Mixed topics on geometry of varieties of Fano type

A Thesis Submitted for the Degree of Doctor of Philosophy

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Abstract

In this thesis, we investigate the deformation properties of Fano threefolds and the birational geometry of foliations. First, we try to find compactification of several families of Fano threefolds. Then we give a description of the connections between foliated minimal models. Finally, we will discuss geometric properties of Fano foliations. This thesis contains results of (1), (26), (19) and some recent independent work.

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1 Introduction

1.1 Minimal models

We work on algebraic varieties over $\mathbb{C}$.

Classification of varieties is one of the central topics in algebraic geometry. As it is difficult to classify them up to isomorphism, birational equivalence was introduced, which reflects geometric properties of varieties on an open subset. Two varieties X and Y are birational if there are two non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$. The minimal model program (MMP), which intends to find a simpler model in each birational class, has been the subject of intensive research since the 1980's.

Definition 1.1 ((8, Definition 3.6.7)). Let $\pi\colon X \to U$ be a projective morphism of normal quasi-projective varieties. Suppose that X is log canonical and let $\phi\colon X \dashrightarrow Y$ be a birational contraction of normal quasi-projective varieties over U, where Y is projective over U.

We say that Y is a minimal model for K_X over U if ϕ is K_X-non-positive and K_Y is nef over U.

'log canonical(lc)' in the above definition is a condition of mild singularity - we will give the definition in 2.4 together with other conditions such as kawamata log terminal(klt) and terminal. We remark that the mild singularity condition is important in running MMP - there are many fundamental tools in birational geometry that require mild singularities (e.g., see (29, Theorem 3.3 - 3.7)).

The existence of minimal models in most cases was solved in (8, Theorem 1.2, Corollary 1.3.3).

1.2 Building blocks

We have **three building blocks** in birational geometry:

- Fano varieties - a projective normal variety X with K_X being anti-ample (e.g. $\mathbb{P}^n$, the Hirzebruch surface $\mathbb{F}_1$, general hypersurfaces in $\mathbb{P}^n$ with degree $\leq n$);

- Calabi-Yau varieties - a projective normal variety X with $K_X \equiv 0$ (e.g. K3 surfaces, general hypersurface in $\mathbb{P}^n$ with degree $= n+1$);

- Canonically polarised varieties - a projective normal variety X with K_X being ample (e.g. general hypersurfaces in $\mathbb{P}^n$ with degree $\geq n+2$).

The birational surgeries here are not arbitrary - they are K-negative, which means they are directed by the canonical class in the following way: they only contract subspaces which are covered by rational curves. Furthermore, for any such curve C on X, we have $K_X \cdot C < 0$.

Since the above three types of varieties are fundamental blocks in birational geometry, it makes sense to try to understand their geometric properties and then classify them if possible. Now we focus on Fano varieties. For simplicity, we start with smooth Fano manifolds and study their deformations:

Question. Given two Fano manifolds X and Y of dimension n, when can we find a morphism $f\colon \mathfrak{X} \to V$ such that $f^{-1}(P) = X$ and $f^{-1}(Q) = Y$ for some $P, Q \in V$ and all other fibres are also Fano manifolds of the same dimension?

In dimension 1 there is only one Fano manifold $\mathbb{P}^1$. In dimension 2, these Fano manifolds, called *Del Pezzo surfaces*, are classified into 9 families. Furthermore, X and Y are in the same family if and only if they have the same self-intersection number for the canonical divisor, i.e. $K_X^2 = K_Y^2$. However, in dimension 3, the above question is much more complicated - according to Mori, Mukai and Iskovskikh's work, there are 105 deformation families.

However, in the above classification, the variety V may be non-complete - in section 2.2, there are examples where $C \cong \mathbb{A}^1$ and the morphism $\mathfrak{X} \to C$ cannot be extended to some $f'\colon \mathfrak{X}' \to \mathbb{P}^1$ such that $(f')^{-1}(\infty)$ is also a Fano manifold.

This problem was solved in any dimension (cf. see (9, Corollary 1.4)). It states that if one fixes the anticanonical volume (which is a generalisation of the above K_X^2 in the surface case) and consider possibly singular Fano varieties that are K-polystable, then one can complete any family of such varieties over a non-complete curve C^0.

To be precise, let (C, P) be a smooth curve with a point $P \in C$. Define $C^0 := C - \{P\}$. Let

$$\pi^0\colon \mathfrak{X}^0 \to C^0$$

be a family of K-polystable varieties. Then there exists a family of K-polystable varieties $\pi\colon \mathfrak{X} \to C$ such that for the following pullback diagram:

$$\begin{array}{ccc} \mathfrak{X} \times_C C^0 & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \pi \\ C^0 & \longrightarrow & C \end{array}$$

and $(\mathfrak{X} \times_C C^0, C^0) \cong (\mathfrak{X}^0, C^0)$. I aim to give an explicit description of the deformation space for threefolds when it is of dimension 1:

Theorem 1.2. Let M_3^{Kps} be the projective moduli space whose closed points are 3-dimensional K-polystable smoothable Fano varieties. Then all one-dimensional components of M_3^{Kps} are isomorphic to $\mathbb{P}^1$.

For definition of the moduli space and related properties, see section 2.4.

1.3 Foliations

Recently, it has been observed that the above MMP process can also be extended to a structure on varieties called **foliations**. On a normal variety X, a foliation $\mathcal{F}$ is a subsheaf of the tangent sheaf T_X satisfying saturated and integrable conditions (see Definition 3.1).

For foliations on threefolds with mild singularity, MMP was established for threefolds in (13) and (12). The following theorem shows the relation between minimal models for corank 1 foliations on threefolds:

Theorem 1.3. Let $(X, \mathcal{F})$ be a $\mathbb{Q}$-factorial foliated pair with F-dlt singularity. Assume $\dim X = 3$. Let

$$\alpha_1 \colon (X, \mathcal{F}) \dashrightarrow (Y_1, \mathcal{F}_1),\ \alpha_2 \colon (X, \mathcal{F}) \dashrightarrow (Y_2, \mathcal{F}_2)$$

be two $K_{\mathcal{F}}$-non-positive birational maps where $(Y_1, \mathcal{F}_1)$ and $(Y_2, \mathcal{F}_2)$ are both $\mathbb{Q}$-factorial and F-dlt. Let $\alpha \colon X_1 \dashrightarrow X_2$ be the induced birational map. Suppose $K_{\mathcal{F}_1}$ and $K_{\mathcal{F}_2}$ are both big and nef.

Then α is composed of a finite sequence of flops.

For the algebraically integrable case, the existence of good minimal models has been proven for foliated triples polarised with ample divisors in any dimension in (17, Theorem 2.1.2). Fano varieties are one of the three building blocks in the MMP. Therefore, understanding their properties is key to understanding all varieties. In the same way, Fano foliations form one of the three basic foliated building blocks as well. Therefore, Fano foliations shed some light on birational geometry of foliations. A fundamental question for Fano foliations is how they behave in families. Before that,a first step is to prove that at least we are able to represent them in families. To be precise, we ask the following question:

Question. Can we find two varieties X, T which are of finite type, together with a morphism $f \colon X \to T$, a foliation $\mathcal{F}$ on X such that for every Fano foliation $(Y, \mathcal{G})$ with some singularity condition, there exists $t \in T$ such that

$$(X_t, \mathcal{F}_t) \cong (Y, \mathcal{G})?$$

The classical picture for Fano pairs is known as the BAB conjecture and was proven by Birkar (7, Theorem 1.1):

Theorem 1.4. Let $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$. Then the varieties X satisfying

- (X, B) has bounded singularities (ϵ-lc) of dimension d for some boundary B;
- $-(K_X + B)$ is big and nef;

form a bounded family.

Similarly, we ask:

Question. Is there a way to **bound Fano foliations** when bounding the singularity?

In fact, people have already noticed that the classical BAB conjecture cannot hold for such foliations. I will discuss possible obstructions in section 4.

Since $-K_{\mathcal{F}}$ is ample for a Fano foliation $(X, \mathcal{F})$, it is natural to look at the behaviour of global sections $D \in H^0(X, \mathcal{O}_X(-mK_{\mathcal{F}}))$ for $m \geq 0$. A **complement** of $\mathcal{F}$ is a divisor $\frac{1}{m}D$ for the above D, such that $(X, \mathcal{F}, \frac{1}{m}D)$ is log canonical. My collaborators and I showed the **existence** of $\mathbb{Q}$-complements. There is also a surprising result that plays a significant role in our proof: for log Calabi-Yau foliations induced by a morphism, the log canonical property can be deduced from **horizontal discrepancy**(19). Geometrically, log canonical property in this case is only determined by the discrepancies of exceptional divisors E over X, such that E intersects a general leaf.

A heuristic reason is: usually for an algebraic family (e.g. Fano fibrations, log Calabi-Yau fibrations), if one only has good singularity on a general fibre, it is hard to control the singularity of every fibre. However, when we have log Calabi-Yau condition, this is true. This provides us with an algebraic insight to prove why Fano foliation can never be induced by morphisms by adding a general anti-canonical $\mathbb{Q}$-section to the boundary. The obstruction to a Fano foliation being induced by a morphism was proved with analytic tools in (5).

In addition, **inversion of adjunction**(Theorem 4.14) is also proven for vertical divisors under algebraically integrable condition, which gives a tool to control the singularity in the lifting process.

In the case of Fano varieties, Birkar first proved the boundedness of complements and used complements to construct bounded embeddings for Fano pairs. For the definition of complements, see Definition 4.4.

However, the existence of complements in the classical case follows directly from Bertini' theorem which is wrong for foliations. The existence is given by the following theorem:

Theorem 1.5. Let $(X, \mathcal{F}, B)$ be an lc Fano foliated triple with $\mathcal{F}$ algebraically integrable such that $\mathcal{F}$ has a normal general leaf.

Then there exists a $\mathbb{Q}$-complement. That is, there exists an effective divisor $D \sim_{\mathbb{Q}} -(K_{\mathcal{F}} + B)$ such that $(X, \mathcal{F}, D)$ is log canonical.

We add some explanations for the conditions in Theorem 1.5. Let X be a normal variety and $\mathcal{F}$ an algebraically integrable foliation on X. By (4, Lemma 3.2), there exists a unique irreducible closed subvariety $W \subseteq \mathrm{Chow}(X)$, such that the general point of W parametrizes the closure of a general leaf of $\mathcal{F}$. Let $\pi\colon U \to W$ and $e\colon U \to X$ be the corresponding universal morphisms defined in (4, Lemma 3.2). Then we may give the following definition:

Definition 1.6. Let X, $\mathcal{F}$, U, W, e and π be as above. We say that $\mathcal{F}$ *has a normal general leaf* if there exists a non-empty open subset $W_0 \subseteq W$ such that for any closed point $p \in W_0$, the closure of $e(\pi^{-1}p)$ is normal.

Following is an example where the condition in Definition 1.6 is not satisfied.

Example 1.7. Let $\mathcal{F}$ be the foliation on $\mathbb{A}^2$ whose general leaf is the curve $\{y^3 = tx^2\}$ removing the point $(0, 0)$ for $t \in \mathbb{C}\backslash\{0\}$. Then $\mathcal{F}$ does not have a normal general leaf.

Next, for boundedness of complements I prove the following theorem in corank one:

Theorem 1.8. Let d be a positive integer and $\mathfrak{R} \subseteq [0, 1]$ a finite set of rational numbers. There there exists a positive integer n, depending only on d and $\mathfrak{R}$ satisfying the following. Assume $(X, \mathcal{F}, B)$ is a foliated triple such that

- X is $\mathbb{Q}$-factorial klt of dimension d;
- $(X, \mathcal{F}, B)$ is a log canonical, corank one, algebraically integrable triple;
- $-(K_{\mathcal{F}} + B)$ is ample;
- $B \in \Phi(\mathfrak{R})$;
- $(X, \mathcal{F}, B)$ satisfies property-$(\diamond)$.

Then there exists an n-complement $K_{\mathcal{F}} + B^+$ for $K_{\mathcal{F}} + B$ with $B^+ \geq B$.

Here n-complement (Definition 4.4) reflects singularities of $|-n(K_{\mathcal{F}}+B)|$.

For the definition of $\Phi(\mathfrak{R})$, see Definition 4.17. Here $B \in \Phi(\mathfrak{R})$ means that the coefficients of B belong to $\Phi(\mathfrak{R})$. Property-($\diamond$), which will be defined precisely in Definition 4.18, means that the general log leaf is dlt - this condition allows us to modify the Fano foliation to a fibration birationally, such that the restricted morphism to a general log leaf is an isomorphism.

2 One-dimensional K-moduli

the content of this section is extracted from (1).

2.1 Preliminaries

Definition 2.1 (volume, cf.(25, Definition 3.2.1))**.** Let X be an irreducible projective variety of dimension n. Let D be an $\mathbb{R}$-divisor on X. Then define the *volume* of D to be

$$\mathrm{vol}_X(D) := \limsup_{m\to\infty} \frac{n!h^0(X, \lfloor mD \rfloor)}{m^n}$$

Definition 2.2. Let X be a normal variety, E a prime divisor over X. We denote the centre of E on X by $\mathrm{cent}_X(E)$.

The next definition is vital for our calculation for various birational invariant to verify K-stability.

Definition 2.3 (Nakayama-Zariski Decomposition, cf. (33, Definition 1.12))**.** Let X be a smooth projective variety, $B \geq 0$ a big $\mathbb{R}$-divisor and let C be a prime divisor. Let

$$\sigma_C(B) := \inf\{\mathrm{mult}_C B' | B' \sim_{\mathbb{Q}} B\}$$

Then write $B = P(B) + N(B)$ where

$$N(B) := \sum_{C \text{ prime}} \sigma_C(B)C, \ P(B) := B - N(B)$$

Here $P(B)$ is called the positive part of B and $N(B)$ is called the negative or fixed part of B. Notice that here $N(B)$ is defined as a class of $\mathbb{R}$-divisors.

Remark.
1. The definition for Nakayama-Zariski decomposition can be extended to pseudo-effective divisors as well (see (33, Definition 1.12), but I only use it when the divisor is big;
2. if $P(B)$ is nef then according to (30, Remark 11.4.8), $\mathrm{vol}_X(B) = \mathrm{vol}_X(P(B)) = P(B)^{\dim X}$.

In birational geometry, the *log discrepancy* is often used to measure singularities:

Definition 2.4. Let (X, Δ) be a *pair*, (i.e., Δ is a $\mathbb{Q}$-divisor on X with $\Delta \geq 0$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier). Let $f : Y \to X$ be a birational projective morphism, then we can write

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

with $f_*\Delta_Y = \Delta$. Let $E \subseteq Y$ be a divisor over X then we define the log discrepancy

$$A(X, \Delta; E) \colon = \text{mult}_E(1 - \Delta_Y)$$

The definition of $A(X, \Delta; E)$ does not depend on the choice of Y. We say that (X, Δ) is

1. *log canonical* (lc) if $A(X, \Delta; E) \geq 0$ for any E over X;
2. *kawamata log terminal* (klt) if $A(X, \Delta; E) > 0$ for any E over X and $\Delta \in [0, 1)$;
3. *terminal* if $A(X, \Delta; E) > 1$ for any E over X.

Suppose $A(X, \Delta; E) = 0$, then

1. E is called an log canonical place (lc place) of (X, Δ);
2. $\text{cent}_X(E)$ is called an log canonical centre (lc centre) of (X, Δ).

Definition 2.5. Let X be a normal projective variety. We say that X is $\mathbb{Q}$-Fano if

1. $-mK_X$ is an ample Cartier divisor for some $m \in \mathbb{N}^+$, and
2. X has klt singularities.

The volume of a Fano variety X is defined as $(-K_X)^{\dim X}$.

Definition 2.6. Let $\sigma \colon Y \to X$ be a projective morphism where Y is a normal variety and $F \subseteq Y$ a prime divisor. Then we define

$$H^0(X, -mK_X - xF) \colon = H^0(Y, \sigma^*\mathcal{O}_X(-mK_X) \otimes (\lfloor -xF \rfloor))$$

where $x \in \mathbb{R}_{\geq 0}$.

Definition 2.7 ((23, Definition 1.3))**.** Let X be a $\mathbb{Q}$-Fano variety with $-rK_X$ being Cartier for some $r \in \mathbb{N}^+$. A prime divisor F over X is called *dreamy* if

$$\bigoplus_{i,j \in \mathbb{N}} H^0(X, -irK_X - jF)$$

is a finitely generated $\mathbb{N}^{\oplus 2}$-algebra over $\mathbb{C}$.

The next invariant measures the singularity of a divisor either globally or around a (possibly not closed) point.

Definition 2.8. Let (X, Δ) be a log canonical pair and $M \geq 0$ an $\mathbb{R}$-Cartier divisor. Then define

$$\text{lct}(X, \Delta; M) :\, = \text{sup}\{t \in \mathbb{R}_{\geq 0} | (X, \Delta + tM) \text{ is log canonical}\}.$$

Let $\eta \in X$ be a scheme point, then the local log canonical threshold is defined as

$$\text{lct}_\eta(X, \Delta; M) :\, = \text{sup}\{t \in \mathbb{R}_{\geq 0} | (X, \Delta + tM) \text{ is log canonical around } \eta\}$$

2.2 Description of families

Smooth Fano threefolds are classified - there are 105 deformation families according to Mori, Mukai and Iskovskikh's work. There are 8 families whose deformation spaces have dimension one (using standard results in the deformation theory of smooth Fano threefolds, one can show that for a smooth Fano threefold X, the moduli dimension at $[X]$ is given by the Euler characteristic $-\chi(X, T_X)$, and this is constant in each deformation family), but 2 of them have no K-polystable element. The other 6 families are as follows:

Family 1. This is $\mathfrak{F}$ 2.24 in Mori-Mukai notation. Every element is a (1,2)-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$. Consider the following family of threefolds over $\mathbb{P}^1$ given in (3, Theorem 4.67):

$$\{\lambda(xu^2 + yv^2 + zw^2) + \mu(xvw + ywu + zuv) = 0\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$$

It is proven in (3, Theorem 4.67) that the fibre over any $[\lambda, \mu] \in \mathbb{P}^1$ is K-polystable and every smooth member belongs to $\mathfrak{F}2.24$.

Family 2. This is $\mathfrak{F}$ 2.25 in Mori-Mukai notation. Every element is the blow up of $\mathbb{P}^3$ along a quartic elliptic curve. Consider the following family of curves over $\mathbb{P}^1$ given in (3, Lemma 4.31):

$$\{\mu(x_0^2 + x_1^2) + \lambda(x_2^2 + x_3^2) = 0, \lambda(x_0^2 - x_1^2) + \mu(x_2^2 - x_3^2) = 0\} \subseteq \mathbb{P}^3 \times \mathbb{P}^1$$

Let $C_{[\lambda,\mu]}$ be the fibre over $[\lambda, \mu]$ and $X_{[\lambda,\mu]}$ the blow up of $\mathbb{P}^3$ along $C_{[\lambda,\mu]}$. Then we have

1. $X_{[\lambda,\mu]}$ is smooth and K-polystable for $[\lambda, \mu] \notin \{[0, 1], [1, 0], [\pm 1, 1], [\pm i, 1]\}$ according to (3, Corollary 4.32);

2. $X_{[\lambda,\mu]}$ is toric and K-polystable with 4 ordinary double points for

$$[\lambda, \mu] \in \{[0, 1], [1, 0], [\pm 1, 1], [\pm i, 1]\}$$

according to (34, Lemma 5.1).

