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Analytical Solution to Optimal Distributed Bipartite Consensus for Heterogeneous Multi-Agent Systems on Coopetition Networks: A Fast Convergent Algorithm

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***Abstract*—This note makes the first attempt to investigate the *optimal* bipartite consensus problem in the general case for heterogeneous multi-agent systems with cooperative-competitive interactions, specifically those containing a spanning tree. Unlike traditional control protocols based on the gradient descent method, our proposed optimal bipartite algorithm demonstrates a fast superlinear convergence speed. The key innovation of this method lies in the design of a fully distributed optimal controller, which is based on a distributed observer utilizing local neighbor information. The analytical solution for the optimal controller is derived with the help of the Riccati equation, aligning with classical optimal control theory. This approach unifies the design method for both the bipartite consensus problem and the well-studied consensus problem. Additionally, the proposed optimal algorithm can be directly extended to homogeneous systems. A numerical example is provided to demonstrate the effectiveness and rapid convergence of our control algorithm.**

***Index Terms*—Distributed optimal bipartite consensus, Heterogeneous multi-agent system, LQ optimal control, Distributed observer.**

I. INTRODUCTION

Over the past few decades, there has been a noticeable increase in research interest in multi-agent systems (MASs). This growing interest is largely motivated by the practical applications of MASs in control theory and various industries, including unmanned vehicles [1], traffic networks [2], distributed sensor networks [3], and smart grids [4], among others [5]. As the society, economy and industrial technology continue to develop rapidly, considerable attention has been drawn to an emerging research direction: the bipartite consensus of MASs. Unlike traditional consensus control, bipartite consensus is designed to guide all agents to converge to a final value of equal magnitude but opposite sign within cooperative and competitive communication networks.

In many complex network environments, it is more reasonable to consider both cooperative and competitive relationships among agents. For instance, in social networks, relationships between pairs of agents may be characterized as friendly or hostile, based on trust or distrust. In economic systems, duopolistic regimes often emerge when companies compete for resources [6]–[8]. Similarly, in the RoboMaster competition, each robot gathers information from both teammates and opponents to make decisions that align with its team. Unlike existing studies on MASs that focus solely on cooperative relationships, signed graph theory is introduced to characterize interaction communications where cooperation and competition coexist among agents. The concept of bipartite consensus has been first introduced in [9], where the necessary and sufficient conditions have been established for bipartite consensus among single-integrator agents under a strongly connected signed graph.

Building on the pioneering work of Altafini [9], the bipartite consensus control problem of MASs has been extensively studied over the past decade. The findings from [9] have been extended to more general contexts including double-integrator dynamics [10], high-order agent dynamics with a single input [11], general multi-input multi-output linear systems [12], and singular MASs [13], among others. Since then, various variants of bipartite consensus have emerged, such as interval bipartite consensus, adaptive bipartite consensus, finite-time bipartite consensus, mean-square bipartite consensus, and approximate optimal consensus based on reinforcement learning (RL) [14]–[18]. It is noteworthy that, in all these studies, the dynamics of MASs are assumed to be the same.

In engineering practice, agents often have different dynamics, making bipartite consensus of heterogeneous MASs a focal point of research in recent years. Utilizing gauge transformations, the authors in [19] demonstrated the equivalence between bipartite output synchronization and the well-studied cooperative output synchronization, providing a sufficient H_∞-criterion for achieving bipartite output consensus. Bipartite output tracking control protocols have been further explored in [20]. However, the lower bound for the coupling gain (required to guarantee bipartite consensus performance) depends on the minimal real part of the nonzero eigenvalues of the Laplacian matrix of the signed graph, which may necessitate global topology information. Although distributed adaptive protocols based on output regulation theory [21] have been proposed in [22], these approaches require the solution of the regulation equation and do not consider performance optimization. As a result, the convergence speed in these related works is *not* optimal.

Regarding the global optimal consensus problem, designing an effective consensus controller poses significant challenges. A primary obstacle is that the design of each controller requires information from non-neighboring agents, which typically leads to a centralized algorithm. This difficulty became apparent in 1968 when Witsenhausen provided a counterexample [23], demonstrating that the solution to linear optimal control problems with decentralized feedback control constraints might be a nonlinear function. To achieve global optimal consensus for MASs, various approaches have been explored. For example, the inverse optimal control method [24], the LQR method [25], [26], and the network approximation technique [27] have been applied to develop distributed controllers based on local neighbors' information. However, the state weight matrix in the corresponding cost function tends to be either overly complex or very specific, and these methods are generally applicable only

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to homogeneous systems with cooperative networks. Currently, the optimal bipartite consensus control problem for continuous-time heterogeneous MASs has primarily focused on using reinforcement learning (RL) techniques [18], [28]–[30] to approximate the solution of the optimal controller, rather than providing an analytical solution. Furthermore, the cost function considered is local, not global. As a result, research on distributed global optimal bipartite consensus in general for discrete-time heterogeneous MASs remains under explored and requires further in-depth investigation.

Motivated by the above discussion, this paper aims to study the distributed optimal bipartite leader-follower consensus for heterogeneous MASs over a general directed signed interaction graph. We propose a unified design framework for a distributed asymptotically optimal bipartite consensus controller using LQ optimal control and distributed observer design. The primary contributions of this paper are as follows.

1) A distributed asymptotically optimal controller is derived with the aid of a distributed observer that incorporates each agent's historical state information. Unlike the methods described in [9], [19], [20], the proposed approach does not require the gauge transformations process and eliminates the need for the solvability assumption of the output regulation equation. This offers a fresh perspective on distributed optimal bipartite consensus control.
2) The precise analytical solution of the optimal controller is obtained by solving the Algebraic Riccati Equation (ARE), and the state weight matrix in the global cost function is also allowed to be any positive definite constant matrix, making it more general compared to those used in [24], [27]. This aligns with classical optimal control methods, demonstrating the versatility and applicability of our approach.
3) The proposed optimal controller can drive all agents to bipartite consensus with a fast superlinear convergence speed, which is superior to traditional bipartite consensus algorithms based on gradient descent. This is supported by theoretical analyses and a numerical example. Additionally, the proposed optimal consensus algorithm is directly applicable to the homogeneous case.

Notations: $\mathbb{R}^{n\times m}$ represents the set of $n \times m$-dimensional real matrices. I_p is the identity matrix with dimension $p \times p$. $\text{diag}\{a_1, a_2, \cdots, a_N\}$ denotes the diagonal matrix with diagonal elements being $a_1, \cdots, a_N$. $\|x\|$ is the 2-norm of a vector x. $\rho(A)$ is the spectral radius of matrix A. A^T and $A^\dagger$ denote the transpose and the Moore-Penrose inverse of a matrix A. $Range(A)$ means the range space of A. $\mathbf{N}(A)$ is the null Space of A. $sign(\cdot)$ is the sign function.

