Journal of Financial Econometrics, 2026, **24(3),** nbag008
https://doi.org/10.1093/jjfinec/nbag008
Article

OXFORD

Enforcing an Admissible Parameter Space for Vector Multiplicative Error Models: The Fundamental Role of Matrix Inequality Constraints

Menelaos Karanasos[1], Yongdeng Xu[2], Stavroula Yfanti[3], and Constantin Zopounidis[4]

[1]Economics and Finance, Brunel University of London, Uxbridge, UK
[2]Cardiff Business School, Cardiff University, Cardiff, UK
[3]School of Business and Management, Queen Mary University of London, London, UK
[4]School of Production, Engineering and Management, Technical University of Crete, Chania, Greece
Address correspondence to M. Karanasos, Economics and Finance, Brunel University of London, Kingston Lane, Uxbridge UB8 3PH, UK, or e-mail: menelaos.karanasos@brunel.ac.uk.

Abstract

We derive an admissible parameter space for vector multiplicative error models (vMEMs), explicitly formulating it in terms of the model's matrix parameters through a set of matrix inequalities. Another key contribution is the adoption of constrained maximum likelihood estimation for the multivariate process, which ensures compliance with these matrix inequalities and addresses the limitations of unconstrained approaches used in previous studies. To demonstrate the effectiveness of the proposed method, we apply it to four empirical cases in financial volatility modeling, emphasizing its practical relevance.

Key words admissible parameter space, constrained maximum likelihood estimation, matrix inequalities, MEM, multivariate volatility modeling, second moment structure
JEL classifications C32, C53, C58, G15

Building on the foundational research of Bollerslev, Engle, and Wooldridge (1988) and Bollerslev (1990) on MGARCH models,[1] subsequent scholarly efforts have developed sophisticated multivariate frameworks designed to capture both conditional and unconditional interdependencies among system variables. The methodological advancements in this field have been marked by significant contributions, particularly the MGARCH specification proposed by Jeantheau (1998) and the vMEM, whose theoretical and empirical implications have been extensively examined by Manganelli (2005),

We would like to thank two anonymous referees for their insightful and constructive feedback, and for their careful reading of the manuscript through three rounds of detailed comments, which significantly improved the quality of the paper.

[1] MGARCH is an abbreviation for multivariate generalized autoregressive heteroscedasticity.

Received: March 23, 2023. **Revised:** December 26, 2025. **Accepted:** March 12, 2026

Cipollini and Gallo (2010), and Cipollini, Engle, and Gallo (2013, 2017).[2] A growing body of research has consistently demonstrated their efficacy and reliability across a range of analytical contexts. Over the years, these two multivariate systems have gained widespread empirical validation in financial and economic time series analysis, with Cipollini and Gallo (2025) providing a comprehensive survey of the latter.

A well-defined parameter space is essential to ensuring the theoretical and practical validity of vMEM. Without appropriate restrictions, these models may become mathematically invalid, leading to erroneous outputs, such as unrealistic predictions—like negative variance values—which should not occur. This underscores that the parameter space is not merely a technical detail but a fundamental aspect of model design that directly impacts both theoretical soundness and practical usability.

Conrad and Karanasos (2010), in the context of modeling volatility interactions using a MGARCH formulation, addressed the challenge of ensuring that the conditional variables remain almost surely positive at all points in time, including both estimated (in-sample) and forecasted (out-of-sample) values. Specifically, they derive the so-called final equation representation of Jeantheau's MGARCH process, expressed through a marginalized univariate series, and subsequently obtain necessary and sufficient nonnegativity conditions in terms of the corresponding parameters.[3] Explicit expressions as functions of the model's matrix parameters were provided only for the bivariate $(1, 1)$ specification. The absence of such expressions for higher dimensions prevented them from incorporating these constraints into the estimation procedure and limited their empirical applications to two dimensions.

This study begins by reformulating the nonnegativity conditions for an N-dimensional system of order $(1, q)$, originally derived by Conrad and Karanasos (2010), and explicitly representing them as matrix inequalities in terms of the model's matrix parameters (see Theorem 1).

Examining two contrasting scenarios, the first involves excessively rigid constraints that force all parameters to remain nonnegative, as in Jeantheau (1998), imposing unnecessary limitations on flexibility and accuracy. At the other end of the spectrum, when no constraints are applied, parameter estimation gains full flexibility, allowing negative values. While this enhances adaptability, it also introduces the risk of unrealistic results, particularly when inherently nonnegative variables receive negative predictions—or, more precisely, when there is a positive probability that they will eventually become negative. We illustrate this concern with examples from Manganelli (2005), Cipollini and Gallo (2010), and Cipollini, Engle, and Gallo (2013, 2017); see Supplementary Appendix F and Section 5.1.

In this vein, we introduce the acronym cvMEM, which carries a dual interpretation. Specifically, the c stands for constrained vMEM in the context of estimation and corrected vMEM when referring to parameterization, reflecting the methodological distinction in our approach. That is, to correct the parameterization used in the three cited studies, we incorporate matrix inequality constraints in vMEM estimation, ensuring a feasible parameter space and thereby preserving both the theoretical and empirical validity of the model.

Consequently, our second key contribution (see Section 5) is the development and implementation of a constrained maximum likelihood (ML) estimation technique for the multivariate process, distinguishing it from the unconstrained approaches applied in prior research such as Conrad and Karanasos (2010) and Cipollini, Engle, and Gallo (2013).[4] A Monte Carlo simulation confirms the superiority of the former over the latter (see Section 4). Third, we establish nonnegativity conditions for asymmetric N-dimensional systems. These constraints introduce additional matrix inequalities,

[2] vMEM stands for vector multiplicative error model.

[3] The research by Nelson and Cao (1992), He and Teräsvirta (1999), Gourieroux (2007), Tsai and Chan (2007, 2008), Nakatani and Teräsvirta (2008, 2009), Conrad (2010), and Conrad and Karanasos (2010) underscores the theoretical significance of deriving such necessary and sufficient conditions. This line of research originated with Nelson's (1991) seminal work. However, there is ongoing debate in the literature and among specialized econometric software users regarding the constraints that should be imposed on the aforementioned parameters. With respect to univariate GARCH models, and the distinction between asymmetry and leverage effects, Stavroyiannis (2018) demonstrates that the approach implemented by various software packages does not consistently align with the "Nelson–Cao" inequality constraints. In particular, Monte Carlo simulations revealed that the estimated parameters were not theoretically coherent with these constraints, which are essential for ensuring positivity of the conditional variables.

[4] The former study employed quasimaximum likelihood (QML), whereas the latter utilized the efficient generalized method of moments (GMM).

extending the results of Conrad and Karanasos (2010). We also derive theoretical insights into the second-order moments of these models, contributing to the literature by expanding on He and Teräsvirta (2004), who focused exclusively on the symmetric case within a GARCH framework. Further details are provided in Supplementary Appendices C.3 and C.4.

He and Teräsvirta (1999), in their analysis of a univariate GARCH model, and Conrad and Karanasos (2010), in the context of a bivariate GARCH(1,1) specification, illustrate that relaxing nonnegativity constraints enhances the flexibility of the autocorrelation function's structure compared to cases where all parameters are strictly nonnegative. Similarly, allowing certain parameters to take negative values is expected to improve the predictive performance of multivariate models. A detailed presentation of the explicit expressions for optimal forecasts is provided in Supplementary Appendices C.1 and C.2, while a Monte Carlo simulation empirically substantiates this claim (see Section 4). We emphasize that analyzing optimal forecasts and the second-moment structure gives rise to two additional sets of constraints: one ensuring the existence of unconditional means (see Remark C.1 in Supplementary Appendix C.1) and the other concerning the existence of unconditional variances (see Remark C.3 in Supplementary Appendix C.3).

The practical relevance and effectiveness of the proposed method are demonstrated through four empirical applications, each conducted on a distinct real-world dataset. For instance, our analysis reveals significant interactions among high–low range volatility, realized volatility, and absolute return in European stock indices, with evidence of negative bidirectional conditional dependencies between the first two (see the first empirical example in Section 5.1). Range volatility captures intraday fluctuations that contain microstructure effects and long-range dependence of asset prices (Alizadeh, Brandt, and Diebold 2002; Chou 2005; Brandt and Diebold 2006). One of our findings—namely, that, high–low range volatility, negatively impacts realized volatility—is consistent with Cipollini, Engle, and Gallo (2013), and suggests that the extreme values involved in the range metric play an incremental informational role in lowering the five-minute intradaily measure.[5]

The second financial application in our study illustrates the unrestricted interactions among the high–low range volatility of 4 European equity markets, with 7 of the 12 conditional mean dynamic interdependencies being negative, while still adhering to nonnegativity constraints.

Furthermore, our empirical analysis demonstrates notable interactions among the intersector volatilities within U.S. and European credit default swap markets. For example, recent conditional default risk volatility in the consumer sector negatively affects current volatility levels in the industrial sector, suggesting a potential stabilizing effect arising from intersectoral interaction.

Finally, our finding that the conditional mean of stock volume exerts a negative influence on that of volatility—observed in two out of the five datasets used in the last empirical example (see Supplementary Appendix E.1)—aligns with the theoretical framework proposed by Wang (2007). Wang argues that foreign purchases reduce market volatility in emerging economies by expanding the investor base, thereby enhances informational efficiency and stabilizes stock prices.

The outline of the paper is as follows. Section 1 introduces the vMEM, a multivariate asymmetric specification with dynamic interdependencies. In Section 2, we review the "one-sided" representation of the process, followed by a formal exposition of the necessary and sufficient nonnegativity conditions in Section 3.2, where they are explicitly formulated as matrix inequality constraints in terms of the model's matrix parameters. To enhance understanding, Section 3.3 presents numerical examples demonstrating these conditions. Section 4 outlines the Monte Carlo simulation, describing its design and findings. Section 5 provides empirical examples along with an out-of-sample forecasting exercise, emphasizing the fundamental role of matrix inequalities in defining a feasible parameter space and, consequently, ensuring model validity. The conclusions are presented in Section 6. A full proof of Theorem 1, including all intermediate steps and necessary assumptions, is provided in Appendix A. A Supplementary Appendix discusses optimal forecasts and the second moment structure of our model, containing proofs and additional details, such as examples illustrating cases where the absence of constraints in the parameter space led to erroneous parameterization.

[5] See Supplementary Appendix E.2.

