In summary, Table VI presents the highest possible minimum weight d_n for an optimal self-dual code of length $n \le 40$, along with the reference for the first known code with this weight. No entry in the last column indicates the first code appears in this correspondence.

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An Improved Upper Bound on the Minimum Distance of Doubly-Even Self-Dual Codes

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***Abstract*—We derive a new upper bound on the minimum distance d of doubly-even self-dual codes of length n. Asymptotically, for n growing, it gives $\lim_{n\to\infty} \sup d/n \le (5-5^{3/4})/10 < 0.165630$, thus improving on the Mallows–Odlyzko–Sloane bound of $1/6$ and our recent bound of 0.166315.**

***Index Terms*—Distance distribution, self-dual codes, upper bounds.**

I. Introduction

Self-dual codes attract a great deal of attention, mainly due to their intimate connections with important problems in algebra, combinatorics, and number theory (see many references in [2], [3], [11], [14], and [16]).

A binary self-dual linear code C of length n and minimum distance d is doubly-even if all its weights are divisible by 4. By a result of Gleason (see e.g., [11, Sec. 19.2]) such codes exist only for n divisible by 8 (for a proof not based on invariant theory see [8]). Let d_n be the maximum distance of a doubly-even self-dual (DESD) code of length n. Here we consider the asymptotic problem of determining

$$\overline{\delta} = \lim_{n\to\infty} \sup \frac{d_n}{n}.$$

By a result of Thompson [17] there exist DESD codes satisfying the Gilbert–Varshamov bound, i.e., $\overline{\delta} \ge H^{-1}(1/2) = 0.110\cdots$. It is generally believed that this bound gives the true value. The Mallows–Odlyzko–Sloane bound [12], [13] yields $\overline{\delta} \le 1/6$. This estimate essentially exploits invariant theory. Recently, using a variant of linear programming approach, we improved it to $\overline{\delta} < 0.166315$.

For unrestricted self-dual codes the best known upper bound is due to Ward [18] and also equals $1/6$ (see also Conway and Sloane [3] and Rains [15] for better bounds for finite lengths).

Our main result here is the following theorem.

Theorem 1:

$$\overline{\delta} \le (5-5^{3/4})/10 < 0.165630. \qquad \square$$

To prove it we use a modification of the linear programming method for upper-bounding individual components of the distance distribution of the DESD codes. We show that under some assumptions about the minimum distance of the code, its distance distribution is upper-bounded by the normalized binomial distribution. This phenomenon for arbitrary codes was discussed in, e.g., [6], [7]. Furthermore, since the upper binomial bound is actually attained at any interval of size $o(n)$, it proves an existence of nonzero component of the distance distribution in the interval of binomiality.

Note, that estimating the range of binomiality requires analysis of properties of certain polynomials and their expansions in the basis of Krawtchouk polynomials. In the previous paper [8] we were able to compute only the zeroth coefficient of the expansion. Here we develop alternative techniques which allow computing the total spectrum, thus yielding a better estimate for the range of binomiality. Since now we possess a complete knowledge about the coefficients of the expansion, new ideas are necessary to achieve further improvements.

II. Basic Relations

We need some notations. In what follows, all logarithms are natural, and *the logarithm of a negative number is understood as its real part* (by this convention we avoid writing the absolute values of the expressions under logarithms). As usual

$$H(x) = -x \ln x - (1-x) \ln(1-x)$$

stands for the natural entropy function.

Let C be a DESD code of the minimum distance d, the relative distance $\delta = d/n$, and let $(B_0, B_1, \cdots, B_n)$ stand for its distance distribution. Clearly, $B_0 = 1$, $B_1 = \cdots = B_{d-1} = 0$, and $B_j = 0$ whenever j is not a multiple of 4. Moreover, the distance

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distribution is symmetric with respect to $n/2$, i.e., $B_j = B_{n-j}$ and $\sum_{i=0}^{n} B_i = |C| = 2^{n/2}$. The distance distribution is invariant under the MacWilliams transform

$$|C|B_i = \sum_{j=0}^{n} B_j P_i(j) \tag{1}$$

where P_i is the corresponding Krawtchouk polynomial of degree i

$$P_i(x) = \sum_{k=0}^{i} (-1)^k \binom{x}{k}\binom{n-x}{i-k} = \frac{(-2)^i}{i!}x^i + \cdots \tag{2}$$

(for properties of Krawtchouk polynomials see, e.g., [1], [5], [9]–[11]).