II. Preliminaries and Problem Formulation

A. Graph theory

A signed directed graph $\mathcal{G}_s = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of nodes (agents), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. $\mathcal{A}_s = [a_{ij}]_{N\times N}$ is weighted signed adjacency matrix. For an edge (i, j), node i is called the parent node, node j is called the child node. If $(i, j) \in \mathcal{E}$, then $a_{ij} \neq 0$, i.e., agents i and j can exchange information; $a_{ij} = 0$, otherwise. Moreover, $a_{ij} > 0$ and $a_{ij} < 0$ mean the cooperative interaction and competitive interaction between agent i and agent j, respectively. The signed directed graph containing the leader 0 is the augmented graph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ where $\bar{\mathcal{V}} = \mathcal{V} \cup \{0\}$ and $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. $\bar{\mathcal{N}}_i = \{j|(i, j) \in \bar{\mathcal{E}}\}$ represents the neighbour set of agent i. A directed graph is said to contain a spanning tree if there exists a node called root such that there exists a directed path from this node to every other node in the graph. The Laplacian matrix $\mathcal{L}_s$ of a signed graph is defined as $\mathcal{L}_s = diag(\sum_{j=1}^{N} |a_{1j}|, \cdots, \sum_{j=1}^{N} |a_{Nj}|) - \mathcal{A}_s$.

Lemma 1. *A signed directed graph $\mathcal{G}$ is structurally balanced if its node set $\mathcal{V}$ can be divided into two disjoint nonempty subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \geq 0$, if $\forall i, j \in \mathcal{V}_q$ or $\forall i, j \in \mathcal{V}_r$, and $a_{ij} \leq 0$, if $\forall i \in \mathcal{V}_q, \forall j \in \mathcal{V}_r$, where $q \neq r$, and $q, r \in \{1, 2\}$.*

B. Problem Formulation

Consider a heterogeneous discrete-time multi-agent system consisting of N agents over a directed graph $\mathcal{G}$ with the dynamics of each agent given by

$$x_i(s+1) = A_i x_i(s) + B_i u_i(s), i = 1, 2, \cdots, N \tag{1}$$

where $x_i(s) \in \mathbb{R}^n$ and $u_i(s) \in \mathbb{R}^m$ are, respectively, the state and the input of each agent, and $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are the coefficient matrices.

The dynamic of the leader is given by

$$x_0(s+1) = A_0 x_0(s), \tag{2}$$

where $x_0 \in \mathbb{R}^n$ is the leader's state, and $A_0 \in \mathbb{R}^{n\times n}$ is the system state matrix.

Define the cost function of the multi-agent system (1) as

$$\begin{aligned} J(\tau, \infty) = \sum_{s=\tau}^{\infty} \Bigg(&\sum_{i=1}^{N} \sum_{j=0}^{N} (x_i(s) - d_i x_j(s))^T Q (x_i(s) - d_i x_j(s)) \\ &+ \sum_{i=1}^{N} u_i^T(s) R_i u_i(s) \Bigg), \end{aligned} \tag{3}$$

where $Q \geq 0$ and $R_i > 0$ are weighting matrices, $d_i = 1, \forall i \in \mathcal{V}_q$, and $d_i = -1, \forall i \in \mathcal{V}_r$, where $q \neq r$.

Assumption 1. *For $i = 1, 2, \cdots, N$, (A_i, B_i) is stabilizable.*

Assumption 2. *The signed directed graph $\mathcal{G}$ is structurally balanced, and graph $\bar{\mathcal{G}}$ has a directed spanning tree with the node 0 as the root.*

Problem 1. *For the heterogeneous MASs* (1)–(2)*, our objective is to design a distributed control protocol $u_i(s)$ that minimizes the cost function* (3) *while ensuring that the system* (1)-(2) *achieves leader-follower bipartite consensus, i.e., for any initial conditions x_{i0},*

$$\lim_{s\to\infty} \|x_i(s) - d_i x_0(s)\| = 0, \quad i = 1, \cdots, N.$$

Remark 1. *It should be noted that the existing method for a similar Problem 1 in continuous-time systems [31] requires a dynamic compensator for each agent, resulting in the necessity to solve multiple Sylvester matrix equations: $A_0 = A_i + B_i U_i$. However, the unique solution U_i may not be exist, for instance, in the case of $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Furthermore, when $A_i = A$ and $B_i = B$, the following bipartite consensus protocol, proposed in [12], [31], is commonly used*

$$u_i(s) = \mu S \sum_{j\in N_i} |a_{ij}|(x_j(s) - sign(a_{ij}) x_i(s)) \tag{4}$$

where $S = R^{-1} B^T P$ is a feedback matrix and coupling gain $\mu \geq \frac{1}{2\min_{i=2,\cdots,N} Re(\lambda_i)}$. It is clear that this bipartite consensus protocol depends on the non-zero eigenvalues of the Laplacian matrix $\mathcal{L}_s$ of signed graph $\mathcal{G}_s$, and the convergence speed is also influenced by these eigenvalues, which cannot be further optimized. In contrast, this paper aims to design a distributed optimal controller $u_i(s)$ for discrete-time multi-agent systems that achieves bipartite consensus while minimizing the global cost function (3).

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III. Distributed optimal bipartite consensus protocol

Define the bipartite state error between agent i and j as:

$$e_{ij}(s) = x_i(s) - d_i x_j(s). \quad (5)$$

Through the foregoing analyses, (5) can be equivalently expressed as

$$e_{ij}(s) = \begin{cases} x_i(s) - x_j(s), & i, j \in \mathcal{V}_1 \text{ or } \mathcal{V}_2 \\ x_i(s) + x_j(s), & i \in \mathcal{V}_1, \text{and } j \in \mathcal{V}_2 \end{cases} \quad (6)$$

Then, the state error system is given by

$$\begin{aligned} e_{ij}(s+1) = & A_i e_{ij}(s) + B_i u_i(s) \\ & + d_i(A_i - A_j)x_j(s) - d_i B_j u_j(s) \end{aligned} \quad (7)$$

with $i = 1, \cdots, N$, $j \in \mathcal{N}_i$ and $B_0 = 0$. Undoubtedly, the relative neighbour error system (7) is not a standard linear system due to the last two non-zero terms. In fact, these extra terms can be viewed as a total disturbance term $r_i(s)$, i.e., $r_i(s) = d_i(A_i - A_j)x_j(s) - d_i B_j u_j(s)$, for $j \in \mathcal{N}_i$. Also, we first apply the feedforward controller to eliminate $r_i(s)$ such that the system (7) is converted into a standard linear system. Consequently, the controller for agent i is given by

$$u_i(s) = u_{ri}(s) + u_{fi}(s), \quad (8)$$

where the first part (the feedforward controller $u_{ri}(s)$) in (8) is designed in the following two ways:

- Case 1: If B_i is an invertible matrix, then

$$u_{ri}(s) = -d_i B_i^{-1}[(A_i - A_j)x_j(s) - B_j u_j(s)], \quad (9)$$

- Case 2: If $Range([A_j - A_i, B_j]) \subseteq Range(B_i)$, then

$$\begin{aligned} u_{ri}(s) = & -d_i B_i^{\dagger}[(A_i - A_j)x_j(s) - B_j u_j(s)] \\ & + (I - B_i^{\dagger} B_i)z(s), \end{aligned} \quad (10)$$

where $(I - B_i^{\dagger} B_i)z(s) \in \mathbf{N}([A_j - A_i, B_j])$ for all $z(s)$.