Family 3. This is $\mathfrak{F}$ 2.22 in Mori-Mukai notation. Every element is the blow up of $\mathbb{P}^3$ along a quartic rational curve. Consider the following family of curves:

$$\{ux^2(\mu x + \lambda y) = vy^2(\mu y + \lambda x)\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1_{[\lambda,\mu]}$$

Let $C_{[\lambda,\mu]}$ be the fibre over $[\lambda, \mu]$. Let $Q \subseteq \mathbb{P}^3$ be a fixed smooth quartic surface, then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and we can identify $C_{[\lambda,\mu]}$ with a curve on Q hence $C_{[\lambda,\mu]} \subseteq \mathbb{P}^3$. Let $X_{[\lambda,\mu]}$ be the blow up of $\mathbb{P}^3$ along $C_{[\lambda,\mu]}$. Then we have

1. $X_{[\lambda,\mu]}$ is smooth and K-polystable for $[\lambda, \mu] \notin \{[1, 0], [\pm 1, 1], [\pm 3, 1]\}$ according to (3, Proposition 4.33);

2. $X_{[\pm 3,1]} \rightsquigarrow X_{[0,1]}$;

3. $X_{[\pm 1,1]} \rightsquigarrow X_{[1,0]}$. $X_{[1,0]}$ has two singular points which are both ordinary double points.

Here by $X_{[\pm 3,1]} \rightsquigarrow X_{[0,1]}$ we mean both $X_{[3,1]}$ and $X_{[-3,1]}$ degenerate to $X_{[0,1]}$.

Family 4. This is $\mathfrak{F}$ 3.12 in Mori-Mukai notation. Every element is the blow up of $\mathbb{P}^3$ along the disjoint union of a line and a twisted cubic curve. Let D be a general conic curve in $\mathbb{P}^2$ and

$$Q\colon = \mathbb{P}^1 \times D \subseteq \mathbb{P}^1 \times \mathbb{P}^2$$

Since $D \cong \mathbb{P}^1$, we can take $C_{[\lambda,\mu]}$ to be the family of curves on Q as described in Family 3. i.e., since $C_{[\lambda,\mu]}$ is a family of curves on $\mathbb{P}^1 \times \mathbb{P}^1$, we may also regard $C_{[\lambda,\mu]}$ as a family of curves on Q via the isomorphism $Q = \mathbb{P}^1 \times D \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $X_{[\lambda,\mu]}$ be the blow up of $\mathbb{P}^2 \times \mathbb{P}^1$ along $C_{[\lambda,\mu]}$. Then we have

1. $X_{[\lambda,\mu]}$ is smooth and K-polystable for $[\lambda, \mu] \notin \{[1, 0], [\pm 1, 1], [\pm 3, 1]\}$ according to (22, Main Theorem);

2. $X_{[\pm 3,1]} \rightsquigarrow X_{[0,1]}$;

3. $X_{[\pm 1,1]} \rightsquigarrow X_{[1,0]}$. $X_{[1,0]}$ has two singular points which are both ordinary double points.

To see that every element can be described as a blow up of $\mathbb{P}^1 \times \mathbb{P}^2$, let X be a threefold in the family. Let $Y \to \mathbb{P}^3$ be the blow up of a line, then there is a natural projection $\pi_1 \colon Y \to \mathbb{P}^1$ which is a $\mathbb{P}^2$-bundle. Similarly, one can check that there is a $\mathbb{P}^1$-bundle $\pi_2 \colon V \to \mathbb{P}^2$ where $V \to \mathbb{P}^3$ is the blow up of a twisted cubic curve. The above morphism $X \to \mathbb{P}^1 \times \mathbb{P}^2$ is in fact $\pi_1 \times \pi_2$.

Family 5. This is $\mathfrak{F}$ 4.13 in Mori-Mukai notation. Every element is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of degree $(1,1,3)$. Let

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = \{[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]\}$$

and $C_{[\lambda,\mu]}$ the following family of curves

$$\{x_0 y_1 - x_1 y_0 = 0, \mu(x_0^3 z_0 - x_1^3 z_1) + \lambda x_0 x_1 (x_1 z_0 - x_0 z_1) = 0\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1_{[\lambda,\mu]}$$

Let $X_{[\lambda,\mu]}$ be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along $C_{[\lambda,\mu]}$. Then we have

1. $X_{[\lambda,\mu]}$ is smooth and K-polystable for $[\lambda,\mu] \notin \{[1,0],[\pm 1,1]\}$ according to (3, 5.22);
2. $X_{[\pm 1,1]} \cong X_{[1,0]}$. $X_{[1,0]}$ has two singular points which are both ordinary double points.

Family 6. This is $\mathfrak{F}$ 3.13 in Mori-Mukai notation. Every element is the complete intersection of three divisors of type $(1,1,0),(1,0,1),(0,1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Let $[\lambda,\mu] \in \mathbb{P}^1$ and consider the following family of threefolds:

$$\{x_0y_0 + x_1y_1 + x_2y_2 = 0, y_0z_0 + y_1z_1 + y_2z_2 = 0, (\lambda+\mu)x_0z_1 + (\mu-\lambda)x_1z_0 - 2x_2z_2 = 0\} \subseteq (\mathbb{P}^2)^3 \times \mathbb{P}^1_{[\lambda,\mu]}$$

and let $X_{[\lambda,\mu]}$ be the fibre over $[\lambda,\mu] \in \mathbb{P}^1$. Then we have

1. $X_{[\lambda,\mu]}$ is smooth and K-polystable for $[\lambda,\mu] \notin \{[1,0],[\pm 1,1]\}$ according to (3, 5.97);
2. $X_{[\pm 1,1]}, X_{[1,0]}$ are K-unstable;
3. every smooth member of $\mathfrak{F}3.13$ corresponds to some fibre in the above family.

Furthermore, the above family can be reparametrized such that $X_{[\pm 1,1]} \rightsquigarrow X'$ where X' is the toric K-polystable threefold given by

$$\{x_0y_1 = x_1y_0, y_1z_2 = y_2z_1, x_0z_2 = x_2z_0\} \subseteq (\mathbb{P}^2)^3$$

Sing(X') consists of 3 ordinary double points.

Now we define a Fano threefold:

$$X'_\infty = \begin{cases} x_2y_3 - x_3y_2 = 0, \\ y_2z_3 - y_3z_2 = 0, \\ x_2z_3 - x_3z_2 = 0, \\ x_1y_1z_3 + x_1y_3z_1 + x_3y_1z_1 + x_3y_2z_3 = 0, \\ x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 + x_2y_3z_2 = 0. \end{cases} \qquad (1)$$

One can check that X'_∞ has a unique singular point hence not isomorphic to X' since X' has three singular points. In the next subsections, it can be proven that X'_∞ is K-polystable as well.

2.3 K-stability for Fanos

Before we give the definition of K-stability, we consider a special degeneration called a *test configuration*.

Definition 2.9. Let X be an n-dimensional $\mathbb{Q}$-Fano variety such that $-rK_X$ is Cartier. A test configuration of $(X, -rK_X)$ consists of

- a normal variety $\mathfrak{X}$ with a $\mathbb{G}_m$-action;
- a flat $\mathbb{G}$-equivariant projective morphism $\pi\colon \mathfrak{X} \to \mathbb{P}^1$, where $\mathbb{G}_m$ acts on $\mathbb{P}^1$ by
$$(t, [x, y]) \mapsto [tx, y];$$
- a $\mathbb{G}_m$-invariant π-ample $\mathbb{Q}$-line bundle $\mathfrak{L}$ on $\mathfrak{X}$ and a $\mathbb{G}_m$-invariant isomorphism:
$$(\mathfrak{X}\backslash\pi^{-1}(0), \mathfrak{L}|_{\mathfrak{X}\backslash\pi^{-1}(0)}) \cong (X \times (\mathbb{P}^1\backslash\{0\}), p_1^*L)$$
where $p_1\colon X \times (\mathbb{P}^1\backslash\{0\}) \to X$ is the first projection.

For such a test configuration, the Donaldson–Futaki invariant is defined as
$$\text{DF}(\mathfrak{X}; \mathfrak{L})\colon = \frac{1}{L^n}(\mathfrak{L}^n \cdot K_{\mathfrak{X}/\mathbb{P}^1} + \frac{n}{n+1}\mathfrak{L}^{n+1}).$$
The above test configuration is called *product type* if we have an isomorphism
$$\mathfrak{X}\backslash\mathfrak{X}_\infty \cong X \times \mathbb{P}^1\backslash\{\infty\}.$$

The Fano variety X is called

- *K-semistable* if DF$(\mathfrak{X}; \mathfrak{L}) \geq 0$ for all test configurations;
- *K-stable* if DF$(\mathfrak{X}; \mathfrak{L}) > 0$ for all test configurations;
- *K-polystable* if it is K-semistable and DF$(\mathfrak{X}; \mathfrak{L}) = 0 \Longleftrightarrow (\mathfrak{X}; \mathfrak{L})$ is of product type.

Definition 2.10 (Pseudo-effective threshold)**.** Let X be a $\mathbb{Q}$-Fano variety and E a prime divisor over X. Let $f\colon Y \to X$ be a model over X such that cent$_Y(E)$ is a divisor. We still use E to denote cent$_Y(E)$. Now define
$$\tau(E)\colon = \sup\{s \in \mathbb{R}_{\geq 0} | f^*(-K_X) - sE \text{ is big}\} < \infty$$

Definition 2.11 (β-invariant, ([23], Definition 1.3))**.** Let X be a $\mathbb{Q}$-Fano variety of dimension n, and E a prime divisor over X. Let $f\colon Y \to X$ be a birational model over X such that $\text{cent}_Y(E)$ is a divisor. We define

$$\beta_X(E)\colon = A_X(E) - S_X(E)$$

where $S_X(E)$ is defined as

$$S_X(E)\colon = \int_0^{\tau(E)} \frac{1}{(-K_X)^n}\text{vol}(f^*(-K_X) - sE)ds$$

Remark. The definitions for $\tau(E)$ and $S_X(E)$ do not depend on the choice of Y where $\text{cent}_Y(E)$ is a divisor. The reason is as follows. Let $g\colon \tilde{Y} \to Y$ be a birational model over Y with $\tilde{f}\colon \tilde{Y} \to X$ the composition with f and $\tilde{E}\colon = g_*^{-1}(E)$. Then $g^*(E) = \tilde{E} + F$ for some exceptional divisor $F \geq 0$ and we have $\tilde{f}^*(-K_X) - s\tilde{E} = g^*(f^*(-K_X) - sE) + sF$. Then $|g^*(f^*(-K_X) - sE) + sF|_{\mathbb{R}} = |g^*(f^*(-K_X) - sE)|_{\mathbb{R}} + sF$. Therefore, we have $\text{vol}_Y(f^*(-K_X) - sE) = \text{vol}_{\tilde{Y}}(\tilde{f}^*(-K_X) - s\tilde{E})$.

The valuative criterion tells us that we may use β-invariant to check K-stability:

Theorem 2.12 (([23], Corollary 1.5, Theorem 1.6),([10], Definition 2.4))**.** Let X be a $\mathbb{Q}$-Fano variety. Then

1. X is K-stable if and only if $\beta_X(E) > 0$ for any dreamy divisor E over X;

2. X is K-semistable if and only if $\beta_X(E) \geq 0$ for any prime divisor E over X.

2.4 K-moduli

Definition 2.13 (([10]))**.** Let $n \in \mathbb{N}^+$ and $V \in \mathbb{R}^+$. Define a functor $\mathcal{M}^{\text{Kss}}_{n,V}\colon \text{Sch}_{\mathbb{C}} \to \text{Set}$ as follows:

$$\mathcal{M}^{\text{Kss}}_{n,V}(S)\colon = \{X \to S \text{ flat morphism whose geometric fibers are } n\text{-dimensional K-semistable}$$
$$\mathbb{Q}\text{-Fano varieties with volume } V\text{, satisfying Kollár's condition }\}.$$

Here Kollár's condition means that $\omega^{[m]}_{X/S}$ is flat over S and commutes with any base change for each $m \in \mathbb{Z}$.

Proposition 2.14 (([37], Theorem 7.25, Theorem 8.14))**.** $\mathcal{M}^{\text{Kss}}_{n,V}$ is an Artin stack of finite type, and admits a separated good moduli space of finite type $M^{\text{Kps}}_{n,V}$, which parametrizes n-dimensional K-polystable Fano varieties with klt singularities that have anticanonical volume V.

2.5 Abban-Zhuang method

To verify K-stability, the Abban-Zhuang theory allows us to check only for invariant divisors.

Definition 2.15. Let (X, Δ) be a pair and $G \subseteq \mathrm{Aut}(X)$. We say that (X, Δ) is G-invariant if Δ is G-invariant, i.e. $g(\Delta) = \Delta$ for any $g \in \mathrm{Aut}(X)$.

In fact, to prove K-stability, it is sufficient to check β-invariants for invariant divisors be the following proposition:

Proposition 2.16 ((38, Corollary 4.14))**.** Let G be an algebraic group and let (X, Δ) be a log Fano pair with a G-action. If for all G-invariant irreducible divisors E over X:

1. $\beta_{(X,\Delta)}(E) \geq 0$, then (X, Δ) is K-semistable;

2. $\beta_{(X,\Delta)}(E) > 0$ and G is reductive, then (X, Δ) is K-polystable.

Now we define another invariant $\delta(E)$ which is closely related to $\beta(E)$.

Definition 2.17. Let X be a $\mathbb{Q}$-Fano variety of dimension n, and E a prime divisor over X. Let $f\colon Y \to X$ be a birational model over X such that $\mathrm{cent}_Y(E)$ is a divisor. We define

$$\delta_X(E)\colon = \frac{A_X(E)}{S_X(E)}$$

Let Z be an irreducible subvariety of X we define the local δ-invariant as

$$\delta_Z(X)\colon = \inf_{Z \subseteq \mathrm{cent}_X(E)} \delta_X(E)$$

Next we define a useful tool called admissible flags.

Definition 2.18 ((31, Introduction,p784))**.** Let X be a variety. An *admissible flag* $Y_\bullet$ of length l is a flag of subvarieties

$$Y_\bullet\colon = Y = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_l$$

such that

1. $\pi\colon Y \to X$ is a birational model over X;

2. Y_i is codimension i in Y;

3. Y_i is smooth at the generic point of Y_l.

The following theorem by Abban and Zhuang provides us with a way to estimate δ-invariants using admissible flags:

Theorem 2.19 ((3, Corollary 1.110)). Let Y be a smooth threefold such that $-K_Y$ is big and nef. Let $P \in C \subseteq H \subseteq Y$ be an admissible flag on Y. Let F be a prime divisor over Y such that $\text{cent}_Y(F) = C$. Take the Nakayama-Zariski decomposition

$$-K_Y - uH = P(u) + N(u),\ u \in [0, \tau_Y(H)]$$

Suppose $P(u)$ is nef for all $u \in [0, \tau_Y(H)]$. Let $d(u) := \text{mult}_C(N(u)|_H)$ and $P(u,v)$ to be the positive part of $(P(u) - vC)|_H$. Then

$$\delta_Y(F) \geq \min\{\frac{1}{S_Y(F)}, \frac{1}{S_Y(W^H_{\bullet,\bullet};C)}\}$$

where $S_Y(W^H_{\bullet,\bullet};C)$ is defined as

$$S_Y(W^H_{\bullet,\bullet};C) := \frac{3}{(-K_Y)^3}\int_0^{\tau(H)} d(u)P(u,0)^2 du + \frac{3}{(-K_Y)^3}\int_0^{\tau(H)}\int_0^{t(u)} P(u,v)^2 du dv$$

where $t(u)$ is the pseudo-effective threshold

$$t(u) := \sup\{v \in \mathbb{R}_{\geq 0} | (P(u) - vC)|_H \text{ is pseudo–effective}\}.$$

Remark. The above theorem applies to cases when one needs to calculate the β-invariant $\beta(F)$ a prime divisor F whose centre on Y is not a point. However, this is enough for my calculation. For the case when $\dim \text{cent}_Y(F) = 0$, there is a similar formula and see (3, Theorem 1.112) for more details.

The next lemma is very useful when we try to study $\delta_Y(F)$ when F is on Y:

Lemma 2.20 ((3, Corollary 1.44)). Let Y be a Fano variety of dimension n and F a prime divisor. Let $Z \subseteq F$ be an irreducible subvariety with generic point η_Z. Let $G \subseteq \text{Aut}(Y)$ be a reductive subgroup. Suppose that Z is G-invariant. Then

$$\delta_Y(F) \geq \frac{n+1}{n}\alpha_{G,Z}(Y)$$

where

$$\alpha_{G,Z}(Y) := \sup\{\text{lct}_{\eta_Z}(X;M) | M \in |-K_Y| \text{ is } G\text{–invariant}\}.$$

2.6 Application to one-dimensional K-moduli

In this subsection I will show how to use Theorem 2.19 to prove $X_{[1,0]}$ in Family 4 is K-polystable. The proofs for the other families are similar.

The next diagram summarises the main idea to prove a given threefold X is K-polystable. First, fix some subgroup $G \subseteq \mathrm{Aut}(X)$ and let F be a G-invariant divisor over X. Second, we choose Y to be a small resolution of X. Since δ-invariant does not depend on the choice of birational models, we may use the model Y to calculate $\delta_X(F)$. As we will use intersection numbers to estimate $\delta_X(F)$, it is easier to handle it on Y than on X since Y is smooth. Finally, we consider the dimension of $\mathrm{cent}_Y(F)$:

- if $\dim \mathrm{cent}_Y(F) = 2$, then we use Lemma 2.20 to estimate $\delta_Y(F)$ with log canonical thresholds;
- if $\dim \mathrm{cent}_Y(F) = 1$, then we choose a prime divisor $S \subseteq Y$ containing $\mathrm{cent}_Y(F)$ and use Lemma 2.19 to estimate $\delta_Y(F)$;
- $\dim \mathrm{cent}_Y(F)$ is never 0 for Family 4 and 6 we treat in this thesis, but when this occurs for other families, one may choose a flag containing $\mathrm{cent}_Y(F)$ and use (3, Theorem 1.112), which is similar with Lemma 2.19, to to estimate $\delta_Y(F)$.

I will show how to apply the above method to Family 4.

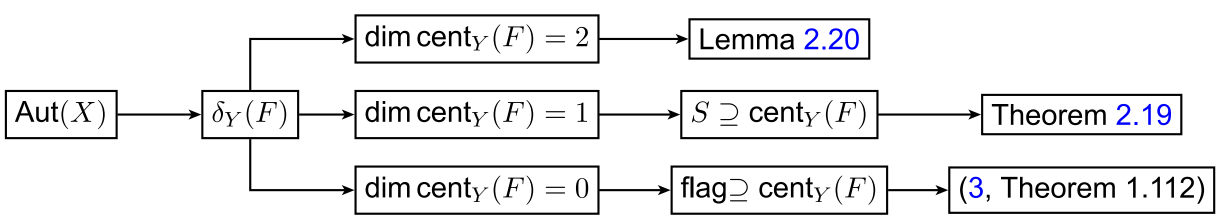


Figure 1: Approach Diagram

Proposition 2.21. $X_{[1,0]}$ in Family 4 is K-polystable.

For simplicity I use X to denote $X_{[1,0]}$. First, we give a description of X as a blow up of $\mathbb{P}^3 = \{[x_0, x_1, x_2, x_3]\}$. Define the following lines in $\mathbb{P}^3$:

- $L\colon = \{x_0 = x_3 = 0\}$;
- $l_1\colon = \{x_0 - x_1 = x_2 = 0\}$;

- $l_2 \colon = \{x_1 = x_2 = 0\}$;
- $l_3 \colon = \{x_2 - x_3 = x_1 = 0\}$.

Then X is the blow up of $\mathbb{P}^3$ along $L \cup l_1 \cup l_2 \cup l_3$.

