Quasi-Consensus Control of Delayed Multi-Agent Systems With Stochastic Communication Protocols and Amplify-and-Forward Relays

Jie Ban, Liangdong Guo, Zidong Wang, Hongbin Cai, and Bing Li

***Abstract*—In this paper, the observer-based quasi-consensus control problem is investigated for a class of discrete-time multi-agent systems subject to time-varying delay. To improve communication quality and extend transmission distance, a stochastic communication protocol and an amplify-and-forward relay mechanism are incorporated. During the process of signal amplification and transmission in the AaF relay, stochastic packet loss is considered, which introduces additional complexity into the system analysis. The main objective is to design a distributed observer-based control strategy capable of handling time-varying delays, stochastic scheduling governed by a Markov chain, and random packet dropouts. Sufficient conditions are derived to guarantee the achievement of quasi-consensus in probability among the agents. These conditions are formulated in terms of matrix inequalities through which the required gain matrices are computed. A simulation example is provided to validate the effectiveness and robustness of the proposed control approach under realistic communication constraints.**

***Index Terms*—Multi-agent systems; time-varying delay; stochastic communication protocol; amplify-and-forward relay; packet losses**

Abbreviations and Notations

$\mathbb{R}^{n\times m}$	The set of all $n \times m$ real matrices
$\mathbb{R}^{n}$	The set of all $n-$dimensional real vector
$\mathbb{Z}^{+}$	The positive integers
I_n	The $n-$dimensional identity matrix
$*$	The symmetric part in the symmetric matrix
$\otimes$	The Kronecker product
$\lambda_{\max}\{G\}$	The maximum eigenvalue of the matrix G
Q^T	The transposition of the matrix Q
$\mathrm{diag}\{\cdots\}$	The block-diagonal matrix
$\mathrm{col}\{\cdots\}$	The column vector
$\mathbb{E}\{\cdot\}$	The expectation operator
$\|x\|$	The Euclidean norm of x
$\|y\|_\infty$	The infinity norm of y
$\vartheta_1 \circ \vartheta_2$	The composition of functions ϑ_1 and ϑ_2
$\|z\|$	The absolute value of z

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I. Introduction

In recent years, with the rapid advancement of intelligent technologies and their integration into modern life, multi-agent systems (MASs) (e.g. robots, unmanned aerial vehicles, and autonomous driving platforms) have been increasingly recognized by the public and extensively adopted in industrial automation, intelligent transportation, environmental monitoring, smart healthcare, disaster response, precision agriculture, and various other real-world applications that demand distributed coordination, autonomous decision-making, and robust system-level cooperation [1]–[4]. These widespread applications are underpinned by significant research advances in consensus control, which has emerged as a central topic in the study of MASs. In recent years, a wide range of theoretical results and practical algorithms have been developed to address various consensus-related challenges [5]–[9].

It is widely acknowledged that time delays are inherent in practical engineering applications arising from factors such as network transmission, information processing, and actuator response. These delays not only impair overall system performance but can also induce instability, thereby presenting significant challenges for real-world implementation [10]–[12]. In response to these issues, considerable research efforts have been devoted to the analysis and control of MASs affected by delays [13]–[15]. Nevertheless, studies specifically addressing MASs with time-varying delays remain relatively limited. As such, further investigation into time-varying delay effects not only enriches the existing theoretical framework but also offers valuable insights for practical system design and implementation in dynamic and uncertain environments.

During signal transmission, data conveyed through shared communication channels is often subject to bandwidth limitations, which may result in data collisions and degraded performance [16]–[19]. To address this issue, appropriate scheduling of data transmission at each time instant is essential. In this context, communication protocols play a critical role in managing access to the communication medium [20]–[22]. In recent years, extensive research has been carried out to examine the influence of communication protocols on the performance of networked control systems [23]–[28]. These existing studies demonstrate that communication protocols can significantly enhance communication quality, mitigate data collisions, and improve the overall performance and reliability of control systems.

Random packet dropout is a frequent and unavoidable phenomenon in communication networks, often caused by bandwidth limitations and network congestion [29]–[31]. As a result, investigating the robustness and control of MASs under random packet dropout has become an important area of study. In recent years, despite substantial progress in this field [32]–[34], research specifically targeting the impact of random packet losses on MAS performance remains relatively limited, and this gap highlights the need for further investigation and forms one of the key motivations for the present study.

Apart from the adverse effects of bandwidth limitations, restricted transmission capacity can significantly limit the communication range in networked systems. To overcome this constraint, relay-based communication protocols have been progressively developed, including the Amplify-and-Forward (AaF) protocol [35], [36], Decode-and-Forward protocol [37], [38], and Filter-and-Forward protocol [39], [40]. The introduction of relay nodes is motivated by practical communication constraints in networked multi-agent systems, where direct long-distance transmission may be unreliable because of limited communication range, signal attenuation, bandwidth constraints, or network congestion. Such situations commonly arise in applications such as unmanned aerial vehicle formations, mobile robot coordination, wireless sensor networks, intelligent transportation systems, and offshore or remote monitoring networks. In these scenarios, relay nodes can assist signal forwarding and maintain effective information exchange among distributed agents [41], [42]. It is noteworthy that, to date, the application of relay-based protocols in MASs has not been reported in the existing literature. The AaF mechanism, while retaining the inherent linear amplification characteristic of AaF relays, offers a tractable framework for quasi-consensus analysis. Therefore, a key objective of this study is to explore and integrate relay communication strategies within MASs.

Building upon the insights gained from existing studies, this paper investigates the quasi-consensus control problem for MASs with time-varying delays under the combined influence of the SCP and the AaF relay protocol. The main challenges addressed in this work are summarized as follows: 1) how to accurately characterize the consensus behavior of MASs affected by time-varying delays? 2) how to develop effective control strategies in the presence of random Markovian parameters and stochastic packet loss? and 3) how to design control gains that guarantee the achievement of quasi-consensus in such settings? To tackle these challenges, a new probabilistic framework for quasi-consensus is proposed, including a formal definition and a supporting lemma tailored to MASs with time-varying delays. An observer-based control strategy is then developed to account for both the Markovian switching and random packet loss phenomena. In addition, suitable gain matrices are designed to ensure that the proposed MAS framework achieves quasi-consensus in probability under the SCP and AaF protocols.

The main contributions of this paper, distinguishing it from prior studies, are summarized as follows.

1) This paper is the first to investigate the observer-based control problem for MASs with time-varying delays under the combined influence of the SCP and the AaF relay protocol. The proposed framework jointly considers scheduling constraints, signal amplification, and packet loss effects introduced by relay-based communication, which have not been simultaneously addressed in the existing literature.
2) A distributed observer-based control strategy is developed, which operates on relative measurement outputs affected by random Markovian parameters and stochastic packet dropout. This structure enables each agent to estimate the required system states locally, thereby enhancing robustness and adaptability under unreliable and dynamically varying network conditions.
3) A novel framework of quasi-consensus in probability is proposed for MASs with time-varying delays, and sufficient conditions are established which not only provide rigorous theoretical guarantees but also offer a tractable means to compute the associated observer and controller gain matrices through convex optimization.

II. Problem Formulation and Preliminaries

A. System Model

Consider a class of MASs consisting of M agents with time-varying delays. The dynamics of the l-th agent are described by the following discrete-time model:

$$\begin{cases} x_l(s+1) = Ax_l(s) + A_\tau x_l(s-\tau(s)) + Bu_l(s) + E\omega_l(s) \\ \bar{y}_l(s) = Cx_l(s) + Dv_l(s) \end{cases} \tag{1}$$

where $x_l(s) \in \mathbb{R}^{n_x}$ and $u_l(s) \in \mathbb{R}^{n_u}$ denote the state and input of the l-th agent, respectively; A, B, C, D, A_τ, and E are known matrices with appropriate dimensions; $\tau(s)$ represents the time-varying delay; $\omega_l(s) \in \mathcal{L}_\infty^{n_\omega}$ and $v_l(s) \in \mathcal{L}_\infty^{n_v}$ are the unknown exogenous disturbances acting on the l-th agent; and the time-varying delay $\tau(s)$ satisfies $\underline{\tau} \leq \tau(s) \leq \overline{\tau}$. The measured output vector is denoted by $\bar{y}_l(s) \in \mathbb{R}^{n_y}$, and for notational convenience, it is expressed as $\bar{y}_l(s) = \mathrm{col}\{\bar{y}_l^1, ..., \bar{y}_l^{n_y}\}$. Note that $\tau(s)$ is introduced to model the internal state delay, whereas the non-ideal observation path is represented through the SCP update mechanism, relay-induced packet dropouts, and channel noise, which are to be discussed later.

B. Graph Theory

For the aforementioned MAS with time-varying delays and consisting of N agents, let $\mathbf{M} \triangleq \{1, 2, \ldots, M\}$ denote the index set of agents. It is assumed that the agents communicate with one another over a directed graph topology denoted by $\mathcal{G}$. The directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is composed of a set of nodes $\mathcal{V}$, a set of edges $\mathcal{E}$, and an adjacency matrix $\mathcal{D} \triangleq (d_{lk})_{M\times M}$, where $l, k \in \mathbf{M}$. An edge $(l, k) \in \mathcal{E}$ indicates that there exists a communication link from agent l to agent k. If $(l, k) \in \mathcal{E}$, then $d_{lk} = 1$; otherwise, $d_{lk} = 0$. Self-loops are not allowed, i.e., i.e., $(l, l) \notin \mathcal{E}$. The neighbor set of agent l is defined as $\mathcal{N}_l \triangleq \{k \mid (l, k) \in \mathcal{E}\}$. The in-degree of node l

is given by $b_l = \sum_{k=1}^{M} d_{lk}$, and the in-degree matrix is $\mathscr{B} \triangleq \text{diag}\{b_1, b_2, \ldots, b_M\}$. Consequently, the Laplacian matrix of the graph is defined as $\mathscr{L} \triangleq \mathscr{B} - \mathscr{D}$.

