 1, ..., T, \tag{4.26}$$

$$z^{min} \le z_t \le z^{max}, \ t = 1, ..., T, \tag{4.27}$$

where f_1, f_2 and f_3 are given respectively by

$$f_1(x, y, z, u) := -\sum_{t=1}^{T} \pi_t(x_t + u_t W + y_t - z_t), \tag{4.28}$$

$$f_2(x, y, z) := \sum_{t=1}^{T} \left(\pi_t^{Up} y_t - \pi_t^{Do} z_t + \sum_{g=1}^{G} \pi_{g,t} x_t + c_t \right), \tag{4.29}$$

$$f_3(x, d, r, u) := -\sum_{t=1}^{T} \left(\pi_t^{De} d_t - \omega_t x_t - \pi_t^{Ri} u_{T+t} r_t \right), \tag{4.30}$$

and $u := (u_1, \ldots, u_{2T})$ is an *uncertain* vector, which resides in an *uncertainty* set U. Here, we assume that $U := \prod_{j=1}^{2T} [\lambda_j, \gamma_j]$, where $\lambda_j \in \mathbb{R}$, $\gamma_j \in \mathbb{R}$ are fixed and $\lambda_j < \gamma_j$ for $j = 1, \ldots, 2T$.

Motivation by Bidding Strategy of Virtual Power Plant. The study of problem (EU3) has been motivated from modeling *virtual power plant* (VPP) in electricity markets (see e.g., [32] for a type of VPP model). In this sense, the objective function (4.28) is to minimize VPP cost for the next day ahead for the regulation markets, where x, y, and z are decision vectors of variables that indicate electricity generated by power generators, electricity purchased from the regulation markets, and electricity sold to the regulation markets, respectively. (Note that x_t, y_t, and z_t refer to x, y, and z at time slot t, respectively.) The vector $(u_1, ..., u_T)$ denotes the percentage of wind farm power output and the vector $(u_{T+1}, ..., u_{2T})$ is the percentage of power generation from other VPP operators. The value of π_t is the market clearing price at time slot t and W is the power generation by wind farms. The objective function (4.29) is to minimize electricity purchasing and generation cost in the regulation markets. The values of π_t^{Up} and π_t^{Do} are the up and down regulation market prices at time slot t, respectively. The value of $\pi_{g,t}$ is the start-up cost of generator g at time slot t. The cost c_t is the VPP marginal cost at time slot t. The objective function (4.30) is to minimize cost by maximizing received revenue and minimizing the electricity purchasing and generation costs, where d_t and r_t are electricity consumed and produced by the VPP at time slot t. The values of π_t^{De} and π_t^{Ri} are the marginal costs that the VPP and rival operators satisfy the electricity demand, respectively. The value of ω_t is the offered price of the VPP at time slot t. For the constraints, (4.19) and (4.20) are the enforced unit ramping limits, where R^{max} and R^{min} are the ramp up and down rates in the VPP, respectively. The constraint (4.24) constrains the electricity balance of the VPP and the constraint (4.23) constrains the produced and purchased electricity which must always be more than the consumed and sold amount. The boundaries constraints for decision variables are explained from (4.25)to (4.27), where X^{max} denotes the maximal budget in the VPP.

Transforming into Multiobjective Semidefinite Programs. Note that the box $U := \prod_{j=1}^{2T}[\lambda_j, \gamma_j]$ can be presented as $U = \text{conv}\{\bar{u}^l \mid l = 1, \ldots, 4^T\}$, where $\bar{u}^l := (\bar{u}_1^l, \ldots, \bar{u}_{2T}^l), l = 1, \ldots, 4^T$ are extreme points of the box U. Moreover, by letting

$D = \text{diag}(-\lambda_1, \ldots, -\lambda_{2T}, \gamma_1, \ldots, \gamma_{2T})$ and $C_i := \text{diag}\begin{pmatrix} e_i^{2T} \\ -e_i^{2T} \end{pmatrix}, i = 1, \ldots, 2T$, the uncertainty set U can be also written as the following matrix inequality:

$$U = \{u := (u_1, \ldots, u_{2T}) \in \mathbb{R}^{2T} \mid D + \sum_{i=1}^{2T} u_i C_i \succeq 0\}. \tag{4.31}$$