1. The vMEM

In this section, we introduce the vMEM (see Cipollini and Gallo 2025, for a survey on MEM). To maintain clarity throughout this paper, we adhere to the following notational conventions:

The set of integers is denoted by $\mathbb{Z}$, while its subset is defined as $\mathbb{Z}_a = \{z \in \mathbb{Z} : z \geq a\}$ for $a \in \mathbb{Z}$. The set of real numbers, along with its subsets of positive and nonnegative real numbers are represented by $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$, respectively. Moreover, $(\Omega, \mathcal{M}, P)$ designates a probability space, and $\mathcal{L}_2(\Omega, \mathcal{M}, P)$, abbreviated as $\mathcal{L}_2$, stands for the Hilbert space of real random variables with finite first two moments. Furthermore, we adopt the convention of using uppercase boldface symbols for square matrices and bold lowercase letters for vectors. Specifically, $\mathbf{z} = [z_i]_{i=1,\dots,N}$ is an $N \times 1$ column vector for $N \in \mathbb{Z}_2$, and $\mathbf{Z} = [z_{ij}]_{i,j=1,\dots,N}$ corresponds to a square matrix of order N. To streamline notation, we will omit subscripts in subsequent expressions unless needed for clarity. The expectation operator $\mathbb{E}$ is applied elementwise, so that $\mathbb{E}(\mathbf{z}) = [\mathbb{E}(z_i)]$. Similarly, the conditional expectation given $\mathcal{M}_{t-1}$, the filtration encompassing all information available through time $t-1$, is applied elementwise, resulting in $\mathbb{E}(\mathbf{z}_t \mid \mathcal{M}_{t-1})$.

We consider an N-dimensional process, $\mathbf{y}_t$, which is assumed to be governed by the following mathematical formulation:

$$\mathbf{y}_t = \boldsymbol{\mu}_t \odot \mathbf{e}_t. \tag{1}$$

In this context, $\boldsymbol{\mu}_t$ is measurable with respect to $\mathcal{M}_{t-1}$. The notation $\odot$ signifies the elementwise Hadamard product.

The random vector $\mathbf{e}_t$ is independent and identically distributed (*i.i.d.*). Moreover, it remains strictly positive for all t, that is, $\mathbf{e}_t > \overrightarrow{\mathbf{0}}$ where $\overrightarrow{\mathbf{0}}$ is the vector of all zeros. Its conditional expectation, given the filtration $\mathcal{M}_{t-1}$, equals the unit vector: $\mathbb{E}(\mathbf{e}_t \mid \mathcal{M}_{t-1}) = \mathbf{j}$. Therefore, the conditional expectation of $\mathbf{y}_t$ given $\mathcal{M}_{t-1}$ satisfies $\mathbb{E}(\mathbf{y}_t \mid \mathcal{M}_{t-1}) = \boldsymbol{\mu}_t$. Furthermore, we assume that $\mathbf{e}_t$ has positive definite conditional covariance matrix $\mathbb{C}(\mathbf{e}_t|\mathcal{M}_{t-1}) = \mathbf{Q}$. The implications for the conditional second moments of $\mathbf{y}_t$ are discussed in detail in Supplementary Appendix C.3.

A key challenge in constructing a valid model is the selection of a precisely defined and well-specified parameter space for the conditional mean vector $\boldsymbol{\mu}_t$ that ensures its positivity for all t, not only within the observed sample (estimated values) but also in future periods (forecasted values). From this point forward, superscripts enclosed in parentheses or brackets (e.g., $(.)^{(m)}$) designate the index position of the corresponding term (e.g., mth term) in a sequence, distinguishing them from exponents representing powers.

The asymmetric version with dynamic interdependencies of the vMEM of order $(1, q)$, as proposed by Cipollini and Gallo (2010) and Cipollini, Engle, and Gallo (2013), is characterized by the following system of equations:

$$\mu_{it} = \omega_i + \sum_{l=1}^{q} \sum_{j=1}^{N} (\alpha_{ij}^{(l)} + \gamma_{ij}^{(l)} s_{j,t-l}) y_{j,t-l} + \sum_{j=1}^{N} \beta_{ij} \mu_{j,t-1},$$

where $i = 1, \dots, N$, $q \in \mathbb{Z}_1$, and $s_{it} = I(x_{it} < 0)$ is an indicator function that takes the value 1 when its argument is true and 0 otherwise. The variable x_{it} represents a signed quantity such as underlying stock returns.[6] The indicator variable s_{it} is a binary random variable, as it takes values of either 1 or 0, depending on the sign of x_{it}. Specifically, its formulation is given by $s_{it} = 0.5[1 - sgn(x_{it})1]$. Its expected value, $\mathbb{E}(s_{it}) = P(x_{it} < 0)$, corresponds to the probability that x_{it} is negative, which is 0.5 if it is a continuous random variable with zero median.

[6] The asymmetry under consideration was initially introduced by Glosten, Jagannathan, and Runkle (1993). Additionally, we consider an alternative asymmetry proposed by Ding, Granger, and Engle (1993); results related to this specification are available upon request.

The model described above can be represented as an N-dimensional system incorporating both unconditional and conditional dynamic interdependencies:

$$(\mathbf{I}-\mathbf{B}L)\boldsymbol{\mu}_t = \boldsymbol{\omega}\sum_{l=1}^{q} L^l\left(\mathbf{A}^{(l)} + \Gamma^{(l)}\mathbf{S}_t\right)\mathbf{y}_t, \tag{2}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{B}$ is a full square matrix of order N, where the off-diagonal elements capture the dynamic interdependencies between the conditional means and L is the lag operator: $L\mathbf{z}_t = \mathbf{z}_{t-1}$; $\boldsymbol{\omega}$ is a vector that contains the constants; $\mathbf{A}^{(l)}$, $l=1,\ldots,q$, are full matrices, whose cross-diagonal elements capture the interactions between the unconditional variables; $\boldsymbol{\Gamma}^{(l)}$ are also full matrices and account for both asymmetric own and cross effects. $\mathbf{S}_t$ is a diagonal matrix with elements s_{it}. Observe that if asymmetries are absent, meaning that $\mathbf{S}_t = \mathbf{0}$ for all t, then the model simplifies to its symmetric counterpart.

Remark 1 *The vMEM specification previously discussed exhibits structural similarities to corresponding MGARCH systems. This resemblance underscores the potential for applying analogous analytical techniques and methodological approaches (i.e., the use of matrix inequality constraints) to both formulations. In their latest research, Karanasos, Xu, and Yfanti (2025b) have introduced a specific designation for this MGARCH framework, which they term: multivariate unrestricted full asymmetric (MUFA). The specification is classified as full because all square matrices are full, and unrestricted, because, as shown below, certain elements of* $\mathbf{B}$ *(including some off-diagonal ones)*, $\mathbf{A}^{(l)}$ *(for* $2\le l\le q$*), and* $\boldsymbol{\Gamma}^{(l)}$ *are not constrained to be strictly positive and may also take negative values. Similarly, the vMEM formulation considered here is designed to accommodate feedback effects between the conditional means, allowing for both positive and negative interactions (refer to Theorem 1 below).*[7]

Next, we will present the identifiability condition for the vMEM; The determinant of a square matrix $\mathbf{Z}$ will be denoted by det$[\mathbf{Z}]$.

Assumption A1 (Identifiability). The formulation of the N-dimensional vMEM of order $(1,q)$ at the true values of the parameters is minimal if $\mathbf{I}-\mathbf{B}L$, $\sum_{l=1}^{q}\mathbf{A}^{(l)}L^l$ (positive signed variables), and $\sum_{l=1}^{q}\left(\mathbf{A}^{(l)}+\boldsymbol{\Gamma}^{(l)}\right)L^l$ (negative signed variables), satisfy the following conditions:

1) $\mathbf{I}-\mathbf{B}L$ is column reduced, that is $det[\mathbf{B}]\neq 0$.
2) The following operators are invertible:
$det[\mathbf{I}-\mathbf{B}z]\neq 0$, $det[\sum_{l=1}^{q}\mathbf{A}^{(l)}z^l]\neq 0$, $det[\sum_{l=1}^{q}(\mathbf{A}^{(l)}+\boldsymbol{\Gamma}^{(l)})z^l]\neq 0$ for $|z|\le 1$.
3) $\sum_{l=1}^{q}\mathbf{A}^{(l)}L^l$ (the case of positive signed variables) and $\mathbf{I}-\mathbf{B}L$ are coprime. That is, any of the greatest common left divisors of $\sum_{l=1}^{q}\mathbf{A}^{(l)}L^l$ and $\mathbf{I}-\mathbf{B}L$ are unimodular (a matrix lag-operator polynomial is called unimodular if its determinant is a nonzero constant independent of the lag operator L). In addition, $\sum_{l=1}^{q}\left(\mathbf{A}^{(l)}+\boldsymbol{\Gamma}^{(l)}\right)L^l$ (negative signed variables) and $\mathbf{I}-\mathbf{B}L$ are coprime as well.

The identifiability of the model in Equation (2) is guaranteed by Assumption A1. In the case of the symmetric MGARCH model, analogous conditions are established in Proposition 3.4 of Jeantheau (1998) and Assumption A1 of Conrad and Karanasos (2010).

[7] In the restricted full (symmetric) formulation of order$(1,1)$ (see Jeantheau, 1998; Ling and McAleer, 2003 for the GARCH case), the matrices $\mathbf{A}$ and $\mathbf{B}$ are full but constrained to contain only nonnegative elements. As noted by Conrad and Karanasos (2010), the assumption that only positive feedback is allowed is appealing because positive constants and parameter matrices with nonnegative coefficients provide a sufficient condition for ensuring the positive definiteness of the conditional covariance matrix in the extended formulation.

As previously highlighted, a key issue lies in determining the necessary and sufficient conditions for the vMEM to ensure a positive $\boldsymbol{\mu}_t$ for all t. This will be the primary focus of the analysis in the following sections.

2. "One-sided" representation

This section presents an important proposition. In particular, we establish the "one-sided" infinite-order expansion of each conditional mean, formulated in terms of convolutions of MEM kernels and the unconditional variables.

We begin by presenting the following definition.

Definition 1. *To facilitate the forthcoming discussion, we define* $\beta(L) = 1 - \sum_{i=1}^{N} \beta_i L^i = \prod_{i=1}^{N}(1 - \phi_i L)$ *as follows:*

$$\beta(L) = \det[\mathbf{I} - \mathbf{B}L], \tag{3}$$

where ϕ_i are the roots of $\beta(z^{-1})$.

Under Assumption A1, where $\beta_N \neq 0$, the $\beta(L)$ constitutes a scalar polynomial of order N. For the sake of consistency and without loss of generality, we shall adhere to the convention: $|\phi_1| \geq |\phi_2| \geq \cdots \geq |\phi_N|$, following Nelson and Cao (1992) and Tsai and Chan (2008).

We now proceed to establish the invertibility condition for the N-dimensional vMEM.