The next lemma describes the automorphism group of X:

Lemma 2.22 (=(1, Lemma 4.3, 4.4))**.** Let $G \colon =< \tau, \mathbb{C}^* >\subseteq$ Aut$(\mathbb{P}^3)$ where

$$\tau[x_0, x_1, x_2, x_3] = [x_3, x_2, x_1, x_0]$$

$$\lambda[x_0, x_1, x_2, x_3] = [\lambda x_0, \lambda x_1, x_2, x_3], \lambda \in \mathbb{C}^*$$

Then the action G lifts to X and Aut$(X) = G$. Furthermore we have

1. there is no G-invariant point or plane on $\mathbb{P}^3$;
2. if $S \subseteq \mathbb{P}^3$ is a G-invariant irreducible quadric surface on $\mathbb{P}^3$, then

   $$S = R \colon = \{x_0x_2 - x_1x_3 = 0\} \text{ or}$$

   $$S = \{ax_0x_3 + bx_1x_2 + c(x_0x_2 + x_1x_3) = 0\}$$

   for some $[a, b, c] \in \mathbb{P}^2$ with $ab \neq c^2$
3. if $C \subseteq \mathbb{P}^3$ is an irreducible G-invariant curve then $C = L_a$ for some $a \in \mathbb{C} \cup \{\infty\}$ where

   $$L_a \colon = \{x_0 = ax_1, x_3 = ax_2\}.$$

Proof. Since $L \cup l_1 \cup l_2 \cup l_3$ is G-invariant, the action $G \curvearrowright \mathbb{P}^3$ lifts to X.

proof of (1): Suppose a closed point $p = [x_0, x_1, x_2, x_3]$ is $\mathbb{C}^*$-invariant, then $p \in L^1 \colon = \{x_0 = x_1 = 0\}$ or $L^2 \colon = \{x_2 = x_3 = 0\}$. However, the action of τ on L^1 is $\tau[0, 0, x_2, x_3] = [x_3, x_2, 0, 0]$. Therefore, p cannot be τ-invariant if $p \in L^1$. Similarly, p cannot be τ-invariant if $p \in L^2$. There are only 4 $\mathbb{C}^*$-invariant planes: $\{x_0 + x_1 = 0\}, \{x_0 - x_1 = 0\}, \{x_2 + x_3 = 0\}, \{x_2 - x_3 = 0\}$., but none of them is τ-invariant.

proof of (2): Since S is τ-invariant, we may assume S is defined by

1. $f(x_0, x_1, x_2, x_3) = a_1(x_0x_1 + x_2x_3) + a_2(x_0x_2 + x_1x_3) + a_3x_0x_3 + a_4x_1x_2$, or
2. $f(x_0, x_1, x_2, x_3) = a_1(x_0x_1 - x_2x_3) + a_2(x_0x_2 - x_1x_3)$.

Then for $\lambda \in \mathbb{C}^*$, $\lambda(S)$ is defined by

1. $\lambda(f) = a_1(x_0x_1 + \lambda^2 x_2x_3) + a_2\lambda(x_0x_2 + x_1x_3) + a_3\lambda x_0x_3 + a_4\lambda x_1x_2$, or

2. $\lambda(f) = a_1(x_0x_1 - \lambda^2 x_2x_3) + a_2\lambda(x_0x_2 - x_1x_3)$.

Therefore, S is either R or defined by

$$f(x_0, x_1, x_2, x_3) = a_2(x_0x_2 + x_1x_3) + a_3x_0x_3 + a_4x_1x_2$$

$$= x_0(a_2x_2 + a_3x_3) + x_1(a_2x_3 + a_4x_2), [a_2, a_3, a_4] \in \mathbb{P}^2$$

Since S is irreducible, we require that $a_2^2 \neq a_3a_4$.

proof of (3): Suppose every point $p \in C$ is $\mathbb{C}^*$-invariant, then $C = L^1$ or L^2 according to proof of (1). However, L^1 and L_2 are not G-invariant, so we may choose $p \in C$ such that $p = [x, y, z, w]$ is not fixed by $\mathbb{C}^*$. Suppose $y \neq 0$ and define $a\colon = x/y$, then $C \subseteq \{x_0 = ax_2\}$. Since C is G-invariant, we have $C = \tau(C) \subseteq \tau\{x_0 = ax_2\} = \{x_3 = ax_1\}$. $\square$

Lemma 2.23 (=(1, Lemma 4.6, 4.7, 4.8))**.** Let F be a prime G-invariant divisor on X then $\beta_X(F) > 0$.

Here is the outline of the proof: first, if there is such a divisor with negative β-invariant, we use lemma 2.20 to prove that there are only finitely many possibility of F. Second, we calculate the β-invariant of all these divisors.

Now we describe some birational contractions on X. Consider the following birational map:

$$\chi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$$

$$[x_0, x_1, x_2, x_3] \mapsto [x_0, x_3] \times [x_1(x_0 - x_1), x_1x_2, x_2(x_3 - x_2)]$$

Then the undefined locus of χ is $L \cup l_1 \cup l_2 \cup l_3$ and the blow up $X \to \mathbb{P}^3$ resolves the indeterminancy:

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ \mathbb{P}^3 & \overset{\chi}{\dashrightarrow} & \mathbb{P}^1 \times \mathbb{P}^2 \end{array}$$

To describe η, let H_{12} and H_{23} be the two hyperplanes in $\mathbb{P}^3$ containing $l_1 \cup l_2$ and $l_2 \cup l_3$ respectively. Let Q be the following quadric in $\mathbb{P}^3$:

$$Q\colon = \{x_0x_2 + x_1x_3 - x_0x_3 = 0\}$$

Then we have

1. $\chi(H_{12}) = \mathbb{P}^1 \times [1,0,0]$, $\chi(H_{23}) = \mathbb{P}^1 \times [0,0,1]$,

2. $\chi(Q)$ is the image of $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^2$, $[u,v] \mapsto [u,v] \times [u^2, uv, v^2]$.

(1) is simply from the definition of χ and H_{12}, H_{23}. To see (2), we may reparametrize Q. Notice that Q is the image of $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, $[u,v] \times [a,b] \mapsto \times[ub, ua+ub, -va, vb]$. So a birational subset of $\chi(Q)$ is parametrized by $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$, $[u,v] \times [a,b] \mapsto [ub, vb] \times [-u^2a(a+b), -uva(a+b), -v^2a(a+b))] = [u,v] \times [u^2, uv, v^2]$ when $b, a, (a+b) \neq 0$.

Denote by $\gamma\colon V \to \mathbb{P}^3$ the blow up of the lines L, ℓ_1, ℓ_3, and by $\phi\colon \widetilde{X} \to V$ the blow up of the proper transform of the line ℓ_2.

Let $\varphi\colon W \to \mathbb{P}^3$ be the blow up of the lines L and ℓ_2, and $\delta\colon \overline{X} \to W$ the blow up of the proper transform of the disjoint lines ℓ_1 and ℓ_3. Then we have a G-equivariant commutative diagram: where $\widetilde{X} \to X$ and $\overline{X} \to X$ are G-equivariant small resolutions of the 3-fold X.

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X & \longleftarrow & \overline{X} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \delta} \\ V & \xrightarrow{\gamma} & \mathbb{P}^3 & \xleftarrow[\varphi]{} & W \end{array}$$

Figure 2: Blow up Diagram

Let E_L, E_1, E_2, E_3 be the π-exceptional divisors that are mapped to L, ℓ_1, ℓ_2, ℓ_3, respectively, let H_L be a general plane in $\mathbb{P}^3$ that contains L, let H_2 be a general plane in $\mathbb{P}^3$ that contains ℓ_2, let H be a general plane in $\mathbb{P}^3$, let $\overline{E}_L$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{Q}$, $\overline{R}$, $\overline{H}_{12}$, $\overline{H}_{23}$, $\overline{H}_L$, $\overline{H}_2$, $\overline{H}$ be the proper transforms on $\overline{X}$ of the surfaces E_L, E_1, E_2, E_3, Q, R, H_{12}, H_{23}, H_L, H_2, H, respectively. Then $\overline{H}$, $\overline{E}_L$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$ generate Pic($\overline{X}$), and their intersections can be described as follows: $\overline{H}^3 = 1$, $\overline{E}_L^3 = \overline{E}_2^3 = -2$, $\overline{E}_1^3 = \overline{E}_3^3 = -1$, $\overline{E}_L^2 \cdot \overline{H} = \overline{E}_1^2 \cdot \overline{H} = \overline{E}_2^2 \cdot \overline{H} = \overline{E}_3^2 \cdot \overline{H} = \overline{E}_2 \cdot \overline{E}_3^2 = \overline{E}_2 \cdot \overline{E}_1^2 = -1$, and other triple intersections are zero. Note that $-K_{\overline{X}} \sim 4\overline{H} - \overline{E}_L - \overline{E}_1 - \overline{E}_2 - \overline{E}_3$ and

$$\begin{array}{lll} \overline{Q} \sim 2\overline{H} - \overline{E}_L - \overline{E}_1 - \overline{E}_3, & \overline{H}_{12} \sim \overline{H} - \overline{E}_1 - \overline{E}_2, & \overline{H}_L \sim \overline{H} - \overline{E}_L, \\ \overline{R} \sim 2\overline{H} - \overline{E}_L - \overline{E}_2, & \overline{H}_{23} \sim \overline{H} - \overline{E}_2 - \overline{E}_3, & \overline{H}_2 \sim \overline{H} - \overline{E}_L. \end{array}$$

Note also that $\overline{E}_L$, $\overline{E}_2$, $\overline{E}_1 + \overline{E}_3$, $\overline{Q}$, $\overline{R}$, $\overline{H}_{12} + \overline{H}_{23}$ are G-invariant and G-irreducible.

proof of Lemma 2.23. For the sake of contradiction, assume that $\beta(F) \leq 0$ for some G-invariant prime divisor on X.

First, we prove that $F = \overline{E}_2, \overline{E}_L$ or $\overline{Q}$. According to Lemma 2.20, there exists $\lambda > \frac{4}{3}$ and an effective divisor $\Delta \geq 0$ such that $-K_{\overline{X}} \sim \lambda F + \Delta$. Suppose $F \neq \overline{E}_2, \overline{E}_L$, then since $F \neq \overline{E}_1 + \overline{E}_3$, we see that $\pi(F)$ is a G-invariant irreducible surface of degree $d \geqslant 2$, because $\mathbb{P}^3$ does not contain G-invariant planes by Lemma 2.22. Then $F \sim d\overline{H} - m\overline{E}_L - r(\overline{E}_1 + \overline{E}_3) - s\overline{E}_2$ for some non-negative integers m, r, s. Then $4 \geqslant \lambda d > \frac{4}{3}d$, so that $d = 2$ and

$$\Delta \sim_{\mathbb{Q}} (4 - 2\lambda)\overline{H} + (m\lambda - 1)\overline{E}_L + (s\lambda - 1)\overline{E}_2 + (r\lambda - 1)(\overline{E}_1 + \overline{E}_3).$$

Let ℓ be a general line in $\mathbb{P}^3$ that intersects the lines ℓ_1 and ℓ_2, and let $\overline{\ell}$ be its proper transform on the 3-fold $\overline{X}$. Then $\overline{\ell} \not\subseteq \mathrm{Supp}(\Delta)$, so that $0 \leqslant \Delta \cdot \overline{\ell} = 2 - 2\lambda + r\lambda$, which implies that $r \neq 0$. Similarly, intersecting Δ with the proper transform of a general line in $\mathbb{P}^3$ that intersects L and ℓ_2, we see that $(m, s) \neq (0, 0)$.

Since $r \neq 0$, the quadric $\pi(F)$ contains ℓ_1 and ℓ_3. Hence, using Lemma 2.22, we get

$$\pi(F) = \big\{ax_0x_3 + bx_1x_2 - a(x_0x_2 + x_1x_3) = 0\big\}$$

for some $[a, b] \in \mathbb{P}^1$ such that $[a, b] \neq [0, 1]$ and $[a, b] \neq [1, 1]$. This gives $\ell_2 \not\subseteq\in \pi(F)$, so that $s = 0$. Then $m \neq 0$, so that $L \subset \pi(F)$. Then $[a, b] = [1, 0]$ and $F = \overline{Q}$.

Second, I only have to show that that $\beta(\overline{E}_2)$, $\beta(\overline{E}_L)$, $\beta(\overline{Q})$ are positive. Observe that $\beta(\overline{E}_L) > 0$ follows from the proof of (22, Lemma 4.2). Nevertheless, let us compute $\beta(\overline{E}_L)$. We let $\mathscr{S} = \overline{E}_L$. Then

$$-K_Y - u\mathscr{S} \sim_{\mathbb{R}} 4\overline{H} - (1 + u)\mathscr{S} - \overline{E}_1 - \overline{E}_2 - \overline{E}_3 \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right)\mathscr{S} + \frac{1}{2}\big(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\big) + 2\overline{H}_L,$$

Since $|\overline{H}_L|$ induces the morphism $\overline{X} \to \mathbb{P}^1$, we have $\overline{H}_L$ is nef and vol$(\overline{H}_L) = 0$. Therefore, it follows from (3, (5.18.1)) that $\tau = \frac{3}{2}$. Moreover, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} \left(\frac{3}{2} - u\right)\mathscr{S} + \frac{1}{2}\big(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\big) + 2\overline{H}_L & \text{if } 0 \leqslant u \leqslant 1, \\ \left(\frac{3}{2} - u\right)\big(\mathscr{S} + \overline{Q} + \overline{H}_{12} + \overline{H}_{23}\big) + 2\overline{H}_L & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u - 1)\big(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\big) & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Now, by integrating $(P(u))^3$ we get $\beta(\overline{E}_L) = 1 - S_Y(\overline{E}_L) = 1 - \frac{37}{56} = \frac{19}{56}$.

Now, we deal with $\overline{Q}$. Set $\mathscr{S} = \overline{Q}$. Since $\overline{Q} \sim 2\overline{H} - \overline{E}_1 - \overline{E}_L - \overline{E}_3$, we have

$$-K_Y - u\mathscr{S} \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right)\mathscr{S} + \frac{1}{2}\big(\overline{E}_L + \overline{E}_1 + \overline{E}_2\big) + \frac{1}{2}\overline{H}_2,$$

so that $\tau = \frac{3}{2}$. Moreover, if $0 \leqslant u \leqslant 1$, then $N(u) = 0$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then $N(u) = (u-1)(\overline{E}_L + \overline{E}_1 + \overline{E}_2)$. Then

$$P(u) \sim_{\mathbb{R}} \begin{cases} \left(\frac{3}{2} - u\right)\mathscr{S} + \frac{1}{2}\left(\overline{E}_L + \overline{E}_1 + \overline{E}_2\right) + \frac{1}{2}\overline{H}_2 & \text{if } 0 \leqslant u \leqslant 1, \\ \left(\frac{3}{2} - u\right)\left(\mathscr{S} + \overline{E}_L + \overline{E}_1 + \overline{E}_2\right) + \frac{1}{2}\overline{H}_2 & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Now, by integrating we get $\beta(\overline{Q}) = 1 - S_Y(\overline{Q}) = 1 - \frac{129}{224} > 0$.

Finally, we proceed to study $\overline{E}_2$. Let $\mathscr{S} = \overline{E}_2$. Then $\tau = 2$, since

$$-K_Y - u\mathscr{S} \sim_{\mathbb{R}} (2-u)\mathscr{S} + \frac{3}{2}(\overline{H}_{12} + \overline{H}_{23}) + \frac{1}{2}(\overline{E}_1 + \overline{E}_3).$$

Moreover, if $u \in [0,1]$, then $N(u) = 0$. If $u \in [1,2]$, then $N(u) = (u-1)(\overline{H}_{12} + \overline{H}_{23})$, so

$$P(u) \sim_{\mathbb{R}} \begin{cases} (2-u)\mathscr{S} + \frac{3}{2}\left(\overline{H}_{12} + \overline{H}_{23}\right) + \frac{1}{2}\left(\overline{E}_1 + \overline{E}_3\right) & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)\mathscr{S} + \frac{5-u}{2}\left(\overline{H}_{12} + \overline{H}_{23}\right) + \frac{1}{2}\left(\overline{E}_1 + \overline{E}_3\right) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Integrating, leads to $S_Y(\overline{E}_2) = \frac{51}{56}$, so that $\beta_{\overline{X}}(\overline{E}_2) > 0$. □

Next, I will discuss the case when dim $Z \geq 1$ where $Z := \text{cent}_Y(F)$.

Lemma 2.24 ((22, Lemma 4.2))**.** Suppose that Z is a curve in E_L. Then $\beta(\mathbf{F}) > 0$.

We now study G-invariant irreducible curves lying on $\overline{E}_2$.

Lemma 2.25 (=(1, Lemma 4.10))**.** Suppose that $\pi(Z) = L_\infty = \ell_2$ (L_∞ is defined in Lemma 2.22 (3)); then $\beta(\mathbf{F}) > 0$.

Proof. Observe that $\overline{Z} \subset \overline{E}_2$. We will soon see that $\overline{Z}$ is a smooth G-irreducible curve, so we let $\mathscr{S} = \overline{E}_2$. Then $S_Y(\mathscr{S}) = \frac{51}{56}$ (see the proof of Lemma 2.23).

As in Lemma 2.22, let $S = \{x_0x_2 + x_1x_3 = 0\}$, and let $\overline{S}$ be its proper transform on $\overline{X}$. Set $C_S = \overline{S}|_{\mathscr{S}}$ and $C_R = \overline{R}|_{\mathscr{S}}$. Using the map $[x_0, x_1, x_2, x_3] \mapsto ([x_0, x_3], [x_1, x_2])$, we can G-equivariantly identify $\delta(\mathscr{S}) = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([x_0, x_3], [x_1, x_2])$ such that the involution τ acts as $([x_0, x_3], [x_1, x_2]) \mapsto ([x_3, x_0], [x_2, x_1])$, and $\Gamma \cong \mathbb{C}^*$ acts as

$$\left([x_0, x_3], [x_1, x_2]\right) \mapsto \left([\lambda x_0, x_3], [\lambda x_1, x_2]\right),$$

where $\lambda \in \mathbb{C}^*$. Then the only G-invariant irreducible curves in the surface $\delta(\mathscr{S})$ are the curves $\delta(C_S) = \{x_0x_2 + x_1x_3 = 0\}$ and $\delta(C_R) = \{x_0x_2 - x_1x_3 = 0\}$, which implies that C_S and C_R are the only G-invariant irreducible curves in $\mathscr{S}$, so that $\mathscr{C} = C_S$ or $\mathscr{C} = C_R$.

The morphism δ in Figure 2 induces a G-equivariant birational morphism $\sigma\colon \mathscr{S} \to \delta(\mathscr{S})$ that blows up the points $([0,1],[1,0])$ and $([1,0],[0,1])$, which are not contained in the curves $\delta(C_S)$ and $\delta(C_R)$. In particular, we see that $\mathscr{S}$ is a sextic del Pezzo surface.

Set $\mathbf{e}_1 = \overline{E}_1|_{\mathscr{S}}$ and $\mathbf{e}_3 = \overline{E}_3|_{\mathscr{S}}$. Then $\mathbf{e}_1$ and $\mathbf{e}_3$ are the σ-exceptional curves such that $\sigma(\mathbf{e}_1) = ([0,1],[1,0])$ and $\sigma(\mathbf{e}_3) = ([1,0],[0,1])$. Let $\mathbf{s}_1$ and $\mathbf{s}_3$ be the proper transforms on $\mathscr{S}$ of the curves $\{x_2 = 0\}$ and $\{x_1 = 0\}$, and $\mathbf{l}_1$ and $\mathbf{l}_3$ be the proper transforms of the curves $\{x_0 = 0\}$ and $\{x_3 = 0\}$, respectively. Then $\sigma(\mathbf{s}_1)$ and $\sigma(\mathbf{s}_3)$ are the sections of the natural projection $\delta(\mathscr{S}) \to \ell_2$ that pass through the points $\delta(\mathbf{e}_1)$ and $\delta(\mathbf{e}_3)$, respectively, and $\sigma(\mathbf{l}_1)$ and $\sigma(\mathbf{l}_3)$ are the fibres of this projection that pass through the points $\delta(\mathbf{e}_1)$ and $\delta(\mathbf{e}_3)$, respectively. Then $C_S \sim C_R \sim \mathbf{s}_1 + \mathbf{l}_1 + 2\mathbf{e}_1 \sim \mathbf{s}_3 + \mathbf{l}_3 + 2\mathbf{e}_3$.