Let $f(x)$ be a polynomial

$$f(x) = \sum_{i=0}^{n} A_i P_i(x)$$

then (see, e.g., [10])

$$A_i = A_i(f) = 2^{-n} \sum_{j=0}^{n} f(j) P_j(i) \tag{3}$$

in particular

$$A_0(f) = 2^{-n} \sum_{j=0}^{n} f(j) \binom{n}{j}.$$

The following lemma of Delsarte [4] is the core of the linear programming approach.

Lemma 1: Let $f(x)$ be a polynomial of degree r

$$f(x) = \sum_{i=0}^{r} A_i P_i(x), \qquad 0 \le r \le n,$$

then

$$|C|A_0 + |C| \sum_{i=d}^{r} A_i B_i = f(0) + f(n) + \sum_{j=d}^{n-d} f(j) B_j. \tag{4}$$

Proof: Calculating $|C| \sum_{i=0}^{r} A_i B_i$, we get the claim from (1). □

We use this lemma with the polynomials

$$\mathcal{B}_h^n(x, k) = \frac{1}{2^{2k}(2k)!} \prod_{i=0}^{k-1} ((n-2x)^2 - h^2 i^2). \tag{5}$$

These polynomials are even in respect to $n/2$. The zeros of $\mathcal{B}_h^n(x, k)$ are $n/2 \pm hi/2$, $i = 0, \cdots, k-1$. Notice that $n/2$ is a root of multiplicity 2.

Let us sketch the proof of the main result. Let

$$\mathcal{B}_h^n(x, k) = \sum_{j=0}^{2k} A_j(n, h, k) P_j^n(x)$$

be the Krawtchouk expansion of the defined polynomials. Then, choosing $h = 8$, $d \le n/2 - 4k$, by the above lemma we can rewrite (4) as

$$\begin{aligned} 2^{n/2} A_0(n, 8, k) &- 2\mathcal{B}_8^n(0, k) \\ &= \sum_{j=d,\, 4|j}^{2k} \left(2\mathcal{B}_8^n(j, k) - 2^{n/2} A_j(n, 8, k)\right) B_j \\ &\quad + 2 \sum_{j=2k+2,\, 4|j}^{n/2-4k} \mathcal{B}_8^n(j, k) B_j. \end{aligned} \tag{6}$$

In Section IV we show that asymptotically when n grows and $k/n < 1/12$

$$\mathcal{B}_8^n(0, k) = o(2^{n/2} A_0(n, 8, k))$$

and for $j/n > (5 - 5^{3/4})/10$

$$2^{n/2} A_j(n, 8, k) = o(\mathcal{B}_8^n(j, k)).$$

This yields

$$2^{n/2} A_0(n, 8, k) = 2(1 - o(1)) \cdot \sum_{j=d,\, 4|j}^{n/2-4k} \mathcal{B}_8^n(j, k) B_j. \tag{7}$$

Therefore, since $\mathcal{B}_8^n(j, k)$ for odd k's is positive in the interval of summation, we obtain that for every j in the interval $[d, n/2 - 4k]$

$$B_j < 2^{n/2-1} \frac{A_0(n, 8, k)}{\mathcal{B}_8^n(j, k)}.$$

Choosing k as an appropriate function of j, we conclude that B_j is, up to a factor polynomial in n, upper-bounded by $2^{-n/2}\binom{n}{j}$. Letting k tend to $n/12$ and plugging the bounds for B_j into (7) we get that there should exist a nonzero (in fact, binomial) component in the distance distribution in the interval

$$\left((5 - 5^{3/4})n/10,\ (5 - 5^{3/4})n/10 + o(n)\right)$$

thus proving the claim.