Assumption A2 (Invertibility). The roots ϕ_i, $i = 1, \ldots, N$ of $\beta(z^{-1})$ in Equation (3) lie inside the unit circle.

Assumption A2 ensures the invertibility of the model specified in Equation (2). Comparable conditions in the MGARCH setting are outlined in Assumption B1 in Jeantheau (1998) and Assumption A1 in Conrad and Karanasos (2010).

Before proceeding, we introduce a general observation that will be applied methodically later in our analysis. In particular, we consider the following proposition, which provides an explicit representation of the multivariate "one-sided" decomposition.

The notation $\mathbf{Z}^k = \prod_{i=1}^{k} \mathbf{Z}$ signifies the matrix power operation, while $adj[\mathbf{Z}]$ denotes the adjoint of $\mathbf{Z}$.

Proposition 1 *Let Assumptions A1 and A2 be satisfied. Then the vMEM$(1, q)$ in Equation (2) admits the multivariate "one-sided" representation:*

$$\boldsymbol{\mu}_t = \frac{adj[\mathbf{I} - \mathbf{B}]\boldsymbol{\omega}}{\beta(1)} + \sum_{k=1}^{\infty} \sum_{s=1}^{\min(q,k)} \mathbf{B}^{k-s} L^k \left(\mathbf{A}^{(s)} + \boldsymbol{\Gamma}^{(s)} \mathbf{S}_t\right) \mathbf{y}_t, \tag{4}$$

where $adj[\mathbf{I} - \mathbf{B}]\boldsymbol{\omega}/\beta(1) = [\mathbf{I} - \mathbf{B}]^{-1}\boldsymbol{\omega}$.

The proof is trivial and, hence, omitted.

3. Admissible parameter space

In this section, the conditions guaranteeing nonnegativity—both necessary and sufficient—are expressed through matrix inequalities that depend explicitly on the model's matrix parameters.

To begin, we introduce a formal claim concerning the asymmetries.

3.1 Claim and notation

Claim 1. *It suffices to demonstrate that the constraints hold under the two extreme scenarios:*

i) *when, for every t, all N signed variables x_{it} are positive, thereby yielding $\mathbf{S}_t = \mathbf{0}$,*
ii) *when, for each t, we have $\mathbf{S}_t = \mathbf{I}$, meaning all signed variables are negative.*

All intermediate cases, in which some signed variables are positive while the remainder are negative, are thereby covered. The result arises because, analogous to the threshold GARCH (TGARCH) specification of Zakoïan (1994), each component $\left(\mathbf{A}^{(l)} + \mathbf{\Gamma}^{(l)}\mathbf{S}_t\right)\mathbf{y}_t$, for $l = 1, \ldots, q$, in Equation (4) can be partitioned as $\mathbf{A}^{(l)}\mathbf{y}_t^{(+)} + \left(\mathbf{A}^{(l)} + \mathbf{\Gamma}^{(l)}\right)\mathbf{y}_t^{(-)}$. Here, $\mathbf{y}_t^{(+)} = (\mathbf{I} - \mathbf{S}_t)\mathbf{y}_t$ and $\mathbf{y}_t^{(-)} = \mathbf{S}_t\mathbf{y}_t$, both of which are constructed to contain only nonnegative entries.[8]

Before stating Theorem 1, we define the notation used in its formulation.

We refer to the elementwise absolute value of $\mathbf{Z}$ as $abs[\mathbf{Z}] = [|z_{ij}|]$. Additionally, $\max[\mathbf{Z}]$ denotes the largest element of $\mathbf{Z}$, and $\log[\mathbf{Z}]$ represents the elementwise logarithm of $\mathbf{Z}$, namely $\log[\mathbf{Z}] = [\log(z_{ij})]$.

Notation 1.

i) *Consider the collection of matrices $\mathbf{Y}^{(n)} = \left[y_{ij}^{(n)}\right]$ for $n = 1, \ldots, N$. For each matrix, $\max[\mathbf{Y}^{(n)}]$ is defined as $\max\left(y_{ij}^{(n)}\right)$ for $i, j = 1, \ldots, N$, meaning it represents the largest element among the N^2 entries of $\mathbf{Y}^{(n)}$,*
ii) *Let $\mathbf{Y}_{\max^{(n)}} = \left[\max_{1 \le n \le N}\left(y_{ij}^{(n)}\right)\right]$. Here, each entry in the ijth position is determined by taking the maximum of the N corresponding values $y_{ij}^{(n)}$, ensuring that the element is the largest among them.*

Notation 2.

(i) The computation of κ_{ij} (and κ), which appear in Condition (C3a) in Theorem 1(B), is accomplished through the following procedure:

(a) Define $\mathbf{H}^{(n)} = [\eta_{ij}^{(n)}]$, for each n in the range $1 \le n \le N$, given by

$$\mathbf{H}^{(n)} = abs\left[\frac{adj[\mathbf{I}\phi_n - \mathbf{B}]\sum_{l=1}^{q}\mathbf{A}^{(l)}}{\sum_{j=1}^{N} j\beta_j\phi_n^{N-(j-1)}}\right].$$

Moreover, we introduce the matrix $\mathbf{H}_{\max}^{(n)} = \left[\max_{2 \le n \le N}\left(\eta_{ij}^{(n)}\right)\right]$, where each (i, j)th entry is the largest value among $\eta_{ij}^{(n)}$ for all n from 2 to N,

(b) We now introduce the matrix $\mathbf{\Phi} = [\varphi_{ij}]$, as specified by

$$\mathbf{\Phi} = \{\log[\mathbf{H}^{(1)}] - \log[(N-1)\mathbf{H}_{\max^{(n)}}]\}[\log(|\phi_2|) - \log(|\phi_1|)]^{-1}.$$

Next, consider the matrix $\mathbf{K} = [\kappa_{ij}]$, where each element κ_{ij} is computed as the smallest integer satisfying $\kappa_{ij} \ge \max\{0, \varphi_{ij}\}$. Also let $\kappa = \max[\mathbf{K}]$, indicating the maximum value within $\mathbf{K}$,

(ii) Similarly, the quantities $\tilde{\kappa}_{ij}$ [stated in Condition (C3b)] and $\tilde{\kappa}$ are derived analogously to those in (i) except that in (a) the term $\sum_{l=1}^{q}\mathbf{A}^{(l)}$ is replaced by $\sum_{l=1}^{q}\left(\mathbf{A}^{(l)} + \mathbf{\Gamma}^{(l)}\right)$.

[8] We appreciate the anonymous referee's insightful comment regarding this point.

3.2 Matrix inequality constraints

Theorem 1 *For the N-dimensional vMEM*$(1, q)$ *satisfying Assumptions A1 and A2, under the mild condition that all N roots* ϕ_i *are distinct, the necessary and sufficient conditions ensuring* $\mu_{it} > 0$ *for every* $i = 1, \ldots, N$ *and all t are given by*

(A) $adj[\mathbf{I} - \mathbf{B}]\boldsymbol{\omega} > \mathbf{0}$

$$(B)\begin{cases} \phi_1 \text{ is real}, \text{ and } \phi_1 > 0 & (C1) \\ adj[\mathbf{I}\phi_1 - \mathbf{B}]\sum_{l=1}^{q}\mathbf{A}^{(l)}\phi_1^{q-l} > \mathbf{0} & (C2a) \\ adj[\mathbf{I}\phi_1 - \mathbf{B}]\sum_{l=1}^{q}\left(\mathbf{A}^{(l)} + \boldsymbol{\Gamma}^{(l)}\right)\phi_1^{q-l} > \mathbf{0} & (C2b) \\ \sum_{s=1}^{\min(q,k)}\mathbf{B}^{k-s}\mathbf{A}^{(s)} \geq \mathbf{0} & (C3a) \\ \sum_{s=1}^{\min(q,k)}\mathbf{B}^{k-s}\left(\mathbf{A}^{(s)} + \boldsymbol{\Gamma}^{(s)}\right) \geq \mathbf{0}. & (C3b) \end{cases}$$

The matrix inequalities detailed in (C3a) must hold for each entry indexed by (i,j)*, ensuring validity for every* $k \in \{1, \ldots, \kappa_{ij}\}$*. Similarly, the conditions in (C3b) must be satisfied for every* (i,j)*-th entry, with k spanning from 1 to* $\tilde{\kappa}_{ij}$*.*

The proof of the aforementioned theorem is detailed in Appendix A.4 (with further information available in Supplementary Appendices A.1–A.3). Theorem 1 establishes that the admissible parameter space for the vMEM is governed by matrix inequality constraints, which serve as a foundational component of its theoretical framework. Consequently, it is imperative that this admissible parameter space be enforced via a constrained estimation approach that integrates these matrix inequality restrictions to ensure the model's validity.

Next, we detail two important remarks that elucidate central aspects of our findings.

Remark 2. *Condition (C1) requires that the inverse root of* $\det[\mathbf{I} - \mathbf{B}L]$ *with the largest modulus, labeled as* ϕ_1*, must belong to the set of positive real numbers. Regarding the last two conditions in part (B), since* $\kappa = \max_{i,j}\{\kappa_{ij}\}$ *and* $\tilde{\kappa} = \max_{i,j}\{\tilde{\kappa}_{ij}\}$ *(see Notation 2) it suffices to verify the nonnegativity constraints specified in Conditions (C3a) and (C3b) for all k from 1 up to* κ *and* $\tilde{\kappa}$*, respectively.*

Remark 3 *As indicated by Tsai and Chan (2008), two crucial points are in order:*

i) *The assumption* $|\phi_1| > |\phi_2|$ *is essential for our constant* κ*, as specified in Notation 2, to be well defined. However, it is important to note that while this distinct-root assumption underpins the precise definition of* κ*, it is not necessary for the validity of Conditions (C1) and (C2a).*
ii) *Apart from a parametric subset of zero Lebesgue measure, the roots in question are inherently distinct, thereby satisfying* $|\phi_1| > |\phi_2|$*. Consequently, from an estimation standpoint, the scenario of encountering equal roots is effectively negligible.*

The forthcoming remarks demonstrate that Conditions (C2a) and (C3a) can be interpreted as a direct matrix-based generalization of the corresponding conditions in Tsai and Chan (2008) to a multivariate setting, thereby preserving the original theoretical insights while adapting them to a higher dimensional framework.