Denote $\tilde{x} := \begin{pmatrix} x \\ y \\ z \\ d \\ r \end{pmatrix} = (x, y, z, d, r) \in \mathbb{R}^{5T}$ and set

$$B^1 := \text{diag}(\underbrace{-x^{min}, \ldots, -x^{min}}_{T}, \underbrace{x^{max}, \ldots, x^{max}}_{T}),$$

$$A_i^1 := \begin{cases} \text{diag}\begin{pmatrix} -e_i^T \\ e_i^T \end{pmatrix}, & i = 1, \ldots, T, \\ 0, & i = T+1, \ldots, 5T, \end{cases}$$

$$B^2 := \text{diag}(\underbrace{R^{max}, \ldots, R^{max}}_{T-1}),\ A_i^2 := \begin{cases} \text{diag}(-e_i^{T-1}), & i = 1, \\ \text{diag}(e_{i-1}^{T-1} - e_i^{T-1}), & i = 2, \ldots, T-1, \\ \text{diag}(e_{i-1}^{T-1}), & i = T, \\ 0, & i = T+1, \ldots, 5T, \end{cases}$$

$$B^3 := \text{diag}(\underbrace{R^{min}, \ldots, R^{min}}_{T-1}),\ A_i^3 := -A_i^2, i = 1, \ldots, 5T,$$

$$B^4 := \text{diag}(\underbrace{-d^{min}, \ldots, -d^{min}}_{T}, \underbrace{d^{max}, \ldots, d^{max}}_{T}),$$

$$A_i^4 := \begin{cases} 0, & i = 1, \ldots, 3T, \\ \text{diag}\begin{pmatrix} -e_{i-3T}^T \\ e_{i-3T}^T \end{pmatrix}, & i = 3T+1, \ldots, 4T, \\ 0, & i = 4T+1, \ldots, 5T, \end{cases}$$

$$B^5 := \text{diag}(\underbrace{-r^{min}, \ldots, -r^{min}}_{T}, \underbrace{r^{max}, \ldots, r^{max}}_{T}),$$

$$A_i^5 := \begin{cases} 0, & i = 1, \ldots, 4T, \\ \text{diag}\begin{pmatrix} -e_{i-4T}^T \\ e_{i-4T}^T \end{pmatrix}, & i = 4T+1, \ldots, 5T. \end{cases}$$

We can see that (4.18)–(4.22) amount to the following linear matrix inequalities, repectively,

$$B^j - \sum_{i=1}^{5T} \tilde{x}_i A_i^j \succeq 0, j = 1, \ldots, 5, \tag{4.32}$$

where we should remind that $\tilde{x}_i = x_i, i = 1, \ldots, T, \tilde{x}_i = y_{i-T}, i = T+1, \ldots, 2T, \tilde{x}_i = z_{i-2T}, i = 2T+1, \ldots, 3T, \tilde{x}_i = d_{i-3T}, i = 3T+1 \ldots, 4T$ and $\tilde{x}_i = r_{i-4T}, i = 4T+1, \ldots, 5T$.

Since $x_t \geq 0$, $y_t \geq 0$, $z_t \geq 0$, $d_t \geq 0$ and $r_t \geq 0$ for all $t = 1, \ldots, T$, (4.23) is equivalent to the following matrix inequalities

$$\begin{pmatrix} x_t + y_t + r_t & z_t + d_t \\ z_t + d_t & x_t + y_t \end{pmatrix} \succeq 0, t = 1, \ldots, T. \tag{4.33}$$