We used the MATHEMATICA package for analytical calculations.

III. Polynomials

Our goal here is to find the Krawtchouk expansion

$$\mathcal{B}_h^n(x, k) = \sum_{j=0}^{2k} A_j(n, h, k) P_j^n(x)$$

and evaluate asymptotics of its coefficients for $h = 8$. In [6] we have found such expressions for $h = 2$ and 4. For $h = 8$ we were only able in [8] to calculate $A_0(n, 8, k)$. Here we further develop our techniques to find $A_j(n, 8, k)$ for every j (actually, only j being multiples of 4 are of interest). Notice, that for odd j the coefficients vanish due to the symmetry of the polynomial in respect to $n/2$.

The starting point is the following lemma [8, Lemma 4].

Lemma 2:

$$\left.\frac{d^k \cos(t \arccos z)}{dz^k}\right|_{z=1} = \frac{1}{(2k-1)!!} \prod_{i=0}^{k-1} (t^2 - i^2).$$ □

Lemma 3: For j even

$$A_j(n,\, h,\, k) = \frac{(-1)^{j/2} h^{2k}}{2^{3k} k!} \left(\frac{d^k}{dz^k} \sin^j \Psi_h \cos^{n-j} \Psi_h \right)\bigg|_{z=1}$$

where

$$\Psi_h = \frac{\arccos z}{h}.$$

Proof: Using the previous lemma we obtain

$$\begin{aligned}\prod_{i=0}^{k-1} &\left((n-2x)^2 - h^2 i^2\right) \\ &= h^{2k} \prod_{i=0}^{k-1} \left(\left(\frac{n-2x}{h} \right)^2 - i^2 \right) \\ &= (2k-1)!! h^{2k} \frac{d^k}{dz^k} \cos((n-2x)\Psi_h)\bigg|_{z=1}.\end{aligned}$$

Now, by (3)

$$\begin{aligned}A_j(n,\, h,\, k) &= \frac{1}{2^n} \sum_{x=0}^{n} \mathcal{B}_h^n(x,\, k) P_x(j) \\ &\quad - \frac{(2k-1)!! h^{2k}}{2^{n+2k}(2k)!} \sum_{x=0}^{n} P_x(j) \frac{d^k}{dz^k} \cos((n-2x)\Psi_h)|_{z=1} \\ &= \frac{h^{2k}}{2^{n+3k} k!} \frac{d^k}{dz^k} \left(\sum_{x=0}^{n} P_x(j) \cos((n-2x)\Psi_h) \right)\bigg|_{z=1}.\end{aligned} \tag{8}$$

Furthermore, since

$$\sum_{x=0}^{n} P_x(j) u^x = (1-u)^x (1+u)^{n-x}$$

we have, with notation $\iota = \sqrt{-1}$

$$\begin{aligned}\sum_{x=0}^{n} &P_x(j) \cos((n-2x)\Psi_h) \\ &= \frac{1}{2} \sum_{x=0}^{n} P_x(j) (\exp(\iota(n-2x)\Psi_h) \\ &\quad + \exp(-\iota(n-2x)\Psi_h)) \\ &= \frac{1}{2} \exp(\iota n \Psi_h)(1 - \exp(-2\iota\Psi_h))^j \\ &\quad \cdot (1 + \exp(-2\iota\Psi_h))^{n-j} \\ &\quad + \frac{1}{2} \exp(-\iota n \Psi_h)(1 - \exp(2\iota\Psi_h))^j \\ &\quad \cdot (1 + \exp(2\iota\Psi_h))^{n-j} \\ &= \frac{1}{2} \exp(\iota n \Psi_h)(-2\iota)^j \sin^j \Psi_h \cdot \exp(-\iota j \Psi_h) \\ &\quad \cdot 2^{n-j} \cos^{n-j} \Psi_h \cdot \exp(-\iota(n-j)\Psi_h) \\ &\quad + \frac{1}{2} \exp(-\iota n \Psi_h)(2\iota)^j \sin^j \Psi_h \cdot \exp(\iota j \Psi_h) \\ &\quad \cdot 2^{n-j} \cos^{n-j} \Psi_h \cdot \exp(\iota(n-j)\Psi_h) \\ &= 2^n \iota^j \sin^j \Psi_h \cdot \cos^{n-j} \Psi_h,\end{aligned}$$