C. SCP Protocol

Information exchange among agents is typically carried out via communication networks. In this paper, the SCP is selected because it is both practically relevant and well suited to the communication framework considered in this paper. This protocol is designed to ensure that only one measurement output is transmitted through the communication channel at each time instant, thereby preventing data collisions.

For the l-th agent, data can be received from only one of its neighboring agents at time s. The agent selected to transmit data through the communication channel at time s is indicated by the random variable $\gamma(s)$. As described in [18], $\gamma(s)$ is modeled as a stochastic process governed by a Markov chain, defined over the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\geq 0}, \mathbf{P})$. Given $\gamma(s) = i$, $\Upsilon \triangleq \{1, \ldots, n_y\}$, the probability of $\gamma(s+1) = j$ is given by the following equation:

$$\mathbf{P}\{\gamma(s+1) = j | \gamma(s) = i\} = p_{ij}, i, j \in \Upsilon$$

where $p_{ij} \geq 0$, $\sum_{j=1}^{n_y} p_{ij} = 1$. The transition probability matrix P can be readily obtained as $\mathrm{P} = (p_{ij})_{n_y \times n_y}$. Let the Kronecker delta function $\delta(i, \gamma(s))$ be the scheduling operator at transmission instant s, defined as follows:

$$\delta(i, \gamma(s)) = \begin{cases} 1, & i = \gamma(s) \\ 0, & \text{otherwise,} \end{cases}$$

the updating matrix is denoted as:

$$\Delta_{\gamma(s)} \triangleq \text{diag}\{\delta(1, \gamma(s)), \ldots, \delta(n_y, \gamma(s))\}.$$

Then, the measurement output scheduled by the SCP protocol can be obtained as:

$$y_l(s) = \Delta_{\gamma(s)} \bar{y}_l(s) + \big(I_{n_y} - \Delta_{\gamma(s)}\big) y_l(s-1). \quad (2)$$

D. AaF Relay Protocol

AaF-relay-based strategies have emerged as a significant area of interest in recent years, attracting substantial attention in the field of networked control. For agent l, the observer-based control framework incorporating the AaF relay mechanism is illustrated in Fig. 1. As shown in the figure, each communication channel is assumed to include a single relay node. The relay divides the communication link between the network and the observer into two segments: Communication network-to-Relay (CTR) and Relay-to-Observer (RTO).

To better reflect the influence of limited bandwidth and network congestion, packet losses in the CTR and RTO stages are considered separately in the analysis. As a result, the final analysis explicitly captures relay amplification, packet dropouts, and communication uncertainty, and sufficient conditions are established to guarantee quasi-consensus in probability.

Considering the impact of network bandwidth and congestion, let us analyze the packet loss separately during the CTR and RTO communication processes. As shown in Fig. 1, the signal is first transmitted by the sensor, forwarded via the communication network under the SCP protocol to the AaF relay, and then amplified and delivered to the observer. The signals received at the relay and the observer are denoted by $r_l(s)$ and $\hat{y}_l(s)$, respectively, and are governed by the following equations:

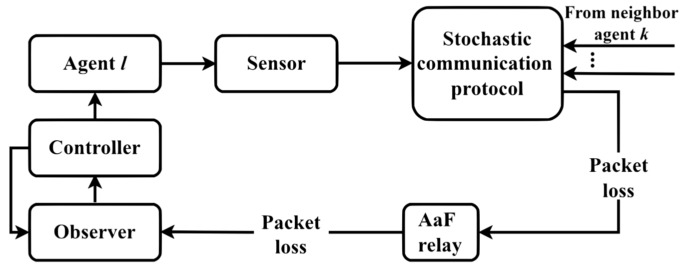

Fig. 1. Observer-based control for agent l over SCP and AaF relay protocols.

$$\begin{cases} r_l(s) = \sqrt{E_{1,l}}\,\alpha_{1l}(s) y_l(s) + L_{1l}(s) n_{1l}(s) \\ \hat{y}_l(s) = \sqrt{E_{2,l}}\,\alpha_{2l}(s) r_l(s) + L_{2l}(s) n_{2l}(s) \end{cases} \quad (3)$$

where $E_{1,l}$ and $E_{2,l}$ are the average signal energies, $L_{1l}(s)$ and $L_{2l}(s)$ are known matrices with proper dimensions. $n_{1l}(s) \in l_2\big([0,T], \mathbf{R}^{n_{1l}}\big)$ and $n_{2l}(s) \in l_2\big([0,T], \mathbf{R}^{n_{2l}}\big)$ are the noise disturbances, where $T > 0$ is known positive integer. $\alpha_{1l}(s)$ and $\alpha_{2l}(s)$ are independent random variables that follow the Bernoulli distributions given below:

$$\mathbf{P}\{\alpha_{1l}(s) = 0\} = 1 - \bar{\alpha}_{1l}, \quad \mathbf{P}\{\alpha_{1l}(s) = 1\} = \bar{\alpha}_{1l}$$
$$\mathbf{P}\{\alpha_{2l}(s) = 0\} = 1 - \bar{\alpha}_{2l}, \quad \mathbf{P}\{\alpha_{2l}(s) = 1\} = \bar{\alpha}_{2l}.$$

Note that the independence assumption for packet losses is introduced for analytical convenience only. In realistic wireless environments, link quality, fading, and congestion may vary with the scheduled channel, and hence a mode-dependent packet-loss model would be more refined. Such an extension is meaningful and would further enhance the realism of the framework.

Remark 1: The combination of the SCP and the AaF relay effectively improves transmission efficiency, enhances communication quality, and mitigates issues such as data collisions due to bandwidth constraints and signal degradation caused by limited transmission range. Moreover, random packet loss and channel noise are considered in both the CTR and RTO communication stages, further aligning the model with realistic conditions. It is also worth noting that the final received signal $\hat{y}_l(s)$, derived from both equations (2) and (3), exhibits increased complexity, which introduces additional challenges to the control design and analysis presented in this study. Note that $E_{1,l}$ and $E_{2,l}$ are adopted as nominal average-energy parameters for modelling and analysis, while adaptive power allocation in AaF relay networks is left for future investigation.

E. Observer-based Controller

The measurement output of an individual agent is typically dependent on the relative states of neighboring agents or external environmental factors, and therefore cannot be directly utilized for control purposes. Accordingly, the relative state and relative measurement output for agent l are defined as follows:

$$\xi_l(s) \triangleq \sum_{k\in\mathcal{N}_l} d_{lk}(x_l(s)-x_k(s)), \tag{4}$$

$$\eta_l(s) \triangleq \sum_{k\in\mathcal{N}_l} d_{lk}(\hat{y}_l(s)-\hat{y}_k(s)). \tag{5}$$

Based on these definitions, the following observer-based controller is proposed:

$$\begin{cases} \hat{\xi}_l(s+1) = A\hat{\xi}_l(s) + A_\tau\hat{\xi}_l(s-\tau(s)) + L_{\gamma(s)}\big(\eta_l(s) \\ \qquad -\bar{\alpha}_{1l}\bar{\alpha}_{2l}\sqrt{E_{1,l}}\sqrt{E_{2,l}}\hat{\eta}_l(s)\big) \\ \qquad +B\sum_{k\in\mathcal{N}_l} d_{lk}(u_l(s)-u_k(s)) \\ \hat{\eta}_l(s) = C\hat{\xi}_l(s) \\ u_l(s) = K_{\gamma(s)}\hat{\xi}_l(s) \end{cases} \tag{6}$$

where $\hat{\xi}_l(s)$ is the relative state estimation, $\hat{\eta}_l(s)$ denotes the relative measurement output estimation, and $L_{\gamma(s)}$, $K_{\gamma(s)}$ are the observer and control gain matrices, respectively, which are to be designed. Note that the current design assumes the realization of $\gamma(s)$ is available to the observer for gain selection, while the packet-dropout variables $\alpha_{1l}(s)$ and $\alpha_{2l}(s)$ only affect the success or failure of data transmission.

By employing the Kronecker product, the closed-loop dynamics of the MAS (1) under the observer-based controller (6) can be compactly expressed as:

$$\begin{cases} x(s+1) = (I_M\otimes A)x(s) + (I_M\otimes A_\tau)x(s-\tau(s)) \\ \qquad +\big(I_M\otimes BK_{\gamma(s)}\big)\hat{\xi}(s) + (I_M\otimes E)\omega(s) \\ \hat{\xi}(s+1) = \big(I_M\otimes A - \bar{\alpha}E_R\otimes L_{\gamma(s)}C\big)\hat{\xi}(s) \\ \qquad +(I_M\otimes A_\tau)\hat{\xi}(s-\tau(s)) + \big(I_M\otimes L_{\gamma(s)}\big)\eta(s) \\ \qquad +\big(\mathcal{L}\otimes BK_{\gamma(s)}\big)\hat{\xi}(s) \end{cases} \tag{7}$$

where

$$\begin{aligned} &x(s) \triangleq \mathrm{col}\{x_1(s), x_2(s), \dots, x_M(s)\} \\ &\hat{\xi}(s) \triangleq \mathrm{col}\{\hat{\xi}_1(s), \hat{\xi}_2(s), \dots, \hat{\xi}_M(s)\} \\ &\omega(s) \triangleq \mathrm{col}\{\omega_1(s), \omega_2(s), \dots, \omega_M(s)\} \\ &\eta(s) \triangleq \mathrm{col}\{\eta_1(s), \eta_2(s), \dots, \eta_M(s)\} \\ &\bar{\alpha} \triangleq \mathrm{diag}\{\bar{\alpha}_{11}\bar{\alpha}_{21}, \dots, \bar{\alpha}_{1M}\bar{\alpha}_{2M}\} \\ &E_R \triangleq \mathrm{diag}\{\sqrt{E_{1,1}}\sqrt{E_{2,1}}, \dots, \sqrt{E_{1,M}}\sqrt{E_{2,M}}\}. \end{aligned}$$

Remark 2: As seen from (3)–(5), the observer-based consensus controller (6) is a distributed controller, whose design relies on both the relative state $\xi_l(s)$ and the relative measurement output $\eta_l(s)$. It inherently captures the joint effects of SCP and AaF. Thus derived, the closed-loop dynamics (7) encompass the aforementioned scenarios of SCP and random packet loss, serving as the basis for analyzing the control performance under the joint influence of SCP and AaF in the next section. It should be noted that this paper assumes the Markov switching process of SCP is independent of the Bernoulli variable $\alpha_{ij}(s)$.