Remark 4 *Conditions (C2a) and (C3a) in Theorem 1 provide a matrix-based extension of the findings of Tsai and Chan (2008) to the multivariate domain. Tsai and Chan demonstrate that, under mild conditions, ensuring the nonnegativity of a specific finite subset of scalar parameters in the infinite ARCH representation of the univariate GARCH model is one component of a set of three conditions that, collectively, are necessary and sufficient for guaranteeing that the remaining parameters are also nonnegative (see eq. (8) in their paper). Analogously, Condition (C3a) in our framework imposes elementwise nonnegativity on only a finite number of matrix parameters in the multivariate "one-sided" representation. Another condition in this set requires that the scalar ARCH polynomial, when evaluated by substituting the lag operator L with* $1/\phi_1$*—the smallest absolute root of the GARCH polynomial—be nonnegative (refer to eq. (7) in their paper). Condition (C2a) in our analysis generalizes this by introducing its matrix equivalent.*

The following proposition derives the reduced form of Theorem 1 when specifically applied to the vMEM(1,1). By specializing the general result to this case, we obtain a simplified formulation that enhances interpretability while preserving the key theoretical insights of the original theorem.

Proposition 2. *For the N-dimensional vMEM*$(1,1)$*, which corresponds to the specification with* $q=1$*, the conditions (B) outlined in Theorem 1 simplify to*

$$(B)\begin{cases} \phi_1 \text{ is real, and } \phi_1 > 0 & (C1) \\ adj[\mathbf{I}\phi_1 - \mathbf{B}]\mathbf{A} > \mathbf{0}, \quad adj[\mathbf{I}\phi_1 - \mathbf{B}](\mathbf{A}+\boldsymbol{\Gamma}) > \mathbf{0} & (\mathrm{C2a,b}) \\ \mathbf{B}^{k-1}\mathbf{A} \geq \mathbf{0}, \quad \mathbf{B}^{k-1}(\mathbf{A}+\boldsymbol{\Gamma}) \geq \mathbf{0}, & (\mathrm{C3a,b}) \end{cases}$$

where $\mathbf{A}^{(1)} \overset{def}{=} \mathbf{A}$ *and* $\Gamma^{(1)} \overset{def}{=} \Gamma$*. As in Theorem 1 the matrix inequalities detailed in Condition (C3a) must hold for each entry indexed by* (i,j)*, ensuring validity for every* $k \in \{1,\dots,\kappa_{ij}\}$*. Similarly, the conditions in (C3b) must be satisfied for every* (i,j)*th entry, with k spanning from 1 to* $\tilde{\kappa}_{ij}$*. Analogously to Theorem 1, the terms* κ_{ij} *and* $\tilde{\kappa}_{ij}$ *are specified by Notation 2, this time with* $q=1$.

In Supplementary Appendix B.2, we provide a demonstration that the matrix inequalities outlined in Theorem 1 can be equivalently expressed in terms of scalar inequalities. For an illustration of this result, we refer the reader to Supplementary Appendix B.3, where we present a detailed example: the trivariate symmetric case of order $(1,1)$. The subsequent section includes numerical examples that demonstrate the practical application of the proposed constraints.

3.3 Numerical examples

In what follows, we graphically illustrate the necessary and sufficient parameter set for the trivariate symmetric system. These examples may be particularly beneficial for researchers who prioritize implementation over theoretical derivations, as they provide a direct approach to applying the constraints in a structured and computationally accessible manner. We discuss four examples (see Tables 1 and 2):

- We consider the case where two off-diagonal elements of **B** are allowed to vary within the range $[-0.5, 0.5]$. As an initial example, we investigate the scenario in which β_{13} and β_{31} are varied. The objective is to determine whether bidirectional negative (conditional) feedback is feasible.
- In the second case, we permit β_{21} and β_{31} (i.e., two parameters from the first column of **B**) to vary. The goal is to examine whether a single variable can exert negative (conditional) influences on the other two.
- In the third scenario, we modify β_{21} and β_{23}, both positioned in the second row of **B**, to check if two conditional variables can negatively affect a third one.
- In the fourth example, we explore whether more than two off-diagonal elements of the **B** matrix can be negative. This is achieved by fixing β_{21} as negative while varying β_{13} and β_{31}.

Table 1 Parameter settings for Examples 1 and 2.

	Example 1	Example 2
$\boldsymbol{\omega}'$	$\begin{pmatrix} 0.149 & 0.074 & 0.124 \end{pmatrix}$	$\begin{pmatrix} 0.153 & 0.106 & 0.103 \end{pmatrix}$
A	$\begin{pmatrix} 0.064 & 0.021 & 0.158 \\ 0.008 & 0.005 & 0.108 \\ 0.028 & 0.043 & 0.198 \end{pmatrix}$	$\begin{pmatrix} 0.075 & 0.017 & 0.123 \\ 0.021 & 0.002 & 0.109 \\ 0.030 & 0.037 & 0.104 \end{pmatrix}$
B	$\begin{pmatrix} 0.790 & 0.032 & \beta_{13} \\ 0.001 & 0.808 & 0.006 \\ \beta_{31} & 0.137 & 0.616 \end{pmatrix}$	$\begin{pmatrix} 0.780 & 0.003 & 0.017 \\ \beta_{21} & 0.901 & 0.022 \\ \beta_{31} & 0.082 & 0.650 \end{pmatrix}$

Table 2 Parameter settings for Examples 3 and 4.

	Example 3	Example 4
$\boldsymbol{\omega}'$	$\begin{pmatrix} 0.162 & 0.114 & 0.117 \end{pmatrix}$	$\begin{pmatrix} 0.214 & 0.184 & 0.164 \end{pmatrix}$
A	$\begin{pmatrix} 0.075 & 0.011 & 0.140 \\ 0.013 & 0.003 & 0.139 \\ 0.023 & 0.044 & 0.201 \end{pmatrix}$	$\begin{pmatrix} 0.078 & 0.012 & 0.171 \\ 0.012 & 0.005 & 0.100 \\ 0.048 & 0.029 & 0.228 \end{pmatrix}$
B	$\begin{pmatrix} 0.744 & 0.002 & 0.051 \\ \beta_{21} & 0.901 & \beta_{23} \\ 0.009 & 0.056 & 0.559 \end{pmatrix}$	$\begin{pmatrix} 0.743 & 0.031 & \beta_{13} \\ -0.028 & 0.851 & 0.053 \\ \beta_{31} & 0.111 & 0.548 \end{pmatrix}$

The choice of the three negative β's is largely guided by the empirical results outlined in Table 5 of Section 5, particularly those related to the S&P 500 and DAX indices in Dataset 1. The figures below illustrate the parameter combinations that satisfy the necessary and sufficient conditions of Theorem 1, along with those required for the existence of the first and second unconditional moments. These conditions are detailed in the Supplementary Appendices:

◁ C.1 (see eq. (C.4) and Remark C.1),

◁ C.3 (see eq. (C.15) in Theorem C.2 and Remark C.3).

We begin by analyzing the implications of Example 1, as illustrated in Figure 1a:

1) *Theorem compliance*—All combinations of β_{13} and β_{31} enclosed within the green solid lines satisfy the conditions established in Theorem 1.
2) *Existence of unconditional moments*—Parameter combinations bounded by the dotted grey (dashed red) lines fulfill the conditions for the existence of the first (second) unconditional moments.
3) *Key finding*—Notably, both off-diagonal elements can simultaneously assume negative values.

Figure 1b provides a visual representation of Example 2. According to the conditions outlined in Theorem 1 negative effects from μ_{1t} to μ_{2t} and μ_{3t} are permissible. The region in the third quadrant that lies above and to the right of all three lines represents the negative parameter set that satisfies all conditions simultaneously. As illustrated in Figure 1c, the parameter configurations in Example 3 satisfy the conditions of Theorem 1, allowing negative interactions from μ_{1t} and μ_{3t} to μ_{2t}. Example 4 reveals an interesting result: three off-diagonal elements in the **B** matrix can be negative while still fulfilling all nonnegativity conditions.

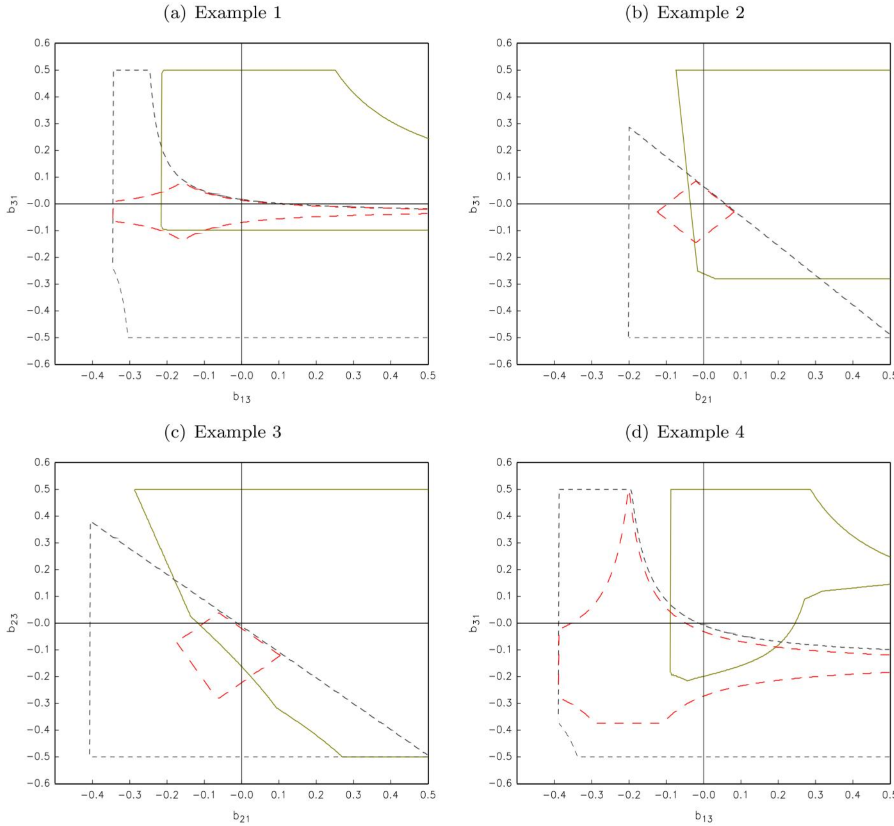

Figure 1 Necessary and sufficient parameter sets for the trivariate symmetric MEM(1, 1) from Examples 1–4. Solid brown lines represent the restrictions implied by Proposition 2. Dotted gray lines represent the restrictions implied by the existence of the unconditional first moment. Dashed red lines represent the restrictions implied by the existence of the unconditional second moment.

4. Monte Carlo simulations

In this section, we employ the symmetric (1, 1) specification and apply Monte Carlo simulations to evaluate the impact of omitting the nonnegativity constraints in Theorem 1. Our focus is on:

i) The potential bias in ML estimates and
ii) The implications for out-of-sample forecast accuracy.