To guarantee the solvability of the liner equation $B_i u_{ri}(s) + r_i(s) = 0$, $u_{ri}(s)$ is designed via Case 1 or Case 2, where Case1 is a special case of Case 2, and an invertible B_i can indeed be replaced by the identity matrix I without loss of generality. It is clear that $u_{ri}(s)$ requires only the state and input information from the neighboring agent j, making it a distributed form of control. Under the feedforward control (9) or (10), the relative neighbor error system (7) is rewritten as the following standard linear system:

$$e_{ij}(s+1) = A_i e_{ij}(s) + B_i u_{fi}(s). \quad (11)$$

The bipartite consensus problem for heterogeneous MASs (1)–(2) is equivalent to the stability problem of the error system (7). In general, the total edge number over the communication graph may exceed N. However, under Assumption 2, it is sufficient to select edges $e_{ij}(s)$ along a spanning tree within the signed graph.

Redefine the global error vector

$$\Xi(s) = \begin{bmatrix} e_{10}^T(s) & \cdots & e_{ij}^T(s) \cdots & e_{N,N-1}^T(s) \end{bmatrix}^T$$

where all $e_{ij}(s)$ terms correspond to the edges of the spanning tree selected as above. Then, the global error system is written as

$$\Xi(s+1) = \tilde{A}\Xi(s) + \mathcal{B}u_f(s), \quad (12)$$

where $u_f(s) = \begin{bmatrix} u_{f1}^T(s) & u_{f2}^T(s) \cdots & u_{fN}^T(s) \end{bmatrix}^T$, $\tilde{A} = diag\{A_1; A_2; \cdots; A_N\}$, and $\mathcal{B} = diag\{B_1; B_2; \cdots; B_N\}$.

The corresponding cost function (3) is reformulated as

$$\begin{aligned} J(\tau, \infty) &= \sum_{s=\tau}^{\infty} \left(\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} e_{ij}^T(s) Q e_{ij}(s) + \sum_{i=1}^{N} u_{fi}^T(s) R_i u_{fi}(s) \right) \\ &= \sum_{s=\tau}^{\infty} [\Xi^T(s) \mathcal{Q} \Xi(s) + u_f^T(s) R u_f(s)], \end{aligned} \quad (13)$$

with $\mathcal{Q} = diag\{Q, \cdots, Q\}$ and $R = diag\{R_1, \cdots, R_N\}$, and $\mathcal{N}_i$ specifically refers to the neighborhood in the spanning tree subgraph, where each node (except the root node) has exactly one parent node.

Remark 2. *Unlike traditional bipartite consensus protocols [9], [11], where gauge transformation is used to convert non-cooperative problems into cooperative ones, this paper addresses the bipartite consensus problem for MASs described by* (1)–(2) *using a novel approach. We solve the problem with a distributed bipartite consensus controller based on LQ optimal control theory and distributed observer design. This approach eliminates the need for gauge transformations and directly tackles the bipartite consensus challenge in a more straightforward and efficient manner.*

By applying the feedforward controller $u_{ri}(s)$ in (9) or (10), the solution to Problem 1 becomes equivalent to solving the optimal control problem for the system (12) with the cost function (13). In particular, when the global error information $\Xi(s)$ is available for all agents, the optimal control can be obtained by the standard LQ regulator (LQR) method stated in the following lemma.

Lemma 2. *[32] Suppose that $\Xi(s)$ is available for all agents. Consider system* (12) *with cost* (13)*. Then, the optimal controller is given by $u_{fi}(s) = K_i \Xi(s)$, which can be formulated in a global form as:*

$$u_f(s) = K_u \Xi(s), \quad (14)$$

where the feedback gain K_u is given by

$$K_u = -(R + \mathcal{B}^T P \mathcal{B})^{-1} \mathcal{B}^T P \tilde{A} \quad (15)$$

and P is the solution of the following ARE

$$P = \tilde{A}^T P \tilde{A} + \mathcal{Q} - \tilde{A}^T P \mathcal{B}(R + \mathcal{B}^T P \mathcal{B})^{-1} \mathcal{B}^T P \tilde{A}. \quad (16)$$

The corresponding optimal cost function is

$$J^*(\tau, \infty) = \Xi^T(\tau) P \Xi(\tau). \quad (17)$$

Moreover, if P is the unique positive definite solution to (16)*, then $\tilde{A} + \mathcal{B}K_u$ is stable.*

Under the centralized controller (14) (the communication topology is a complete graph), the multi-agent system (1)–(2) achieves bipartite consensus. However, in practical communication networks, the global error information $\Xi(s)$ is not accessible to all agents. Therefore, we will design a distributed controller based on an observer to overcome this limitation.

A. Distributed optimal bipartite consensus protocol design

In this subsection, we design a novel distributed bipartite consensus controller that leverages a distributed observer to estimate the global information $\Xi(s)$. Since the control input $u_{fi}(s)$ is not shared with other agents, the distributed optimal control problem of the system (12) is transformed into a decentralized control problem, which can be reformulated as follows:

$$\Xi(s+1) = \tilde{A}\Xi(s) + \sum_{i=1}^{N} \mathcal{B}_i u_{fi}(s), \quad (18)$$

$$Y_i(s) = H_i \Xi(s), i = 1, \cdots, N \quad (19)$$

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where $Y_i(s)$ is the local measurement for agent i, $\mathcal{B}_i$ consists of $0_{n\times m}$ and B, and H_i is a $\{0, I_n\}$ matrix whose specific forms depend on the interaction among agents.