Recall that $\mathbf{e}_1$, $\mathbf{e}_3$, $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{l}_1$, $\mathbf{l}_3$ are all (-1)-curves in $\mathscr{S}$, they generate the Mori cone $\overline{\mathrm{NE}(\mathscr{S})}$. Note that $\overline{H}|_{\mathscr{S}} \sim \mathbf{l}_1 + \mathbf{e}_1$ and $\overline{E}_2|_{\mathscr{S}} \sim -\mathbf{s}_1 + \mathbf{l}_1$, and we have $\overline{H}_{12}|_{\mathscr{S}} = \mathbf{s}_1$ and $\overline{H}_{23}|_{\mathscr{S}} = \mathbf{s}_3$. So, using the description of $P(u)$ and $N(u)$ obtained in the proof of Lemma 2.23, we get

$$P(u)\big|_{\mathscr{S}} \sim_{\mathbb{R}} \begin{cases} 3\mathbf{e}_1 - \mathbf{e}_3 + (3-u)\mathbf{l}_1 + (u+1)\mathbf{s}_1 & \text{if } 0 \leqslant u \leqslant 1, \\ (4-u)\mathbf{e}_1 + (u-2)\mathbf{e}_3 + (3-u)\mathbf{l}_1 + (3-u)\mathbf{s}_1 & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(u)\big|_{\mathscr{S}} = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)(\mathbf{s}_1 + \mathbf{s}_3) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

In particular, we see that $\mathscr{C} \not\subset \mathrm{Supp}(N(u)|_{\mathscr{S}})$ for every $u \in [0,2]$.

Now, intersecting $P(u)|_{\mathscr{S}} - v\mathscr{C}$ with $\mathbf{e}_1$, $\mathbf{e}_3$, $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{l}_1$, $\mathbf{l}_3$, we find $P(u,v)$ and $N(u,v)$ for $u \in [0,2]$ and $v \in [0, t(u)]$. If $u \in [0,1]$, then $t(u) = 1$,

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (3-2v)\mathbf{e}_1 - \mathbf{e}_3 + (3-u-v)\mathbf{l}_1 + (u-v+1)\mathbf{s}_1 & \text{if } 0 \leqslant v \leqslant u, \\ (3-2v)\mathbf{e}_1 - \mathbf{e}_3 + (3-2v)\mathbf{l}_1 + (u-v+1)\mathbf{s}_1 & \text{if } u \leqslant v \leqslant 1, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u, \\ (v-u)\big(\mathbf{l}_1 + \mathbf{l}_3\big) & \text{if } u \leqslant v \leqslant 1. \end{cases}$$

If $u \in [1,2]$, then $t(u) = \frac{1}{2}$, $N(u,v) = 0$ and $P(u,v) \sim_{\mathbb{R}} P(u)|_{\mathscr{S}} - v\mathscr{C}$. This gives

$$\big(P(u,v)\big)^2 \sim_{\mathbb{R}} \begin{cases} 4 - 2u^2 + 2v^2 + 4u - 8v & \text{if } u \in [0,1], v \in [0,u], \\ 4(1-v)(1+u-v) & \text{if } u \in [0,1], v \in [u,1], \\ 2(1-2v)(5-2u-2v) & \text{if } u \in [1,2], v \in [0,0.5]. \end{cases}$$

Now, by integrating we get $S(W^{\mathscr{S}}_{\bullet,\bullet};\mathscr{C}) = \frac{9}{28}$, so that $\beta(\mathbf{F}) > 0$ by Theorem 2.20. □

For $a \in \mathbb{C}^*$, let $\Pi_a = \{x_0 - ax_1 = x_3 - ax_2\} \subset \mathbb{P}^3$. Then $L_a \subset \Pi_a$, the plane Π_a does not contain L, ℓ_1, ℓ_2, ℓ_3, and neither $\ell_1 \cap \ell_2$ nor $\ell_2 \cap \ell_3$ lie on Π_a. Set $P_1 = \Pi_a \cap \ell_1$, $P_2 = \Pi_a \cap \ell_2$, $P_3 = \Pi_a \cap \ell_3$, $P_4 = \Pi_a \cap L$. Let $\overline{\Pi}_a$ be the preimage on $\overline{X}$ of the plane Π_a. Then $\varphi \circ \delta$ in diagram 2 induces a birational morphism $\overline{\Pi}_a \to \Pi_a$ that is a blow up of P_1, P_2, P_3, P_4.

Lemma 2.26 (=(1, Lemma 4.11))**.** If $a \notin \{1, 2\}$, no three of P_1, P_2, P_3, and P_4 are collinear, and none of them lies on L_a.

When $a = 1$, no three of P_1, P_2, P_3, and P_4 are collinear, P_1 and P_3 lie on L_1, but P_2 and P_4 do not. When $a = 2$, then P_1, P_3 and P_4 lie on the line $\Pi_2 \cap \{x_0 - x_1 + x_2 = 0\}$, but P_2 does not, and none of P_1, P_2, P_3 or P_4 lies on L_2.

Proof. When $a = 1$, we have $P_1 = [1, 1, 0, 0]$, $P_2 = [1, 0, 0, 1]$, $P_3 = [0, 0, 1, 1]$, $P_4 = [0, 1, 1, 0]$. Therefore, we have

$$P_1, P_3 \in L_1 = \{x_0 - x_1 = x_2 - x_3 = 0\}$$

whereas P_2, P_4 does not belong to L_1.

When $a = 2$, we have $P_1 = [1, 1, 0, -1]$, $P_2 = [1, 0, 0, 1]$, $P_3 = [-1, 0, 1, 1]$, $P_4 = [0, 1, 1, 0]$. Therefore, we have

$$P_1, P_3, P_4 \in \{x_0 - x_1 + x_2 = 0\}$$

but $P_2 \notin \{x_0 - x_1 + x_2 = 0\}$. □

Thus, if $a \neq 2$, $\overline{\Pi}_a$ is quintic del Pezzo surface, while if $a = 2$, $\overline{\Pi}_a$ is a weak quintic del Pezzo surface. In both cases, we let $Y = \overline{X}$ and $\mathscr{S} = \overline{\Pi}_a$. Then

$$-K_Y - u\mathscr{S} \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right)\mathscr{S} + \frac{1}{2}(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}) + \frac{1}{2}\overline{H}_L.$$

Therefore, it follows from (3, (5.18.1)) that $\tau = \frac{3}{2}$. Moreover, if $0 \leqslant u \leqslant 1$, then $N(u) = 0$. Furthermore, if $1 \leqslant u \leqslant \frac{3}{2}$, then $N(u) = (u-1)(\overline{Q} + \overline{H}_{12} + \overline{H}_{23})$. Thus, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} \left(\frac{3}{2} - u\right)\mathscr{S} + \frac{1}{2}\left(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\right) + \frac{1}{2}\overline{H}_L & \text{if } 0 \leqslant u \leqslant 1, \\ \left(\frac{3}{2} - u\right)\left(\mathscr{S} + \overline{Q} + \overline{H}_{12} + \overline{H}_{23}\right) + \frac{1}{2}\overline{H}_L & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

By integrating we obtain $S_Y(\mathscr{S}) = \frac{227}{448}$.

Now, let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ be exceptional curves of the blow up $\mathscr{S} \to \Pi_a$ that are mapped to the points P_1, P_2, P_3, P_4, respectively. Then $\overline{E}_1|_{\mathscr{S}} = \mathbf{e}_1$, $\overline{E}_2|_{\mathscr{S}} = \mathbf{e}_2$, $\overline{E}_3|_{\mathscr{S}} = \mathbf{e}_3$, $\overline{E}_L|_{\mathscr{S}} = \mathbf{e}_4$. Set $\mathbf{h} = \overline{H}|_{\mathscr{S}}$. Then

$$P(u)\big|_{\mathscr{S}} \sim_{\mathbb{R}} \begin{cases} (4-u)\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 & \text{if } 0 \leqslant u \leqslant 1, \\ (8-5u)\mathbf{h} - (3-2u)\big(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\big) - (2-u)\mathbf{e}_4 & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Note that $L_a \not\subset H_{12} \cup H_{23}$ for every $a \in \mathbb{C}^*$. Moreover, one has $L_a \subset Q$ if and only if $a = 2$.

Lemma 2.27 (=(1, Lemma 4.12))**.** Suppose that $\pi(Z) = L_a$ for $a \in \mathbb{C} \setminus \{0, 1, 2\}$. Then $\beta(\mathbf{F}) > 0$.

Proof. Let $\mathscr{C} = \overline{Z}$. Note that $\mathscr{C} \sim \mathbf{h}$. Arguing as in the proof of (22, Lemma 4.1), we get

$$t(u) = \begin{cases} 2-u & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{5-3u}{2} & \text{if } 1 \leqslant u \leqslant \frac{7}{5}, \\ 6-4u & \text{if } \frac{7}{5} \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Moreover, if $0 \leqslant u \leqslant 1$, then $P(u,v) \sim_{\mathbb{R}} (4-u-v)\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4$ for $v \in [0, 2-u]$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$ and $0 \leqslant v \leqslant 3-2u$, then

$$P(u,v) \sim_{\mathbb{R}} (8-5u-v)\mathbf{h} - (3-2u)\big(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\big) - (u-2)\mathbf{e}_4.$$

Finally, if $1 \leqslant u \leqslant \frac{3}{2}$ and $3-2u \leqslant v \leqslant t(u)$, then

$$P(u,v) \sim_{\mathbb{R}} (17-11u-4v)\mathbf{h} - (6-4u-v)\big(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\big) - (11-7u-3v)\mathbf{e}_4.$$

This gives $S(W^{\mathscr{S}}_{\bullet,\bullet}; \mathscr{C}) = \frac{753}{1120}$, so that $\beta(\mathbf{F}) > 0$ by Theorem 2.20, since $S_Y(\mathscr{S}) = \frac{227}{448}$. ◻

Lemma 2.28 (=(1, Lemma 4.13))**.** Suppose that $\pi(Z) = L_1$. Then $\beta(\mathbf{F}) > 0$.

Proof. Let $\mathscr{C} = \overline{Z}$. Then $\mathscr{C} \sim \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3$. Moreover, if $0 \leqslant u \leqslant 1$, then $t(u) = 3-u$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then $t(u) = 6-4u$. Furthermore, if $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-u-v)\mathbf{h} + (v-1)\big(\mathbf{e}_1 + \mathbf{e}_3\big) - \mathbf{e}_2 - \mathbf{e}_4 & \text{if } 0 \leqslant v \leqslant 1, \\ (4-u-v)\mathbf{h} - \mathbf{e}_2 - \mathbf{e}_4 & \text{if } 1 \leqslant v \leqslant 2-u, \\ (3-2u-2v)\big(2\mathbf{h} - \mathbf{e}_2 - \mathbf{e}_4\big) & \text{if } 2-u \leqslant v \leqslant 3-u. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$ and $0 \leqslant v \leqslant 3-2u$, then

$$P(u,v) \sim_{\mathbb{R}} (8-5u-v)\mathbf{h} - (3-2u-v)\big(\mathbf{e}_1 + \mathbf{e}_3\big) - (3-2u)\mathbf{e}_2 - (u-2)\mathbf{e}_4.$$

Finally, if $1 \leqslant u \leqslant \frac{3}{2}$ and $3-2u \leqslant v \leqslant 6-4u$, then

$$P(u,v) \sim_{\mathbb{R}} (11-7u-2v)\mathbf{h}-(6-4u-v)\mathbf{e}_2-(5-3u-v)\mathbf{e}_4.$$

Therefore, we have

$$S\big(W^{\mathscr{S}}_{\bullet,\bullet};\mathscr{C}\big)=\frac{3}{28}\int\limits_0^1\int\limits_0^1 u^2+2uv-v^2-8u-4v+12dvdu+$$
$$+\frac{3}{28}\int\limits_1^{2-u}\int\limits_1^{2-u} u^2+2uv+v^2-8u-8v+14dvdu+\frac{3}{28}\int\limits_{2-u}^{3-u}\int\limits_{2-u}^{3-u} 2(3-u-v)^2dvdu+$$
$$+\frac{3}{28}\int\limits_1^{\frac{3}{2}}\int\limits_0^{3-2u} 12u^2+2uv-v^2-40u-4v+33dvdu+\frac{3}{28}\int\limits_1^{\frac{3}{2}}\int\limits_{3-2u}^{6-4u} 2(6-4u-v)(5-3u-v)dvdu=\frac{31}{32}.$$

Thus, it follows from Theorem 2.20 that $\beta(\mathbf{F})>0$, because $S_Y(\mathscr{S})=\frac{227}{448}$. □

Lemma 2.29 (=(1, Lemma 4.14))**.** Suppose that $\pi(Z)=L_2$. Then $\beta(\mathbf{F})>0$.

Proof. Let us introduce several curves on the surface $\mathscr{S}=\overline{\Pi}_2$ as follows:

- let $\mathbf{l}_{12}$ be the proper transform of the line in Π_2 that contains P_1 and P_2,
- let $\mathbf{l}_{23}$ be the proper transform of the line in Π_2 that contains P_2 and P_3,
- let $\mathbf{l}_{24}$ be the proper transform of the line in Π_2 that contains P_2 and P_4,
- let $\mathbf{l}$ be the proper transform on $\mathscr{S}$ of the line $\Pi_2\cap\{x_0+x_1-x_2=0\}$.

On $\mathscr{S}$, we have $\mathbf{l}_{12}\sim\mathbf{h}-\mathbf{e}_1-\mathbf{e}_2$, $\mathbf{l}_{23}\sim\mathbf{h}-\mathbf{e}_2-\mathbf{e}_3$. $\mathbf{l}_{24}\sim\mathbf{h}-\mathbf{e}_2-\mathbf{e}_4$, $\mathbf{l}\sim\mathbf{h}-\mathbf{e}_1-\mathbf{e}_3-\mathbf{e}_4$. Note that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{l}_{12}$, $\mathbf{l}_{23}$, $\mathbf{l}_{24}$ are all (-1)-curves in $\mathscr{S}$, and $\mathbf{l}$ is the unique (-2)-curve in the surface $\mathscr{S}$. By (20, Proposition 8.5), these curves generate the Mori cone $\overline{\mathrm{NE}(\mathscr{S})}$.

Now, let $\mathscr{C}=\overline{Z}$. Then $\mathscr{C}\sim\mathbf{h}$. Moreover, intersecting the divisors under consideration with the curves $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{l}_{12}$, $\mathbf{l}_{23}$, $\mathbf{l}_{24}$, $\mathbf{l}$, we see that

$$t(u)=\begin{cases}\frac{7-3u}{3} & \text{if } 0\leqslant u\leqslant 1,\\ \frac{10-6u}{3} & \text{if } 1\leqslant u\leqslant \frac{4}{3},\\ 6-4u & \text{if } \frac{4}{3}\leqslant u\leqslant \frac{3}{2}.\end{cases}$$

Furthermore, if $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-u-v)\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 & \text{if } 0 \leqslant v \leqslant 1-u, \\ \frac{3-u-v}{2}\big(3\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4\big) - \mathbf{e}_2 & \text{if } 1-u \leqslant v \leqslant 2-u, \\ \frac{7-3u-3v}{2}\big(3\mathbf{h} - \mathbf{e}_1 - 2\mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4\big) & \text{if } 2-u \leqslant v \leqslant \frac{7-3u}{3}, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1-u, \\ \frac{v+u-1}{2}\mathbf{l} & \text{if } 1-u \leqslant v \leqslant 2-u, \\ \frac{v+u-1}{2}\mathbf{l} + (u+v-2)\big(\mathbf{l}_{12} + \mathbf{l}_{23} + \mathbf{l}_{24}\big) & \text{if } 2-u \leqslant v \leqslant \frac{7-3u}{3}. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then $P(u,v)$ is $\mathbb{R}$-rationally equivalent to

$$\begin{cases} \frac{16-10u-3v}{2}\mathbf{h} - \frac{6-4u-v}{2}\big(\mathbf{e}_1 + \mathbf{e}_3\big) - (3-2u)\mathbf{e}_2 - \frac{4-2u-v}{2}\mathbf{e}_4 & \text{if } 0 \leqslant v \leqslant 3-2u, \\ \frac{22-14u-5v}{2}\mathbf{h} - \frac{6-4u-v}{2}\big(\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\big) - \frac{10-6u-3v}{2}\mathbf{e}_4 & \text{if } 3-2u \leqslant v \leqslant 2-u, \\ \frac{10-6u-3v}{2}\big(\mathbf{h} - \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_4\big) & \text{if } 2-u \leqslant v \leqslant \frac{10-6u}{3}, \end{cases}$$

and

$$N(u,v) = \begin{cases} \frac{v}{2}\mathbf{l} & \text{if } 0 \leqslant v \leqslant 3-2u, \\ \frac{v}{2}\mathbf{l} + (v+2u-3)\mathbf{l}_{24} & \text{if } 3-2u \leqslant v \leqslant 2-u, \\ \frac{v}{2}\mathbf{l} + (v+2u-3)\mathbf{l}_{24} + (u+v-2)\big(\mathbf{l}_{12} + \mathbf{l}_{23}\big) & \text{if } 2-u \leqslant v \leqslant \frac{10-6u}{3}. \end{cases}$$

Likewise, if $\frac{4}{3} \leqslant u \leqslant \frac{3}{2}$ and $0 \leqslant v \leqslant 3-2u$, then

$$P(u,v) \sim_{\mathbb{R}} \frac{16-10u-3v}{2}\mathbf{h} - \frac{6-4u-v}{2}\big(\mathbf{e}_1 + \mathbf{e}_3\big) - (3-2u)\mathbf{e}_2 - \frac{4-2u-v}{2}\mathbf{e}_4$$

and $N(u,v) = \frac{v}{2}\mathbf{l}$. Finally, if $\frac{4}{3} \leqslant u \leqslant \frac{3}{2}$ and $3-2u \leqslant v \leqslant 6-4u$, then

$$P(u,v) \sim_{\mathbb{R}} \frac{22-14u-5v}{2}\mathbf{h} - \frac{6-4u-v}{2}\big(\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\big) - \frac{10-6u-3v}{2}\mathbf{e}_4$$

and $N(u,v) = \frac{v}{2}\mathbf{l} + (v+2u-3)\mathbf{l}_{24}$.