We conduct a comparative analysis of three estimation scenarios:

I Incorporating matrix inequality constraints within the estimation process.
II Enforcing Bollerslev's sufficient conditions, which restrict all parameters to non-negative values.
III Implementing an unconstrained estimation method.

In the context of the vMEM, the data generating process (DGP) is defined as follows. The disturbance vector $\mathbf{e}_t$ is drawn from a multivariate log-normal distribution, where the unit vector $\mathbf{j}$ serves as the conditional expectation, and the conditional covariance matrix is given by $\mathbf{Q} = [q_{ij}]$.

The findings are derived from Monte Carlo simulations conducted with 1000 repetitions, each utilizing a sample size of 1000 observations.

Table 3 The mean and RMSE of ML estimates.

	True	Mean			RMSE		
		Case I	Case II	Case III	Case I	Case II	Case III
ω_{11}	0.214	0.244	0.185	0.245	0.328	0.216	0.417
ω_{12}	0.184	0.239	0.246	0.279	0.319	0.353	0.390
ω_{13}	0.164	0.106	0.194	0.049	0.293	0.209	0.370
α_{11}	0.078	0.077	0.070	0.079	0.082	0.082	0.095
α_{12}	0.012	0.014	0.012	0.017	0.019	0.027	0.045
α_{13}	0.200	0.202	0.192	0.211	0.212	0.208	0.226
α_{21}	0.012	0.028	0.020	0.031	0.044	0.042	0.063
α_{22}	0.005	0.009	0.009	0.010	0.015	0.019	0.039
α_{23}	0.100	0.088	0.068	0.106	0.113	0.117	0.150
α_{31}	0.150	0.152	0.148	0.151	0.153	0.150	0.158
α_{32}	0.029	0.029	0.030	0.030	0.034	0.034	0.049
α_{33}	0.120	0.113	0.113	0.113	0.119	0.119	0.125
β_{11}	0.743	0.747	0.637	0.743	0.769	0.669	0.790
β_{12}	0.031	0.018	0.113	0.024	0.206	0.221	0.307
β_{13}	−0.060	−0.064	0.021	−0.076	0.147	0.070	0.215
β_{21}	−0.020	0.041	0.046	0.108	0.219	0.127	0.334
β_{22}	0.851	0.792	0.752	0.726	0.823	0.777	0.792
β_{23}	0.053	−0.003	0.078	−0.096	0.178	0.169	0.299
β_{31}	−0.120	−0.196	0.003	−0.259	0.317	0.015	0.422
β_{32}	0.111	0.204	0.024	0.269	0.342	0.050	0.450
β_{33}	0.548	0.562	0.470	0.572	0.574	0.473	0.601

The true parameter values are reported in the first column. Case I imposes the matrix inequality constraints of Theorem 1. Case II enforces Bollerslev's sufficient conditions. In Case III, no constraints are imposed. In this case, approximately 20 out of the 1000 simulations exhibit negative conditional means during the simulation/optimization process; these cases are disregarded. RMSE stands for the root mean squared error.

Table 4 RMSE for the out-of-sample forecasting.

Model	$k=1$			$k=5$			$k=20$		
	Case I	Case II	Case III	Case I	Case II	Case III	Case I	Case II	Case III
$\mathbb{E}(\boldsymbol{\mu}_{1,t+k}\|\mathcal{M}_t)$	0.142	0.314	1.017	4.572	4.534	14.887	9.903	6.667	48.354
$\mathbb{E}(\boldsymbol{\mu}_{2,t+k}\|\mathcal{M}_t)$	0.270	0.493	1.310	2.888	3.131	29.258	7.623	4.823	26.537
$\mathbb{E}(\boldsymbol{\mu}_{3,t+k}\|\mathcal{M}_t)$	0.113	0.182	0.659	3.013	3.300	26.813	8.194	4.932	27.808

$k=1,5,20$ are 1-, 5-, and 20-day-ahead forecasting, respectively. Cases I, II, and III are as in Table 3. To obtain the forecasts, we use eq. (C.2) in Supplementary Appendix C.1.

It is important to note that if all parameter values in the DGP are nonnegative, there should be no discrepancies among the three estimation approaches.

In our specified DGP, three elements of the **B** matrix $(\beta_{13}, \beta_{21}, \beta_{31})$ are assigned negative values while preserving the matrix inequalities outlined in Theorem 1.

The corresponding parameter values are presented in the first column of Table 3. The estimation method incorporating constraints exhibits lower bias compared to the other two approaches. Conversely, the unconstrained estimation displays the poorest performance, as indicated by both higher bias and greater root mean squared error (RMSE).

The RMSEs for out-of-sample predictions are presented in Table 4. When forecasting one step ahead, the estimation method incorporating matrix inequalities produces the most accurate results.

At the five-step-ahead horizon, the first two estimation approaches perform equally well. For 20-step-ahead forecasting, the model estimated under Bollerslev's sufficient conditions outperforms the others, primarily due to its ML estimates exhibiting lower standard deviations. In contrast, the unconstrained estimation approach performs the worst.

5. Empirical analysis

In this section, we conduct an estimation of trivariate and four-variate cvMEM$(1,2)$:

$$(\mathbf{I}-\mathbf{B}L)\boldsymbol{\mu}_t=\boldsymbol{\omega}+\left(L\left(\mathbf{A}^{(1)}+\boldsymbol{\Gamma}^{(1)}\mathbf{S}_t\right)+L^2\mathbf{A}^{(2)}\right)\mathbf{y}_t. \quad (5)$$

Here, we reiterate that $\mathbf{y}_t=\boldsymbol{\mu}_t\odot\mathbf{e}_t$ (see Equation (1)), and that $\mathbf{y}_t$ is the vector containing the observed series.[9] As previously stated, the vMEM assumes that the stochastic vector $\mathbf{e}_t>\overrightarrow{\mathbf{0}}$ (and, consequently, $\mathbf{y}_t>\overrightarrow{\mathbf{0}}$) follows an *i.i.d.* distribution with a conditional expectation given by the unit vector $\mathbf{j}$. This implies that $\boldsymbol{\mu}_t=\mathbb{E}(\mathbf{y}_t\mid\mathcal{M}_{t-1})$, with a corresponding conditional covariance matrix $\mathbf{Q}$ (refer to Supplementary Appendix C.3 for additional discussion). The components of $\mathbf{y}_t$ may comprise three distinct measures of volatility—namely, high–low range volatility, absolute return and realized volatility—for a single asset. Alternatively, they can serve as a proxy for volatility across multiple financial markets, particularly high-low range volatility. In another context, they may consist of squared returns from three sectoral CDS (Credit Default Swap) indices. Finally, the components can also include intraday trading duration, volume and volatility.

Following the methodology proposed by Taylor and Xu (2017), we adopt a multivariate log-normal (conditional) distribution to model the innovation vector $\mathbf{e}_t$: $\mathbf{e}_t\mid\mathcal{M}_{t-1}\sim\ln N(\mathbf{j},\mathbf{Q})$. The log-likelihood function, derived from the observed data $\boldsymbol{y}_t$, is given by

$$l(\theta)=\sum\nolimits_{t=1}^{T}\ln f(y_t|\boldsymbol{\theta}),$$

where

$$\ln f(y_t\mid\boldsymbol{\theta})=-\frac{N}{2}\ln(2\pi)-\frac{1}{2}\ln|\mathbf{Q}|-\sum\nolimits_{i=1}^{N}\ln y_{it}-\frac{1}{2}(\ln\boldsymbol{y}_t-\ln\boldsymbol{\mu}_t-\boldsymbol{\xi})'\mathbf{Q}^{-1}(\ln\boldsymbol{y}_t-\ln\boldsymbol{\mu}_t-\boldsymbol{\xi}). \quad (6)$$

Here, we define $\boldsymbol{\xi}-1/2\mathbf{q}$, where $\mathbf{q}$ represents the vector consisting of the diagonal elements of $\mathbf{Q}$. Additional details can be found in Supplementary Appendix D.[10] In the subsequent analysis, we estimate four cvMEMs—three trivariate and one four-variate—based on data availability and model feasibility.

We enforce the matrix inequalities stated in Theorem 1, thereby incorporating parameter space restrictions in conjunction with sample data through constrained ML estimation. To incorporate conditions (C3a) and (C3b) into the constrained ML estimation, we proceed as follows:

1) We begin by setting κ and $\tilde{\kappa}$ equal to N and impose the associated constraints during estimation.
2) After obtaining the parameter estimates, we compute the implied values of κ and $\tilde{\kappa}$ and verify ex post whether conditions (C3a) and (C3b) are satisfied.
3) If either condition is not satisfied, we increment κ and $\tilde{\kappa}$ by one and repeat the checking procedure until the conditions are fulfilled.

[9] In Supplementary Appendix C, for the vMEM$(1,1)$, we present explicit solutions for the optimal predictors and the second moment structure.

[10] An alternative estimation method was proposed by Cipollini, Engle, and Gallo (2013). They bypassed the specification of the conditional distribution of the errors and used only the first two conditional moments of the errors by employing an efficient GMM estimation approach. Through a simulation study, Taylor and Xu (2017) demonstrated that both the QML and GMM estimation techniques are consistent and that the efficiency loss of QML estimation compared to GMM estimation due to misspecification of the error distribution is negligible.

In all empirical applications, initiating the procedure with $\kappa = \tilde{\kappa} = N$ has been sufficient, as the ex post checks consistently confirm compliance with the constraints. Our use of an iterative strategy is motivated by the fact that a full imposition of (C3a) and (C3b) would produce an excessively large number of constraints, especially when the implied value of $\tilde{\kappa}$ (or of κ) is high, thereby increasing computational complexity and reducing optimization stability. Moreover, because (C3a) and (C3b) depend on power functions of κ and $\tilde{\kappa}$, these terms vanish beyond a threshold under stationarity, rendering additional constraints immaterial in practice. For this reason, we employ a simplified stepwise procedure that reduces computational burden and streamlines estimation.

5.1 The cvMEM estimation results

Our analysis reveals that a critical aspect of the vMEM was overlooked in the three cited papers. The lack of a formally defined admissible parameter space led to an unconstrained estimation procedure. Consequently, the estimated parameters fell outside the theoretically feasible range (see Supplementary Appendix F), creating a nonzero probability that variables that should remain strictly nonnegative could take on negative values over time. This fundamental issue is central to Theorem 1, which underscores the necessity of enforcing appropriate (matrix inequality) constraints to ensure parameter admissibility and maintain model validity.[11]

This section presents three empirical applications that illustrate the flexibility and practical relevance of the proposed framework. Each example focuses on a distinct setting, ranging from asset-level volatility modeling to cross-market and sectoral dynamics. The details of each case are discussed below. The estimation results (based on Equations (5) and (6)) are presented in Tables 5–7.