Then, for each $t \in \{1, \ldots, T\}$, by denoting

$$A_{i,t} := \begin{cases} \mathrm{diag}(-1, -1), & i = t, T + t, \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & i = 2T + t, 3T + t, \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, & i = 4T + t, \\ 0, & i \in \{1, \ldots, 5T\} \backslash \{t, T + t, 2T + t, 3T + t, 4T + t\}, \end{cases},$$

we see that (4.33) is nothing but $-\sum_{i=1}^{5T} \tilde{x}_i A_{i,t} \succeq 0, t = 1, \ldots, T$, which can be further written as

$$B^6 - \sum_{i=1}^{5T} \tilde{x}_i A_i^6 \succeq 0, \tag{4.34}$$

where $B^6 := 0$ and $A_i^6 := \mathrm{diag}(A_{i,1}, \ldots, A_{i,T}), i = 1, \ldots, 5T$. For each $u \in U$, put

$$B^7(u) := \mathrm{diag}(u_1 W, \ldots, u_T W),$$
$$A_i^7(u) := \begin{cases} \mathrm{diag}\left(-e_i^T\right), & i = 1, \ldots, T, \\ \mathrm{diag}\left(-e_{i-T}^T\right), & i = T + 1, \ldots, 2T, \\ \mathrm{diag}\left(e_{i-2T}^T\right), & i = 2T + 1, \ldots, 3T, \\ \mathrm{diag}\left(-e_{i-3T}^T\right), & i = 3T + 1, \ldots, 4T, \\ \mathrm{diag}\left(u_{i-3T} e_{i-4T}^T\right), & i = 4T + 1, \ldots, 5T, \end{cases}$$

$$B^8(u) := \mathrm{diag}(\pi_1 u_1 W, \ldots, \pi_T u_T W, X^{max} - \pi_1 u_1 W, \ldots, X^{max} - \pi_T u_T W),$$
$$A_i^8(u) := \begin{cases} \mathrm{diag}\left(\pi_i \begin{pmatrix} -e_i^T \\ e_i^T \end{pmatrix}\right), & i = 1, \ldots, T, \\ \mathrm{diag}\left(\pi_{i-T} \begin{pmatrix} -e_{i-T}^T \\ e_{i-T}^T \end{pmatrix}\right), & i = T + 1, \ldots, 2T, \\ \mathrm{diag}\left(\pi_{i-2T} \begin{pmatrix} e_{i-2T}^T \\ -e_{i-2T}^T \end{pmatrix}\right), & i = 2T + 1, \ldots, 3T, \\ 0, & i = 3T + 1, \ldots, 5T. \end{cases}$$

Then, (4.24) and (4.25) become the following linear matrix inequalities, repectively,

$$B^j(u) - \sum_{i=1}^{5T} \tilde{x}_i A_i^j(u) \succeq 0, \ j = 7, 8. \tag{4.35}$$

Similarly, by letting

$$B^9 := \operatorname{diag}(\underbrace{-y^{min}, \dots, -y^{min}}_{T}, \underbrace{y^{max}, \dots, y^{max}}_{T}),$$

$$A_i^9 := \begin{cases} 0, & i = 1, \dots, T, \\ \operatorname{diag}\begin{pmatrix} -e_{i-T}^T \\ e_{i-T}^T \end{pmatrix}, & i = T+1, \dots, 2T, \\ 0, & i = 2T+1, \dots, 5T, \end{cases}$$

$$B^{10} := \operatorname{diag}(\underbrace{-z^{min}, \dots, -z^{min}}_{T}, \underbrace{z^{max}, \dots, z^{max}}_{T}),$$

$$A_i^{10} := \begin{cases} 0, & i = 1, \dots, 2T, \\ \operatorname{diag}\begin{pmatrix} -e_{i-2T}^T \\ e_{i-2T}^T \end{pmatrix}, & i = 2T+1, \dots, 3T, \\ 0, & i = 3T+1, \dots, 5T, \end{cases}$$

we see that (4.26) and (4.27) are respectively equivalent to the following linear matrix inequalities

$$B^j(u) - \sum_{i=1}^{5T} \tilde{x}_i A_i^j(u) \succeq 0, \; j = 9, 10.$$

This, together with (4.32), (4.34) and (4.35), shows that the constraints (4.18)–(4.27) are written as the following linear matrix inequality

$$B(u) - \sum_{i=1}^{5T} \tilde{x}_i A_i(u) \succeq 0,$$

where $u \in U$, $B(u) := \operatorname{diag}\big(B^1, \dots, B^6, B^7(u), B^8(u), B^9, B^{10}\big)$ and $A_i(u) := \operatorname{diag}\big(A_i^1, \dots, A_i^6, A_i^7(u), A_i^8(u), A_i^9, A_i^{10}\big)$ for $i = 1, \dots, 5T$.