A new distributed controller for agent i is designed as

$$u^*_{fi}(s) = K_i\hat{\Xi}_i(s), \tag{20}$$

where $\hat{\Xi}_i(s)$ is a distributed observer for each agent based on the available information from itself and its neighbor agents. Totally different from the traditional observer, $\hat{\Xi}_i(s)$ is designed as follows:

$$\begin{aligned}\hat{\Xi}_i(s+1) = &\tilde{A}\hat{\Xi}_i(s) + \mathcal{B}_1K_1\hat{\Xi}_i(s) + \cdots \\ &+ \mathcal{B}_{i-1}K_{i-1}\hat{\Xi}_i(s) + \mathcal{B}_i u^*_{fi}(s) \\ &+ \mathcal{B}_{i+1}K_{i+1}\hat{\Xi}_i(s) + \cdots + \mathcal{B}_N K_N\hat{\Xi}_i(s) \\ &+ L_i(Y_i(s) - H_i\hat{\Xi}_i(s)),\end{aligned} \tag{21}$$

where $K_i = \begin{bmatrix}0 & \cdots & I & 0\cdots & 0\end{bmatrix} K_u$ is the subpart of K_u obtained by solving ARE (16), L_i is the observer gain to be determined later to ensure the stability of the observers. Specifically, for agent i, the input $u_{fj}(s), i \neq j$ is unknown, so we replace it by the observer information $K_j\hat{\Xi}_i(s)$, and this forms the innovation of our design method.

The following Theorem 1 demonstrates that the stability of the designed observer can be guaranteed. Furthermore, it shows that the multi-agent system (1) achieves leader-follower bipartite consensus under the proposed distributed controller.

Theorem 1. *Let Assumption 2 hold. Consider the global error system* (18) *and the distributed control laws* (20)-(21)*. If there exist observer gains L_i $(i = 1, \cdots, N)$ such that the matrix*

$$A_c = \begin{bmatrix}\Theta_1 & -\mathcal{B}_2K_2 & \cdots & -\mathcal{B}_NK_N \\ -\mathcal{B}_1K_1 & \Theta_2 & \cdots & -\mathcal{B}_NK_N \\ \vdots & \vdots & \ddots & \vdots \\ -\mathcal{B}_1K_1 & \cdots & -\mathcal{B}_{N-1}K_{N-1} & \Theta_N\end{bmatrix} \tag{22}$$

is asymptotically stable with $\Theta_i = \tilde{A} + \mathcal{B}K_u - \mathcal{B}_iK_i - L_iH_i$, then the observers (21) *are asymptotically stable, i.e.,*

$$\lim_{k\to\infty} \|\hat{\Xi}_i(s) - \Xi(s)\| = 0. \tag{23}$$

Moreover, if the Riccati equation (16) *has a positive definite solution P, under the distributed controllers* (9)*,* (20) *with* (21)*, the MASs* (1)–(2) *can achieve leader-follower bipartite consensus.*

Proof. Define the observer error vector $\tilde{\Xi}_i(s) = \Xi(s) - \hat{\Xi}_i(s)$. Then, combining system (18) with observers (21), one obtains

$$\begin{aligned}\Xi(s+1) = &(\tilde{A} + \mathcal{B}K_u)\Xi(s) - \mathcal{B}_1K_1\tilde{\Xi}_1(s) \\ &- \mathcal{B}_2K_2\tilde{\Xi}_2(s) - \cdots - \mathcal{B}_NK_N\tilde{\Xi}_N(s),\end{aligned} \tag{24a}$$

$$\begin{aligned}\tilde{\Xi}_i(s+1) = &(\tilde{A} + \mathcal{B}K_u - \mathcal{B}_iK_i - L_iH_i)\tilde{\Xi}_i(s) \\ &- \mathcal{B}_1K_1\tilde{\Xi}_1(s) - \cdots - \mathcal{B}_{i-1}K_{i-1}\tilde{\Xi}_{i-1}(s) \\ &- \mathcal{B}_{i+1}K_{i+1}\tilde{\Xi}_{i+1}(s) - \cdots - \mathcal{B}_NK_N\tilde{\Xi}_N(s),\end{aligned} \tag{24b}$$

Combining equations (24a) and (24b) yields

$$\begin{bmatrix}\Xi(s+1) \\ \tilde{\Xi}(s+1)\end{bmatrix} = \bar{A}_c\begin{bmatrix}\Xi(s) \\ \tilde{\Xi}(s)\end{bmatrix}, \tag{25}$$

where $\tilde{\Xi}(s) = \begin{bmatrix}\tilde{\Xi}_1^T(s), \tilde{\Xi}_2^T(s), \cdots, \tilde{\Xi}_N^T(s)\end{bmatrix}^T$, $\bar{A}_c = \begin{bmatrix}\tilde{A} + \mathcal{B}K_u & \Psi \\ 0 & A_c\end{bmatrix}$, $\Psi = \begin{bmatrix}-\mathcal{B}_1K_1 & \cdots & -\mathcal{B}_NK_N\end{bmatrix}$ and A_c in (22) is derived.

Obviously, if there exist matrices L_i such that the closed-loop matrix A_c is asymptotically stable, then observer errors $\tilde{\Xi}(s)$ converge to zero as $s \to \infty$, i.e., Eq. (23) holds. Furthermore, since P is the positive definite solution to Riccati equation (16), we know that $\tilde{A} + \mathcal{B}K_u$ is stable and, based on the LQ control theory, the leader-follower consensus of multi-agent system (1) can be achieved, i.e., Problem 1 is solved. The proof is now complete. □

Remark 3. *Unlike the cases of complete graphs and undirected graph presented in [27], [33], the proposed control strategy is indeed distributed for general directed graphs, as it relies solely on local information of neighboring agents. On the one hand, the optimal control gain matrix K_i, as the subpart of the global feedback gain matrix K_u obtained by solving the ARE* (16)*, relies solely on the system matrices $\tilde{A}, \mathcal{B}$ of the global relative error system* (12) *constructed by the neighbor error $e_{ij}(s)$ along with a spanning tree; hence, K_i is obtained under the distributed information without needing to know the Laplacian matrix's nonzero eigenvalues (which is global information). On the other hand, the distributed observer $\hat{\Xi}_i(s)$ is designed to estimate the full $\Xi(s)$ based on the local observer data.*

Based on Theorem 1, the next task is to appropriately determine the observer gain matrix L_i to ensure the stability of the observer error system (24b), which can be rewritten as:

$$\tilde{\Xi}(s+1) = A_c\tilde{\Xi}(s) = (A_w - LH)\tilde{\Xi}(s), \tag{26}$$

where

$$A_w = \begin{bmatrix}\tilde{A} + \sum_{j=2}^{N}\mathcal{B}_jK_j & -\mathcal{B}_2K_2 & \cdots & -\mathcal{B}_NK_N \\ -\mathcal{B}_1K_1 & \tilde{A} + \sum_{j=1, j\neq 2}^{N}\mathcal{B}_jK_j & \cdots & -\mathcal{B}_NK_N \\ \vdots & \vdots & \ddots & \vdots \\ -\mathcal{B}_1K_1 & \cdots & \cdots & \tilde{A} + \sum_{j=1}^{N-1}\mathcal{B}_jK_j\end{bmatrix}$$

$$H = diag\{H_1, H_2, \cdots, H_N\}, \quad L = diag\{L_1, L_2, \cdots, L_N\}.$$

In addition, the convergence speed of $N+1$ agents depends on the spectral radius of $\tilde{A} + \mathcal{B}K_u$ and A_c. Since the optimal feedback gain matrix K_u has been given by (15), in order to achieve the goal of rapid convergence, we need to adjust the observer gain matrices $L_i, i = 1, 2, \cdots, N$ such that the spectral radius of A_c is as small as possible.