where at the last step the following identities have been used:

$$\begin{aligned}1 - \exp(2\iota\varphi) &= 2\iota \sin\varphi \cdot \exp(\iota\varphi) \\ 1 + \exp(2\iota\varphi) &= 2 \cos\varphi \cdot \exp(\iota\varphi).\end{aligned}$$

Plugging it into(8) we obtain the claim. □

To derive the explicit formulas for the coefficients $A_j(n,\, 8,\, k)$ we exploit the following values of $A_j(n,\, 4,\, k)$ obtained in [8, Lemma 3].

Lemma 4: For j even

$$A_j(n,\, 4,\, k) = \frac{n-2j}{2^{2k}(n-2k-j)} \binom{n/2-k-j/2}{k-j/2}.$$

□

Comparing the two previous lemmas we get

Corollary 1:

$$\begin{aligned}\frac{d^k}{dz^k} &\sin^j \Psi_4 \cos^{n-j} \Psi_4 \bigg|_{z=1} \\ &= \frac{(-1)^{j/2} k!(n-2j)}{2^{3k}(n-2k-j)} \binom{n/2-k-j/2}{k-j/2}.\end{aligned}$$

□

Theorem 2: For even j, $j \le 2k-2$,

$$\begin{aligned}A_j(n,\, 8,\, k) &= \frac{1}{(2k-j)2^{n/2}} \sum_{i=0}^{n/2-j} (i-j) \\ &\quad \cdot \binom{n/2-j}{i} \binom{i/2-k-1}{k-j/2-1} \\ A_{2k}(n,\, 8,\, k) &= 2^{-2k}.\end{aligned}$$

Proof: The expression for $A_{2k}(n,\, 8,\, k)$ follows from(2) by comparing the leading coefficients. Now

$$\begin{aligned}\sin^j \Psi_8 \cos^{n-j} \Psi_8 &= \frac{1}{2^j} \sin^j \Psi_4 \cos^{n-2j} \Psi_8 \\ &= \frac{1}{2^{n/2}} \sin^j \Psi_4 (1 + \cos \Psi_4)^{n/2-j} \\ &= \frac{1}{2^{n/2}} \sum_{i=0}^{n/2-j} \binom{n/2-j}{i} \sin^j \Psi_4 \cos^i \Psi_4.\end{aligned}$$

Furthermore, by Corollary 1

$$\begin{aligned}\frac{d^k}{dz^k} &\sin^j \Psi_8 \cos^{n-j} \Psi_8 \bigg|_{z=1} \\ &= \frac{(-1)^{j/2}}{2^{n/2}} \sum_{i=0}^{n/2-j} \binom{n/2-j}{i} \frac{k!(i-j)}{2^{3k}(i-2k)} \binom{i/2-k}{k-j/2}\end{aligned}$$

and the result follows by Lemma 3. □

Let us consider now the situation of growing k and n, and estimate the asymptotics of $A_j(n,\, 8,\, k)$ and $\mathcal{B}_8^n(x,\, k)$.