A fourth one involves a trivariate system of intraday trading duration, stock volume, and volatility. We use the same dataset as in Manganelli (2005): five stock tickers and their corresponding names: AVT: AVNET INC, COX: COX COMMUNICATION, CP: CDN PACIFIC, DLP: DELTA & PINELAND, GAP: GREAT A&P. While he conducted an equation-by-equation estimation of the model, our method estimates the full framework concurrently.[12] For brevity, we defer this empirical example to Supplementary Appendix E.1.

5.1.1 Three volatility measures

The first example incorporates three volatility metrics—namely, high–low range volatility, absolute return, and realized volatility—and analyzes the performance of five major indices: Dow Jones Industrial Average (DJ30), Standard & Poor's 500 (S&P500), Nasdaq 100 **(**NASD100**)**, FTSE100 (FTSE), and DAX 30 (DAX). This framework was originally proposed by Engle and Gallo (2006) and later estimated by Cipollini, Engle, and Gallo (2013).[13] While we replicate the variable selection from Cipollini, Engle, and Gallo (2013), our analysis is based on a distinct dataset and sample period—January 1, 2010–November 19, 2021—to provide new insights into the interactions among different volatility measures. The realized volatility data are sourced from the Oxford-Man Institute of Realized Volatility Lab, which stopped providing updates in 2022. For the purpose of comparison, the dataset employed by Cipollini, Engle, and Gallo (2013) has also been adopted; refer to Supplementary Appendices E.2 and E.3 for additional information.

Our results reveal substantial dynamic interactions among the three different volatility measurements—daily high–low range, absolute return, and realized volatility—in agreement with the work of Cipollini, Engle, and Gallo (2013) (see Table 5). The range captures intraday price fluctuations (Alizadeh, Brandt, and Diebold 2002; Chou 2005; Brandt and Diebold 2006; Martens and Van Dijk

[11] The empirical analysis presented below does not aim to replicate the work of the three referenced studies by employing identical estimation techniques (namely, GMM) or by strictly adhering to their exact specifications. However, constrained estimation can be applied seamlessly to both ML and GMM. In the latter framework, these constraints give rise to an optimization problem in which a quadratic form of the moment conditions is minimized subject to parameter constraints expressed as matrix inequalities.

[12] See Subsection 4.1 in Manganelli (2005) for a concise description of the data preparation process in both his paper and ours.

[13] For the description of their dataset, see Section 4 of Cipollini, Engle, and Gallo (2013).

Table 5 Trivariate MEM(1,2) of daily high–low range volatility, absolute return, and realized volatility.

	S&P500			DJ30			DAX			FTSE			NASD100		
$\mathbf{A}^{(1)}$	**0.085**	0.006	**0.044**	**0.089**	0.006	0.034	0.037	0.006	**0.043**	**0.060**	0.005	**0.040**	**0.117**	0.017	0.086
	(3.77)	(0.60)	(2.63)	(3.58)	(0.54)	(.)	(1.63)	(0.80)	(2.42)	(2.43)	(.)	(3.35)	(2.85)	(1.40)	(0.91)
	0.053	0.013	0.027	0.015	0.001	0.019	0.035	0.007	**0.065**	0.003	0.004	0.013	0.062	0.012	0.060
	(0.82)	(0.33)	(0.49)	(0.41)	(.)	(1.29)	(1.25)	(.)	(1.96)	(0.04)	(0.08)	(0.50)	(0.28)	(0.16)	(0.28)
	0.084	0.026	**0.264**	**0.056**	**0.033**	**0.185**	**0.084**	0.004	**0.098**	0.027	**0.034**	**0.102**	0.062	**0.062**	0.437
	(2.60)	(1.68)	(3.92)	(2.34)	(2.49)	(5.07)	(4.02)	(0.42)	(2.40)	(0.79)	(2.75)	(2.85)	(0.67)	(2.16)	(1.40)
B	**0.785**	0.039	**−0.043**	**0.739**	0.021	**−0.013**	**0.843**	0.028	**−0.040**	**0.818**	0.013	−0.017	**0.728**	**0.040**	−0.087
	(11.76)	(0.70)	(2.16)	(12.94)	(0.98)	(2.03)	(17.65)	(0.57)	(1.98)	(11.16)	(0.43)	(1.20)	(3.01)	(2.20)	(0.59)
	0.003	**0.912**	−0.029	0.003	**0.847**	−0.001	0.000	**0.943**	−0.067	0.004	**0.889**	0.001	0.002	**0.878**	−0.031
	(0.02)	(18.79)	(0.34)	(.)	(40.89)	(.)	(.)	(26.26)	(1.52)	(0.03)	(13.40)	(0.04)	(0.21)	(6.71)	(0.12)
	−0.036	0.060	**0.620**	0.003	0.011	**0.651**	−0.110	0.061	**0.777**	−0.019	0.004	**0.772**	0.005	0.038	**0.453**
	(0.23)	(0.50)	(6.17)	(0.14)	(.)	(14.55)	(1.02)	(0.69)	(16.13)	(0.15)	(0.06)	(16.57)	(0.05)	(0.11)	(3.66)
$\mathbf{\Gamma}^{(1)}$	**0.073**	**0.064**	0.045	**0.141**	**0.207**	**0.260**	**0.075**	0.089	**0.130**	**0.097**	**0.150**	**0.184**	**0.070**	0.097	−0.009
	(5.02)	(1.97)	(1.28)	(8.14)	(7.45)	(6.15)	(6.29)	(1.59)	(6.07)	(5.18)	(4.03)	(4.59)	(2.41)	(0.60)	(0.06)
$\mathbf{A}^{(2)}$	−0.003	−0.002	−0.002	−0.006	−0.001	−0.017	0.010	−0.007	0.001	−0.020	−0.004	−0.015	−0.018	−0.008	−0.036
	(0.11)	(0.06)	(0.05)	(0.26)	(0.02)	(0.42)	(0.57)	(0.36)	(0.04)	(1.02)	(0.09)	(0.45)	(0.61)	(0.14)	(0.21)

1st row: high–low range volatility; 2nd row: absolute return; 3rd row: realized volatility. Bollerslev–Wooldridge robust absolute *t*-statistics are shown in parentheses. Variables significant at the 5% confidence level are formatted in bold. The parameters ω_i and the q_{ij}, where $i, j = 1, \ldots, N$, are not reported but they are available upon request. The symbol "(.)" indicates that *t*-statistics are not reported because the associated coefficients are estimated at the boundary (i.e., zero) under the nonnegativity constraints, for which standard errors—and thus *t*-statistics—are not well defined.

Table 6 Four-variate MEM (1, 2) of daily volatility in four European equity markets.

	Restricted estimation				Unrestricted estimation			
$\mathbf{A}^{(1)}$	**0.083**	**0.109**	**0.031**	0.010	**0.101**	0.041	**0.067**	**0.073**
	(4.21)	(6.37)	(2.53)	(0.71)	(3.30)	(1.62)	(3.21)	(2.41)
	0.004	**0.152**	0.023	0.022	0.022	**0.079**	**0.063**	**0.088**
	(0.24)	(7.26)	(1.89)	(1.41)	(0.77)	(2.78)	(2.96)	(2.77)
	0.018	**0.059**	**0.056**	0.023	0.030	0.014	**0.082**	**0.070**
	(1.28)	(3.92)	(3.21)	(1.35)	(1.26)	(0.66)	(3.60)	(2.84)
	0.010	**0.046**	**0.020**	**0.082**	0.022	0.008	**0.043**	**0.119**
	(0.80)	(3.50)	(2.09)	(4.60)	(1.06)	(0.42)	(2.79)	(4.77)
B	**0.924**	**−0.147**	−0.009	0.004	**0.897**	−0.029	−0.063	**−0.114**
	(83.20)	(6.52)	(0.87)	(0.32)	(25.15)	(0.85)	(1.83)	(2.22)
	—	**0.767**	—	—	−0.030	**0.902**	−0.061	**−0.127**
		(39.32)			(0.80)	(26.46)	(1.71)	(2.34)
	−0.016	**−0.081**	**0.904**	−0.008	−0.036	−0.002	**0.871**	**−0.095**
	(1.09)	(4.10)	(43.64)	(0.35)	(1.15)	(0.06)	(31.08)	(2.37)
	−0.008	**−0.065**	0.008	**0.861**	−0.028	0.003	−0.023	**0.779**
	(0.60)	(3.63)	(0.64)	(39.69)	(0.97)	(0.13)	(0.89)	(17.94)
$\mathbf{\Gamma}^{(1)}$	**0.027**	**0.025**	**0.051**	**0.044**	**0.025**	**0.023**	**0.050**	**0.040**
	(5.52)	(5.03)	(7.79)	(7.10)	(5.40)	(5.14)	(7.85)	(6.32)
$\mathbf{A}^{(2)}$	**−0.026**	0.016	0.008	−0.007	−0.027	0.012	0.005	0.003
	(1.97)	(0.96)	(0.44)	(0.41)	(1.63)	(0.71)	(0.30)	(0.14)

The 1st, 2nd, 3rd, and 4th rows correspond to FR, DE, UK, and CH, respectively. Bollerslev–Wooldridge robust absolute *t*-statistics are shown in parentheses. Variables significant at the 5% confidence level are formatted in bold. The diagonal elements of the respective diagonal matrices are reported in the rows of $\mathbf{\Gamma}^{(1)}$ and $\mathbf{A}^{(2)}$. The parameters ω_i and the q_{ij}, where $i, j = 1, \ldots, N$, are not reported but are available upon request.

Table 7 Trivariate MEM (1, 2) of CDS volatility.