III. Main Result

A. Deviation Error System Model

In this section, an augmented deviation error system is established to facilitate the subsequent dynamic analysis of the closed-loop multi-agent system.

The state deviation and estimation error for the l-th agent are defined as:

$$e_x^l(s) \triangleq x_l(s) - \bar{x}(s), \quad e_\xi^l(s) \triangleq \xi_l(s) - \hat{\xi}_l(s)$$

where $\bar{x}(s) = \frac{1}{M}\sum_{l=1}^M x_l(s)$ represents the average state. It is straightforward to obtain

$$\bar{x}(s) = \frac{1}{M}(\mathbf{1}\otimes I_{n_x})x(s)$$

where $\mathbf{1} \triangleq \mathrm{col}_M\{1, 1, \dots, 1\}^T$.

By using (2), (3), (5), and the definition of $e_x^l(s)$, the relative measurement output $\eta(s)$ can be computed as:

$$\begin{aligned} \eta(s) =& (\mathcal{L}\otimes I_{n_y})\hat{y}(s) \\ =& \big(\alpha(s)E_R\mathcal{L}\otimes\Delta_{\gamma(s)}C\big)e_x(s) \\ &+ \big(\alpha(s)E_R\mathcal{L}\otimes\Delta_{\gamma(s)}D\big)v(s) \\ &+ \big(\alpha(s)E_R\mathcal{L}\otimes(I_{n_y}-\Delta_{\gamma(s)})\big)y(s-1) \\ &+ \big(\alpha_2(s)E_2L_1(s)\mathcal{L}\otimes I_{n_1}\big)n_1(s) \\ &+ \big(L_2(s)\mathcal{L}\otimes I_{n_2}\big)n_2(s) \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\alpha(s) \triangleq \mathrm{diag}\{\alpha_{11}(s)\alpha_{21}(s), \alpha_{12}(s)\alpha_{22}(s), \dots, \alpha_{1M}(s)\alpha_{2M}(s)\} \\ &\alpha_2(s) \triangleq \mathrm{diag}\{\alpha_{21}(s), \alpha_{22}(s), \dots, \alpha_{2M}(s)\} \\ &E_2 \triangleq \mathrm{diag}\{\sqrt{E_{2,1}}, \sqrt{E_{2,2}}, \dots, \sqrt{E_{2,M}}\} \\ &v(s) \triangleq \mathrm{col}\{v_1(s), v_2(s), \dots, v_M(s)\} \\ &n_1(s) \triangleq \mathrm{col}\{n_{11}(s), n_{12}(s), \dots, n_{1M}(s)\} \\ &n_2(s) \triangleq \mathrm{col}\{n_{21}(s), n_{22}(s), \dots, n_{2M}(s)\} \\ &L_1(s) \triangleq \mathrm{col}\{L_{11}(s), L_{12}(s), \dots, L_{1M}(s)\} \\ &L_2(s) \triangleq \mathrm{col}\{L_{21}(s), L_{22}(s), \dots, L_{2M}(s)\}. \end{aligned}$$

By combining (7) with the definition of $e_\xi^l(s)$ and substituting (8) into it, the following deviation error dynamics can be derived:

$$\begin{cases} e_x(s+1) = \big(I_M\otimes A + \mathfrak{M}\mathcal{L}\otimes BK_{\gamma(s)}\big)e_x(s) \\ \qquad +(\mathfrak{M}\otimes E)\omega(s) + (I_M\otimes A_\tau)e_x(s-\tau(s)) \\ \qquad -\big(\mathfrak{M}\otimes BK_{\gamma(s)}\big)e_\xi(s) \\ e_\xi(s+1) = \big(\bar{\alpha}E_R\mathcal{L}\otimes L_{\gamma(s)}C - \alpha(s)E_R\mathcal{L}\otimes L_{\gamma(s)}\Delta_{\gamma(s)} \\ \qquad \times C\big)e_x(s) + (I_M\otimes A_\tau)e_\xi(s-\tau(s)) \\ \qquad +\big(I_M\otimes A - \bar{\alpha}E_R\otimes L_{\gamma(s)}C\big)e_\xi(s) \\ \qquad -\big(\alpha(s)E_R\mathcal{L}\otimes L_{\gamma(s)}(I_{n_y}-\Delta_{\gamma(s)})\big)y(s-1) \\ \qquad +\big(\mathcal{L}\mathfrak{M}\otimes E\big)\omega(s) - \big(L_2(s)\mathcal{L}\otimes L_{\gamma(s)}\big)n_2(s) \\ \qquad -\big(\alpha(s)E_R\mathcal{L}\otimes L_{\gamma(s)}\Delta_{\gamma(s)}D\big)v(s) \\ \qquad -\big(\alpha_2(s)E_2L_1(s)\mathcal{L}\otimes L_{\gamma(s)}\big)n_1(s) \end{cases} \tag{9}$$

where

$$\begin{aligned} &e_x(s) \triangleq \mathrm{col}\{e_x^1(s), \dots e_x^M(s)\} \\ &e_\xi(s) \triangleq \mathrm{col}\{e_\xi^1(s), \dots e_\xi^M(s)\} \end{aligned}$$

$$\mathfrak{M} \triangleq \begin{bmatrix} \frac{M-1}{M} & -\frac{1}{M} & \cdots & -\frac{1}{M} \\ -\frac{1}{M} & \frac{M-1}{M} & \cdots & -\frac{1}{M} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M} & -\frac{1}{M} & \cdots & \frac{M-1}{M} \end{bmatrix}$$

Define the augmented state and disturbance vectors as $\varphi(s) \triangleq \operatorname{col}\{e_x(s), e_\xi(s), \eta(s-1), y(s-1)\}$ and $\psi(s) \triangleq \operatorname{col}\{\omega(s), v(s), n_1(s), n_2(s)\}$, respectively. Then, the augmented closed-loop system can be expressed as:

$$\varphi(s+1) = \mathcal{A}_{\gamma(s)}\varphi(s) + \mathcal{A}_\tau \varphi(s-\tau(s)) + \mathcal{E}_{\gamma(s)}\psi(s) \quad (10)$$

where

$$\mathcal{A}_{\gamma(s)} \triangleq \begin{bmatrix} \mathcal{A}^{11}_{\gamma(s)} & \mathcal{A}^{12}_{\gamma(s)} & 0 & 0 \\ \mathcal{A}^{21}_{\gamma(s)} & \mathcal{A}^{22}_{\gamma(s)} & 0 & \mathcal{A}^{24}_{\gamma(s)} \\ \mathcal{A}^{31}_{\gamma(s)} & 0 & 0 & \mathcal{A}^{34}_{\gamma(s)} \\ \mathcal{A}^{41}_{\gamma(s)} & 0 & 0 & \mathcal{A}^{44}_{\gamma(s)} \end{bmatrix}$$

$$\mathcal{A}^{11}_{\gamma(s)} \triangleq I_M \otimes A + \mathfrak{M}\mathcal{L} \otimes BK_{\gamma(s)}$$

$$\mathcal{A}^{12}_{\gamma(s)} \triangleq -\mathfrak{M} \otimes BK_{\gamma(s)},\ \mathcal{A}^{31}_{\gamma(s)} \triangleq \alpha(s)E_R\mathcal{L} \otimes \Delta_{\gamma(s)}C$$

$$\mathcal{A}^{21}_{\gamma(s)} \triangleq \bar{\alpha}E_R\mathcal{L} \otimes L_{\gamma(s)}C - \alpha(s)E_R\mathcal{L} \otimes L_{\gamma(s)}\Delta_{\gamma(s)}C$$

$$\mathcal{A}^{22}_{\gamma(s)} \triangleq I_M \otimes A - \bar{\alpha}E_R \otimes L_{\gamma(s)}C$$

$$\mathcal{A}^{24}_{\gamma(s)} \triangleq -\alpha(s)E_R \otimes L_{\gamma(s)}(I_{n_y} - \Delta_{\gamma(s)})$$

$$\mathcal{A}^{34}_{\gamma(s)} \triangleq \alpha(s)E_R \otimes (I_{n_y} - \Delta_{\gamma(s)})$$

$$\mathcal{A}^{41}_{\gamma(s)} \triangleq \mathcal{L} \otimes \Delta_{\gamma(s)}C, \mathcal{A}^{44}_{\gamma(s)} \triangleq \mathcal{L} \otimes (I_{n_y} - \Delta_{\gamma(s)})$$

$$\mathcal{A}_\tau \triangleq \begin{bmatrix} I_M \otimes A_\tau & 0 & 0 & 0 \\ 0 & I_M \otimes A_\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_{\gamma(s)} \triangleq \begin{bmatrix} \mathcal{E}^{11}_{\gamma(s)} & 0 & 0 & 0 \\ \mathcal{E}^{21}_{\gamma(s)} & \mathcal{E}^{22}_{\gamma(s)} & \mathcal{E}^{23}_{\gamma(s)} & \mathcal{E}^{24}_{\gamma(s)} \\ 0 & \mathcal{E}^{32}_{\gamma(s)} & \mathcal{E}^{33}_{\gamma(s)} & \mathcal{E}^{34}_{\gamma(s)} \\ 0 & \mathcal{E}^{42}_{\gamma(s)} & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}^{11}_{\gamma(s)} \triangleq \mathfrak{M} \otimes E,\ \mathcal{E}^{22}_{\gamma(s)} \triangleq -\alpha(s)E_R\mathcal{L} \otimes L_{\gamma(s)}\Delta_{\gamma(s)}D$$

$$\mathcal{E}^{21}_{\gamma(s)} \triangleq \mathcal{L}\mathfrak{M} \otimes E,\ \mathcal{E}^{23}_{\gamma(s)} \triangleq -\alpha_2(s)E_2L_1(s)\mathcal{L} \otimes L_{\gamma(s)}$$

$$\mathcal{E}^{24}_{\gamma(s)} \triangleq -L_2(s)\mathcal{L} \otimes L_{\gamma(s)},\ \mathcal{E}^{32}_{\gamma(s)} \triangleq \alpha(s)E_R\mathcal{L} \otimes \Delta_{\gamma(s)}D$$

$$\mathcal{E}^{33}_{\gamma(s)} \triangleq \alpha_2(s)E_2L_1(s)\mathcal{L} \otimes I_{n_1}$$

$$\mathcal{E}^{34}_{\gamma(s)} \triangleq L_2(s)\mathcal{L} \otimes I_{n_2},\ \mathcal{E}^{42}_{\gamma(s)} \triangleq \mathcal{L} \otimes \Delta_{\gamma(s)}D$$