Theorem 3: For $\xi < 1/2 - 4\kappa$,

$$\begin{aligned}\frac{1}{n} \ln \mathcal{B}_8^n(\xi n,\, \kappa n) &= (\kappa + 1/8 - \xi/4) \ln(1 + 8\kappa - 2\xi) \\ &\quad + (\kappa - 1/8 + \xi/4) \ln(1 - 8\kappa - 2\xi) \\ &\quad - 2\kappa \ln(4\kappa) + o(1).\end{aligned}$$

Proof: Notice that

$$\mathcal{B}_8^n(x,\, k) = n 2^{4k-3} \binom{n/8 - x/4 + k - 1}{2k}.$$

The Stirling approximation accomplishes the proof. □

For our purposes we will need only odd k. Indeed, only in this case $\mathcal{B}_8^n(x,\, k) \ge 0$ for all x divisible by 4. So, from now on we assume k to be **odd**.

Denote, for n divisible by 8, j divisible by 4 and odd k,

$$S_j(i) = \binom{n/2-j}{i} \binom{i/2-k-1}{k-j/2-1}.$$

The following notation is used in the sequel, $k = \kappa n$, $j = \xi n$, $i = \eta n$, $y = 1 - 2\xi$, and $\rho = \sqrt{y^2 - 16\kappa + 128\kappa^2}$.

Lemma 5: Let $\kappa < \sqrt{2}/12$, then for sufficiently large n the function $|S_{\xi n}(\eta n)|$ has two local maxima in η, one at

$$\eta_1 = \frac{y(8\kappa + y - \rho)}{4(1 - 4\kappa + y)}$$

and another at

$$\eta_2 = \frac{y(8\kappa + y + \rho)}{4(1 - 4\kappa + y)},$$

where $y = 1 - 2\xi$.

The first maximum is the absolute maximum for $\kappa > 1/12$, otherwise, the second maximum is the absolute one. For $\kappa = 1/12$ they are asymptotically equal.

Proof: Let

$$\sigma_\xi(\eta) = \lim_{n \to \infty} \frac{1}{n} \ln S_{\xi n}(\eta n).$$

Observe that $S_j(i)$ may be negative only for odd i's and $2k + 3 \le i \le 4k - j - 3$, that is, $2\kappa \le \eta \le 4\kappa - \xi$.

Using Stirling approximation and the convention that logarithm is understood as its real part, we obtain

$$\begin{aligned}\sigma_\xi(\eta) = &-\eta \ln \eta + (\eta/2 - \kappa) \ln(\eta/2 - \kappa)\\ &+ (1/2 - \xi) \ln(1/2 - \xi) - (1/2 - \eta - \xi)\\ &\cdot \ln(1/2 - \eta - \xi) - (\kappa - \xi/2) \ln(\kappa - \xi/2)\\ &+ (2\kappa - \eta/2 - \xi/2) \ln(2\kappa - \eta/2 - \xi/2).\end{aligned}$$

Now

$$\frac{d\sigma_\xi(\eta)}{d\eta} = \frac{1}{2} \ln \frac{(\eta - 2\kappa)(1 - 2\eta - 2\xi)^2}{4\eta^2(4\kappa - \eta - \xi)}. \tag{9}$$

To find extrema we equate the square of the expression under the logarithm to 1, which gives after substitution $y = 1 - 2\xi$,

$$\begin{aligned}&\left(8\eta^3 + \eta^2(2 - 24\kappa - 6y) + \eta(8\kappa y + y^2) - 2\kappa y^2\right)\\ &\cdot \left(\eta^2(8\kappa - 2y - 2) + \eta(8\kappa y + y^2) - 2\kappa y^2\right) = 0.\end{aligned}$$

The roots of the second factor are exactly η_1 and η_2 from the claim. The first factor has only one real root corresponding to the minimum. Indeed, since always $2\kappa \ge \xi$, it can be directly verified that the derivative of the first factor in η is positive for $\kappa < \sqrt{2}/12$. Moreover, the only real root of the first factor belongs to the interval $[2\kappa, 4\kappa - \xi]$, since the first factor is negative at the left end of the interval and is positive at the right one. Now, it is easy to check that $\eta_1 < 2\kappa$ and $\eta_2 > 4\kappa - \xi$. Computing the second derivatives at η_1 and η_2 we convince that there are two maxima. Indeed, the second derivative of $\sigma_\xi(\eta)$ in η is

$$\frac{2\eta^2(1 - 4\kappa + y) + \eta y(1 - 20\kappa - y) - 4\kappa y(1 - 8\kappa + y)}{2\eta(\eta - 2\kappa)(y - 2\eta)(1 + 2\eta - 8\kappa - y)}.$$