	US BANKS, CM, FB			EU BANKS, CM, FB		
$\mathbf{A}^{(1)}$	**0.184**	—	0.003	**0.208**	0.004	—
	(5.71)		(0.69)	(7.04)	(0.60)	
	0.032	**0.150**	**0.051**	0.011	**0.158**	**0.045**
	(2.12)	(4.53)	(2.09)	(1.54)	(3.93)	(1.96)
	0.059	0.016	**0.143**	0.001	**0.046**	**0.131**
	(2.12)	(0.81)	(3.53)	(0.26)	(2.56)	(4.09)
B	**0.845**	—	—	**0.897**	—	—
	(28.34)			(16.02)		
	—	**0.963**	**−0.168**	—	**0.915**	**−0.086**
		(34.01)	(2.03)		(19.80)	(2.04)
	0.042	0.033	**0.451**	—	−0.009	**0.858**
	(0.83)	(0.46)	(3.23)		(0.26)	(11.41)
$\mathbf{\Gamma}^{(1)}$	**−0.055**	**−0.032**	−0.063	**−0.052**	**−0.051**	**−0.048**
	(2.52)	(2.12)	(1.63)	(2.92)	(2.00)	(2.58)
$\mathbf{A}^{(2)}$	−0.008	**−0.084**	0.057	−0.092	−0.032	−0.050
	(0.20)	(2.70)	(0.65)	(1.66)	(0.79)	(1.89)

The 1st, 2nd, and 3rd rows correspond to Banks, CM, and FB, respectively. Bollerslev–Wooldridge robust absolute *t*-statistics are shown in parentheses. Variables significant at the 5% confidence level are formatted in bold. The diagonal elements of the respective diagonal matrices are reported in the rows of $\mathbf{\Gamma}^{(1)}$ and $\mathbf{A}^{(2)}$. The parameters ω_i and the q_{ij}, where $i, j = 1, \ldots, N$, are not reported but are available upon request.

2007). In the referenced study, the estimated $\mathbf{A}^{(1)}$ matrix includes negative elements, thus violating the non-negativity conditions (see Table F1 in Section F of the Supplementary Appendix). Nevertheless, the findings of Cipollini, Engle, and Gallo (2013) suggest the presence of negative cross effects among the three conditional means. If the **B** matrix is restricted to nonnegative entries, as in the BEKK formulation by Noureldin, Shephard, and Sheppard (2012), some essential dynamic interdependencies among the volatility measures may be lost, leading to potentially less precise forecasts. Accordingly, we allow some elements of the **B** matrix to be negative while still upholding the matrix inequalities specified in Theorem 1.[14] In effect, our estimation involves a less restricted model, augmented by its own asymmetric impacts, which are often statistically significant. Indeed, our findings reveal that β_{13} and β_{31} may both be negative, as demonstrated in three cases (S&P500, DAX and FTSE), yet the nonnegativity conditions remain satisfied (see Table E6 in Supplementary Appendix E.4). This finding is particularly noteworthy because it shows that the matrix inequalities allow negative conditional cross effects to occur in both directions. This stands in sharp contrast to a bivariate restricted (or even unrestricted) system, where negative bidirectional feedback is strictly forbidden (see Conrad and Karanasos 2010). In three cases, the negative parameter β_{13} is significant. Interestingly, across all five datasets, the conditional mean of the absolute returns is independent of fluctuations in the other two conditional means. The same applies to realized volatility. Lastly, the own asymmetries are, in most cases, positive and significant.

5.1.2 High–low range in four equity markets

In the second empirical case, Cipollini and Gallo (2010) estimated a multivariate process of four variables using daily high–low range data from four European stock markets (United Kingdom, France [FR], Germany [DE], and Switzerland [CH]). Building on Cipollini and Gallo (2010), we employ the same dataset but use a more recent sample—from January 1, 2010 to January 31, 2025—to extend the analysis to the present.

The links between the high–low ranges of four European equity markets are detailed in Table 6. Interestingly, 5 of the 12 off-diagonal elements of $\mathbf{A}^{(1)}$ are both positive and significant. For example, daily range in the U.K. market affects the conditional means of the French (FR) and Swiss (CH) markets (see the third column). Crucially, 7 of the 12 cross-effect elements in the **B** matrix are negative, with three attaining statistical significance, while the nonnegativity conditions outlined in Theorem 1 remain satisfied (refer to Table E7 in Supplementary Appendix E.4). Notably, the French equation (first row) includes two negative off-diagonal parameters, while in the U.K. equation (third row), all three are negative. Moreover, the observed increase in German market volatility corresponds to a decline in the three other conditional means (in the second column, all three off-diagonal elements are negative and significant). Lastly, all four own asymmetric influences are significant, while among the four diagonal elements of $\mathbf{A}^{(2)}$, two are negative, with one being statistically significant.

For comparison purposes, we also estimate an unrestricted model. Matrix $\mathbf{A}^{(1)}$ consists entirely of positive elements, while all off-diagonal elements of matrix **B** are negative, with the exception of a single entry (see the last four columns of Table 6).[15] Although both models allow for negative interdependencies, the constrained model imposes matrix inequalities to ensure that parameter values remain within the defined admissible space.

[14] As noted earlier, for our constrained ML estimation presented in Table 4, we imposed the matrix inequality constraints detailed in Theorem 1, specifically Conditions (A), (C1) and (C2a, C2b) in (B), as well as Conditions (C3a, C3b) in (B) for $\kappa, \tilde{\kappa} = 1, 2, 3$. After completing the estimation, we computed the values of κ and $\tilde{\kappa}$ using the formulae in Notation 2, denoting these estimates as $\widehat{\kappa}$ and $\widehat{\tilde{\kappa}}$, respectively. We then verified whether the conditions (C3) for $\kappa = 4, \ldots, max(\widehat{\kappa}, \widehat{\tilde{\kappa}})$ were satisfied for all five indices. In general, for an N-dimensional system, one should impose conditions (C3a, C3b) in Theorem 1(B) for $\kappa = 1, \ldots, N$. For large values of N, see Footnote 18 regarding the curse of dimensionality.

[15] The estimated value of ϕ_1 is complex, resulting in a violation of the matrix inequality constraints, in particular Condition C1 of Theorem 1.

5.1.3 Volatility in credit default swaps

The third empirical illustration incorporates three sectoral CDS indices from markets in the United States and the European Union (EU). Our sample covers a 20-year period from January 3, 2005, until January 31, 2025, of daily sectoral index returns based on 5-year corporate spreads for the following sectors: Banks, Construction & Materials (CM), and Food & Beverage (FB). We select one core financial industry (Banks) and two nonfinancial sectors (from industrials and consumer goods) to investigate cross-sectoral volatility interactions in corporate credit risk.

Table 7 reports the estimation of the two trivariate systems. We find significant interactions in corporate credit risk across the three different sectors under investigation.

In the U.S. case, positive unconditional influences from Banks are observed in the two nonfinancial sectors (as reflected in the significant estimated parameters $\alpha_{21}^{(1)}$ and $\alpha_{31}^{(1)}$), whereas no cross-sector effects are observed in the financial sector (based on the insignificance of $\alpha_{12}^{(1)}$ and $\alpha_{13}^{(1)}$). This indicates that the credit or lending process connecting financial and nonfinancial industries is characterized by a unidirectional impact from lenders to borrowers. Increased lender's default risk raises borrowers' credit risk volatility. However, in the European market, no significant unconditional interactions are observed between the financial and nonfinancial sectors. Regarding the interaction between industrial and consumer sectors, we observe positive bidirectional unconditional effects in the EU market (as indicated by significant $\alpha_{23}^{(1)}$ and $\alpha_{32}^{(1)}$ estimates), whereas in the U.S. market, the impact flows only from FB to CM. Moreover, most $\mathbf{A}^{(2)}$ elements are negative; however, only one is statistically significant, specifically for the U.S. industrial sector.

In the **B** matrix, our findings are consistent across both markets. We observe only one significant negative impact from consumer goods to industrials. The recent (conditional) default risk volatility of the consumer sector reduces the current volatility in the industrial sector.

Turning to the (own) asymmetric effects, they are predominantly negative and statistically significant.[16] This reflects the fact that, in CDS markets, negative shocks to credit spreads (i.e., negative returns) increase spread volatility, but the magnitude of this effect is smaller than that of the volatility increase associated with positive returns—contrary to equities, where negative shocks exacerbate the second conditional moment even further.

CDS spread volatility increases in both rising and falling spread environments, but it is more pronounced when spreads are increasing (i.e., credit is deteriorating). This asymmetry can be explained by market dynamics, risk perceptions, and liquidity. When CDS spreads increase, it typically signals rising credit risk or fear of default. This environment is highly uncertain, with news or data causing sharp, unpredictable reactions, hence higher volatility. When CDS spreads decrease, the market is pricing in improved credit conditions. But good news tends to be more gradual and less dramatic, so while volatility does rise (due to changes in positioning, liquidity, etc.), it does not spike as much.

Finally, the matrix inequality constraints of Theorem 1 are satisfied (see Table E8 in Supplementary Appendix E.4). For comparison purposes, we have also estimated unrestricted versions of the two models. The results are reported in Supplementary Appendix E.5 (see Table E9). It is evident that the nonnegativity conditions of Theorem 1 are violated.

5.2 The fundamental role of matrix inequality constraints

A critical issue to consider is whether performing unconstrained estimation—apart from yielding estimates that extend beyond the feasible parameter domain, and, therefore, attributing a positive probability to the occurrence of negative values in variables that should remain strictly nonnegative—also creates empirical shortcomings. Such inconsistencies undermine the empirical relevance of the unconstrained model and suggest potential limitations in its predictive reliability, particularly in long-term forecasting.

To investigate this issue, we conduct an out-of-sample forecasting evaluation using the second dataset, where volatility forecasts are generated based on high–low range measurements. For

[16] The variable x_{it} in the asymmetric term represents the spread returns.

Table 8 Out-of-sample forecasting in four European stock markets.

	$k=1$		$k=5$		$k=22$	
	Case I	**Case II**	**Case I**	**Case II**	**Case I**	**Case II**
	Panel A: RMSE for the out-of-sample forecasting					
$\mu_{1,t+k\|\mathcal{M}_t}$	3.899	6.872	4.413	15.652	5.183	69.629
$\mu_{2,t+k\|\mathcal{M}_t}$	3.491	3.436	3.871	13.288	5.284	48.512
$\mu_{3,t+k\|\mathcal{M}_t}$	4.164	3.841	3.766	16.194	3.792	50.239
$\mu_{4,t+k\|\mathcal{M}_t}$	2.804	2.707	2.812	9.697	2.970	29.596
	Panel B: Number of negative volatility forecasts					
$\mu_{1,t+k\|\mathcal{M}_t}$	0	0	0	695	0	962
$\mu_{2,t+k\|\mathcal{M}_t}$	0	0	0	454	0	734
$\mu_{3,t+k\|\mathcal{M}_t}$	0	0	0	166	0	419
$\mu_{4,t+k\|\mathcal{M}_t}$	0	0	0	102	0	399

Case I imposes the matrix inequality constraints of Theorem 1. In Case II, no constraints are imposed.

estimation, we use the first 2705 observations, reserving the final 1000 data points for out-of-sample prediction. The out-of-sample forecast horizon begins in mid-2020, coinciding with the outbreak of the COVID-19 pandemic, which led to pronounced volatility in stock market returns. Forecasts are generated for horizons of 1, 5, and 22 steps ahead, corresponding to one-day, one-week, and one-month-ahead volatility projections. Each model is reestimated at every 20th observation using a rolling window of 3260 data points to generate out-of-sample forecasts. We estimate the model using two approaches: first, by incorporating the matrix inequalities outlined in Theorem 1 (Case I), and second, without any imposed constraints (Case II). Table 8 presents the RMSE values for out-of-sample forecasts, facilitating a comparative assessment of both estimation strategies. Additionally, in Panel B of the table, we report the count of negative forecasts for each horizon, enabling an evaluation of the constraint's effectiveness in maintaining forecast feasibility.