Remark 3: Based on the definitions of $e^l_x(s)$ and $e^l_\xi(s)$ in Eq. (5), and by utilizing the Kronecker product, (8) and (9) can be readily derived, leading to the final augmented closed-loop system (10). Due to the scheduling behavior imposed by the SCP protocol, the augmented closed-loop system (10) becomes a stochastic system with Markovian switching parameters. Furthermore, the AaF relay protocol introduces packet dropout effects, further contributing to the system's stochastic nature. Although these combined effects increase the complexity of system analysis, they improve communication reliability and extend transmission range. Since the augmented system (10) encapsulates both the state and observer deviation dynamics, its asymptotic analysis is crucial for establishing the quasi-consensus performance of the MAS defined in (1).

B. Consensus Analysis

To facilitate the subsequent theoretical development, the following definition and lemma are presented.

Definition 1: The MAS with time-varying delay (1) is said to achieve quasi-consensus with probability 1-ϵ if, for a constant $\epsilon \in (0,1)$, there exists a positive constant $\mathcal{B}(\epsilon, \|\psi(s)\|_\infty)$ satisfying

$$\mathbf{P}\left\{\lim_{s\to\infty} \|x_l(s) - x_k(s)\| \le \mathcal{B}(\epsilon, \|\psi(s)\|_\infty)\right\} \ge 1 - \epsilon.$$

Lemma 1: For any positive integer s, assume that there are functions V: $\mathbb{Z}^+ \times \mathbb{R}^{2n_x+2n_y} \to \mathbb{R}^+$, $\vartheta_1(t)$, $\vartheta_2(t)$, $\vartheta_3(t) \in \mathcal{K}_\infty$, and $\vartheta_4(t) \in \mathcal{K}$ such that

1) $\vartheta_1(\|\tilde{\varphi}(s)\|) \le V(s, \varphi(s)) \le \vartheta_2(\|\tilde{\varphi}(s)\|)$, where $\varphi(s) \in R^{2n_x+2n_y}$, $\tilde{\varphi}(s) = \operatorname{col}\{\varphi(s), \varphi(s-1), \ldots, \varphi(s-\bar{\tau})\}$;
2) $\mathbb{E}\{\triangle V(s, \varphi(s))\} \le -\mathbb{E}\{\vartheta_3(\|\tilde{\varphi}(s)\|_\infty)\} + \vartheta_4(\|\psi(s)\|)$;
3) $\vartheta_3 \circ \vartheta_2^{-1}(t) \in \mathcal{VK}_\infty$, and $(Id - \vartheta_3 \circ \vartheta_2^{-1})(t) \in \mathcal{K}$ where Id represents the identity function.

Then, there exist functions $\vartheta(t) \in \mathcal{KL}$ and $\widehat{\varepsilon}(t) \in \mathcal{K}_\infty$ with $\widehat{\varepsilon}(t) \le \vartheta_3 \circ \vartheta_2^{-1}(t)$ such that

$$\mathbf{P}\left\{\|\tilde{\varphi}(s)\| \le \vartheta(\|\tilde{\varphi}(0)\|, s) + \frac{\breve{\vartheta}(\|\psi(s)\|_\infty)}{\epsilon}\right\} \ge 1 - \epsilon$$

where $\epsilon \in (0,1)$, $\breve{\vartheta}(t) = \vartheta_1^{-1} \circ \widehat{\varepsilon}^{-1} \circ \vartheta_4(t)$. In particular, when $\mathbb{E}\{s, V(\varphi(s))\} > \widehat{\varepsilon}^{-1} \circ \vartheta_4(\|\psi(s)\|_\infty)$, $\vartheta(t)$ can be derived from

$$\mathbb{E}\{s, V(\varphi(s))\} \le \vartheta(\|\varphi(0)\|, s).$$

Proof: The proof follows a procedure similar to that of Theorem 1 in [18] and is therefore omitted here. ■

Remark 4: It should be pointed out that the functions $\mathcal{K}_\infty$, $\mathcal{K}$, $\mathcal{KL}$, and $\mathcal{VK}_\infty$ are defined in [43]. Unlike the result in [43], which primarily focuses on quasi-consensus analysis of MASs without delays, Lemma 1 in this paper is developed for MASs with time-varying delays as described in (1). In comparison with the existing literature, this result not only preserves the effectiveness of quasi-consensus analysis but also provides a tractable framework for addressing coordination challenges in MASs under realistic time-delay conditions.

For simplicity, we denote $\gamma(s)$ by γ as no ambiguity arises. The following theorem provides the quasi-consensus condition for the time-varying delayed MAS (1) under the SCP and AaF relay protocols.

For simplicity, $\gamma(s)$ is abbreviated as γ since no ambiguity arises. The following theorem gives the quasi-consensus condition for the time-varying delayed MAS (1) under the SCP and AaF relay protocols.

Theorem 1: Consider the time-varying delayed MAS (1) with the observer-based controller (6) operating over a directed communication graph $\mathcal{G}$. Given a constant $\epsilon > 0$ and the gain matrices L_γ and K_γ, suppose that there exist positive definite matrices $Q \in \mathbb{R}^{M(2n_x+2n_y)\times M(2n_x+2n_y)}$, $P_\gamma \in \mathbb{R}^{M(2n_x+2n_y)\times M(2n_x+2n_y)}$, $R \in \mathbb{R}^{M(n_\omega+n_v+n_1+n_2)\times M(n_\omega+n_v+n_1+n_2)}$, and symmetric matrices with full rank $S_1, S_2 \in \mathbb{R}^{M(2n_x+2n_y)\times M(2n_x+2n_y)}$, such that

$$\Xi_\gamma^T \bar{P}_\gamma \Xi_\gamma + \Sigma_\gamma < 0 \quad (11)$$

where

$$\bar{P}_i = \sum_{j=1}^{n_y} p_{ij} P_j, (i, j \in \Upsilon), \quad \Xi_\gamma = \left[\mathcal{A}_\gamma, \mathcal{A}_\tau, \mathcal{E}_\gamma\right]$$

$$\Sigma_\gamma = \begin{bmatrix} -P_\gamma + (1 + \bar{\tau} - \underline{\tau})Q + S_1 & 0 & 0 \\ 0 & -Q + S_2 & 0 \\ 0 & 0 & -R \end{bmatrix}.$$

Then, the time-varying delayed MAS (1) achieves the quasi-consensus defined in Definition 1. Moreover, there exist positive constants $\hat{c}$, ς, and $\mathcal{K}-$function $\hat{\vartheta}(t)$ satisfying

$$\mathbf{P}\left\{\|x_l(s) - x_k(s)\| \leq \sqrt{2}\hat{\vartheta}(\|\varphi(0)\|)e^{-\hat{c}s} + \frac{\varsigma\|\psi(s)\|_\infty}{\epsilon}\right\} \geq 1 - \epsilon.$$

The upper bound on the steady-state consensus error between agents is given by

$$\mathcal{B}(\epsilon, \|\psi(s)\|_\infty) = \frac{\varsigma}{\epsilon}\|\psi(s)\|_\infty.$$

Proof: Consider the following Lyapunov-Krasovskii functional:

$$V(s) = V_1(s) + V_2(s) + V_3(s) \tag{12}$$

where

$$V_1(s) = \varphi^T(s)P_\gamma\varphi(s)$$
$$V_2(s) = \sum_{\mu=s-\tau(s)}^{s-1} \varphi^T(\mu)Q\varphi(\mu)$$
$$V_3(s) = \sum_{d=s-\bar{\tau}+1}^{s-\underline{\tau}} \sum_{\mu=d}^{s-1} \varphi^T(\mu)Q\varphi(\mu).$$

It can be shown that the combined terms $V_2(s) + V_3(s)$ satisfy

$$\begin{aligned} &V_2(s) + V_3(s) \\ =&\sum_{d=s-\bar{\tau}+1}^{s-1} \varphi^T(\mu)Q\varphi(\mu) + \sum_{d=s-\bar{\tau}+2}^{s-1} \varphi^T(\mu)Q\varphi(\mu) + \\ &\ldots + \sum_{d=s-\underline{\tau}}^{s-1} \varphi^T(\mu)Q\varphi(\mu) + \sum_{d=s-\tau(s)}^{s-1} \varphi^T(\mu)Q\varphi(\mu). \end{aligned}$$

For the term $\sum_{d=s-\tau(s)}^{s-1} \varphi^T(\mu)Q\varphi(\mu)$, the following bounds hold:

$$\sum_{d=s-\tau(s)}^{s-1} \varphi^T(\mu)Q\varphi(\mu) \leq \sum_{d=s-\bar{\tau}}^{s-1} \varphi^T(\mu)Q\varphi(\mu) \tag{13}$$

$$\sum_{d=s-\tau(s)}^{s-1} \varphi^T(\mu)Q\varphi(\mu) \geq \sum_{d=s-\underline{\tau}}^{s-1} \varphi^T(\mu)Q\varphi(\mu). \tag{14}$$