Substituting $\eta = \eta_1$ and $\eta = \eta_2$ we get

$$\frac{16\rho(\varepsilon - 12\varepsilon\kappa - \rho)(1 - 4\kappa + y)^3}{y(4\kappa + y - 1)(2 - 16\kappa - \varepsilon\rho + y)^2(8\kappa + \varepsilon\rho + y)^2}$$

where $\varepsilon = 1$ corresponds to η_1, and $\varepsilon = -1$ corresponds to η_2. Since $y = 1 - 2\xi \ge 1 - 4\kappa$, then $4\kappa + y - 1 \ge 0$. Therefore, the sign of the second derivative coincides with the sign of

$$\varepsilon - 12\varepsilon\kappa - \rho.$$

Checking that

$$\rho^2 - (1 - 12\kappa)^2 = y^2 - (1 - 4\kappa)^2 \ge 0$$

we conclude that in both cases it is negative.

The only thing left to be proved is that the absolute maximum is at η_2 for $\kappa < 1/12$ and at η_1 otherwise. Consider the function $\nu = \sigma_\xi(\eta_2) - \sigma_\xi(\eta_1)$. For $\kappa = 1/12$ we compute $\nu = 0$.

Differentiating in y we get

$$\frac{d\nu}{dy} = \frac{1}{4} \ln \frac{(1 - 12\kappa + \rho)^2}{(1 - 4\kappa + y)(-1 + 4\kappa + y)}.$$

The condition for ν to be increasing in y is that

$$\frac{2(1 - 12\kappa)(1 - 12\kappa + \rho)}{(1 - 4\kappa + y)(-1 + 4\kappa + y)} < 0.$$

Since all the terms but $(1 - 12\kappa)$ are positive we conclude that ν increases in y for $\kappa > 1/12$ and decreases otherwise. Therefore, it attains the minimum value at $y = 1 - 4\kappa$ for $\kappa < 1/12$ and the maximum value for $\kappa > 1/12$.

Consider the case $\kappa < 1/12$. For $y = 1 - 4\kappa$ we have the equation at the bottom of this page. This function decreases in κ and equals 0 at $\kappa = 1/12$. So, for $\kappa < 1/12$, $\nu < 0$. For $\kappa > 1/12$, the proof is similar. □

Remark: Actually, the constraint on κ, $\kappa < \sqrt{2}/12$ in the claim can be omitted. We used it to simplify the proof.

Theorem 4: Let $0 < \kappa < \sqrt{2}/12$. Then

$$\begin{aligned}&\frac{1}{n} \ln A_{\xi n}(n, 8, \kappa n)\\ &\quad= \frac{y}{4} \ln \frac{1 - 12\kappa + \varepsilon\rho}{-1 + 4\kappa + y}\\ &\qquad- \frac{1}{4} \ln \frac{(2 - 16\kappa + \rho - \varepsilon y)(2 - 16\kappa - \rho + \varepsilon y)}{16(1 - 4\kappa + y)(-1 + 4\kappa + y)}\\ &\qquad+ \kappa \ln \frac{(\rho - \varepsilon y)^2(2 - 16\kappa + \rho - \varepsilon y)(2 - 16\kappa - \rho + \varepsilon y)}{256\kappa^2(1 - 4\kappa + y)(-1 + 4\kappa + y)}\\ &\qquad+ o(1)\end{aligned}$$

where $\varepsilon = 1$ for $0 < \kappa \le 1/12$ and $\varepsilon = -1$ otherwise. Moreover, for sufficiently large n, $8|n$, odd k, and $4|j$, $A_j(n, 8, k)$ is positive for $\kappa < 1/12$. For $\kappa > 1/12$ its sign coincides with the sign of $(\eta_1 - \xi)$.