The following lemma presents an approach that uses LMI techniques for the optimal design of the distributed observers.

Lemma 3. *For $U_c = diag\{U_{c1}, U_{c2}, \cdots, U_{cN}\}$ and $W = diag\{W_1, W_2, \cdots, W_N\}$, assume that there exist matrices U_c, W, S and parameter α such that*

$$S = S^T > 0, \quad U_c = U_c^T > 0, \quad \alpha I - U_c > 0, \tag{27a}$$

$$\begin{bmatrix}U_c - S & (U_cA_w - WH)^T \\ U_cA_w - WH & U_c\end{bmatrix} > 0 \tag{27b}$$

with $L_i = U_{ci}^{-1}W_i$. Then, the optimal observer gains can be designed by solving the following optimization problem

$$\min_{L_i, U_c, S} \alpha \qquad subject \quad to \quad (27) \tag{28}$$

Proof. If there exist a symmetric positive definite matrix U_c satisfying the Lyapunov inequality

$$(A_w - LH)^TU_c(A_w - LH) - U_c < 0,$$

then, equivalently, there exists a positive definite matrix $S^T = S > 0$ such that

$$(A_w - LH)^TU_c(A_w - LH) - U_c < -S. \tag{29}$$

In this case, the system (26) is asymptotically stable. It is very difficult to solve the inequality (29) directly because it requires the simultaneous selection of U_c and the gain L. In this case, we take $L = U_c^{-1}W$, and it is easy to prove that the inequality (29) is equivalent to

$$A_w^T U_c - H^T W^T + U_c A_w - WH + S < 0. \tag{30}$$

Then, utilizing the Schur's complement lemma, the inequality (27) holds. Now, we consider the cost function

$$J_L = \sum_{k=0}^{T_N} \tilde{\Xi}(s)^T S \tilde{\Xi}(s) \tag{31}$$

By taking (29) into consideration, we derive

$$\begin{aligned}&\tilde{\Xi}(s+1)^T U_c \tilde{\Xi}(s+1) - \tilde{\Xi}(s)^T U_c \tilde{\Xi}(s)\\ &= \tilde{\Xi}(s)^T (A_c^T U_c A_c - U_c)\tilde{\Xi}(s) < -\tilde{\Xi}(s)^T S \tilde{\Xi}(s)\end{aligned}$$

and then

$$J_L \le \tilde{\Xi}^T(0) U_c \tilde{\Xi}(0) - \tilde{\Xi}^T(T_N) U_c \tilde{\Xi}(T_N) \le \tilde{\Xi}^T(0) U_c \tilde{\Xi}(0)$$

with $T_N \to \infty$, $\tilde{\Xi}^T(T_N) U_c \tilde{\Xi}(T_N) \to 0$. This inequality implies that one can minimize the cost function (31) by minimizing the bound $\tilde{\Xi}^T(0) U_c \tilde{\Xi}(0)$. Note that $\tilde{\Xi}^T(0) U_c \tilde{\Xi}(0) \le \|\tilde{\Xi}(0)\|^2 \|U_c\| \le M_0^2 \|U_c\|$, where M_0 is the upper bound of the initial value $\tilde{\Xi}(0)$. Therefore, the optimal observer gain L is derived by minimizing the maximum eigenvalue of U_c, i.e., the minimization problem (28) is solved. □

Remark 4. *The parameter L_i in distributed observer* (21) *to minimize the spectral radius of A_c is obtained by solving the optimization* (28) *in Lemma 3. Yet, due to the constraints on the diagonal structure of U_c and W, the derived observer gain L_i by the LMI technique is a suboptimal solution.*

To analyze the asymptotical optimal property of the corresponding cost function, we first derive the cost function under the newly proposed distributed controller (20). After that, we discuss the difference between this new cost function and the cost function under the centralized optimal control (14).

In order to facilitate the analysis, we denote

$$\mathcal{M}_1 = (\tilde{A} + \mathcal{B}K_u)^T P \Psi - \begin{bmatrix} K_1^T R_1 K_1 & \cdots & K_N^T R_N K_N \end{bmatrix},$$

$$\mathcal{M}_2 = \begin{bmatrix} K_1^T R_1 K_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_N^T R_N K_N \end{bmatrix} + \Psi^T P \Psi,$$

$$\Delta J(\tau, \infty) = \sum_{s=\tau}^{\infty} \begin{bmatrix} \Xi(s) \\ \tilde{\Xi}(s) \end{bmatrix}^T \begin{bmatrix} 0 & \mathcal{M}_1 \\ \mathcal{M}_1^T & \mathcal{M}_2 \end{bmatrix} \begin{bmatrix} \Xi(s) \\ \tilde{\Xi}(s) \end{bmatrix}.$$

Theorem 2. *Under the proposed distributed controllers* (20) *and* (21) *with L_i $(i = 1, 2, \cdots, N)$ selected such that the matrix A_c in* (22) *is asymptotically stable, the corresponding cost function is given by*

$$J^\star(\tau, \infty) = \Xi^T(s) P \Xi(s) + \Delta J(\tau, \infty) \tag{32}$$

where $\Delta J(\tau, \infty)$ is the cost difference between the cost function $J^\star$ in (32) *and the cost function $J^*(\tau, \infty)$ in* (17)*. In particular, the proposed new distributed bipartite consensus controller approximates the optimal centralized controller.*

Proof. The proof follows a similar approach to that of Theorem 2 in [34]. To save space, we omit the detailed steps here. □

B. Comparison with conventional bipartite consensus algorithms

In this subsection, we will discuss the superiority of the newly proposed bipartite consensus algorithm, particularly in terms of convergence speed and performance index. By comparing these aspects with existing methods, we aim to highlight the advantages of our approach in achieving faster and more efficient consensus among agents in multi-agent systems.