Based on (13) and (14), the following inequality can be derived:

$$\tilde{\varphi}^T(s)\Omega_1^\gamma\tilde{\varphi}(s) \leq V(s) \leq \tilde{\varphi}^T(s)\Omega_2^\gamma\tilde{\varphi}(s) \tag{15}$$

where

$$\begin{aligned} \Omega_1^\gamma = &\begin{bmatrix} P_\gamma & 0 & \ldots & 0 & 0 \\ 0 & (\bar{\tau} - \underline{\tau})Q & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & Q & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \\ &+ \mathrm{diag}\{0, \underbrace{Q, \ldots, Q}_{\underline{\tau}}, 0, \ldots, 0\} \end{aligned}$$

$$\Omega_2^\gamma = \begin{bmatrix} P_\gamma & 0 & 0 & \ldots & 0 \\ 0 & (\bar{\tau} - \underline{\tau} + 1)Q & 0 & \ldots & 0 \\ 0 & 0 & (\bar{\tau} - \underline{\tau})Q & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & Q \end{bmatrix}.$$

Construct the following two $\mathcal{K}_\infty$ functions

$$\vartheta_1(t) = \min_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega_1^\gamma\}\}t^2$$
$$\vartheta_2(t) = \max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}\}t^2.$$

Then, $V(s)$ in (12) can be derived as:

$$\vartheta_1(\|\tilde{\varphi}(s)\|) \leq V(s) \leq \vartheta_2(\|\tilde{\varphi}(s)\|). \tag{16}$$

It is obvious that the condition 1) of Lemma 1 is satisfied.

Next, the differences and expectations of $V_1(s)$, $V_2(s)$, and $V_3(s)$ along the solution trajectories of the augmented system (10) are computed as follows:

$$\begin{aligned} &\mathbb{E}\{V_1(s+1) - V_1(s)\} \\ =&\mathbb{E}\Big\{\big(\mathcal{A}_\gamma\varphi(s) + \mathcal{A}_\tau\varphi(s - \tau(s)) + \mathcal{E}_\gamma\psi(s)\big)^T \bar{P}_\gamma\big(\mathcal{A}_\gamma\varphi(s) \\ &+ \mathcal{A}_\tau\varphi(s - \tau(s)) + \mathcal{E}_\gamma\psi(s)\big) - \varphi^T(s)P_\gamma\varphi(s)\Big\} \end{aligned} \tag{17}$$

$$\begin{aligned} &\mathbb{E}\{V_2(s+1) - V_2(s)\} \\ =&\mathbb{E}\Big\{\sum_{\mu=s+1-\tau(s+1)}^{s} \varphi^T(\mu)Q\varphi(\mu) - \sum_{\mu=s-\tau(s)}^{s-1} \varphi^T(\mu)Q\varphi(\mu)\Big\} \\ \leq&\mathbb{E}\Big\{\varphi^T(s)Q\varphi(s) - \varphi^T(s - \tau(s))Q\varphi(s - \tau(s)) \\ &+ \sum_{\mu=s-\bar{\tau}+1}^{s-\underline{\tau}} \varphi^T(\mu)Q\varphi(\mu)\Big\} \end{aligned} \tag{18}$$

$$\begin{aligned} &\mathbb{E}\{V_3(s+1) - V_3(s)\} \\ =&\mathbb{E}\Big\{(\bar{\tau} - \underline{\tau})\varphi^T(s)Q\varphi(s) - \sum_{\mu=s-\bar{\tau}+1}^{s-\underline{\tau}} \varphi^T(\mu)Q\varphi(\mu)\Big\}. \end{aligned} \tag{19}$$

By combining (11) and (17)-(19), we have

$$\begin{aligned} &\mathbb{E}\{V(s+1) - V(s)\} \\ <&\mathbb{E}\Big\{- \varphi^T(s)S_1\varphi(s) - \varphi^T(s - \tau(s))S_2\varphi(s - \tau(s)) \\ &+ \psi^T(s)R\psi(s)\Big\} \\ \leq& - \min\big\{\{|\lambda(S_1)|\}_{\min}, \{|\lambda(S_2)|\}_{\min}\big\} \\ &\times \mathbb{E}\{\|\varphi(s) + \varphi(s - \tau(s))\|^2\} + \lambda_{\max}\{R\}\mathbb{E}\{\|\psi(s)\|^2\}. \end{aligned} \tag{20}$$

Assuming

$$\|\varphi(s) + \varphi(s - \tau(s))\|^2 \geq \|\tilde{\varphi}(s)\|_\infty^2,$$

it can be inferred that

$$\begin{aligned}&\mathbb{E}\{V(s+1)-V(s)\}\\&\leq -\min\Big\{\{|\lambda(S_1)|\}_{\min},\{|\lambda(S_2)|\}_{\min}\Big\}\mathbb{E}\{\|\tilde{\varphi}(s)\|_\infty^2\}\\&\quad+\lambda_{\max}\{R\}\mathbb{E}\{\|\psi(s)\|^2\}. \end{aligned}\tag{21}$$

Choose the $\mathscr{K}_\infty$ functions as follows:

$$\vartheta_3(t)=\varepsilon t^2,\ \vartheta_4(t)=\lambda_{\max}\{R\}t^2$$

where

$$\begin{aligned}0<\varepsilon<\min\Big\{&\{|\lambda(S_1)|\}_{\min},\{|\lambda(S_2)|\}_{\min},\\&\min\nolimits_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega_1^\gamma\}\}\Big\}.\end{aligned}$$

Then, it follows that

$$\mathbb{E}\{V(s+1)-V(s)\}\leq-\vartheta_3(\|\tilde{\varphi}(s)\|_\infty)+\vartheta_4(\|\psi(s)\|)\tag{22}$$

The above (22) indicates that condition 2) of Lemma 1 is satisfied.

According to the definitions of $\vartheta_2(t)$ and $\vartheta_3(t)$, we obtain

$$\begin{aligned}&\vartheta_3\circ\vartheta_2^{-1}(t)=\frac{\varepsilon}{\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}}t\\&\Big(Id-\vartheta_3\circ\vartheta_2^{-1}\Big)(t)=\Big(1-\frac{\varepsilon}{\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}}\Big)t,\end{aligned}$$

and it follows that $\vartheta_3\circ\vartheta_2^{-1}(t)\in\mathcal{VK}_\infty$, $\Big(Id-\vartheta_3\circ\vartheta_2^{-1}\Big)(t)\in\mathscr{K}$, which means that the condition 3) of Lemma 1 is satisfied.

In consideration of (15) and the inequality $\|\tilde{\varphi}(s)\|_\infty\leq\|\tilde{\varphi}(s)\|\leq\sqrt{\theta}\|\tilde{\varphi}(s)\|_\infty$ with $\theta=M(2n_x+n_y)$, it can be deduced that

$$V(s)\leq\vartheta_2(\sqrt{\theta}\|\tilde{\varphi}(s)\|_\infty).\tag{23}$$

Subsequently, by applying (22), one obtains

$$\begin{aligned}\mathbb{E}\{\triangle V(s)\}\leq&-\frac{\varepsilon}{\theta\min_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega_1^\gamma\}}\mathbb{E}\{V(s)\}\\&+\vartheta_4(\|\psi(s)\|).\end{aligned}\tag{24}$$

Define

$$\widehat{\varepsilon}(t)\triangleq\frac{\varepsilon}{\theta\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}\}}t.$$

When $\mathbb{E}\{V(s)\}>\theta\widehat{\varepsilon}^{-1}(\vartheta_4(\|\psi(s)\|_\infty))$, there exists a constant $\sigma\in(0,1)$ such that

$$\vartheta_4(\|\psi(s)\|)\leq\frac{(1-\sigma)\varepsilon}{\theta\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}}\mathbb{E}\{V(s)\},\tag{25}$$

which leads to

$$\mathbb{E}\{\triangle V(s)\}\leq-\frac{\sigma\varepsilon}{\theta\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}}\mathbb{E}\{V(s)\}.\tag{26}$$

Therefore, it follows that

$$\begin{aligned}\mathbb{E}\{V(s+1)\}\leq&\Big(1-\frac{\sigma\varepsilon}{\theta\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}}\Big)\mathbb{E}\{V(s)\}\\=&c\mathbb{E}\{V(s)\},\end{aligned}\tag{27}$$

where

$$c\triangleq1-\frac{\sigma\varepsilon}{\theta\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}}\in(0,1).$$

By mathematical induction, it can be derived

$$\mathbb{E}\{V(s)\}\leq c^s\mathbb{E}\{V(0)\}\tag{28}$$

It can be obtained from (15) and (28) that

$$\mathbb{E}\{\|\tilde{\varphi}(s)\|^2\}\leq c^s\frac{\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}\}}{\min_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega_1^\gamma\}\}}\mathbb{E}\{\|\tilde{\varphi}(0)\|^2\}.\tag{29}$$

We let

$$\tilde{c}\triangleq\ln\frac{1}{c},\ \hat{\vartheta}(\|\tilde{\varphi}(0)\|)\triangleq\frac{\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}\}}{\min_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega_1^\gamma\}\}}\mathbb{E}\{\|\tilde{\varphi}(0)\|^2\}.$$

and define the $\mathscr{KL}-$function $\vartheta(t)$ and $\mathscr{K}_\infty$ function $\breve{\vartheta}(t)$ as follows:

$$\begin{aligned}&\vartheta(t)\triangleq c^t\frac{\max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega_2^\gamma\}\}}{\min_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega_1^\gamma\}\}}\mathbb{E}\{\|\tilde{\varphi}(0)\|^2\}\\&\breve{\vartheta}(t)\triangleq\vartheta_1^{-1}\circ\widehat{\varepsilon}^{-1}\circ\vartheta_4(t).\end{aligned}$$