Now, the problem (EU3) is rewritten in the form of our multiobjective semidefinite programming problem (USP) as the following one:

$$\min_{\tilde{x} \in \mathbb{R}^{5T}} \left\{ \left(c^1(u)^\top \tilde{x} + \beta^1(u), c^2(u)^\top \tilde{x} + \beta^2(u), c^3(u)^\top \tilde{x} + \beta^3(u)\right) \mid B(u) - \sum_{i=1}^{5T} \tilde{x}_i A_i(u) \succeq 0 \right\}, \tag{BUP}$$

where $u \in U$ which is given by (4.31) and the affine maps $c^j : \mathbb{R}^{2T} \to \mathbb{R}^{5T}$, $\beta^j : \mathbb{R}^{2T} \to \mathbb{R}$, $j = 1, 2, 3$ are given respectively by

$$c_0^1 := (-\pi_1, \dots, -\pi_T, -\pi_1, \dots, -\pi_T, \pi_1, \dots, \pi_T, 0_{2T}),$$

$$c_0^2 := \left(\sum_{g=1}^{G} \pi_{g,1}, \dots, \sum_{g=1}^{G} \pi_{g,T}, \pi_1^{Up}, \dots, \pi_T^{Up}, -\pi_1^{Do}, \dots, -\pi_T^{Do}, 0_{2T}\right),$$

$$c_0^3 := \left(\omega_1, \dots, \omega_T, 0_{2T}, -\pi_1^{De}, \dots, -\pi_T^{De}, 0_T\right), c_i^3 := \begin{cases} 0_{5T}, & i = 1, \dots, T, \\ (0_{4T}, \pi_{i-T}^{Ri} e_{i-T}^T), & i = T+1, \dots, 2T, \end{cases}$$

$c_i^1 := c_i^2 := 0_{5T}, i = 1, \ldots, 2T, \beta_0^1 := \beta_0^3 := 0, \beta_i^1 := \begin{cases} -W\pi_i, & i = 1, \ldots, T, \\ 0, & i = T+1, \ldots, 2T, \end{cases}$

$\beta_0^2 := \sum_{t=1}^{T} c_t, \beta_i^2 := \beta_i^3 := 0, i = 1, \ldots, 2T.$

Semidefinite Programming Reformulations. Given $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3_+ \backslash \{0\}$, one considers a corresponding robust scalarized problem of (BUP) defined by

$$\inf_{\tilde{x} \in \mathbb{R}^{5T}} \Big\{ \sum_{j=1}^{3} \alpha_j \max_{u \in U} \{ c^j(u)^\top \tilde{x} + \beta^j(u) \} \mid B(u) - \sum_{i=1}^{5T} \tilde{x}_i A_i(u) \succeq 0, \ \forall u \in U \Big\}. \tag{BP$_\alpha$}$$

In this setting, an SDP reformulation problem of (BP$_\alpha$) is given by

$$\begin{aligned} \inf_{(\tilde{y}, Z^1, Z^2, Z^3)} \Bigg\{ & \sum_{j=1}^{3} \alpha_j \big((c_0^j)^\top \tilde{y} + \beta_0^j + \mathrm{Tr}(DZ^j) \big) \\ & \mid \bar{B}^l - \sum_{i=1}^{5T} \tilde{y}_i \bar{A}_i^l \succeq 0, \ l = 1, \ldots, 4^T, \tilde{y} := (\tilde{y}_1, \ldots, \tilde{y}_{5T}) \\ & (c_r^j)^\top \tilde{y} + \beta_r^j + \mathrm{Tr}(C_r Z^j) = 0, \\ & Z^j \in S^{4T}, Z^j \succeq 0, r = 1, \ldots, 2T, j = 1, 2, 3 \Bigg\}, \end{aligned} \tag{BP$^*_\alpha$}$$

where $\bar{A}_i^l := A_i(\bar{u}^l), \bar{B}^l := B(\bar{u}^l), l = 1, \ldots, 4^T, i = 1, \ldots, 5T.$

According to the SDP reformulation schemes in Corollary 3.2, we assert that, for a given $\alpha \in \mathrm{int}\mathbb{R}^3$ (resp., $\alpha \in \mathbb{R}^3_+ \backslash \{0\}$), if $(\tilde{x}, Z^1, Z^2, Z^3)$ is a solution of (BP$^*_\alpha$), then $\tilde{x}$ is a robust (resp., weak) efficient solution of (BUP), which means that (x, y, z, d, r) is a robust (resp., weak) efficient solution of problem (EU3).