Proof: As we mentioned, $S_j(i)$ can be negative only for odd i's and $2\kappa \le \eta \le 4\kappa - \xi$. Since the points of maxima, η_1 and η_2 do not belong to this interval, then, by Theorem 2, for sufficiently large n, the value of $A_j(n, 8, k)$ is determined by the maximum of $S_j(i)$ and the sign of $(i - j)$ for the optimal i. Notice that

$$\eta_2 - \xi = \frac{-2 + 8\kappa + \rho y + 3y^2}{4(1 - 4\kappa + y)}$$

is increasing in y, so its minimum is attained at $y = 1 - 4\kappa$ and equals $(1 - 12\kappa)/4 > 0$. So, for $\kappa < 1/12$, we have $\eta_2 > \xi$ and the value of $S_j(i)$ corresponding to η_2 is positive. The result now follows from the previous lemma. □

IV. Proof of the Main Result

Lemma 6: For $\kappa < 1/12$

$$\mathcal{B}_8^n(0, k) = o(2^{n/2} A_0(n, 8, k)), \tag{10}$$

and for

$$\begin{aligned}(5 - 5^{3/4})/10 \le \xi \le 1/2 - 4\kappa\\ (5 - 5^{3/4})/20 \le \kappa < 1/12\\ 2^{n/2} A_j(n, 8, k) = o(\mathcal{B}_8^n(j, k)).\end{aligned} \tag{11}$$

$$\nu = \frac{(1 + 4\kappa)\ln 2 + (1 - 8\kappa)\ln(1 - 8\kappa) - (1 - 4\kappa)\ln(1 - 4\kappa) + 4\kappa \ln \kappa}{2}.$$

Proof: Denote

$$\beta_\xi(\kappa) = \lim_{n\to\infty} \frac{1}{n} \ln \mathcal{B}_8^n(\xi n, \kappa n)$$
$$\alpha_\xi(\kappa) = \lim_{n\to\infty} \frac{1}{n} \ln A_{\xi n}(n, 8, \kappa n)$$
$$\Delta_\xi(\kappa) = \beta_\xi(\kappa) - \alpha_\xi(\kappa) - \ln 2/2.$$

Using results of the previous section we obtain $\Delta_\xi(1/12) = 0$.

We start from proving that $\Delta_0(\kappa) < 0$ for $\kappa < 1/12$. Denote

$$\rho_0 = \sqrt{1 - 16\kappa + 128\kappa^2}.$$

We get

$$\frac{d\Delta_0(\kappa)}{d\kappa} \ln \frac{(1+8\kappa)(1+\rho_0)^2(3-16\kappa+\rho_0)(-1+16\kappa+\rho_0)}{256(1-8\kappa)^3\kappa^2}.$$

As it is easy to check the expression under the logarithm is greater than 1, so the maximum, $\Delta_0(\kappa) = 0$, is attained at $\kappa = 1/12$, thus proving (10).

Let us prove (11). For the range of ξ and κ defined in the claim we have $1 - 4\kappa \le y \le 5^{-1/4}$.

Computing derivative in y

$$\frac{d\Delta_\xi(\kappa)}{dy} = -\frac{1}{8} \ln \frac{(1-12\kappa+\rho)^2(y-8\kappa)}{(4\kappa+y-1)^2(y+8\kappa)}$$

we conclude that its sign is determined by the sign of

$$8\kappa - 96\kappa^2 + 8\kappa\rho + 8\kappa y - \rho y - y^2.$$

Direct checking confirms that the last expression is negative, i.e., $\Delta_\xi(\kappa)$ decreases in y. Therefore, $\Delta_\xi(\kappa)$ attains its minimum at the minimal value of ξ, that is, for $y = 5^{-1/4}$.