If $1 \leqslant u \leqslant \frac{3}{2}$, then $\mathscr{C} \subset \mathrm{Supp}(N(u))$ and $\mathrm{ord}_{\mathscr{C}}(N(u)|_{\mathscr{S}}) = (u-1)$. Thus, we have

$$S\big(W^{\mathscr{S}}_{\bullet,\bullet};\mathscr{C}\big) = \frac{3}{28}\int\limits_{1}^{\frac{3}{2}}(12u^2-40u+33)(u-1)du+$$

$$+\frac{3}{28}\int\limits_{0}^{1}\int\limits_{0}^{1-u} u^2+2uv+v^2-8u-8v+12dvdu+\frac{3}{28}\int\limits_{0}^{1}\int\limits_{1-u}^{2-u}\frac{3u^2+6uv+3v^2-18u-18v+25}{2}dvdu+$$

$$+\frac{3}{28}\int\limits_{0}^{1}\int\limits_{2-u}^{\frac{7-3u}{3}}\frac{(7-3u-3v)^2}{2}dvdu+\frac{3}{28}\int\limits_{1}^{\frac{4}{3}}\int\limits_{0}^{3-2u}\frac{24u^2+20uv+3v^2-80u-32v+66}{2}dvdu+$$

$$+\frac{3}{28}\int\limits_{1}^{\frac{4}{3}}\int\limits_{3-2u}^{2-u}\frac{(14-8u-5v)(6-4u-v)}{2}dvdu+\frac{3}{28}\int\limits_{1}^{\frac{4}{3}}\int\limits_{2-u}^{\frac{10-6u}{3}}\frac{(10-6u-3v)^2}{2}dvdu+$$

$$+\frac{3}{28}\int\limits_{\frac{4}{3}}^{\frac{3}{2}}\int\limits_{0}^{3-2u}\frac{24u^2+20uv+3v^2-80u-32v+66}{2}dvdu+\frac{3}{28}\int\limits_{\frac{4}{3}}^{\frac{3}{2}}\int\limits_{3-2u}^{6-4u}\frac{(14-8u-5v)(6-4u-v)}{2}dvdu.$$

This gives $S(W^{\mathscr{S}}_{\bullet,\bullet};\mathscr{C}) = \frac{2885}{4032} < 1$. Then $\beta(\mathbf{F}) > 0$ by Theorem 2.20, since $S_Y(\mathscr{S}) = \frac{227}{448} < 1$. □

This finishes the proof that X is K-polystable. And with the same method, one can prove the following theorem:

Theorem 2.30 ((1)). $X_{[1,0]}$ in Family 3, 4, 5, and X'_∞ in Family 6 is K-polystable.

proof of theorem 1.2. For Family 3, there exists a K-polystable family over $\mathbb{P}^1$, so we have a surjective morphism $\phi : \mathbb{P}^1 \to M^{\mathsf{Kps}}_{3.12}$. Since every fibre over the $\mathbb{P}^1$ family has worst ordinary double points, they are terminal and hence have unobstructed $\mathbb{Q}$-Gorenstein deformations according to (36, Theorem 1.2). So $M^{\mathsf{Kps}}_{3.12}$ is normal hence smooth according to (27, Remark 2.4). Therefore, it is isomorphic to $\mathbb{P}^1$. The same proof applies for Family 1,2,4,5.

The case for Family 6 is slightly different: we only have some K-polystable families over $\mathbb{A}^1$ rather than $\mathbb{P}^1$, so we do not get the surjective morphism ϕ defined above. Therefore, we use a different strategy here. We will prove that $[X']$ and $[X'_\infty]$ are the only two singular members in $M^{\mathsf{Kps}}_{3.13}$. Since they have terminal klt singularities, we may use (27, Remark 2.4) to imply that $M^{\mathsf{Kps}}_{3.13}$ is smooth.

According to the description in section 2.2, there exists a family over $\mathbb{P}^1$ such that if we denote the fibre over $[\lambda,\mu]$ by $X_{[\lambda,\mu]}$ then

1. $X_{[\lambda,\mu]}$ is smooth and K-polystable when $[\lambda,\mu] \neq [\pm 1, 1], [1,0]$;

2. $X_{[\pm1,1]} \rightsquigarrow X'$;

3. $X_{[1,0]}$ is K-unstable.

So there exists a surjective morphism $\phi : \mathbb{P}^1 \to M^{\mathsf{Kps}}_{3.13}$ such that

1. $\phi([\lambda,\mu]) = [X_{[\lambda,\mu]}]$ for $[\lambda,\mu] \neq [\pm1,1],[1,0]$;

2. $\phi([\pm1,1]) = [X']$.

To prove that $[X'_\infty] \in M^{\mathsf{Kps}}_{3.13}$, we consider the following family:

$$\begin{cases} \mathtt{Pf}_1 = (b-a)x_2y_3 - (b+a)x_3y_2 - 2ax_1y_1, \\ \mathtt{Pf}_2 = (b-a)y_2z_3 - (b+a)y_3z_2 - 2ay_1z_1, \\ \mathtt{Pf}_3 = (b-a)x_2z_3 - (b+a)x_3z_2 + 2ax_1z_1, \\ \mathtt{Pf}_4 = bx_1y_1z_3 + bx_1y_3z_1 + bx_3y_1z_1 + bx_3y_2z_3, \\ \mathtt{Pf}_5 = x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 + x_2y_3z_2, \end{cases} \tag{2}$$

Let $Y_{[a,b]}$ be the fibre over $[a,b] \in \mathbb{P}^1$, we have

1. $Y_{[a,b]}$ is a smooth member of Family 6 for $[a,b] \neq [\pm1,1],[0,1]$;

2. $Y_{[0,1]} \cong X'_\infty$.

So according to Theorem 2.30, $[X'_\infty] \in M^{\mathsf{Kps}}_{3.13}$. Since $X'_\infty \ncong X'$, $[X'_\infty] = \phi([1,0])$. Therefore $M^{\mathsf{Kps}}_{3.13}$ is smooth everywhere and $M^{\mathsf{Kps}}_{3.13} \cong \mathbb{P}^1$. □

3 Foliations on threefolds

3.1 Preliminaries

The content of this section is extracted from (26). However, the proof in this paper is simplified in a way suggested by the examiner.

Throughout, we work on normal algebraic varieties of dimension 3 if not specified otherwise.

Definition 3.1. Let X be a normal variety. A foliation $\mathcal{F}$ on X is a coherent subsheaf of $\mathcal{T}_X$, such that

- $\mathcal{F}$ is saturated, i.e. $\mathcal{T}_X/\mathcal{F}$ is torsion free;
- $\mathcal{F}$ is closed under Lie bracket.

The reflexive hull $N_{\mathcal{F}} := (\mathcal{T}_X/\mathcal{F})^{**}$ is called the normal bundle of $\mathcal{F}$. The dual $N^*_{\mathcal{F}} := \mathcal{H}om(N_{\mathcal{F}}, \mathcal{O}_X)$ is called the conormal bundle of $\mathcal{F}$.

Definition 3.2. The *rank* of a foliation $\mathcal{F}$ on X is defined as the generic rank of the coherent sheaf $\mathcal{F}$ as an $\mathcal{O}_X$-module:

$$\text{rank}(\mathcal{F}) := \text{rank}_{K(X)}\mathcal{F} \otimes K(X)$$

The *corank* of $\mathcal{F}$ is defined as

$$\text{corank}(\mathcal{F}) := \dim X - \text{rank}(\mathcal{F}).$$

Definition 3.3. Suppose $\mathcal{F}$ is a rank r foliation on a normal variety X. Notice that there exists an open embedding $j : X_0 \hookrightarrow X$ such that X_0 is smooth, codim$(X \backslash X_0) \geq 2$ and $\mathcal{F}$ is locally free on X_0, so $\wedge^r \mathcal{F}$ is an invertible sheaf on X_0. We define the canonical divisor of $\mathcal{F}$ to be any divisor $K_{\mathcal{F}}$ on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong j_*(\wedge^r \mathcal{F})$ (See (28, (1.6.3)) for comparison with the classical case). The singular locus of $\mathcal{F}$, denoted by Sing$(\mathcal{F})$, is the co-support of the ideal sheaf defined by the image of the induced map $(\Omega^r_X \otimes \mathcal{O}_X(-K_X)^{**} \to \mathcal{O}_X$.

Definition 3.4. Let X be a variety and $B \geq 0$ a $\mathbb{Q}$-divisor on X. Suppose X is equipped with a foliation $\mathcal{F}$. We say that $(X, B, \mathcal{F})$ is a foliated triple (or $(\mathcal{F}, B)$ is a foliated pair on X) if $K_{\mathcal{F}} + B$ is $\mathbb{Q}$-Cartier.

We now recall the definitions of foliation singularities as well as the notions of invariance, tangency, transversality of subvarieties from (13). We need some preliminary definitions: first we would like to recall that rational maps onto foliated varieties induce natural foliation structures on the domain varieties:

Definition 3.5. The transform of a foliation $\mathcal{F}$ on a normal variety X under a dominant rational map $\phi : Y \dashrightarrow X$ is defined as follows. There exist non-empty open subsets $U \subset X$ and $V \subset Y$ such that the restriction $\phi|_V : V \to U$ is a morphism. Let $\mathcal{F}_U$ denote the restriction of $\mathcal{F}$ to U . Then the morphism $N^*_{\mathcal{F}_U} \to \Omega^1_U$ induces a morphism $\phi|^*_U N^*_{\mathcal{F}_U} \to \Omega^1_V$ and therefore a foliation $\mathcal{G}_V$ on V. In order to extend $\mathcal{G}_V$ to the whole of Y, we let $\mathcal{G}$ denote the saturated subsheaf of T_Y whose restriction to V is $\mathcal{G}_V$. It immediately follows that $\mathcal{G}$ is closed under the Lie bracket as it is so on a dense open subset. We refer to $\mathcal{G}$ as the **induced** foliation on Y. If $\phi : Y \to X$ is a morphism, then the induced foliation is called the **pullback** foliation.

Definition 3.6. Let X be a normal variety and $\mathcal{F}$ a rank r foliation on X. A subvariety $S \subset X$ is called $\mathcal{F}$-**invariant** (or invariant if no confusion can arise) if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$ we have:

$$\partial(I_{S\cap U}) \subset I_{S\cap U} \tag{3}$$

where $I_{S\cap U}$ denotes the ideal sheaf of $S \cap U$ in U.

If $\Delta \subset X$ is a prime divisor, one defines $\epsilon(\Delta) = 1$ if Δ is not invariant in the above sense and $\epsilon(\Delta) = 0$ otherwise.

Definition 3.7. Let X be a normal variety and $\mathcal{F}$ a foliation on X. we say that $\mathcal{F}$ has **non-dicritical** singularities if for any closed point $q \in X$ and any birational morphism $\pi : \tilde{X} \to X$ such that $\pi^{-1}(q)$ is a divisor, we have that each component of $\pi^{-1}(q)$ is $\pi^{-1}\mathcal{F}$-invariant. Where $\pi^{-1}\mathcal{F}$ denotes the pullback foliation on $\tilde{X}$.

Example 3.8. Any smooth foliation is non-dicritical. A typical dicritical foliation is $(\mathbb{A}^2_{(x,y)}, \mathcal{F})$ where $\mathcal{F}$ is generated by $x\partial_x + y\partial_y$ - the leaves are lines passing through $(0, 0)$ with $(0, 0)$ removed.

Definition 3.9. Let X be a normal projective variety equipped with a non-dicritical corank 1 foliation $\mathcal{F}$.

We call a subvariety $W \subset X$ **tangent** to $\mathcal{F}$ if for any birational morphism $\pi : \tilde{X} \to X$ and any divisor E on X such that E dominates W, we have that E is $\tilde{\mathcal{F}}$-invariant, where $\tilde{\mathcal{F}}$ denotes the pullback foliation on $\tilde{X}$.

Otherwise, we call $W \subset X$ transverse to $\mathcal{F}$.

Definition 3.10. For a birational projective morphism $\pi : \tilde{X} \to X$ and a foliated pair $(\mathcal{F}, \Delta)$ on X, let $\tilde{\mathcal{F}}$ be the pullback foliation on $\tilde{X}$ and $\tilde{\Delta}$ the strict transform of Δ on $\tilde{X}$. We then write:

$$K_{\tilde{\mathcal{F}}} + \tilde{\Delta} = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a(\mathcal{F}, \Delta; E)E \tag{4}$$

where the sum runs over all prime exceptional divisors on $\tilde{X}$. The rational numbers $a(\mathcal{F}, \Delta; E)$ are called discrepancies. Here we have $\pi_* K_{\tilde{\mathcal{F}}} = K_{\mathcal{F}}$ since all the E are π-exceptional divisors.

Given a normal variety X and a foliated pair $(\mathcal{F}, \Delta)$ on X we call $(\mathcal{F}, \Delta)$ terminal (respectively canonical, log canonical) if $a(\mathcal{F}, \Delta; E) > 0$ (respectively $\geq 0, \geq -\epsilon(E)$) for every exceptional prime divisors E.

For a not necessarily closed point $P \in X$ we say $(\mathcal{F}, \Delta)$ is terminal (respectively canonical, log-canonical) at P if for all birational morphisms $\pi : \tilde{X} \to X$ and any π-exceptional divisor E on $\tilde{X}$ satisfying cent$_X(E) \ni P$, we have that $a(\mathcal{F}, \Delta; E) > 0$ (≥ 0, respectively $\geq -\epsilon(E)$).

The foliated log discrepancy of E is defined as $A(\mathcal{F}, \Delta; E) := a(\mathcal{F}, \Delta; E) + \epsilon(E)$.

There are two more important concepts related to singularities that will be needed in this paper: For the reader's convenience we recall here the definition of F-dlt-singularities and the notion of log-smoothness.

We first need to clarify the notion of simple singularities, the type of singularities found on foliated log resolutions.

Definition 3.11. Let $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. We say that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies the non-resonant condition if for any $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ satisfying $a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n = 0$, we have $a_1 = a_2 = \cdots = a_n = 0$.

Definition 3.12. [(13, Definition 2.8)] Let $\mathcal{F}$ denote a corank 1 foliation on a smooth variety X of dimension n. We call $p \in X$ a **simple singularity** for $\mathcal{F}$ provided that, in formal coordinates $x_1, .., x_n$ around p, $N^*_{\mathcal{F}}$ is generated by a 1-form which is in one of the following forms for some $1 \leq r \leq n$:

- There are complex numbers $\lambda_1, ..., \lambda_r$ which satisfy the non-resonant condition and such that

$$\omega = \prod_1^r x_i \sum_{i=1}^r \lambda_i \frac{dx_i}{x_i} \tag{5}$$

- There is an integer $k \leq r$ such that

$$\omega = \prod_1^r x_i \sum_{i=1}^r p_i \frac{dx_i}{x_i} + \psi(x_1^{p_1}, ...x_k^{p_k}) \sum_{i=1}^r \lambda_i \frac{dx_i}{x_i} \tag{6}$$

where $p_1, ..., p_k$ are positive integers without a common factor and $\psi(y)$ is a formal power series which is not a unit and the numbers $\lambda_2, ..., \lambda_r \in \mathbb{C}^*$ satisfy the non-resonant condition.

The integer r is referred to as dimension type of the singularity. It is worth noting that the definition can be extended to pairs in the following way:
If (X, D) is normal crossings and $\mathcal{F}$ is a corank 1 foliation on X, then we say that $\mathcal{F}$ has simple singularities adapted to D if $\mathcal{F}$ has simple singularities and for every $p \in X$ we may choose formal coordinates around p as above and such that $D \cup \{\prod_1^r x_i = 0\}$ is also normal crossing at p.
Lastly, we define a **stratum** of Sing$(\mathcal{F})$ as a closed subvariety $Z \subset$ Sing$(\mathcal{F})$ such that for $p \in Z$ and for coordinates $x_1, ..., x_r$ as above in the formal neighbourhood of p, we have that Z is a stratum of $\{\prod_1^r x_i = 0\}$.

Definition 3.13. Let X be a normal variety and $\mathcal{F}$ a corank 1 foliation on X, we say that $(\mathcal{F}, \Delta)$ is foliated log smooth if:

1. (X, Δ) is log smooth;
2. $\mathcal{F}$ has simple singularities;
3. if S denotes the support of $\mathcal{F}$-non-invariant components of Δ, $p \in S$ is a closed point and $\Sigma_1, ..., \Sigma_k$ are $\mathcal{F}$-invariant (possibly formal) divisors passing through p, then $S \cup \Sigma_1 \cup ... \cup \Sigma_k$ is a normal crossings divisor at p.

Definition 3.14. Let $(X, \mathcal{F}, \Delta)$ be a foliated triple and $\pi\colon \tilde{X} \to X$ a birational morphism. Let $\tilde{\mathcal{F}}$ be the induced foliation on $\tilde{X}$ and $\tilde{\Delta}\colon = \pi_*^{-1}\Delta$. We say that π is a foliated log resolution of $(X, \mathcal{F}, \Delta)$ if

1. every component of Exc(π) has codimension 1;
2. $(\tilde{X}, \tilde{\mathcal{F}}, \tilde{\Delta} + \sum E)$ is foliated log smooth where the sum $\sum E$ runs over all π-exceptional divisors.

Definition 3.15. Let X be a normal variety and $\mathcal{F}$ a corank 1 foliation on X. Let $(X, \mathcal{F}, \Delta)$ be a foliated triple.

We call $(X, \mathcal{F}, \Delta)$ foliated divisorial log terminal (F-dlt) if

1. Each irreducible component of Δ is generically transverse to the foliation $\mathcal{F}$ and has coefficients at most one;

2. There is a foliated log resolution $\pi : Y \to X$ of $(\mathcal{F}, \Delta)$ which only extracts divisors E of discrepancy $a(\mathcal{F}, \Delta; E) > -\epsilon(E)$.

Definition 3.16. Given a germ $p \in X$ with a corank 1 foliation $\mathcal{F}$ such that p is a singular point of $\mathcal{F}$ we call a (formal) hypersurface germ $p \in S$ a (formal) separatrix if it is invariant under $\mathcal{F}$.

Next, we clarify the notions of foliated minimal models as well as log flips and flops:

Definition 3.17 ((13, Section 10)). Let $(\mathcal{F}, \Delta)$ be an F-dlt foliated pair on a projective normal variety X. A minimal model of $(X, \mathcal{F}, \Delta)$ is a $K_{\mathcal{F}}$-non-positive birational map $f \colon X \dashrightarrow X'$ such that if $\mathcal{F}'$ is the transformed foliation on X', then

1. X' is $\mathbb{Q}$-factorial and klt and $\mathcal{F}'$ is non-dicritical;
2. $\mathcal{F}'$ is F-dlt and $K_{\mathcal{F}'}$ is nef;
3. if E is a f^{-1}-exceptional divisor, then E is $\mathcal{F}'$-invariant and $a(\mathcal{F}; E) = 0$.

The following definitions are similar to (29, Definition 3.33, 6.10).

Definition 3.18. Let X be a normal variety and $\mathcal{F}$ a corank 1 foliation on X. Let D be an effective $\mathbb{Q}$-divisor on X such that $K_{\mathcal{F}} + D$ is $\mathbb{Q}$-Cartier.

A $(K_{\mathcal{F}} + D)$-flipping contraction is a proper birational morphism $f \colon X \to Y$ to a normal variety Y such that Exc(f) has codimension at least two in X and -$(K_{\mathcal{F}} + D)$ is f-ample. A normal variety X^+ together with a proper birational morphism $f^+ \colon X^+ \to Y$ is called a $(K_{\mathcal{F}} + D)$-flip of f if:

1. $(K_{\mathcal{F}^+} + D^+)$ is $\mathbb{Q}$-Cartier, where D^+ is the birational transform of D on X^+ and $\mathcal{F}^+$ the induced foliation;
2. $(K_{\mathcal{F}^+} + D^+)$ is f^+-ample;
3. Exc(f^+) has codimension at least two in X^+.

Definition 3.19. Let X be a normal variety with klt singularities. Let $\mathcal{F}$ be a F-dlt foliation on X. A flopping contraction is a proper birational morphism $f \colon X \to Y$ to a normal variety Y such that Exc(f) has codimension at least two in X and $K_{\mathcal{F}}$ is f- numerically trivial. We say that $f^+ \colon X^+ \to Y$ is the flop of f if f is a $(K_{\mathcal{F}} + D)$-flipping contraction for some D and f^+ is the $(K_{\mathcal{F}} + D)$-flip.