- **Faster convergence speed.** In fact, from the closed-loop system (25), one has

$$\left\| \begin{bmatrix} \Xi(s) \\ \tilde{\Xi}(s) \end{bmatrix} \right\| \le \rho(\bar{A}_c) \left\| \begin{bmatrix} \Xi(s-1) \\ \tilde{\Xi}(s-1) \end{bmatrix} \right\|,$$

  where $\rho(\bar{A}_c)$ represents the larger spectral radius of $\tilde{A} + \mathcal{B}K_u$ and A_c. In particular, $\tilde{A} + \mathcal{B}K_u$ is the closed-loop system matrix obtained by the optimal feedback controller (15), that is, $\Xi(s+1) = (\tilde{A} + \mathcal{B}K_u)\Xi(s)$ while $\Xi(s+1)^T \mathcal{Q} \Xi(s+1)$ is minimized as in (3), so the modulus of the eigenvalues for $\tilde{A} + \mathcal{B}K_u$ is minimized in a certain sense. Besides, based on the optimization in Lemma 3, we can appropriately select L_i such that the upper bound of the spectral radius $\rho(A_c)$ is as small as possible. From these perspectives, $\rho(\bar{A}_c)$ minimized, which contrasts with conventional consensus algorithms where the maximum eigenvalue of the matrix $\bar{A}_c$ is not minimized and is instead determined by the eigenvalues of the Laplacian matrix $\mathcal{L}$. It is important to note that conventional consensus algorithms are typically based on the gradient descent method, which has a relatively low convergence speed. In contrast, the newly proposed algorithm is based on optimal control theory and exhibits a fast superlinear convergence speed [35]. Therefore, the proposed approach can achieve faster convergence than conventional algorithms as demonstrated in the simulation example.
- **Asymptotic optimality.** The cost difference $\Delta J(\tau, \infty)$ between the new distributed controller (20) and the centralized optimal control (14) is provided in Theorem 2. As $s \to \infty$, this cost difference approaches zero. In other words, the cost function corresponding to the proposed distributed controllers (20) is asymptotically optimal, meaning that as time progresses, the performance of the distributed controller converges to that of the centralized optimal control. This demonstrates that the proposed distributed approach achieves near-optimal performance in the long run.

Remark 5. *It should be noted that the proposed distributed optimal consensus protocol is also applicable to the* ***homogeneous multi-agent systems****, i.e., $A_i = A_0 = A$ and $B_i = B, i = 1, 2, \cdots, N$. Under this circumstance, the relative state error* (7) *is reduced to*

$$e_{ij}(s+1) = Ae_{ij}(s) + Bu_i(s) - d_i B u_j(s).$$

With the edges chosen along with a spanning tree, the global error system is

$$\Xi(s+1) = \mathbf{A}\Xi(s) + \mathbf{B}u(s) \tag{33}$$

with

$$\mathbf{A} = I_{N-1} \otimes A, \quad \mathbf{B} = \begin{bmatrix} B & d_1 B & 0 & \cdots & 0 \\ 0 & B & d_2 B & \cdots & 0 \\ \cdots & \cdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & B & d_{N-1} B \end{bmatrix},$$

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and the system (33) *is reformulated as*

$$\Xi(s+1) = \mathbf{A}\Xi(s) + \sum_{i=1}^{N} \tilde{\mathbf{B}}_i u_i(s),$$
$$\mathcal{Y}_i(s) = \mathcal{H}_i \Xi(s), i = 1, \cdots, N$$

where $\mathcal{Y}_i(s)$ *is the local measurement information for agent* i*,* $\tilde{\mathbf{B}}_i$ *is the subpart of the matrix* $\mathbf{B}$*, and* $\mathcal{H}_i$ *is a* $\{0, I_n\}$ *matrix, whose specific forms depend on the interaction among agents.*

Following a similar design process as in (21)*, we can design a distributed asymptotically optimal controller based on an observer, i.e.,* $u_i(s) = K_i \hat{\Xi}_i(s)$*, which effectively addresses and solves Problem 1.*

Remark 6. *Unlike the related bipartite consensus works [20], [36], the proposed distributed optimal consensus controller does not require the design of a reference internal model or the computation of a solution to a Sylvester equation. This makes the developed bipartite consensus algorithm simpler and more general. In particular, when* $d_i = 1, \forall i \in \mathcal{V}$*, the proposed method can also be applied to solve the consensus problem in cooperative networks, as demonstrated in our recent work [34]. More importantly, the asymptotic optimality of the distributed controller is also guaranteed, making it superior to traditional methods in terms of convergence performance.*

IV. Numerical Simulation

In this section, we validate the proposed theoretical results through the following numerical example.

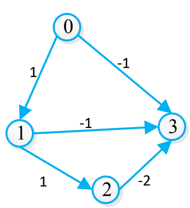

Fig. 1. Communication topology among four agents

Example 1. *Consider the multi-agent system consisting of three heterogeneous follower agents with the system matrices:*

$$A_1 = \begin{bmatrix} 1.1 & 1 \\ -2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0.5 & 0 \\ -2 & -0.6 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, B_3 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}$$

The leader's state trajectory is generated by a sinusoidal trajectory generator with dynamics given by

$$A_0 = \begin{bmatrix} cos(0.5) & sin(0.5) \\ -sin(0.5) & cos(0.5) \end{bmatrix}.$$

The interactions of agents are given in Fig.1, which satisfies Assumption 2. Each agent only receives neighbor error information, we choose the edges $\{e_{01}, e_{12}, e_{23}\}$ *along a spanning tree. The measurement matrix* H_i *is*

$$H_1 = \begin{bmatrix} I_2 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix}, H_3 = \begin{bmatrix} 0 & 0 & I_2 \end{bmatrix}.$$

Set $Q = R_1 = R_2 = R_3 = I_2$*. According to ARE* (16) *and Lemma 3, the feedback gains* K_i *and the observer gain* L_i *can be obtained, respectively. Fig. 2 shows that the observer error vectors* $\tilde{\Xi}_i(s)$ *under the proposed controller* (20) *converge to zero. The state trajectories of the three followers and the leader are depicted in Fig. 3, and the bipartite consensus error trajectories are displayed in Fig. 4. It is evident that the states of agents 1 and 2 asymptotically track the leader's state* $x_0(s)$*, while agent 3 asymptotically converges to the leader's opposite state* $-x_0(s)$ *within 5 steps. This result aligns well with the findings in Theorem 1. To further verify the convergence performance, using the same initial conditions, we apply the traditional consensus algorithms proposed in [9] and [20], the temporal evolution of the states for the leader and followers is displayed in Fig.5. This figure implies the bipartite consensus is achieved within 25 steps, which is slower than our proposed algorithm. Additionally, Figs. 6 and 7 show the evolutionary trends of absolute errors for agents using the new method and the old method, respectively. Clearly, by comparison, the newly proposed method achieves bipartite consensus with a faster convergence speed.*

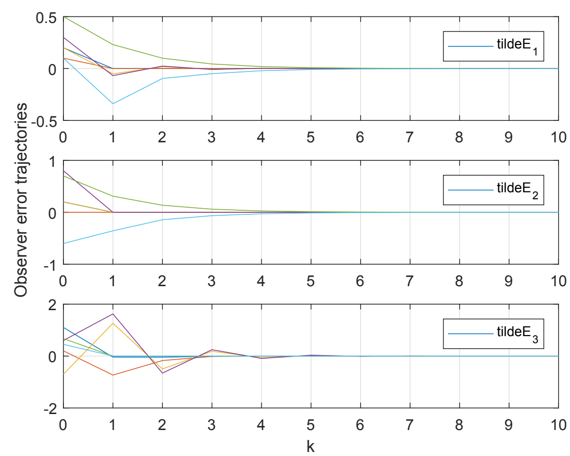

Fig. 2. The distributed observer error trajectories $\tilde{\Xi}_i(s)$.