Now, we differentiate in κ getting

$$\frac{d\Delta_\xi(\kappa)}{d\kappa} = \ln \frac{16(y-8\kappa)(1-4\kappa+y)(-1+4\kappa+y)(8\kappa+y)}{(\rho-y)^2(2-16\kappa+\rho-y)(2-16\kappa-\rho+y)}.$$

One can check that for $y = 5^{-1/4}$ this derivative is negative for κ in the considered interval, and thus the minimum of $\Delta_\xi(\kappa)$ is attained for $y = 5^{-1/4}$ and $\kappa = 1/12$. This minimum equals 0, therefore, for $\kappa < 1/12$, the function $\Delta_\xi(\kappa)$ is positive. □

Theorem 5: If there exists a DESD code C with $\delta = d/n > (5 - 5^{3/4})/10$ then

$$\frac{1}{n} \ln B_j < (1 + o(1)) \ln \frac{\binom{n}{j}}{2^{n/2}}.$$

Proof: Let $\kappa < 1/12$. Substitute the polynomial $\mathcal{B}_8^n(x, k)$ into (4). From the previous lemma we have

$$2^{n/2} A_0(n, 8, k) = (1 - o(1)) \sum_{j=d}^{n-d} \mathcal{B}_8^n(j, k) B_j$$
$$= 2(1 - o(1)) \sum_{j=d}^{n/2-4k} \mathcal{B}_8^n(j, k) B_j.$$

Recall, that since k is chosen to be odd, all summands in the right-hand side are nonnegative. Therefore, for $j \in [d, n/2 - 4k]$

$$B_j \le (1 + o(1)) \frac{2^{n/2}}{2\mathcal{B}_8^n(j,k)}$$

or

$$\frac{1}{n} \ln B_{\xi n} \le \frac{\ln 2}{2} - \beta_\xi(\kappa) + o(1).$$

Given $j = \xi n$ choose

$$\kappa = \frac{(1-2\xi)^2(\xi^2 + (1-\xi)^2)}{8(\xi^4 + (1-\xi)^4)} = \frac{y^2(1+y^2)}{2(1+6y^2+y^4)}.$$

Here

$$8\kappa \le y \le 1 - 2\delta$$

that is,

$$8\kappa \le y \le 5^{-1/4}.$$

Direct verification shows that for chosen κ it always holds.

By Theorem 4 we have by straightforward calculations that

$$\alpha_0(\kappa) - \beta_\xi(\kappa) + \ln 2/2 = H(\xi) - \frac{\ln 2}{2}. \qquad \square$$

Proof of Theorem 1: Assume the contrary, namely, that there exists a code with $\delta > (5 - 5^{3/4})/10$. Then, by the previous theorem, and (4) we have

$$\alpha_0(\kappa) + \frac{\ln 2}{2} = \frac{1}{n} \ln \left(\sum_{j=d}^{n-d} \mathcal{B}_8^n(j, k) \frac{\binom{n}{j}}{2^{n/2}} \right) + o(1)$$
$$= \max_{\xi \in [\delta, 1/2]} (\beta_\xi(\kappa) + H(\xi) - \ln 2/2) + o(1).$$

Notice, that the maximum of the right-hand side is attained at $\xi = \delta$. Indeed, substituting $y = 1 - 2\xi$, we have that the derivative of the right-hand side in y is

$$\frac{1}{8} \ln \frac{(1-y)^4(y+8\kappa)}{(1+y)^4(y-8\kappa)}.$$

It is easy to check that the derivative tends to 0 for $y \to 5^{-1/4}$ and $\kappa \to 1/12$. Hence, at this point the maximum is attained. It equals $(5 \ln 5)/24$. So, for $\delta > (5 - 5^{3/4})/10$ and κ sufficiently close to $1/12$ we have

$$\alpha_0(\kappa) + (\ln 2)/2 < \frac{5 \ln 5}{24}.$$

However, $\alpha_0(1/12) + (\ln 2)/2 = (5 \ln 5)/24$, a contradiction. □

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