We recall that a divisor F on X is called numerically f-trivial for the birational contraction morphism $f \colon X \to Y$ if for every curve C contracted by f, we have $F \cdot C = 0$.

Definition 3.20. Let X be a normal variety. Then for two $\mathbb{Q}$-divisors E_1 and E_2 on X we define

$$\min\{E_1, E_2\} := \sum_{F \subseteq \text{Supp}(E_1) \cup \text{Supp}(E_2)} \min\{\text{mult}_F(E_1), \text{mult}_F(E_2)\} F.$$

In the proof of theorem 1.3 we will make use of the following two results from (13):

Theorem 3.21 ((13),Theorem 9.4)**.** Let X be a normal projective 3-fold with klt singularities. Let $\mathcal{F}$ be a corank 1 foliation on X. Let Δ be a $\mathbb{Q}$-divisor such that $(\mathcal{F}, \Delta)$ is a F-dlt pair. Let $A \geq 0$ and $B \geq 0$ be $\mathbb{Q}$-divisors such that $\Delta = A + B$ and A ample. Assume $K_{\mathcal{F}} + \Delta$ is nef.

Then $K_{\mathcal{F}} + \Delta$ is semi-ample.

Theorem 3.22 ((13),Lemma 3.24)**.** Let X be a normal projective 3-fold and $\mathcal{F}$ a corank1 foliation on X . Let $(\mathcal{F}, \Delta)$ be an F-dlt pair such that $\lfloor \Delta \rfloor = 0$ and let A be an ample $\mathbb{Q}$-divisor on X.

Then there is an effective $\mathbb{Q}$-divisor $A^{'} \sim_{\mathbb{Q}} A$ such that:

1. $(\mathcal{F}, \Delta + A')$ is also F-dlt;
2. $\lfloor \Delta + A' \rfloor = 0$;
3. The support of A' does not contain any log canonical centre of $(\mathcal{F}, \Delta)$.

Remark.
1. For the above two theorems we removed the non-dicritical condition in (13) since F-dlt condition implies non-dicritical singularity ((13, Theorem 11.3));
2. we recall that the definition of lc centre and lc place for foliated pairs is similar with the classical case (Definition 2.4). Let E be a divisor over X such that $a(\mathcal{F}, \Delta; E) = -\epsilon(E)$, then E is called an lc place of $(\mathcal{F}, \Delta)$ and $\text{cent}_X(E)$ is called an lc centre of $(\mathcal{F}, \Delta)$.

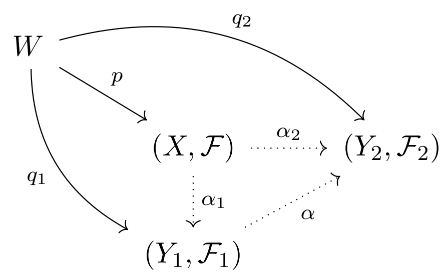
3.2 Proof of Theorem 1.3

Sticking to the notation in the statement of Theorem 1.3, we first show that the birational map α relating the two foliated minimal models is an isomorphism in codimension one.

We start by setting up the notation: Let $(Y_1, \mathcal{F}_1)$ and $(Y_2, \mathcal{F}_2)$ be 2 foliated minimal models of a common F-dlt foliated threefold pair $(X, \mathcal{F})$, such that X is $\mathbb{Q}$-factorial with klt singularities. Denote by $\alpha : (Y_1, \mathcal{F}_1) \dashrightarrow (Y_2, \mathcal{F}_2)$ the map connecting the 2 minimal models and by $\alpha_i : (X, \mathcal{F}) \dashrightarrow (Y_i, \mathcal{F}_i)$ a choice of MMP-steps run through in order to obtain the respective minimal models.

We recall that the steps of the foliated MMP are $K_{\mathcal{F}}$-negative and hence so are the above defined $\alpha_i' s$. For the definition of D-negativity with respect to a divisor D, we refer to (14, Page 3).

Thus there is a common log resolution W of X, Y_1 and Y_2 such that for the following commutative diagram we have:



$$p^*(K_{\mathcal{F}}) = q_1^*(K_{\mathcal{F}_1}) + E_1 = q_2^*(K_{\mathcal{F}_2}) + E_2 \tag{7}$$

where $E_i \geq 0$ is q_i-exceptional and $p^{-1}\mathrm{Exc}(\alpha_i) \subseteq \mathrm{supp} E_i$.

Lemma 3.23 (=(26, Lemma 3.1))**.** With the above notation, we have that α is an isomorphism in codimension 1.

Proof. Suppose $E_1 \neq E_2$ then without loss of generality we assume that , $\overline{E}_1 := E_1 - \min\{E_1, E_2\} > 0$, $\overline{E}_2 := E_2 - \min\{E_1, E_2\} \geq 0$. According to the negativity lemma (e.g. (8, Lemma 3.6.2)) we can find a curve $C \subseteq$ Supp , $\overline{E}_1$ such that C is not contained in Supp $\overline{E}_2$ and $C \cdot \overline{E}_1 < 0$. Then we get

$$q_2^*(K_{\mathcal{F}_2}) = q_1^*(K_{\mathcal{F}_1}) + E_1 - E_2 = q_1^*(K_{\mathcal{F}_1}) + \overline{E}_1 - \overline{E}_2$$

and

$$q_2^*(K_{\mathcal{F}_2}) \cdot C = q_1^*(K_{\mathcal{F}_1}) \cdot C + (\overline{E}_1 - \overline{E}_2) \cdot C < 0$$

contradicting the nefness of $\mathcal{F}_i$.

Hence we conclude that , $\overline{E}_1 = \overline{E}_2$ which implies that q_1 and q_2 contract the same divisors, thus α is an isomorphism in codimension 1. □

Kawamata's strategy to prove that minimal models are connected by flops is to choose an ample divisor L' on minimal model Y_2 and consider its strict transform L on minimal model Y_1. As klt-ness is an open property, for a small number l, (Y_1, lL) is still klt and Kawamata can thus assume that the divisor $K_{Y_1} + lL$ is not nef because otherwise $\mathbb{Q}$-factoriality of Y_2 and the classical base point free theorem (29, Theorem 3.3) would imply that α is an isomorphism. This allows Kawamata then to run a $(K_{Y_1} + lL)$-MMP on Y_1 to finally reach the desired conclusion.

In the foliated case treated here, although there is as well an analogue of the classical basepoint-free theorem for corank1-foliations on 3-folds (see subsection 3.1), its invocation is not as straightforward as is in the original proof. The main difficulties arise from the requirement in the foliated version of the basepoint-free theorem that $(\mathcal{F}_1, lL)$ must be F-dlt. A priori it is not immediate that this condition can be satisfied.

The core of the present argument thus consists in the demonstration that a careful choice of the ample divisor L' on Y_2 guarantees that $(\mathcal{F}_1, lL)$ becomes F-dlt.

Invoking (3.22), we know that we can find an ample L' on Y_2 and a sufficiently small real number l such that $(\mathcal{F}_2, lL')$ is again F-dlt. The next few Lemmata in the present paper lay the ground for the proof that L' can be chosen such that also its strict transform on Y_1 satisfies $(\mathcal{F}_1, lL)$ F-dlt.

Remark. Notice that the Bertini-type theorem 3.22 does not directly apply to $(\mathcal{F}_1, lL)$ as L is no longer ample.

Let $(X, \mathcal{F})$, $(Y_1, \mathcal{F}_1)$, $(Y_2, \mathcal{F}_2)$ and α be as defined in Theorem 1.3. The next lemma states that Exc(α) does not contain any log canonical centre of $\mathcal{F}_1$.

Lemma 3.24. Notation as above. There is no lc centre of $(Y_1, \mathcal{F}_1)$ which is contained in

$$\text{Exc}(\alpha) := \{x \in Y_1 | \alpha \text{ is not an isomorphism around } x\}.$$

Proof. Let E be a divisor over Y_1 such that E is an lc place of $(Y_1, \mathcal{F}_1)$. Let $Z_1 := \text{cent}_{Y_1}(E)$ be the corresponding lc centre on Y_1. Suppose Z_1 is not contained in the isomorphic locus of α_1, we may assume that α_1 is a divisorial or flipping contraction where $Z_1 \subseteq \text{Exc}(\alpha_1)$. According to (13, Lemma 2.7) (for the classical case see (29, Lemma 3.38)), E is not an lc place of $(Y_1, \mathcal{F}_1)$. Therefore, α_1^{-1} is an isomorphism around Z_1. Similarly, α_2^{-1} is an isomorphism around Z_2 where $Z_2 := \text{cent}_{Y_2}(E)$, so α is an isomorphism around Z_1. □

Next, we prove that F-dlt is an open condition around non-lc centres.

Lemma 3.25. Let $(X, \mathcal{F}, B)$ be a corank 1 foliated triple, $L \geq 0$ a $\mathbb{Q}$-divisor on X such that

1. X is a $\mathbb{Q}$-factorial threefold;
2. $(\mathcal{F}, B)$ is F-dlt;
3. Supp(L) does not contain any lc centre of $(\mathcal{F}, B)$.

Then there exists $\delta > 0$ such that $(\mathcal{F}, B + \delta L)$ is F-dlt.

Proof. Since $(\mathcal{F}, B)$ is F-dlt, it is lc according to (13, Remark 3.7). According to (13, Remark 3.2) and (11), there exists a foliated log resolution $f\colon Y \to X$, such that

1. $(Y, \mathcal{G}, B_Y + L_Y + \sum E_i)$ is foliated log smooth where the sum $\sum E_i$ runs over all f-exceptional divisors;

2. $A(\mathcal{F}, B; E_i) \geq 0$ for any exceptional E_i,

where $\mathcal{G}$ is the induced foliation on Y and $B_Y \colon = f_*^{-1}B$ and $L_Y \colon = f_*^{-1}L$ are the strict transforms. Therefore, we may write

$$K_{\mathcal{G}} + B_Y + \sum \epsilon(E_i)E_i = f^*(K_{\mathcal{F}} + B) + \sum A(\mathcal{F}, B; E_i)E_i.$$

Then for any $\delta > 0$, we write

$$K_{\mathcal{G}} + B_Y + \delta L_Y + \sum \epsilon(E_i)E_i = f^*(K_{\mathcal{F}} + B + \delta L) + \sum (A(\mathcal{F}, B; E_i) - \text{mult}_{E_i}(f^*L))E_i$$

where $L_Y \colon = f_*^{-1}L$. Since $\text{Supp}(L)$ does not contain any lc centre of $(X, \mathcal{F}, B)$, we have $\text{mult}_{E_i}(f^*L) = 0$ for any E_i with $A(\mathcal{F}, B; E_i) = 0$. So there exists $0 < \delta \ll 1$, such that $(A(\mathcal{F}, B; E_i) - \text{mult}_{E_i}(f^*L)) \geq 0$. Therefore, $(X, \mathcal{F}, B + \delta L)$ is lc hence $(X, \mathcal{F}, B + \frac{\delta}{2}L)$ and $(X, \mathcal{F}, B)$ have the same lc places and lc centres. Then $(X, \mathcal{F}, B + \frac{\delta}{2}L)$ is log smooth around generic points of lc centres and we may use (13, Remark 3.2) and (11) to get a foliated log resolution which is sequence of blow ups on non-lc centres of $(X, \mathcal{F}, B + \frac{\delta}{2}L)$, so$(X, \mathcal{F}, B + \frac{\delta}{2}L)$ is F-dlt. □

Proposition 3.26. For any curve C_1 lying in the exceptional locus of α, we have $K_{\mathcal{F}_1} \cdot C_1 = 0$.

Proof. By rigidity Lemma (e.g. (29, Lemma 1.6)), there exists a curve $L \subseteq W$ such that $p(L) = C_1$ and $q(L)$ is a point. Therefore, let k be the degree of $L \to C_1$, we have $K_{\mathcal{F}_1} \cdot kC_1 = p^*K_{\mathcal{F}_1} \cdot L = q^*K_{\mathcal{F}_2} \cdot L = 0$ □

proof of Theorem 1.3. According to Theorem 3.22, there exists an ample $\mathbb{Q}$-divisor $L_2 > 0$ on Y_2 such that

1. $\text{Supp}(L_2)$ does not contain any lc centre of $(Y_2, \mathcal{F}_2)$;

2. $(Y_2, \mathcal{F}_2, L_2)$ is F-dlt.

According to Lemma 3.24, $\text{Supp}(L_1)$ does not contain any lc centre of $(Y_1, \mathcal{F}_1)$ where $L_1 \colon = \alpha_*^{-1}(L_2)$. According to Lemma 3.25, there exists $0 < \delta \ll 1$ such that $(Y_1, \mathcal{F}_1, \delta L_1)$ is F-dlt.

Let $\beta\colon (Y_1, \mathcal{F}_1, \delta L_1) \dashrightarrow (Y_0, \mathcal{F}_0, \delta L_0)$ be a ($K_{\mathcal{F}_1} + \delta L_1$)-MMP where $(Y_0, \mathcal{F}_0, \delta L_0)$ is the output.

Next, we prove that β is a sequence of $K_{\mathcal{F}_1}$-flops. Let $h\colon Y_1 \to \overline{Y}_1$ be the first contraction of the MMP which contracts a curve C_1. Suppose C_1 is not contained in Exc(α), then there exists $L_2' \sim_{\mathbb{Q}} L_2$ such that $\alpha_*(C_1) \nsubseteq$ Supp(L_2') then $C_1 \nsubseteq$ Supp(L_1') where $L_1' : = \alpha_*^{-1} L_2'$. Therefore, $C_1 \cdot L_1 = C_1 \cdot L_1' \geq 0$. According to Proposition 3.26, $(K_{\mathcal{F}_1} + \delta L_1) \cdot C_1 = (K_{\mathcal{F}_1} + \delta L_1') \cdot C_1 \geq 0$, we get a contradiction with the MMP condition. So h is a flipping contraction since Exc(α) has dimension 1. Let h^+ be the flipped contraction:

$$\begin{array}{ccc} (Y_1, \mathcal{F}_1, \delta L_1) & \overset{\beta_1}{\dashrightarrow} & (Y_1^+, \mathcal{F}_1^+, \delta L^+) \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle h^+} & \\ & (\overline{Y}_1, \overline{\mathcal{F}}_1, \delta \overline{L}) & \end{array}$$

Suppose $K_{\mathcal{F}_1^+}$ is not h^+-trivial, then $K_{\mathcal{F}_1^+} \sim_{h^+} a(K_{\mathcal{F}_1^+} + \delta L_1^+)$ for some $a \neq 0$ since $(K_{\mathcal{F}_1^+} + \delta L_1^+)$ is h^+-ample. Therefore, $K_{\mathcal{F}_1} \sim_h a(K_{\mathcal{F}_1} + \delta L_1) \neq 0$ for some $a \neq 0$ which is a contradiction with Proposition 3.26. So β_1 is a $K_{\mathcal{F}_1}$-flop. So $K_{\mathcal{F}^+}$ is also nef. According to (13, Theorem 6.4), $(Y_1^+, \mathcal{F}_1^+, \delta L^+)$ is still F-dlt, so we may repeat the above process, so β is a sequence of flops.

Finally, we prove that the induced birational map $\beta_0 \colon Y_0 \dashrightarrow Y_2$ is an isomorphism. Since $K_{\mathcal{F}_2}$ is nef and L_2 is ample, $K_{\mathcal{F}_2} + \delta L_2$ is ample as well. So β_0 is induced by $|K_{\mathcal{F}_0} + \delta L_0|_{\mathbb{Q}}$. However, Theorem 3.21 implies that $(K_{\mathcal{F}_0} + \delta L_0)$ is base point free, so β_0 is a birational morphism which is an isomorphism in codimension 1. Since Y_2 is $\mathbb{Q}$-factorial, β_0 is an isomorphism.

□

4 Algebraically integrable foliations with positive curvature

The content of this section is extracted from (19) except for Theorem 1.8. In this section we only treat foliations that are **algebraically integrable** (See definition below).

4.1 Preliminaries

Definition 4.1 (Algebraic Integrability)**.** Let $\mathcal{F}$ be a foliation on a normal variety X. We say that $\mathcal{F}$ is algebraically integrable if there exists a dominant rational map $f\colon X \dashrightarrow Y$ such that $\mathcal{F}$ is the transform (See Definition 3.5) of $\mathcal{G}$ via f, where $\mathcal{G}$ is the trivial foliation given by closed points on Y.

From now on all the foliations we treat will be algebraically integrable.

Definition 4.2. Let X be a normal variety and $(\mathcal{F}, B)$ be a foliated pair on X. We say that $(\mathcal{F}, B)$ is a *Fano foliated pair* (or $(X, \mathcal{F}, B)$ a *Fano foliated triple*) if the following conditions are satisfied:

1. $-(K_{\mathcal{F}} + B)$ is ample and
2. $(\mathcal{F}, B)$ is log canonical.

We call $(X, \mathcal{F}, B)$ *weak Fano* if it satisfies (2) above, and $-(K_{\mathcal{F}} + B)$ is big and nef.

Definition 4.3. Let $(X, \mathcal{F}, B)$ be a foliated triple. Then we define

$$\text{Nlc}(X, \mathcal{F}, B) := \{\text{cent}_E(X) | a(\mathcal{F}, B; E) > \epsilon(E)\}$$

where the closed subvariety $\text{cent}_E(X) \subseteq X$ is the centre of E on X.

We recall the definition of complements for classical pairs.

Definition 4.4. Let (X, B) be a pair. Then an n-complement is of the form $K_X + B^+$ where $B^+ \geq 0$ is an effective $\mathbb{Q}$-divisor such that

- (X, B^+) is log canonical,
- $n(K_X + B^+) \sim 0$, and
- $nB^+ \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$.

A $\mathbb{Q}$-complement B^+ of (X, B) is an effective $\mathbb{Q}$-divisor $B^+ \geq B$ satisfying $K_X + B^+ \sim_{\mathbb{Q}} 0$. The condition $B^+ \geq B$ is called the monotonic condition.

For the existence of n-complements for Fano pairs with bounded n, we refer to (6, Theorem 1.10).

Remark. In our case we only consider the monotonic case. In fact, we only deal with the following two cases:

1. when we consider $\mathbb{Q}$-complement, the monotonicity is by definition;

2. when we consider n-complement for some bounded n, we always assume that $B \in \Phi(\mathfrak{R})$ for some finite set $\mathfrak{R} \subseteq [0,1]$. (The hyperstandard set $\Phi(\mathfrak{R})$ will be defined in Definition 4.17). Therefore, if B^+ is an n-complement and $\mathfrak{R} \subseteq \mathcal{P}_n$, then we have $B^+ \geq B$. (See (35, 4.3.1, Corollary 4.2.7) for the proof as well as the definition of $\mathcal{P}_n$).

Similarly, we define n-complements for foliations.

Definition 4.5. Let $(X, \mathcal{F}, B)$ be a foliated triple. Then an *n-complement* is of the form $K_{\mathcal{F}} + B^+$ such that

- $(X, \mathcal{F}, B^+)$ is log canonical,
- $n(K_{\mathcal{F}} + B^+) \sim 0$, and
- $nB^+ \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$.