Numerical Simulations. We use a dataset collected by Australian Energy Market Operator (AEMO)[1] to test the efficacy of the proposed semidefinite programming reformulations. This is done by showing how to locate robust Pareto solutions of problem (EU3) via its SDP reformulation problem (BP$^*_\alpha$).

We simulate a set of possible combinations of weights α_j, where $\alpha_j \in [0.00001, 1]$, $j = 1, 2, 3$ with $\sum_{j=1}^{3} \alpha_j = 1$ and obtain robust Pareto solutions that are shown in Fig. 1. As we can see from Fig. 1, with different weights for three objective functions, the model provides various possible tasks for different corresponding costs, which are commonly found in a multiobjective model as these solutions are trade-offs between their objectives. The robust Pareto solutions found for the underlying problem empower the decision-maker to more readily identify preferred (optimal) trade-off trends. Furthermore, the decision-maker gains trade-off (or revenues value) stability due to the fact that the obtained Pareto solutions are *robust* in the sense that they are immune from uncertainty factors in inputs of the problem or fluctuating trading circumstances. For example, the rival operators may alter their bidding strategies in a dynamic market.

We also test for a combination of weights α_j, where $\alpha_j \in \{1, 2, 3\}$, $j = 1, 2, 3$ and compare the proposed SDP reformulation and (direct) *worst case/scenario* and *best case/scenario* approaches, which are done by solving the problem (EU3) directly with some fixed values of u from the uncertainty set U. The comparison between the SDP reformulation and the two

[1] https://www.aemo.com.au/energy-systems/electricity/national-electricity-market-nem/data-nem/aggregated-data

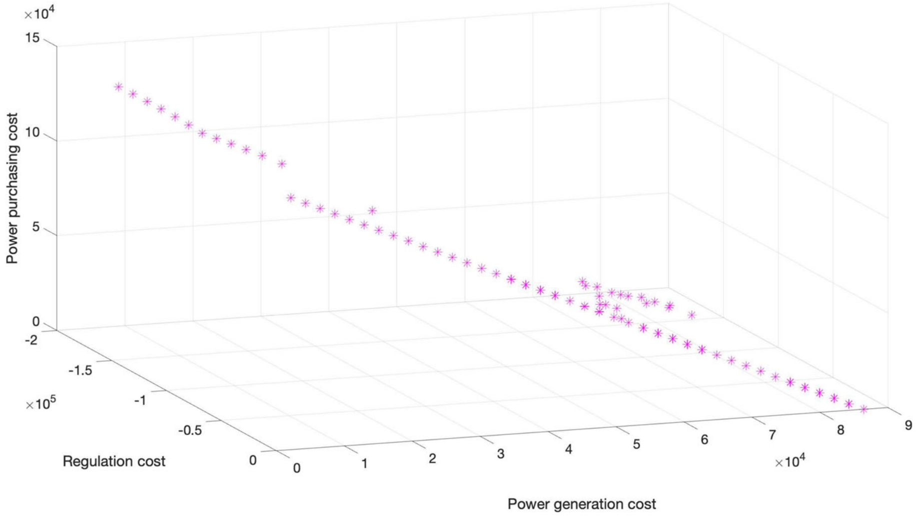


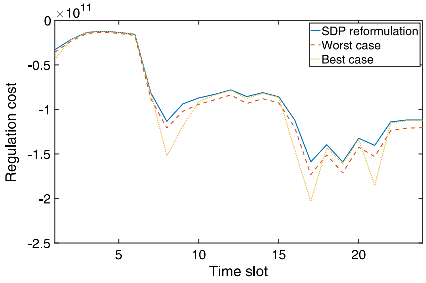
Fig. 1 Robust Pareto solutions for (EU3) with different weight combinations