Based on the definition of $\hat{\vartheta}(\|\tilde{\varphi}(0)\|)$, it can be easily obtained that

$$\vartheta(t)=\hat{\vartheta}(\|\tilde{\varphi}(0)\|)e^{-\tilde{c}t}.$$

According to Lemma 1, there exist a constant $\epsilon\in(0,1)$ and a $\mathscr{K}_\infty$ function $\breve{\vartheta}(t)$ such that

$$\mathbf{P}\left\{\|\tilde{\varphi}(s)\|\leq\hat{\vartheta}(\|\tilde{\varphi}(0)\|)e^{-\tilde{c}s}+\frac{\breve{\vartheta}(\|\psi(s)\|_\infty)}{\epsilon}\right\}\geq1-\epsilon.\tag{30}$$

Furthermore, it is known that

$$\|\tilde{\varphi}(s)\|^2\geq\|\varphi(s)\|^2\geq\frac{\|x_l(s)-x_k(s)\|^2}{2}.\tag{31}$$

Then, by taking the limit as $s\to\infty$ in (30) and applying (31), one obtains

$$\mathbf{P}\left\{\lim_{s\to\infty}\|x_l(s)-x_k(s)\|\leq\frac{\breve{\vartheta}(\|\psi(s)\|_\infty)}{\epsilon}\right\}\geq1-\epsilon,$$

which indicates that the time-varying delayed MAS (1) achieves quai-consensus with probability at least $1-\epsilon$. Moreover, the upper bound of the consensus error between agents can be expressed as $\mathcal{B}(\epsilon,\|\psi(s)\|_\infty)$, where ς is obtained through $\varsigma\|\psi(s)\|_\infty=\vartheta_1^{-1}\circ\widehat{\varepsilon}^{-1}\circ\vartheta_4(\|\psi(s)\|_\infty)$. ■

C. Design of the Controller

In this section, we will design the gain matrices $K_{\gamma(s)}$ and $L_{\gamma(s)}$ in controller (6) such that the time-varying delayed MAS (1) reaches quasi-consensus in probability $1-\epsilon$. Prior to the design procedure, the matrix B is assumed to have full column rank and is decomposed using singular value decomposition as follows:

$$B=U^{\mathrm{T}}\begin{pmatrix}\Sigma\\0\end{pmatrix}Y^{\mathrm{T}}.$$

Theorem 2: Consider the directed communication graph $\mathcal{G}$ and a constant $\epsilon>0$. The time-varying delayed MAS (1), together with the observer-based controller (6), achieves quasi-consensus in probability $1-\epsilon$ if there exist positive definite matrices $P_\gamma^{11}\in\mathbb{R}^{n_u\times n_u}$, $P_\gamma^{11}\in\mathbb{R}^{(n_x-n_u)\times(n_x-n_u)}$, $P_\gamma^2\in$

$\mathbb{R}^{n_x\times n_x}$, $P^3_\gamma, P^4_\gamma \in \mathbb{R}^{n_y\times n_y}$, $Q_1, Q_2 \in \mathbb{R}^{Mn_x\times Mn_x}$, $Q_3, Q_4 \in \mathbb{R}^{Mn_y\times Mn_y}$, $R_1 \in \mathbb{R}^{Mn_\omega\times Mn_\omega}$, $R_2 \in \mathbb{R}^{Mn_v\times Mn_v}$, $R_3 \in \mathbb{R}^{Mn_{n_1}\times Mn_{n_1}}$, $R_4 \in \mathbb{R}^{Mn_{n_2}\times Mn_{n_2}}$, symmetric matrices with full rank $S_{i1}, S_{i2} \in \mathbb{R}^{Mn_x\times Mn_x}$, $S_{i3}, S_{i4} \in \mathbb{R}^{Mn_y\times Mn_y}$, $i = 1, 2$, and any matrices W_γ, Z_γ with appropriate dimensions, such that the following inequalities are satisfied:

$$\Upsilon_\gamma \triangleq \begin{bmatrix} \Upsilon^{11}_\gamma & \Upsilon^{12}_\gamma \\ * & \Upsilon^{22}_\gamma \end{bmatrix} < 0 \tag{32}$$

$$-Q_j + S_{2j} < 0 \tag{33}$$

$$-P^j_\gamma + (1+\overline{\tau}-\underline{\tau})Q_j + S_{1j} < 0, j = 1,2,3,4 \tag{34}$$

where

$$\begin{aligned}
&P^1_\gamma \triangleq U^T \mathrm{diag}\{P^{11}_\gamma, P^{22}_\gamma\}U \\
&\Upsilon^{11}_\gamma \triangleq \mathrm{diag}\Big\{ - I_M \otimes P^1_\gamma + (1+\overline{\tau}-\underline{\tau})Q_1 + S_{11}, \\
&\qquad - I_M \otimes P^2_\gamma + (1+\overline{\tau}-\underline{\tau})Q_2 + S_{12}, \\
&\qquad - I_M \otimes P^3_\gamma + (1+\overline{\tau}-\underline{\tau})Q_3 + S_{13}, \\
&\qquad - I_M \otimes P^4_\gamma + (1+\overline{\tau}-\underline{\tau})Q_4 + S_{14}, \\
&\qquad - Q_1 + S_{21}, -Q_2 + S_{22}, -Q_3 + S_{23}, \\
&\qquad - Q_4 + S_{24}, -R_1, -R_2, -R_3, -R_4 \Big\}
\end{aligned}$$

$$\tilde{\Upsilon}^{12}_\gamma \triangleq \begin{bmatrix} \Gamma^1_\gamma & \Gamma^5_\gamma & \Gamma^{13}_\gamma & \Gamma^{18}_\gamma \\ \Gamma^2_\gamma & \Gamma^6_\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Gamma^7_\gamma & \Gamma^{14}_\gamma & \Gamma^{19}_\gamma \\ \Gamma^3_\gamma & 0 & 0 & 0 \\ 0 & \Gamma^8_\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma^4_\gamma & \Gamma^9_\gamma & 0 & 0 \\ 0 & \Gamma^{10}_\gamma & \Gamma^{15}_\gamma & \Gamma^{20}_\gamma \\ 0 & \Gamma^{11}_\gamma & \Gamma^{16}_\gamma & 0 \\ 0 & \Gamma^{12}_\gamma & \Gamma^{17}_\gamma & 0 \end{bmatrix}$$

$$\begin{aligned}
&\Gamma^1_\gamma \triangleq I_M \otimes A^T P^1_\gamma + \mathcal{L}^T \mathfrak{M}^T \otimes W_\gamma B^T,\ \Gamma^3_\gamma \triangleq I_M \otimes A^T_\tau P^1_\gamma \\
&\Gamma^2_\gamma \triangleq -\mathfrak{M}^T \otimes W_\gamma B^T,\ \Gamma^4_\gamma \triangleq \mathfrak{M}^T \otimes E^T P^1_\gamma \\
&\Gamma^5_\gamma \triangleq \bar{\alpha} E_R \mathcal{L}^T \otimes C^T (I_{n_y} - \Delta_\gamma) Z_\gamma,\ \Gamma^8_\gamma \triangleq I_M \otimes A^T_\tau P^2_\gamma \\
&\Gamma^6_\gamma \triangleq I_M \otimes A^T P^2_\gamma - \bar{\alpha} E_R \otimes C^T Z_\gamma,\ \Gamma^9_\gamma \triangleq \mathfrak{M}^T \mathcal{L}^T \otimes E^T P^2_\gamma \\
&\Gamma^7_\gamma \triangleq -\bar{\alpha} E_R \otimes (I_{n_y} - \Delta_\gamma) Z_\gamma,\ \Gamma^{12}_\gamma \triangleq -L_2(s)\mathcal{L}^T \otimes Z_\gamma \\
&\Gamma^{11}_\gamma \triangleq -\bar{\alpha}_2 E_2 L_1(s) \mathcal{L}^T \otimes Z_\gamma,\ \Gamma^{14}_\gamma \triangleq \bar{\alpha} E_R \otimes (I_{n_y} - \Delta_\gamma) P^3_\gamma \\
&\Gamma^{13}_\gamma \triangleq \bar{\alpha} E_R \mathcal{L}^T \otimes C^T \Delta_\gamma P^3_\gamma,\ \Gamma^{10}_\gamma \triangleq -\bar{\alpha} E_R \mathcal{L}^T \otimes D^T \Delta_\gamma Z_\gamma \\
&\Gamma^{15}_\gamma \triangleq \bar{\alpha} E_R \mathcal{L}^T \otimes D^T \Delta_\gamma P^3_\gamma,\ \Gamma^{16}_\gamma \triangleq \bar{\alpha}_2 E_2 L_1(s) \mathcal{L}^T \otimes P^3_\gamma \\
&\Gamma^{17}_\gamma \triangleq L_2(s) \mathcal{L}^T \otimes P^3_\gamma,\ \Gamma^{18}_\gamma \triangleq \mathcal{L}^T \otimes C^T \Delta_\gamma P^4_\gamma \\
&\Gamma^{19}_\gamma \triangleq \mathcal{L}^T \otimes (I_{n_y} - \Delta_\gamma) P^4_\gamma,\ \Gamma^{20}_\gamma \triangleq \mathcal{L}^T \otimes D^T \Delta_\gamma P^4_\gamma \\
&\Upsilon^{12}_\gamma \triangleq \left(\sqrt{p_{\gamma 1}}\tilde{\Upsilon}^{12}_\gamma, \ldots, \sqrt{p_{\gamma n_y}}\tilde{\Upsilon}^{12}_\gamma\right) \\
&\Upsilon^{22}_\gamma \triangleq \mathrm{diag}\Big\{P_1 - 2P_\gamma, \ldots, P_{n_y} - 2P_\gamma\Big\} \\
&P_\gamma \triangleq \mathrm{diag}\Big\{I_M \otimes P^1_\gamma, I_M \otimes P^2_\gamma, I_M \otimes P^3_\gamma, I_M \otimes P^4_\gamma\Big\}.
\end{aligned}$$