Example 4.6 (Unbounded log canonical Fano foliations=(19, Example 3.9))**.** Let $n \geq 1$, $\mathbb{F}_n$ be the Hirzebruch surface and E be the negative section. Let $\mathcal{G}_n$ be the foliation induced by the canonical fibration on $\mathbb{F}_n$ and $\pi\colon \mathbb{F}_n \to S_n$ be the contraction of E. Let $\mathcal{F}_n = \pi_*\mathcal{G}_n$ be the pushforward of $\mathcal{G}_n$. Then we have the following:

1. $(S_n, \mathcal{F}_n)$ is log canonical but not klt,

2. $-K_{\mathcal{F}_n}$ is ample, and

3. S_n has only one singularity, which is ϵ-lc if and only if $\epsilon \geq \frac{2}{n}$.

Similarly, let $C \subseteq \mathbb{P}^N$ be a smooth curve and $P \in \mathbb{P}^N$ a closed point such that $P \notin C$. We can consider the cone S over C with vertex P. Then S is a ruled surface. Let $\mathcal{F}$ be the foliation induced by rulings of the cone. Then $\mathcal{F}$ is log canonical and has a non-invariant lc place; S has only one singularity which is possibly non-rational.

Now we give the definition of property-$(*)$ modification. It will play an analogous role to that of dlt modification for classical pairs.

Definition 4.7. Let $g : X \to Z$ be an equidimensional contraction between normal varieties. We define the ramification divisor $R(g)$ as follows:

$$R(g) := \sum_{P \text{ prime}} (g^*(P) - g^{-1}(P)).$$

Here $g^{-1}(P)$ is the reduced divisor of $g^*(P)$. Let $V \subseteq X$ be an irreducible subvariety. When g is not necessarily equidimensional, we say that V is horizontal over Z if $g(\eta_V) = \eta_Z$, where η_V and η_Z are the generic points of V and Z respectively. We say V is vertical over Z if $g(\eta_V) \neq \eta_Z$.

Definition 4.8 ((2, Definition 2.13))**.** Let (X, B) be a pair, Z a normal variety, $g : X \to Z$ a projective contraction such that (X, B) is generically log canonical over Z. Let B_v be the vertical part of B. We say that $(X/Z, B)$ satisfies property-$(*)$ if

1. there exists a reduced divisor Σ_Z on Z such that (Z, Σ_Z) is log smooth and $B_v = g^{-1}(\Sigma_Z)$;
2. for any closed point $z \in Z$ and reduced divisor $\Sigma \geq \Sigma_Z$ such that (Z, Σ) is log smooth at z, $(X, B + g^*(\Sigma - \Sigma_Z))$ is log canonical around $g^{-1}(z)$.

Definition 4.9. Let $(X, \mathcal{F}, B)$ be a foliated triple. We say that $(X, \mathcal{F}, B)$ satisfies property-$(*)$ if there exists an $\mathcal{F}$-invariant $\mathbb{Q}$-divisor $G \geq 0$ and a projective contraction $g : X \to Z$ such that

1. $(X/Z, B + G)$ satisfies property-$(*)$;
2. $\mathcal{F}$ is induced by g.

We remark that the above two definitions for property-$(*)$ are for relative pair and foliated triples respectively.

Definition 4.10 (Property-$(*)$ modification, (2, Definition 3.8))**.** Let $(X, \mathcal{F}, B)$ be a foliated triple such that B contains no $\mathcal{F}$-invariant component. A property-$(*)$ modification of $(X, \mathcal{F}, B)$ is a birational morphism

$$f\colon (Y, \mathcal{G}, B') \to (X, \mathcal{F}, B)$$

such that

1. Y is klt;

2. $B' \colon = f_*^{-1}(B) + \sum \epsilon(E)E$ where E runs over all f-exceptional divisors, $(Y, \mathcal{G}, B')$ is log canonical and satisfies property-$(*)$;

3. $K_{\mathcal{G}} + B' + F = f^*(K_{\mathcal{F}} + B)$ for some f-exceptional $F \geq 0$ and $f(F)$ does not contain any log canonical centre of $(X, \mathcal{F}, B)$;

4. $\mathcal{G}$ is induced by an equidimensional morphism $g \colon Y \to Z$.

Next, we introduce definition and some notation for foliation adjunction.

Definition 4.11 ((15, Proposition-Definition 3.7))**.** Let X be a variety, $\mathcal{F}$ a foliation of rank r on X. Let $\iota \colon D \hookrightarrow X$ be an integral subscheme of codimension one. Let $n \colon S \to D$ be the normalisation. Suppose that $K_{\mathcal{F}} + \Delta$ and D are $\mathbb{Q}$-Cartier.

Then, there exists a canonically defined restricted foliation $\mathcal{F}_S$ on S and a canonically defined $\mathbb{Q}$-divisor $\mathrm{Diff}_S(\mathcal{F}, \Delta) \geq 0$, called the different, such that

$$n^*(K_{\mathcal{F}} + \Delta + \epsilon(D)D) \sim_{\mathbb{Q}} K_{\mathcal{F}_S} + \mathrm{Diff}_S(\mathcal{F}, \Delta).$$

We prove adjunction and inversion of adjunction for invariant divisors in this case:

Proposition 4.12 (=(18, Theorem 2.4.1), Adjunction)**.** Let $(X, \mathcal{F}, B)$ be a foliated triple. Suppose $S \subseteq X$ a prime invariant divisor with normalisation $\nu \colon \tilde{S} \to S$. Assume that $(X, \mathcal{F}, B)$ is log canonical, then $(\tilde{S}, \mathcal{F}_{\tilde{S}}, \mathrm{Diff}_{\tilde{S}}(\mathcal{F}, B))$ is log canonical.

Proof. We give a direct proof here, which is essentially the same as the one given in (18).

By (2, Theorem 3.10), we may take a property-$(*)$ modification[1] of $(X, \mathcal{F}, B)$, which is denoted by $f \colon (Y, \mathcal{G}, B') \to (X, \mathcal{F}, B)$. Since the modification is an output of a $K_{\mathcal{G}'} + D$-MMP on some $\mathbb{Q}$-factorial model, we may assume Y to be $\mathbb{Q}$-factorial as well. Then we have

$$K_{\mathcal{G}} + B' = f^*(K_{\mathcal{F}} + B)$$

Let T be the strict transform of S with normalisation $\tilde{T} \to T$. By taking the restriction to $\tilde{T}$, we have

$$K_{\mathcal{G}_{\tilde{T}}} + \mathrm{Diff}_{\tilde{T}}(\mathcal{G}, B') = f|_{\tilde{T}}^*(K_{\mathcal{F}_{\tilde{S}}} + \mathrm{Diff}_{\tilde{S}}(\mathcal{F}, B))$$

Since the left hand side is log canonical by (2, Proposition 3.2), so is the right hand side. ☐

[1] As was brought to our attention by the examiner, the statement of (2, Theorem 3.10) does not hold in the generality stated in the current version of the paper, however the statement holds in the case considered here and in Theorem 4.14.

Next, we prove an inversion of adjunction theorem for foliated pairs.

We will need the following lemma first.

Lemma 4.13 (=(19, Lemma 5.3))**.** Let $\pi\colon Y \to Z$ be an equidimensional fibration between normal varieties with $\mathcal{F}$ the induced foliation on Y such that $\dim Y = n$ and $\mathcal{F}$ has rank r. Let $H_Z \subseteq Z$ be a prime divisor and assume that

- S and T are two irreducible components of $\pi^{-1}(H_Z)$,
- $G \subseteq S \cap T$ is a component of $S \cap T$,
- Y is smooth around the generic point η_G of G, and
- $\pi(G) = H_Z$.

Then $G \subseteq \mathrm{Sing}(\mathcal{F})$.

Proof. Since the statement is local around the generic point η_G, we can assume that:

- Both Y and Z are affine and smooth.
- H_Z is smooth, Ω_Z is locally free and generated by $\mathrm{d}z_1, \mathrm{d}z_2, \ldots, \mathrm{d}z_{n-r}$ where $H_Z := \{z_1 = 0\}$.
- $\pi^\sharp(z_1) = x^{n_1}y^{n_2}f$ where x, y are the local generators of S and T respectively, $n_1, n_2 \geq 1$ and $f \neq 0$ a holomorphic function.

Suppose $G \nsubseteq \mathrm{Sing}(\mathcal{F})$. Since $\pi^\sharp(z_1) = x^{n_1}y^{n_2}f$, there are at least two separatrices passing through a general closed point $w \in G$. This is a contradiction with the Frobenius theorem. □

Theorem 4.14 (Inversion of adjunction=(19, Theorem 5.4))**.** Let $(X, \mathcal{F}, B)$ be a foliated triple such that B does not contain $\mathcal{F}$-invariant component and coefficients of B is contained in $[0, 1]$. Let $S \subseteq X$ be a prime invariant divisor with normalisation $\nu\colon S^\nu \to S$ such that $(S^\nu, \mathcal{F}_{S^\nu}, \mathrm{Diff}_{S^\nu}(\mathcal{F}, B))$ is log canonical. Then

$$\mathrm{Nlc}(X, \mathcal{F}, B) \cap S = \varnothing.$$

Proof. Suppose $\mathrm{Nlc}(X, \mathcal{F}, B) \cap S \neq \varnothing$ for the sake of contradiction.

By (2, Theorem 3.10), there is a property-$(*)$ modification $f\colon Y \to X$ such that $(Y, \mathcal{G}, B_Y)$ satisfies property-$(*)$ with

$$K_{\mathcal{G}} + B_Y + F = f^*(K_{\mathcal{F}} + B) \tag{8}$$

where $B_Y = f_*^{-1}B + \sum \epsilon(E_i)E_i$ and F is effective and f-exceptional. Let $\pi\colon Y \to Z$ be the morphism inducing $\mathcal{G}$ on Y. Let $T := f_*^{-1}(S)$ be the strict transform of S. By the assumption that $\mathrm{Nlc}(X, \mathcal{F}, B) \cap S \neq \varnothing$, we have $\mathrm{Supp}(F) \cap T \neq \varnothing$. Let $\rho\colon T^\rho \to T$ be the normalisation of T. We have the following commutative diagram:

$$\begin{array}{ccccccc} T^\rho & \xrightarrow{\rho} & T & \hookrightarrow & (Y, \mathcal{G}, \tilde{B} + \tilde{E}) & \xrightarrow{\pi} & Z \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & \nearrow & \\ S^\nu & \xrightarrow{\nu} & S & \hookrightarrow & (X, \mathcal{F}, B) & & \end{array}$$

Now we consider the adjunction of equation 8 to S and T:

$$K_{\mathcal{G}_{T^\rho}} + \mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho} = f^*(K_{\mathcal{F}_{S^\nu}} + \mathrm{Diff}_{S^\nu}(\mathcal{F}, B)). \tag{9}$$

Since we have the hypothesis that the restriction $(S^\nu, \mathcal{F}_{S^\nu}, \mathrm{Diff}(\mathcal{F}, B))$ is log canonical, so is

$$(T^\rho, \mathcal{G}_{T^\rho}, \mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}).$$

In particular, we have therefore

$$\mathrm{mult}_G(\mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}) \leq \epsilon(G)$$

for any prime divisor $G \subseteq T^\rho$. Since $\mathrm{Supp}(F) \cap T \neq \varnothing$ and Y is $\mathbb{Q}$-factorial, we may take G to be the strict transform of a component of $\mathrm{Supp}(F) \cap T$ such that $\mathrm{codim}_{T^\rho}(G) = 1$ and $G = \mathrm{Supp}(F_1|_{T^\rho})$ where $F_1 \subseteq \mathrm{Supp}(F)$. Let $a = \mathrm{mult}_{F_1} F$. Note that $a > 0$, $\mathrm{mult}_G(F|_{T^\rho}) = a$, and

$$\begin{aligned} \epsilon(G) &\geq \mathrm{mult}_G(\mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}) \\ &= \mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) + \mathrm{mult}_G((B_Y + F)|_{T^\rho}) \\ &\geq \mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) + \mathrm{mult}_{F_1}(B_Y + F) \\ &\geq \mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) + \epsilon(F_1) + a \\ &\geq a > 0. \end{aligned}$$

Thus, $\epsilon(G) = 1$, $\epsilon(F_1) = 0$, and

$$\mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) \leq 1 - a < 1. \tag{10}$$

Since Y is $\mathbb{Q}$-factorial klt, we have that Y has quotient singularity around codimension 2 generic points. Therefore, we may take $p\colon (Y', \mathcal{G}') \to (Y, \mathcal{G})$ to be a quasi-étale cover around η_G, with $(T')^\rho$ the normalisation of $T'\colon = p_*^{-1}T$, $G' := p_*^{-1}G$, such that Y' is smooth around G'. Then $K_{\mathcal{G}'}, T'$ and

F' are Cartier divisors around $\eta_{G'}$ where $F' = p^{-1}F$. Therefore, $G' \subseteq \text{Sing}(\mathcal{G}')$ by Lemma 4.13. Let e be the ramification index of p along G'. By (21, Lemma 3.4) (see also (16, Proposition 2.2)), we have

$$\text{mult}_{G'}\text{Diff}_{T'}(\mathcal{G}') = e\text{mult}_G(\text{Diff}_T\mathcal{G}) - (e-1)$$

As $G' \subseteq \text{Sing}(\mathcal{G}')$, we may use (15, Proposition 3.13(1)) to get $\text{mult}_{G'}\text{Diff}_{T'}(\mathcal{G}') > 0$. Since $K_{\mathcal{G}'}$ is Cartier around $\eta_{G'}$ and so is $K_{\mathcal{G}_{(T')^\rho}}$, we have that $\text{Diff}_{T'}(\mathcal{G}')$ is a Cartier divisor and $\text{mult}_{G'}\text{Diff}_{T'}(\mathcal{G}') \geq 1$ and thus $\text{mult}_G\text{Diff}_T(\mathcal{G}) \geq 1$, which contradicts the inequality 10. □

We will use the following lemma several times, which asserts that for a log Calabi-Yau foliated triple $(X, \mathcal{F}, B)$ induced by a morphism, if $\text{Nlc}(X, \mathcal{F}, B) \neq \varnothing$, it must contain a horizontal component.

Lemma 4.15 (Horizontal Principle=(19, Proposition 5.5))**.** Let $(X, \mathcal{F}, B)$ be a foliated triple such that $K_{\mathcal{F}} + B \sim_{\mathbb{Q}} 0$. Assume that $\mathcal{F}$ is induced by some morphism $g : X \to Z$. Let $\text{Nlc}(X, \mathcal{F}, B) \neq \varnothing$ be the union of all non-log canonical centres of $(X, \mathcal{F}, B)$. Then there exists a component L of $\text{Nlc}(X, \mathcal{F}, B)$ such that $g(L) = Z$.

Proof. For the sake of contradiction, assume that $\text{Nlc}(X, \mathcal{F}, B) \neq \varnothing$ and it contains no horizontal component. Let $(Y, \mathcal{G}, B_Y + E_h + F) \to (X, \mathcal{F}, B)$ be a property-$(*)$ modification of $(X, \mathcal{F}, B)$, then

$$K_{\mathcal{G}} + B_Y + E_h + F = f^*(K_{\mathcal{F}} + B) \sim_{\mathbb{Q}} 0$$

where E_h is the sum of all $\mathcal{G}$-horizontal divisors and the support of $F > 0$ is the union of non-lc places. Assume $\mathcal{G}$ is induced by $g : Y \to Z$ for some Z with $0 < \dim Z < \dim X$. According to (17, Theorem 2.1.2), we may run a $K_{\mathcal{G}} + B_Y + E_h$-MMP. Since $K_{\mathcal{G}} + B_Y + E_h \sim_{\mathbb{Q}} -F < 0$, this MMP ends with a foliated MFS, say $(\tilde{Y} \to T)/Z$ with induced foliated triple $(\tilde{Y}, \tilde{\mathcal{G}}, B_{\tilde{Y}} + \tilde{E}_h + \tilde{F})$. Notice that by definition of MFS, $-(K_{\tilde{\mathcal{G}}} + \tilde{E}_h + B_{\tilde{Y}}) \sim_{\mathbb{Q}} \tilde{F} \geq 0$ is ample over T. However, F is vertical over Z, so $\tilde{F}$ is vertical over Z as well hence also vertical over T. Therefore, $\tilde{F}$ cannot be relatively ample and we get a contradiction. □

4.2 Existence of $\mathbb{Q}$-complements

Now we are able to show the existence of $\mathbb{Q}$-complements for algebraically integrable Fano foliations.

proof of Theorem 1.5. Let $f\colon (Y, \mathcal{G}, B_Y+E) \to (X, \mathcal{F}, B)$ be a Property $(*)$ modification of $(X, \mathcal{F}, B)$ such that $\mathcal{G}$ is induced by $\pi\colon Y \to Z$ and

$$K_{\mathcal{G}} + B_Y + E = f^*(K_{\mathcal{F}} + B)$$

where $B_Y = f_*^{-1}B$ is the strict transform of B on Y and $E = \sum \epsilon(E_i)E_i$ is the sum of all non-invariant f-exceptional divisors. We fix a general closed point $z \in Z$ such that $F := \pi^{-1}(z)$ is normal and irreducible. We may assume that Y is $\mathbb{Q}$-factorial, so we may also assume that F is normal since z is general. Let $G := f(F)$ be the image. By assumption G is normal and we have the following diagram

$$\begin{array}{ccccc} F & \xhookrightarrow{\iota_F} & (Y, \mathcal{G}, B_Y + E) & \xrightarrow{\pi} & Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & \nearrow & \\ G & \xhookrightarrow{\iota_G} & (X, \mathcal{F}, B) & & \end{array}$$

Since $z \in Z$ is a general point, we may assume that z is in the smooth locus of Z. Therefore, we may choose a pair $(Z, \sum_{i=1}^{r=\dim Z} H_i)$ such that

1. $(Z, \sum_{i=1}^{r=\dim Z} H_i)$ is simple normal crossing around z;
2. $\bigcap_{i=1}^{k} H_i$ is normal around $\pi^{-1}z$ for $k = 1, 2, \ldots, r$.

By applying adjunction (Proposition 4.12) for r times, we get an lc pair $(F, \mathrm{Diff}_F(\mathcal{G}, B_Y + E))$ satisfying

$$K_F + \mathrm{Diff}_F(\mathcal{F}, B) = (K_{\mathcal{G}} + B_Y + E)|_F = (\iota_F \circ f)^*(K_{\mathcal{F}} + B).$$

Define $\mathrm{Diff}_G(\mathcal{F}, B) \colon = g_*\mathrm{Diff}_F(\mathcal{G}, B_Y + E)$ so $K_G + \mathrm{Diff}_G(\mathcal{F}, B)$ is $\mathbb{Q}$-Cartier and

$$K_G + \mathrm{Diff}_G(\mathcal{F}, B) = g_*(K_F + \mathrm{Diff}_F(\mathcal{F}, B)) = g_*(g \circ \iota_G)^*(K_{\mathcal{F}} + B) = \iota_G^*(K_{\mathcal{F}} + B)$$

is anti-ample. Thus there exists an effective divisor Δ_G on G such that

$$K_G + \mathrm{Diff}_G(\mathcal{F}, B) + \Delta_G \sim_{\mathbb{Q}} 0$$

and $(G, \mathrm{Diff}_G(\mathcal{F}, B) + \Delta_G)$ is lc. Let $m_0 \in \mathbb{N}$ such that $m_0\Delta_G \sim -m_0(K_G + \mathrm{Diff}_G(\mathcal{F}, B))$ and $L := -m(K_{\mathcal{F}} + B)$ where m is sufficiently divisible by m_0. Then

$$L|_G = (\iota_G \circ \nu)^*L = -m(K_G + \mathrm{Diff}_G(\mathcal{F}, B)) \sim m\Delta_G$$

and thus, $m\Delta_G \in |L|_G|$. Therefore, there is a non-zero section $\alpha_G \in H^0(G, \mathcal{O}_G(L|_G))$ such that $m\Delta_G = \mathrm{Zeroes}(\alpha_G)$.