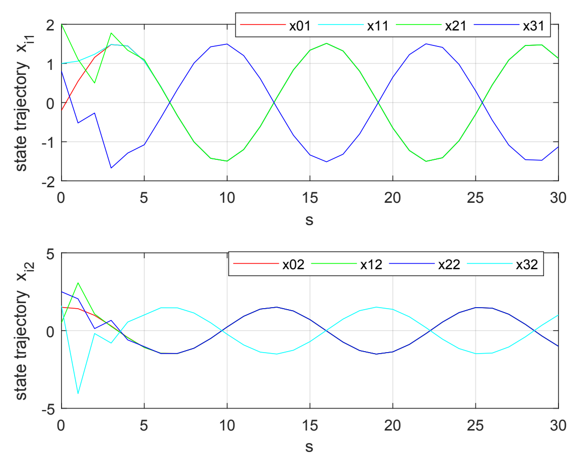

Fig. 3. State trajectories of each agent $x_i(s)$ by the proposed method.

V. Conclusion

This note has established a unified design framework for the bipartite consensus of heterogeneous MASs under a signed directed graph. The distributed optimal bipartite consensus protocol, which minimizes a general global cost function, has been derived using LQ optimal control and observer design incorporating the local neighbour's information. The corresponding analytical solution for the optimal controller has been obtained by solving certain AREs. Through theoretical analysis and a numerical example, it has been shown that the proposed method achieves a faster convergence speed

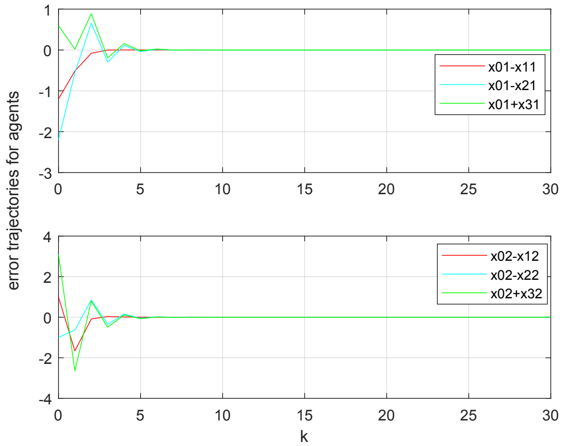

Fig. 4. Bipartite state error trajectories by the proposed method.

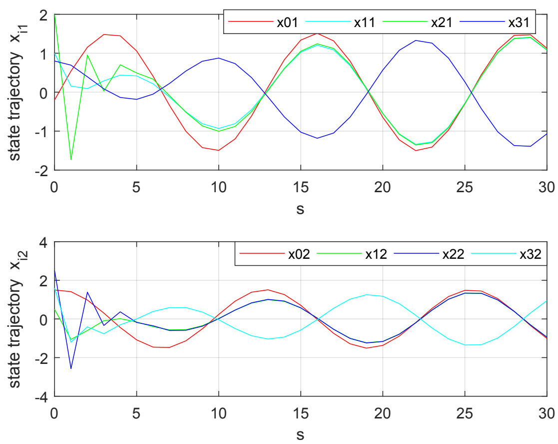

Fig. 5. State trajectories of each agent $x_i(s)$ by the existing method.

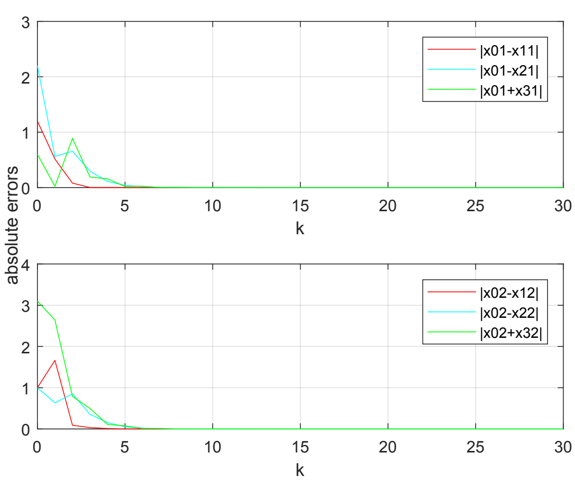

Fig. 6. The evolutionary trend of absolute errors for agents

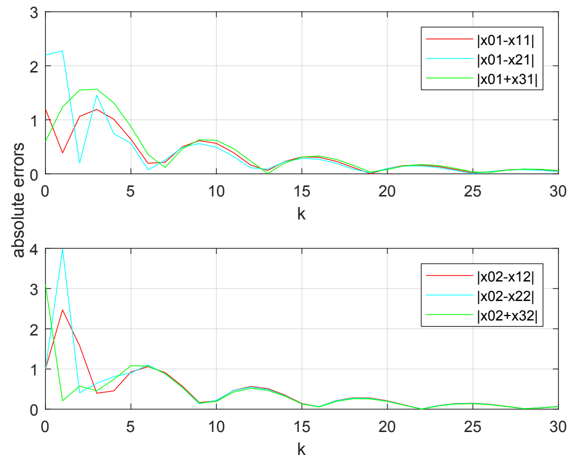

Fig. 7. The evolutionary trend of absolute errors for agents

compared to traditional bipartite consensus methods. Furthermore, this new approach can be directly extended to homogeneous systems with competitive networks. Note that the observer gains solved through the optimization problem (28) require global system dynamics. Therefore, a primary focus of future work will be on developing distributed methods to compute these gains. Additionally, the assumption that all subsystems share the same state-space dimension is somewhat restrictive. Future research will also explore optimal output consensus for more general heterogeneous MASs where agents have different state-space dimensions.

References

[1] X. Yang, W. Wang, and P. Huang, "Distributed optimal consensus with obstacle avoidance algorithm of mixed-order uavsusvsuuvs systems," *ISA Transactions*, vol. 107, pp. 270–286, Dec. 2020.

[2] M. Xu, K. An, L. H. Vu, Z. Ye, J. Feng, and E. Chen, "Optimizing multi-agent based urban traffic signal control system," *Journal of Intelligent Transportation Systems*, vol. 23, no. 4, pp. 357–369, Oct. 2018.

[3] W. Yu, G. Chen, Z. Wang, and W. Yang, "Distributed consensus filtering in sensor networks," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 39, no. 6, pp. 1568–1577, Dec. 2009.