Furthermore, for any $\gamma \in \Upsilon$, the gain matrices K_γ and L_γ are designed by

$$K_\gamma = (B^T P^1_\gamma B)^{-1} B^T B W_\gamma,\ L_\gamma = (P^2_\gamma)^{-1} Z_\gamma$$

and the upper bound $\mathcal{B}(\epsilon, \|\psi(s)\|_\infty)$ is calculated by

$$\mathcal{B}(\epsilon, \|\psi(s)\|_\infty) = \sqrt{\frac{\theta \max_{\gamma\in\Gamma}\{\lambda_{\max}\{\Omega^\gamma_2\}\} R_{\lambda_{\max}}}{\epsilon^2 \varepsilon \min_{\gamma\in\Gamma}\{\lambda_{\min}\{\Omega^\gamma_1\}\}}}$$

where

$$R_{\lambda_{\max}} = \max\big\{\lambda_{\max}\{R_1\}, \lambda_{\max}\{R_2\}, \lambda_{\max}\{R_3\}, \{R_4\}\big\}.$$

Proof: For any $i, j \in \Upsilon$, consider the following inequality:

$$(P_i - P_j) P_j^{-1} (P_i - P_j) \geq 0.$$

Then, it follows that

$$-P_i P_j^{-1} P_i \leq P_j - 2P_i. \tag{35}$$

By incorporating (35) into the analysis in the proof of Theorem 1, it can be readily verified that inequality (35) holds. As the derivation closely parallels the procedure established in Theorem 1, the detailed steps are omitted here for brevity. ■

Remark 5: So far, a comprehensive study has been conducted on the observer-based quasi-consensus control problem for MASs with time-varying delays under the combined influence of the SCP and AaF relay mechanism. Compared with existing methods, the proposed strategy is distinguished by its unified treatment of delayed MASs under the joint influence of SCP scheduling, AaF relay transmission, and stochastic packet dropouts, which provides a more general and practically relevant communication framework. A distributed observer-based controller is developed to enable local state estimation from relative measurement outputs under random Markovian parameters and unreliable communication conditions. In addition, a probabilistic quasi-consensus analysis framework is established, and constructive LMI-based conditions are derived in Theorems 1 and 2 for the design of the gain matrices $L_{\gamma(s)}$ and $K_{\gamma(s)}$. Therefore, the proposed method not only extends existing observer-based quasi-consensus results to a more realistic setting with time-varying delays and relay-assisted communication, but also provides a tractable and implementable design procedure.

IV. Illustrative Example

In this section, a simulation example is provided to demonstrate the effectiveness of the observer-based control strategy (6). Consider a class of MASs with time-varying delays consisting of five agents and the associated Laplacian matrix is given as:

$$\mathcal{L} = \begin{bmatrix} 0.8 & 0 & 0 & 0 & -0.8 \\ -0.5 & 1 & 0 & -0.5 & 0 \\ -0.6 & 0 & 0.6 & 0 & 0 \\ -0.5 & 0 & 0 & 1 & -0.5 \\ 0 & -0.8 & 0 & 0 & 0.8 \end{bmatrix}.$$

For MAS (1), the parameters are chosen as follows:

$$A = \begin{bmatrix} 0.99 & -0.45 \\ -0.10 & -0.73 \end{bmatrix}, B = \begin{bmatrix} 0.2 \\ 0.12 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.05 & 0 \\ 0 & 0.82 \end{bmatrix}, D = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.3 \end{bmatrix}$$

$$A_\tau = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, \mathrm{P} = \begin{bmatrix} 0.55 & 0.45 \\ 0.4 & 0.6 \end{bmatrix}$$
$$L_{1l}(s) = 0.1, L_{2l}(s) = 0.08, E_{1,l} = 1, E_{2,l} = 1$$
$$\overline{\alpha}_{1l} = 0.9, \overline{\alpha}_{2l} = 0.95, \overline{\tau} = 3, \underline{\tau} = 1, E = \begin{bmatrix} 0.06 \\ 0.24 \end{bmatrix}$$

Using the MATLAB toolbox, the gain matrices K_γ and L_γ satisfying the conditions (32)-(34) are obtained as:

$$K_1 = \begin{bmatrix} -1.7284 & 0.8632 \end{bmatrix}, L_1 = \begin{bmatrix} 0.9099 & -0.1649 \\ -0.0581 & -0.2664 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -0.7797 & 1.5696 \end{bmatrix}, L_2 = \begin{bmatrix} 0.2742 & -0.5047 \\ -0.0287 & -0.7798 \end{bmatrix}.$$

To evaluate performance, the following exogenous disturbances and initial conditions are applied:

$$\omega_1(s) = 0.8\cos(1.2s), \omega_2(s) = 0.8\cos(0.5s)$$
$$\omega_3(s) = 0.8\cos(1.3s), \omega_4(s) = 0.8\cos(0.5s)$$
$$\omega_5(s) = 0.4\sin(0.4s) + 0.4\cos(0.7s)$$
$$v_1(s) = \begin{bmatrix} 0.8\sin(1.2s) \\ 0.8\sin(1.2s) \end{bmatrix}, v_2(s) = \begin{bmatrix} 0.8\sin(1.7s) \\ 0.8\sin(1.7s) \end{bmatrix}$$
$$v_3(s) = \begin{bmatrix} 0.8\cos(s) \\ 0.8\cos(s) \end{bmatrix}, v_4(s) = \begin{bmatrix} 0.8\cos(0.5s) \\ 0.8\cos(0.5s) \end{bmatrix}$$
$$v_5(s) = \begin{bmatrix} 0.4\sin(0.8s) + 0.4\cos(0.6s) \\ 0.4\sin(0.8s) + 0.4\cos(0.6s) \end{bmatrix}$$
$$x_1(0) = \begin{bmatrix} 1 \\ -6.5 \end{bmatrix}, x_2(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, x_3(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$x_4(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}, x_5(0) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \hat{\xi}_l(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \eta_l(-1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with $x_l(-3) = x_l(-2) = x_l(-1) = x_l(0)$ for $l = 1, \dots, 5$. The remaining parameters are set as follows:

$$\epsilon = 0.05, \quad n_{1l}(s) = \frac{0.1\cos(s)}{(s+1)^2}, \quad n_{2l}(s) = \frac{0.15\cos(s)}{(s+1)^2}$$

The simulation results supporting the conclusions of this paper are shown in Figs. 2-6, demonstrating the robustness of the proposed method. Under the observer-based controller

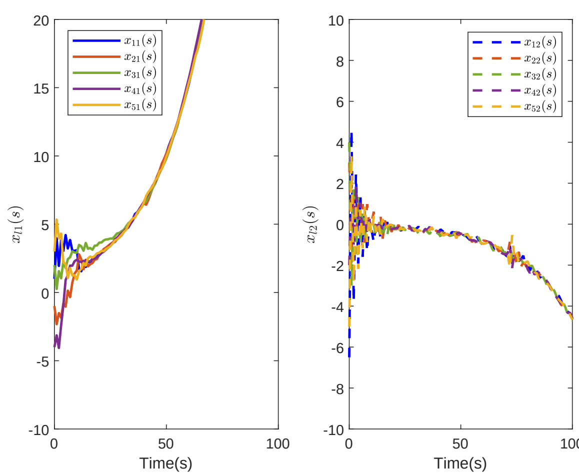

Fig. 2. Trajectories of the state $x_{lk}(s)$.

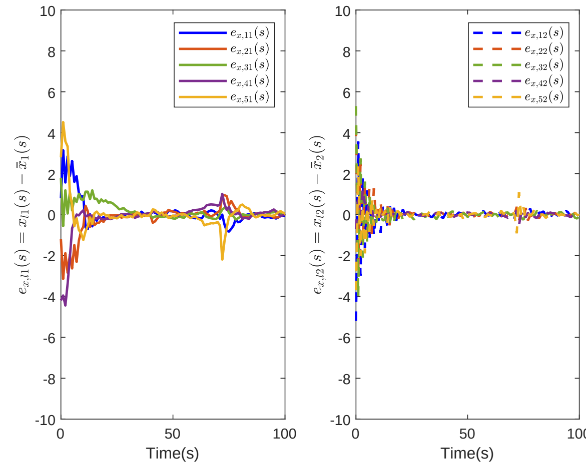

Fig. 3. Trajectories of state deviation $e_{x,lk}(s)$.

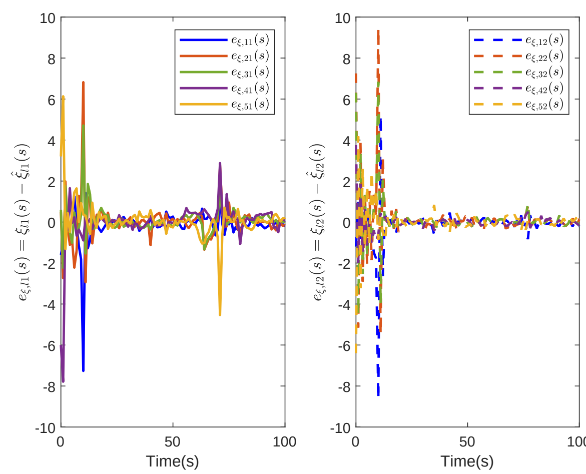

Fig. 4. Trajectories of the estimation error $e_{\xi,lk}(s)$.