We then consider the following short exact sequence on X:

$$0 \to \mathcal{O}_X(L-G) \to \mathcal{O}_X(L) \to \mathcal{O}_G(L) \to 0.$$

We recall that $L = -m(K_{\mathcal{F}} + B)$ is an ample divisor. Then by Serre vanishing theorem, we have $H^1(X, \mathcal{O}_X(L-G)) = 0$ for m sufficiently large and thus, the following morphism is surjective:

$$H^0(X, \mathcal{O}_X(L)) \twoheadrightarrow H^0(G, \mathcal{O}_G(L)).$$

Let $\alpha \in H^0(X, \mathcal{O}_X(L))$ be a lifting of α_G. Then we have an effective divisor $D := \text{Zeroes}(\alpha)$ and thus $K_{\mathcal{F}} + B + \Delta \sim_{\mathbb{Q}} 0$ where $\Delta = \frac{1}{m}D$.

Note that $K_{\mathcal{G}} + B_Y + E + f^*\Delta = f^*(K_{\mathcal{F}} + B + \Delta)$. By adjunction we have

$$\begin{aligned} K_{\mathcal{G}_F} + \text{Diff}_F(\mathcal{G}, B_Y + E + f^*\Delta) &= g^*(K_G + \text{Diff}_G(\mathcal{F}, B + \Delta)) \\ &= g^*(K_G + \text{Diff}_G(\mathcal{F}, B) + \Delta_G). \end{aligned}$$

Since $(G, \text{Diff}_G(\mathcal{F}, B) + \Delta_G)$ is log canonical, so is $(\mathcal{G}_F, \text{Diff}_F(\mathcal{G}, B_Y + E + f^*\Delta))$.

According to the inversion of adjunction (Theorem 4.14), we have

$$\text{Nlc}(\mathcal{G}, B_Y + E + f^*\Delta) \cap F = \varnothing$$

and thus $\text{Nlc}(\mathcal{G}, B_Y + E + f^*\Delta)$ cannot be horizontal as $F := \pi^{-1}(D_Z)$ is irreducible. Hence, by Proposition 4.15, $\text{Nlc}(\mathcal{G}, B_Y + E + f^*\Delta) = \varnothing$. Therefore, $(Y, \mathcal{G}, B_Y + E + f^*\Delta)$ is log canonical and so is $(X, \mathcal{F}, B + \Delta)$.

□

We now make note of a couple of immediate corollaries of the proof of the above proposition. We will not make further use of these in the present paper, but they appear to be of independent interest.

The first can be seen as log-extension of a classification result obtained in (5).

We notice that Proposition 4.15 provides us with an alternative algebraic proof of (5, Proposition 3.14) (see also (32, Proposition 10.3) for a proof using generalised foliated quadruples):

Proposition 4.16 (=(19, Proposition 5.8))**.** Let $(X, \mathcal{F}, B)$ be a Fano foliated triple with corank at least one, then $\mathcal{F}$ is not induced by a morphism.

Proof. Suppose, for the sake of contradiction, that $\mathcal{F}$ is induced by a morphism $\pi\colon X \to Z$. We will then use induction on the corank c. When $c = 1$, we have dim $Z = 1$. We fix a general point $z_0 \in Z$ and $F_0 \colon = \pi^{-1}(z_0)$.

Let $z_1 \in Z \setminus \{z_0\}$ be another general point and $F_1 \colon = \pi^{-1}(z_1)$ such that F_1 is irreducible and normal. Note that $(F_1, \mathrm{Diff}_{F_1}(\mathcal{F}, B))$ is a Fano pair, hence there exists a $\mathbb{Q}$-complement D_{z_1} such that $K_{F_1} + \mathrm{Diff}_{F_1}(\mathcal{F}, B) + D_{z_1} \sim_{\mathbb{Q}} 0$. Since $-(K_{\mathcal{F}} + B)$ is ample, there exists an $m \gg 0$ such that $L := -m(K_{\mathcal{F}} + B) - F_0$ is very ample.

Note that $mD_{z_1} \sim L|_{F_1}$. We consider the following short exact sequence on X:

$$0 \to \mathcal{O}_X(L - F_1) \to \mathcal{O}_X(L) \to \mathcal{O}_{F_1}(L) \to 0.$$

By Serre vanishing theorem, we have the induced surjective morphism for $m \gg 0$:

$$H^0(X, \mathcal{O}_X(L)) \twoheadrightarrow H^0(F_z, \mathcal{O}_{F_z}(L)).$$

Let $mD \in H^0(X, \mathcal{O}_X(L))$ be a lifting of mD_{z_1}. Then the triple $(X, \mathcal{F}, B + D + \frac{1}{m}F_0)$ has the following properties:

- $K_{\mathcal{F}} + B + D + \frac{1}{m}F_0 \sim_{\mathbb{Q}} 0$.
- $(F_1, \mathrm{Diff}_{F_1}(\mathcal{F}, B + D + \frac{1}{m}F_0)) = (F_1, \mathrm{Diff}_{F_1}(\mathcal{F}, B) + D_{z_1})$ is log canonical since $z_0 \neq z_1$.
- $(X, \mathcal{F}, B + D + \frac{1}{m}F_0)$ is obviously not log canonical since $F_0 > 0$ is a vertical divisor.

Thus, we get a contradiction with Proposition 4.15.

Now we suppose the theorem holds for the cases when corank $\leq c-1$. We fix a general Cartier divisor $H_Z > 0$ on Z and $S := \pi^* H_Z$. Then $(S, \mathcal{F}_S, \mathrm{Diff}_S(\mathcal{F}, B))$ is an lc Fano triple with corank $c - 1$ and $\mathcal{F}_S$ is a foliation induced by the morphism $\pi|_S \colon S \to H_Z$, which contradicts the induction hypothesis. $\square$

4.3 Proof of theorem 1.8

The hyperstandard set associated to a set is defined as follows:

Definition 4.17 (Hyperstandard sets)**.** Let $\mathfrak{R} \subseteq [0,1]$, then we define the hyperstandard multiplicities associated to $\mathfrak{R}$ as

$$\Phi(\mathfrak{R}) := \{1 - \frac{r}{m} | r \in \mathfrak{R}, m \in \mathbb{N}^*\}.$$

They are coefficients obtained when we apply adjunction hence will be useful when we make induction on dimension. For details, see (25) for the classical case and (15) for the foliated case.

Definition 4.18. Let $(X, \mathcal{F}, B)$ be a foliated triple. We say that $(X, \mathcal{F}, B)$ satisfies property-($\diamond$) if the closure of a *general log leaf* (see e.g. (5, Definition 3.6)) is dlt. To be precise, let $x \in X \backslash (\mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{F}))$ be a general point and F the leaf passing through x. Then $(F, \mathrm{Diff}_F(X, \mathcal{F}, B))$ is dlt.

As we will see later, the above condition allows to resolve the Fano foliation's 'twisted' locus. Following is a an example from (5, Theorem 1.3) satisfying property-($\diamond$)

Example 4.19. Let $n \in \mathbb{N}_{\geq 3}$. Let $V \subseteq \mathbb{P}^{n+1}$ be a codimension 2 linear subspace, $L \subseteq \mathbb{P}^{n+1}$ a line such that V and L are in general position. Let $Q^n \subseteq \mathbb{P}^{n+1}$ be a general hypersurface of degree 2. Let $\mathcal{F}$ be the foliation on $\mathbb{P}^{n+1}$ induced by the projection from V to L. Let $\mathcal{G}$ be the induced foliation on Q^n. Then $(Q^n, \mathcal{G})$ is a Fano foliation satisfying property-($\diamond$). In fact, the general log leaf is isomorphic to an lc pair (F, B) where $F \cong Q^{n-1}$ and $B \cong Q^{n-1} \cap V$ is smooth. Therefore, (F, B) is dlt.

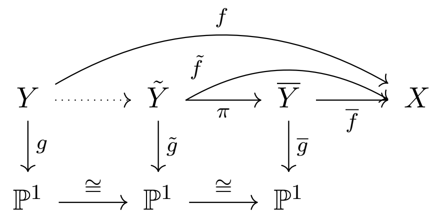
Proposition 4.20. Let $(X, \mathcal{F}, B)$ be as in Theorem 1.8. Then there exists a foliated triple $\overline{f} : (\overline{Y}, \overline{\mathcal{G}}, \overline{B} + \overline{E}_h) \to (X, \mathcal{F}, B)$ over $(X, \mathcal{F}, B)$ such that

1. $\overline{\mathcal{G}}$ is induced by a moprhism $\overline{g} : \overline{Y} \to \mathbb{P}^1$;
2. $\overline{f}$ is birational such that $K_{\overline{\mathcal{G}}} + \overline{B} + \overline{E}_h = \overline{f}^*(K_{\mathcal{F}} + B)$;
3. $(\overline{Y}, \overline{B} + \overline{E}_h + \overline{M})$ is log canonical, where $\overline{M}$ is the sum of two general $\overline{g}$-fibres;
4. $\overline{g}$ has integral fibres. In particular, $R(\overline{g}) = 0$.

Proof. Step 1. In this step we construct an intermediate model $\tilde{Y}$. Let $(Y, \mathcal{G}, B_Y + E_h)$ be a property-($*$) modification of $(X, \mathcal{F}, B)$ such that Y is induced by $g : Y \to Z$. Since $X \dashrightarrow Z$ is not a morphism by Proposition 4.16, there exists $x \in X$ such that $g(f^{-1}(x)) = Z$. Therefore, there exists an irreducible component $L \subseteq f^{-1}(x)$ such that $g(L) = Z$. According to (24, Corollary 1.6), L is rationally chain connected, so Z is unirational. Since Z is normal and dim $Z = 1$, $Z \cong \mathbb{P}^1$. Let E_v be the reduced vertical part of Exc(g). We run a $(K_{\mathcal{G}} + \{\tilde{B}\} + (1-\epsilon)(\tilde{E}_h + \lfloor \tilde{B} \rfloor))$-MMP over X and get an output $(\tilde{Y}, \tilde{\mathcal{G}}, \tilde{B} + \tilde{E}_h)$ for $0 < \epsilon \ll 1$. Let $\tilde{f}$ be the induced morphism $\tilde{f} : \tilde{Y} \to X$. Then $(\tilde{Y}, \tilde{\mathcal{G}}, \{\tilde{B}\} + (1-\epsilon)(\tilde{E}_h + \lfloor \tilde{B} \rfloor))$ satisfies property-($*$) for any $0 < \epsilon \ll 1$. In particular, $(\tilde{Y}, \tilde{\mathcal{G}}, \tilde{B} + \tilde{E}_h + \tilde{g}^*(z))$ is log canonical for any general $z \in Z$. Let $\tilde{F}_z := \tilde{g}^{-1}(z)$ for such general $z \in Z$. Since $(X, \mathcal{F}, B)$ satisfies property ($\diamond$), we have

$$(\tilde{F}_z, \mathrm{Diff}_{\tilde{F}_z}(\tilde{\mathcal{G}}, \tilde{B} + \tilde{E}_h)) \cong (F_z, \mathrm{Diff}_{F_z}(\mathcal{F}, B))$$

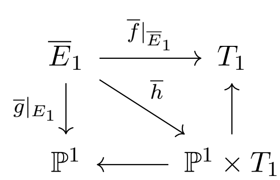
is dlt where $F_z = \tilde{f}(\tilde{F}_z)$ is the leaf on X over $z \in Z$.



Step 2. In this step we construct $\overline{Y}$. Since $-(K_{\tilde{\mathcal{G}}}+\tilde{B}+\tilde{E}_h)=-\tilde{f}^*(K_{\mathcal{F}}+B)$ is semiample and big, so is $-(K_{\tilde{\mathcal{G}}}+\tilde{B}+\tilde{E}_h)+\tilde{g}^*L_Z$ for sufficiently ample divisor L_Z. Let $\overline{Y}$ be the normalisation of the image of the morphism induced by $|-(K_{\tilde{\mathcal{G}}}+\tilde{B}+\tilde{E}_h)+\tilde{g}^*L_Z|_{\mathbb{Q}}$, then a curve $\tilde{C}$ is contracted the morphism $\pi:\tilde{Y}\to\overline{Y}$ if and only if it is contracted by both $\tilde{f}$ and $\tilde{g}$. Observe that we still have the crepant condition $K_{\overline{\mathcal{G}}}+\overline{E}_h+\overline{B}=\overline{f}^*(K_{\mathcal{F}}+B)$. Furthermore, according to the definition of $\overline{Y}$, for any $z\in Z$ and irreducible component $\overline{F}\subseteq\overline{g}^{-1}(z)$, $\overline{f}:\overline{F}^{\nu}\to F^{\nu}$ is an isomorphism, where $F:=f(F)$ is the image of $\overline{F}$. According to Step 1, Exc(π) does not contain any component of Supp$(\tilde{E}_h)$.

Step 3. In this step we prove that every $\overline{\mathcal{G}}$-invariant divisor intersects Supp$(\overline{E}_h)$. Suppose not, then there exists $z\in Z$ such that for some irreducible component $\overline{F}\subseteq\overline{g}^{-1}(z)$, we have $\overline{F}\cap\overline{E}_h=\varnothing$. Define $F:=\overline{f}(F)$. Since X is $\mathbb{Q}$-factorial, $(X,\mathcal{F},B+\epsilon F)$ is a generically log canonical Fano triple for $0<\epsilon\ll 1$. According to theorem 1.3, there exists $L\in|-(K_{\mathcal{F}}+B+\epsilon F)|$, such that $(X,\mathcal{F},B+\epsilon F+L)$ is generically log canonical. By taking a property-$(*)$ modification of $(X,\mathcal{F},B+\epsilon F+L)$, we get a contradiction with Lemma 4.15.

Step 4. In this step we prove that $\overline{g}$ has integral fibres. First we prove $R(\overline{g})=0$. The proof for irreducibility is similar. Suppose $R(\overline{g})\neq 0$ for the sake of contradiction and assume that $\overline{F}$ is a component of $\overline{g}^*(z)$ for some $z\in Z$ such that mult$_{\overline{F}}(\overline{g}^*(z))\geq 2$. According to Step 3, we can assume that $\overline{F}\cap$ Supp$\overline{E}_h\neq\varnothing$. Let $\overline{E}_1$ be an irreducible component of $\overline{E}_h$. Define $T_1:=(\overline{f}(\overline{E}_h))^{\nu}$. Consider the following diagram



Since $\overline{h}$ is by definition injective and that dim $\overline{E}_1=$ dim$(\mathbb{P}^1\times T_1)$, according to Zariski Main Theorem, $\overline{h}:\overline{E}_1\to\mathbb{P}^1\times T_1$ is an isomorphism. In particular, $\overline{g}^*(p)\cap\overline{E}_1$ is reduced. But we get a contradiction with the assumption that mult$_F(\overline{g}^*(p))\geq 2$ and $F\cap\overline{E}_1\neq\varnothing$.

Step 5. In this step we prove that $(\overline{Y}, \overline{B}+\overline{E}_h+\overline{M})$ is a log canonical pair where $\overline{M} = \overline{g}^*(z_1+z_2)$ is the pullback of two general points on Z. First, we have $-(K_{\overline{\mathcal{G}}}+\overline{B}+\overline{E}_h) = -\overline{f}^*(K_{\mathcal{F}}+B)$ is $\mathbb{Q}$-Cartier. Since $R(\overline{g}) = 0$, we have $K_{\mathcal{G}} = K_{\overline{Y}} - \overline{g}^*(K_Z)$. Therefore, $-(K_{\overline{Y}}+\overline{B}+\overline{E}_h)$ is $\mathbb{Q}$-Cartier, i.e. $(\overline{Y}, \overline{B}+\overline{E}_h)$ is a pair. Second, fix any $z \in Z$ and an irreducible component $\overline{F} \subseteq \overline{g}^{-1}(z)$. As $\mathcal{O}_{\overline{Y}}(K_{\overline{\mathcal{G}}}) \cong \omega_{\overline{Y}} \otimes \mathcal{O}_{\overline{Y}}(M)$ where $\mathcal{O}_{\overline{Y}}(M)|_{\overline{F}} = 0$, we have $(\overline{F}, \mathrm{Diff}_{\overline{F}}(\overline{Y}, \overline{B}+\overline{E}_h+\overline{M})) = (\overline{F}, \mathrm{Diff}_{\overline{F}}(\overline{\mathcal{G}}, \overline{B}+\overline{E}_h))$ is log canonical. By inversion of adjunction, $(\overline{Y}, \overline{B}+\overline{B}+\overline{M})$ is log canonical outside $\mathrm{Supp}(\overline{M})$. However, π is an isomorphism over a general point $z \in Z$, $(\tilde{Y}, \tilde{\mathcal{G}}, \{\tilde{B}\}+(1-\epsilon)(\tilde{E}_h+\lfloor\tilde{B}\rfloor))$ satisfies property-$(*)$ for any $0 < \epsilon \ll 1$, so we get the desired result.

□

Now I can give the proof of Theorem 1.8

proof of Theorem 1.8. Let $(\overline{Y}, \overline{\mathcal{G}}, \overline{B}+\overline{E}_h)$ be the model obtained in Proposition 4.20. Since $\overline{g}$ has integral fibres, we have $K_{\overline{\mathcal{G}}} = K_{\overline{Y}} - \overline{g}^*(\mathbb{P}^1) \sim_{\mathbb{Q}} K_{\overline{Y}} + \overline{M}$ where $\overline{M} := \overline{g}^*(z_1+z_2)$ is a nef divisor and z_1, z_2 are two general points on $\mathbb{P}^1$. Then

$$K_{\overline{Y}}+\overline{B}+\overline{E}_h+\overline{M} = K_{\overline{\mathcal{G}}}+\overline{B}+\overline{E}_h = \overline{f}^*(K_{\mathcal{F}}+B) = \overline{f}^*\overline{f}_*\overline{f}^*(K_{\mathcal{F}}+B)$$

$$= \overline{f}^*\overline{f}_*(K_{\overline{\mathcal{G}}}+\overline{B}+\overline{E}_h) = \overline{f}^*\overline{f}_*(K_{\overline{Y}}+\overline{B}+\overline{E}_h+\overline{M}) = \overline{f}^*(K_X+B+M)$$

Therefore, $(X, B+M)$ is an lc pair. There exists a bounded n, which only depends on $\mathfrak{R}$ and d such that there exists an n-complements $K_X+B+L+M$ where $L \in |-K_X-B-M|_{\mathbb{Q}}$.

Now we prove that L defines a foliated n-complement, i.e. $(X, \mathcal{F}, B+L)$ is foliated lc. Since $(X, B+M)$ is log canonical by definition, and components of $\overline{E}_h$ are lc places, $\overline{f}^*L = \overline{f}_*^{-1}L$. We denote $L_{\overline{Y}} := \overline{f}_*^{-1}L$, then we have

$$K_{\overline{Y}}+\overline{B}+\overline{E}_h+L_{\overline{Y}}+\overline{M} = \overline{f}^*(K_X+B+L+M)$$

and $(\overline{Y}, \overline{B}+\overline{E}_h+L_{\overline{Y}}+\overline{M})$ is lc and

$$a(\overline{Y}, \overline{B}+\overline{E}_h+L_{\overline{Y}}+\overline{M}; E) = a(\overline{Y}, \overline{\mathcal{G}}, \overline{B}+\overline{E}_h+L_{\overline{Y}}; E) \geq -1$$

for any $\overline{\mathcal{G}}$-horizontal divisor E. However, $(\overline{Y}, \overline{\mathcal{G}}, \overline{B}+\overline{E}_h+L_{\overline{Y}})$ is a log Calabi-Yau foliated triple, so according to Lemma 4.15, it is log canonical globally. So $(X, \mathcal{F}, B+L)$ is log canonical since

$$K_{\overline{\mathcal{G}}}++\overline{B}+\overline{E}_h+L_{\overline{Y}} = \overline{f}^*(K_{\mathcal{F}}+B+L).$$

□

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