direct approaches for solving the problem (EU3) is shown in Fig. 2. As can be seen from Figs. 2a and c , the regulation cost and power purchasing cost have an analogous trend with the fluctuating market price and demand. However, the proposed SDP method always makes more profits in the regulation market with a similar or less cost on electricity purchasing than in the worst case. Moreover, the SDP reformulation closely tracks the best case performance, maintaining regulation revenues within a narrow margin of the best case while offering significant robustness compared to the worst case. From Fig. 2b, the power generation cost by the proposed method is much less and more stable than the worst case method. In particular, the SDP approach yields generation costs only slightly above the best case scenario but with substantially reduced variability, demonstrating a balanced trade-off between optimality and robustness. Similarly, for power purchasing cost, the SDP method achieves costs nearly indistinguishable from the best case and outperforms the worst case throughout all time slots.

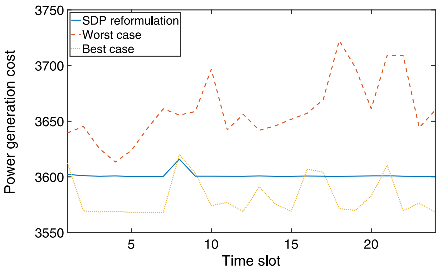
Consequently, the proposed semidefinite programming reformulations are capable of solving the bidding strategy of virtual power plant problem under uncertain wind farm power for a full day ($T = 24$) assuming that the electricity spot market updates their prices on an hourly basis. The proposed conic reformulations perform well for the bidding strategy of virtual power plant problems when the dimension of variables is small (e.g., $T = 24$). However, as T increases, the number of new variables in the reformulation models also grows leading to a significant computational burden for the proposed conic reformulation schemes when applied to higher-dimensional real-world problems.

5 Conclusions and further remarks

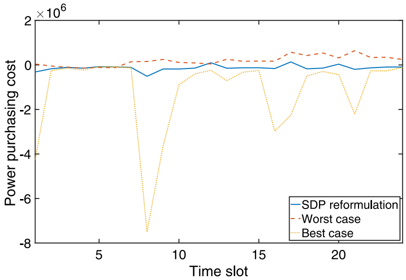
We have presented verifiable linear conic conditions for robust (weak) efficiency of a conic multiobjective linear optimization problem under affinely uncertainty data. It has been shown that the obtained optimality conditions become linear matrix inequalities for the prominent class of multiobjective semidefinite programming problems or second-order conic conditions for the special class of multiobjective second-order cone programming problems. We have



(a) Regulation costs



(b) Power generation costs



(c) Power purchasing costs

Fig. 2 The proposed SDP, worst case and best case approaches for solving (EU3)

proposed (scalar) conic reformulation schemes for solving the conic robust multiobjective linear optimization problem and shown how a robust weak/efficient solution of the uncertain conic multiobjective linear program can be calculated by solving their conic reformulation problems. In particular, we have shown that for multiobjective semidefinite programming and multiobjective second-order cone programming problems their corresponding conic reformulations can be solved by using semidefinite programming and second-order cone programming reformulations, respectively.

Numerical examples are provided to show how the proposed conic programming reformulation schemes can be employed to locate robust weak/efficient solutions for uncertain conic multiobjective optimization problems including a model arisen from practical applications. The numerical simulations show that the proposed conic uncertain multiobjective problem is potentially capable of modeling practical problems involving data uncertainties and the corresponding conic reformulation shemes are able to generate multiple robust efficient solutions for such problems. As a result, the proposed conic uncertain/robust multiobjective models and associated conic reformulation schemes not only empower the decision-maker to more readily locate preferred (optimal) trade-off trends but also enable the opportunity to *stably* achieve corresponding (optimal) trade-off values under the presence of uncertainty in inputs of the actual problem.

It would be of interest to perform a comprehensive analysis of the proposed conic programming reformulation models with recent advanced robust optimization methods, particularly in terms of computational efficiency and solution quality. Moreover, it is worth seeing how we can develop and apply the these conic reformulation schemes to solve other practical problems, such as the internet routing problem under traffic uncertainty [15] or the energy supply system of [29], where the problem data often involve uncertainties.

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