(6), the state trajectories of all agents are depicted in Fig. 2, while the state errors and estimation errors are illustrated in Figs. 3 and 4, respectively, where the subscripts lk represents the k-th component of agent l. Figs. 2-4 demonstrate the effectiveness of the proposed control strategy, showing that the agents can achieve practical consensus, with both estimation errors and state errors converging to zero. Figs. 5 and 6 illustrate the occurrence of random packet loss for each agent during the CTR and RTO stages, respectively. In both figures, the horizontal axis represents time, while the vertical axis represents the random variables $\alpha_{1l}(s)$ (Fig. 5) and $\alpha_{2l}(s)$ (Fig. 6) for each agent l with $l = 1, 2, \dots, 5$. The markers indicate instances when $\alpha_{1l}(s) = 1$ or $\alpha_{2l}(s) = 1$, which means that agent l experiences packet loss at that moment.

V. Conclusion

This paper is the first to investigate the quasi-consensus in probability problem for MASs with time-varying delays under the combined influence of AaF relay and SCP networks. An observer-based control strategy has been proposed, wherein

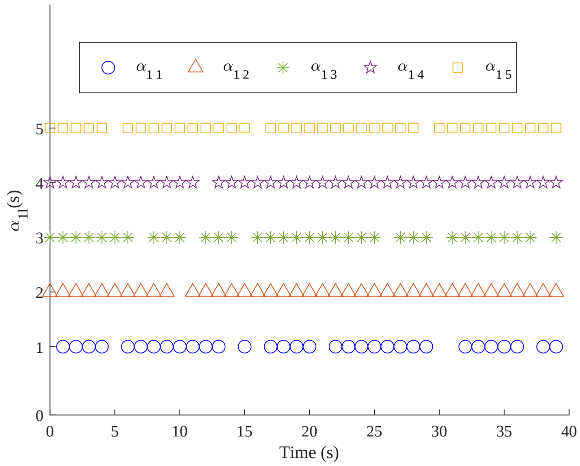

Fig. 5. The occurrence of packet loss events for each agent during the CTR stage, i.e., $\alpha_{1l}(s) = 1$ for $l = 1, \ldots, 5$.

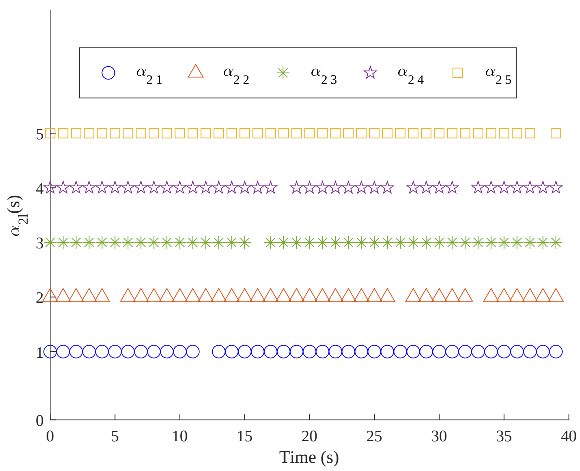

Fig. 6. The occurrence of packet loss events for each agent during the RTO stage, i.e., $\alpha_{2l}(s) = 1$ for $l = 1, \ldots, 5$.

measurement signals are first scheduled via the SCP protocol and then transmitted through AaF relays. Random measurement losses during AaF transmission have also been explicitly accounted for. Under these combined effects, the original MAS with time-varying delays has been reformulated as a stochastic difference equation featuring both Markov switching and packet dropouts. To address the quasi-consensus objective, a probabilistic framework has been established based on Lyapunov theory, and sufficient conditions for achieving quasi-consensus in probability have been derived in terms of linear matrix inequalities, which have also facilitated the design of the corresponding controller gain matrices. Finally, a simulation example has been presented to validate the theoretical results and demonstrate the effectiveness of the proposed method. Future research will focus on 1) applying the developed theoretical framework to practical engineering scenarios [44]–[46]; 2) integrating dynamic quantization mechanisms to address remaining challenges in distributed control of networked MASs [47], [48]; and 3) developing a fully distributed design for large-scale MASs from the viewpoint of scalability and implementation efficiency.

Appendix

The derivations of (8)–(10) are given as follows.

From the definition of the relative measurement output $\eta_l(s) \triangleq \sum_{k\in\mathfrak{n}_l} d_{lk}(\hat{y}_l(s) - \hat{y}_k(s))$, substituting $\hat{y}_l(s)$ and $r_l(s)$ leads to:

$$\begin{aligned}
\eta_l(s) =& \sum_{k\in\mathfrak{n}_l} d_{lk}\Big(\sqrt{E_{2,l}}\alpha_{2l}(s)r_l(s) + L_{2l}(s)n_{2l}(s) \\
&- \sqrt{E_{2,k}}\alpha_{2k}(s)r_k(s) - L_{2k}(s)n_{2k}(s)\Big) \\
=& \sum_{k\in\mathfrak{n}_l} d_{lk}\Big(\alpha_{1l}(s)\alpha_{2l}(s)\sqrt{E_{2,l}}\sqrt{E_{1,l}}y_l(s) \\
&+ \sqrt{E_{2,l}}\alpha_{2l}(s)L_{1l}(s)n_{1l}(s) + L_{2l}(s)n_{2l}(s) \\
&- \alpha_{1k}(s)\alpha_{2k}(s)\sqrt{E_{2,k}}\sqrt{E_{1,k}}y_k(s) \\
&- \sqrt{E_{2,k}}\alpha_{2k}(s)L_{1k}(s)n_{1k}(s) - L_{2k}(s)n_{2k}(s)\Big)
\end{aligned}$$

Then, by applying properties of the Kronecker product, (8) can be readily obtained. Subsequently, we recall the definitions of the state deviation $e_x^l(s)$ and the estimation error $e_\xi^l(s)$, and combining them with (7) and $\bar{x}(s) = \frac{1}{M}(\mathbf{1} \otimes I_{n_x})x(s)$, we obtain:

$$\begin{aligned}
e_x(s+1) =& x(s+1) - \frac{1}{M}(\mathbf{1} \otimes I_{n_x})x(s+1) \\
=& \big(I_M \otimes A + \mathfrak{M}\mathcal{L} \otimes BK_{\gamma(s)}\big)e_x(s) \\
&+ (\mathfrak{M} \otimes E)\omega(s) + (I_M \otimes A_\tau)e_x(s-\tau(s)) \\
&- \big(\mathfrak{M} \otimes BK_{\gamma(s)}\big)e_\xi(s).
\end{aligned}$$

For $e_\xi(s+1)$ in (9), we substitute $x(s+1)$, $\hat{\xi}(s+1)$, and $\eta(s)$ as follows:

$$\begin{aligned}
e_\xi(s+1) =& \xi(s+1) - \hat{\xi}(s+1) \\
=& (\mathcal{L} \otimes I_{n_x})\Big((I_M \otimes A)x(s) + (I_M \otimes A_\tau)x(s-\tau(s)) \\
&+ \big(I_M \otimes BK_{\gamma(s)}\big)\hat{\xi}(s) + (I_M \otimes E)\omega(s)\Big) \\
&- \Big(\big(I_M \otimes A - \bar{\alpha}E_R \otimes L_{\gamma(s)}C\big) \\
&\times \hat{\xi}(s) + (I_M \otimes A_\tau)\hat{\xi}(s-\tau(s)) + \big(I_M \otimes L_{\gamma(s)}\big) \\
&\times \Big(\big(\alpha(s)E_R\mathcal{L} \otimes \Delta_{\gamma(s)}C\big)e_x(s) \\
&+ \big(\alpha(s)E_R\mathcal{L} \otimes \Delta_{\gamma(s)}D\big)v(s) \\
&+ \big(\alpha(s)E_R\mathcal{L} \otimes (I_{n_y} - \Delta_{\gamma(s)})\big)y(s-1) \\
&+ \big(\alpha_2(s)E_2L_1(s)\mathcal{L} \otimes I_{n_1}\big)n_1(s) \\
&+ \big(L_2(s)\mathcal{L} \otimes I_{n_2}\big)n_2(s)\Big) + \big(\mathcal{L} \otimes BK_{\gamma(s)}\big)\hat{\xi}(s)\Big) \\
=& \big(\bar{\alpha}E_R\mathcal{L} \otimes L_{\gamma(s)}C - \alpha(s)E_R\mathcal{L} \otimes L_{\gamma(s)}\Delta_{\gamma(s)} \\
&\times C\big)e_x(s) + (I_M \otimes A_\tau)e_\xi(s-\tau(s)) \\
&+ \big(I_M \otimes A - \bar{\alpha}E_R \otimes L_{\gamma(s)}C\big)e_\xi(s) \\
&- \big(\alpha(s)E_R\mathcal{L} \otimes L_{\gamma(s)}(I_{n_y} - \Delta_{\gamma(s)})\big)y(s-1) \\
&+ (\mathcal{L}\mathfrak{M} \otimes E)\omega(s) - \big(L_2(s)\mathcal{L} \otimes L_{\gamma(s)}\big)n_2(s) \\
&- \big(\alpha(s)E_R\mathcal{L} \otimes L_{\gamma(s)}\Delta_{\gamma(s)}D\big)v(s) \\
&- \big(\alpha_2(s)E_2L_1(s)\mathcal{L} \otimes L_{\gamma(s)}\big)n_1(s).
\end{aligned}$$

Based on the above derivations and the definitions of $\varphi(s)$, $\psi(s)$, $\mathscr{A}\gamma(s)$, $\mathscr{A}\tau$, and $\mathscr{E}\gamma(s)$, the following augmented system can be obtained:

$$\varphi(s+1) = \mathscr{A}_{\gamma(s)}\varphi(s) + \mathscr{A}_{\tau}\varphi(s-\tau(s)) + \mathscr{E}_{\gamma(s)}\psi(s)$$

This concludes the proof.

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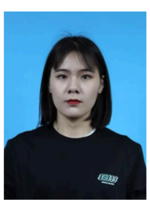

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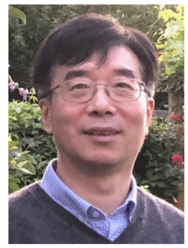

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