[4] W. Zhang and Y. Xu, "Distributed optimal control for multiple microgrids in a distribution network," *IEEE Transactions on Smart Grid*, vol. 10, no. 4, pp. 3765–3779, Jul. 2019.

[5] A. Amirkhani and A. H. Barshooi, "Consensus in multi-agent systems: a review," *Artificial Intelligence Review*, vol. 55, no. 5, pp. 3897–3935, Nov. 2021.

[6] A. V. Proskurnikov, A. S. Matveev, and M. Cao, "Opinion dynamics in social networks with hostile camps: Consensus vs. polarization," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1524–1536, Jun. 2016.

[7] V. Amelkin, F. Bullo, and A. K. Singh, "Polar opinion dynamics in social networks," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5650–5665, Nov. 2017.

[8] F. Liu, D. Xue, S. Hirche, and M. Buss, "Polarizability, consensusability, and neutralizability of opinion dynamics on coopetitive networks," *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3339–3346, Aug. 2019.

[9] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.

[10] S. Miao and H. Su, "Bipartite consensus for second-order multiagent systems with matrix-weighted signed network," *IEEE Transactions on Cybernetics*, vol. 52, no. 12, pp. 13 038–13 047, Dec. 2022.

[11] M. E. Valcher and P. Misra, "On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions," *Systems and Control Letters*, vol. 66, pp. 94–103, Apr. 2014.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication.
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[12] H. Zhang and J. Chen, "Bipartite consensus of multi-agent systems over signed graphs: State feedback and output feedback control approaches: Bipartite consensus of multi-agent systems over signed graphs," *International Journal of Robust and Nonlinear Control*, vol. 27, no. 1, pp. 3–14, Apr. 2016.

[13] L. Zhang and G. Zhang, "Bipartite consensus for descriptor multiagent systems with antagonistic interactions," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 67, no. 11, pp. 2602–2606, Nov. 2020.

[14] D. Meng, M. Du, and Y. Jia, "Interval bipartite consensus of networked agents associated with signed digraphs," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3755–3770, Dec. 2016.

[15] J. Hu, Y. Wu, L. Liu, and G. Feng, "Adaptive bipartite consensus control of highorder multiagent systems on coopetition networks," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 7, pp. 2868–2886, Jan. 2018.

[16] H. Wang, W. Yu, G. Wen, and G. Chen, "Finite-time bipartite consensus for multi-agent systems on directed signed networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 65, no. 12, pp. 4336–4348, Dec. 2018.

[17] J. Hu, Y. Wu, T. Li, and B. K. Ghosh, "Consensus control of general linear multiagent systems with antagonistic interactions and communication noises," *IEEE Transactions on Automatic Control*, vol. 64, no. 5, pp. 2122–2127, May 2019.

[18] Z. Peng, J. Hu, K. Shi, R. Luo, R. Huang, B. K. Ghosh, and J. Huang, "A novel optimal bipartite consensus control scheme for unknown multi-agent systems via model-free reinforcement learning," *Applied Mathematics and Computation*, vol. 369, p. 124821, Mar. 2020.

[19] F. Adib Yaghmaie, R. Su, F. L. Lewis, and S. Olaru, "Bipartite and cooperative output synchronizations of linear heterogeneous agents: A unified framework," *Automatica*, vol. 80, pp. 172–176, Jun. 2017.

[20] T. Han and W. X. Zheng, "Bipartite output consensus for heterogeneous multi-agent systems via output regulation approach," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 68, no. 1, pp. 281–285, jan 2021.

[21] Y. Su and J. Huang, "Cooperative output regulation of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 1062–1066, Apr. 2012.

[22] Q. Liang, Y. Wu, J. Hu, and Y. Zhao, "Bipartite output synchronization of heterogeneous time-varying multi-agent systems via edge-based adaptive protocols," *Journal of the Franklin Institute*, vol. 357, no. 17, pp. 12 808–12 824, Nov. 2020.

[23] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal on Control*, vol. 6, no. 1, pp. 131–147, Feb. 1968.

[24] T. Feng, J. Zhang, Y. Tong, and H. Zhang, "Consensusability and global optimality of discrete-time linear multiagent systems," *IEEE Transactions on Cybernetics*, vol. 52, no. 8, pp. 8227–8238, aug 2022.

[25] H. Sun, Y. Liu, F. Li, and X. Niu, "Distributed lqr optimal protocol for leader-following consensus," *IEEE Transactions on Cybernetics*, vol. 49, no. 9, pp. 3532–3546, Sep. 2019.

[26] Z. Zhang, W. Yan, and H. Li, "Distributed optimal control for linear multiagent systems on general digraphs," *IEEE Transactions on Automatic Control*, vol. 66, no. 1, pp. 322–328, Jan. 2021.

[27] F. Chen and J. Chen, "Minimum-energy distributed consensus control of multiagent systems: A network approximation approach," *IEEE Transactions on Automatic Control*, vol. 65, no. 3, pp. 1144–1159, Mar. 2020.

[28] H. Meng, D. Pang, J. Cao, Y. Guo, and A. U. K. Niazi, "Optimal bipartite consensus control for heterogeneous unknown multi-agent systems via reinforcement learning," *Applied Mathematics and Computation*, vol. 476, p. 128785, Sep. 2024.

[29] L. Yan, J. Liu, G. Lai, C. L. Philip Chen, Z. Wu, and Z. Liu, "Adaptive critic learning-based optimal bipartite consensus for multiagent systems with prescribed performance," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 36, no. 3, pp. 5417–5427, Mar. 2025.

[30] B. Liang, Y. Wei, and W. Yu, "Adaptive optimal bipartite consensus control for heterogeneous multiagent systems," *IEEE Transactions on Control of Network Systems*, vol. 11, no. 4, pp. 2263–2275, Dec. 2024.

[31] Q. Jiao, H. Zhang, S. Xu, F. L. Lewis, and L. Xie, "Bipartite tracking of homogeneous and heterogeneous linear multi-agent systems," *International Journal of Control*, vol. 92, no. 12, pp. 2963–2972, May 2018.

[32] J. M. B.D.O. Anderson, *Linear optimal control*. Prentice Hall, 1971.

[33] Y. Cao and W. Ren, "Optimal linear-consensus algorithms: An lqr perspective," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 40, no. 3, pp. 819–830, Jun. 2010.

[34] L. Zhang, J. Xu, H. Zhang, and L. Xie, "A solution to optimal consensus of multiagent systems," *International Journal of Robust and Nonlinear Control*, Aug. 2025.

[35] H. Wang, Y. Xu, Z. Guo, and H. Zhang, "Superlinear optimization algorithms," *arXiv preprint arXiv:2403.11115*, 2024.

[36] F. A. Yaghmaie, R. Su, F. L. Lewis, and L. Xie, "Multiparty consensus of linear heterogeneous multiagent systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5578–5589, Nov. 2017.