For 1-step-ahead forecasts, models incorporating constraints demonstrate slightly superior predictive accuracy relative to their unconstrained counterparts. Nonetheless, the differences remain minimal, suggesting that the two modeling approaches yield largely comparable results. Importantly, both models generate strictly positive volatility forecasts.

Conversely, when forecasting 5 and 22 steps ahead, notable discrepancies arise. Constrained models, which adhere to the nonnegativity conditions, systematically predict positive volatility values. In contrast, unconstrained models produce a considerable proportion of negative forecasts, an outcome that is theoretically anticipated, as indicated in Theorem 1. Take FR for example, among the 1000 forecasts generated, the unconstrained model predicts negative volatility values in 695 instances for the $k=5$ and in 962 instances for $k=22$. This leads to poor predictive performance, as reflected in the larger RMSE of the unconstrained model.[17] Figure 2 presents the graphs of out-of-sample forecasts for 1-step- and 22-step-ahead horizons. In the unrestricted estimation, negative volatility predictions are observed for the four markets.

Overall, both estimation methods exhibit comparable performance in 1-step-ahead forecasting. However, at longer forecasting horizons, significant issues arise when the matrix inequalities are not enforced. The omission of these constraints leads to a reduced predictive accuracy and the occurrence of infeasible volatility estimates, undermining the model's reliability.

[17] To further evaluate forecasting performance, we examine our first empirical example, in which volatility forecasts are based on three different measures, by conducting an out-of-sample forecasting exercise using the final 1000 observations. The results closely align with those in Table 7. While 1-step-ahead forecasts show minimal differences in predictive accuracy between the 2 estimation approaches, notable discrepancies emerge in the 5-step- and 22-step-ahead forecasts, where the unconstrained model performs worse (owing to space limitations, the results have been omitted from this paper but are available upon request).

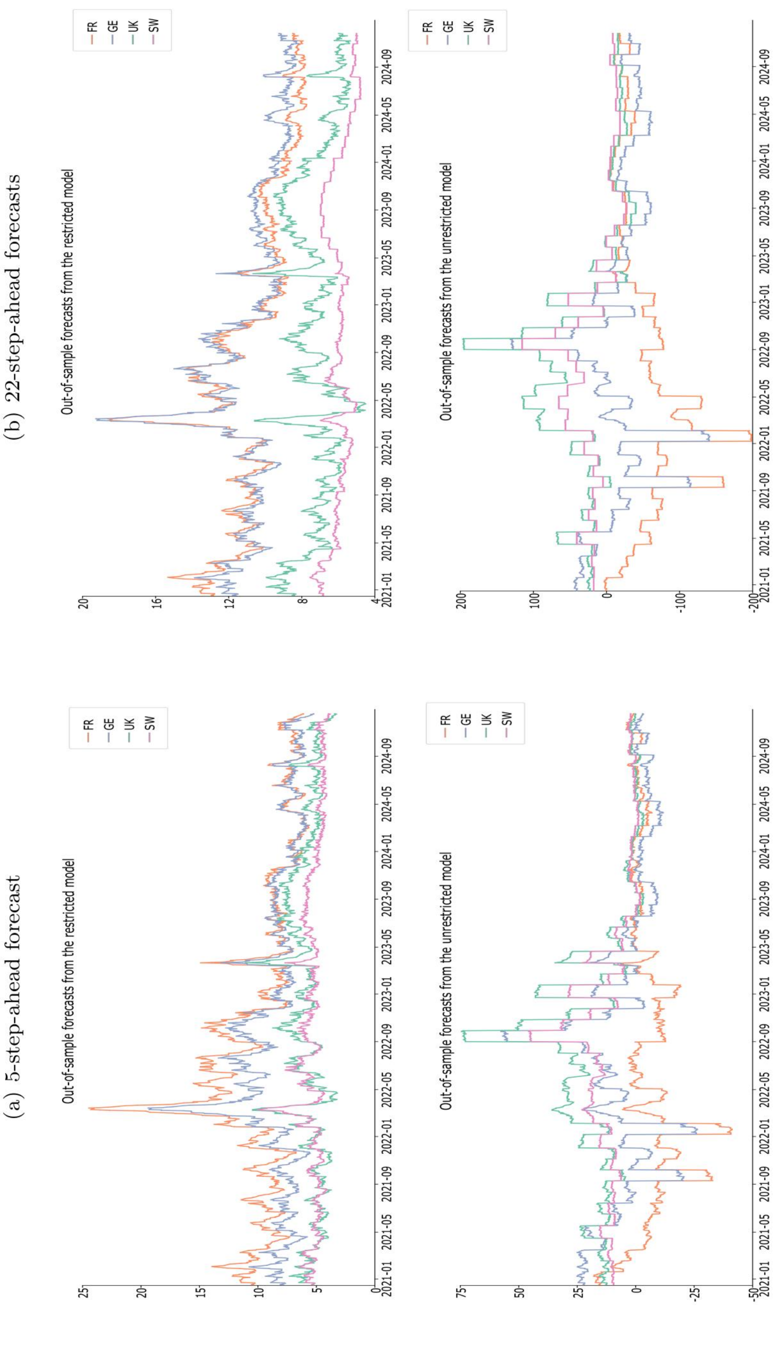

Figure 2 Out-of-sample forecasting for four European stock markets.

Table 9 Impulse response function absorption times (four European equity markets).

Shocks/Response	Case I				Case II			
	FR	DE	UK	CH	FR	DE	UK	CH
FR	29.3	29.9	30.9	30.4	43.8	59.3	49.5	56.1
DE	28.1	28.8	12.0	29.4	34.2	60.1	43.3	56.3
UK	25.9	26.6	28.0	27.4	29.0	55.9	38.3	52.0
CH	25.3	26.0	27.3	26.7	28.5	53.6	37.5	49.9
Average	27.2	27.8	24.6	28.5	33.9	57.2	42.2	53.6

The table reports the absorption time, measured in days, for a shock to decay and for the system to reach its long-run equilibrium. We define the cutoff as 10^{-7}, meaning that the shock effect is considered fully absorbed once it falls below this threshold. Case I imposes the matrix inequality constraints of Theorem 1. In Case II, no constraints are imposed.

Finally, our analysis reveals that the unconstrained model exhibits greater shock persistence, as it absorbs shocks more slowly than the constrained model (see Table 9). Compared to the constrained model, the unconstrained model accommodates a broader range of positive and negative coefficient combinations. This dynamic generates opposing forces, with some components driving the system toward equilibrium, while others push it away. Consequently, this interplay leads to oscillatory behavior, delaying the full dissipation of shocks. Conversely, the constrained model sets certain parameters to zero rather than allowing negative values, ensuring that most components contribute in the same direction. This enhances shock absorption, leading to faster mean reversion.

6. Conclusions

The admissible parameter space for vMEM is determined by restructuring the necessary and sufficient nonnegativity conditions for an N-dimensional system of order (1, q), originally derived by Conrad and Karanasos (2010). These constraints are explicitly formulated as matrix inequalities, using the model's matrix parameters. The results in Theorem 1 are critically important, as an infeasible parameter range assigns a nonzero probability to states where variables that should remain strictly nonnegative instead take on negative values. Furthermore, we derived nonnegativity constraints for asymmetric frameworks, leading to additional matrix inequalities and extending the results of Conrad and Karanasos (2010).

Accordingly, our second major contribution was the development and application of a constrained ML estimation approach for the vMEM, representing a key advancement over the unconstrained methodologies previously used by Manganelli (2005), Conrad and Karanasos (2010), Cipollini and Gallo (2010), and Cipollini, Engle, and Gallo (2013, 2017).

In this context, we introduced the acronym cvMEM, which serves a dual purpose. The constrained vMEM estimation corrects the improper parameterization of vMEM in prior research, ensuring consistency with the theoretical model's nonnegativity conditions.

In this study, we adopted the ML estimation technique. A potential avenue for future research is to incorporate matrix inequalities into the GMM estimation procedure and apply constrained optimization over an admissible parameter space to minimize deviations of moment conditions from zero. An important but challenging direction for future research is to address the quadratic growth in computational complexity as N increases (see, e.g., Otranto and Domianello 2026) and to implement constrained ML or GMM estimation in high-dimensional vMEMs.[18]

[18] The curse of dimensionality presents a critical challenge in high-dimensional vMEM. Specifically, the number of parameters to be estimated in a specification of order $(1,1)$ is given by $N+3N^2$. For example, in a system with 10 variables, this results in 310 parameters. As the number of parameters grows quadratically, the computational

Variable selection is another important area of research in the vMEM literature (see, e.g., Cipollini and Gallo 2010; Cattivelli and Gallo 2020). Variable selection techniques can help mitigate certain challenges associated with high dimensionality by eliminating irrelevant predictors. However, they should be accompanied by constrained estimation. The integration of variable selection and matrix inequality constraints presents an intriguing research question for future exploration.

It is important to emphasize that unconstrained estimation methods are appropriate only when the variables are log-transformed (see, e.g., Hautsch 2008; Taylor and Xu 2017). However, universally applying log transformations to all N variables could introduce unnecessary rigidity into the model specification.

Alternatively, adopting the novel mixture formulation proposed by Karanasos, Xu, and Yfanti (2025b) could serve as an effective mechanism for mitigating certain restrictive inequalities. Specifically, logarithmic transformations can be selectively applied to a subset of the N equations based on appropriate selection criteria. This generalized formulation encompasses existing models, such as log-GARCH (Francq, Jiménez-Gamero, and Meintanis 2017; Francq and Sucarrat 2017) and the exponential specification of vMEM (e.g., Hautsch 2008). A detailed investigation of this approach is left for future research.

It is also important to emphasize that our methodology is broadly applicable beyond vMEM (as well as MGARCH/HEAVY[19] models), extending naturally to other multivariate frameworks, such as VARMA structures, where system variables must remain nonnegative. Three illustrative examples include:

- The multivariate autoregressive conditional double Poisson model, introduced by Heinen and Rengifo (2007), is structured as a VARMA-type system and is suitable for analyzing correlated count data in a multivariate setting.
- The multivariate conditional autoregressive range model, developed by Fernandes, de Sá Mota, and Rocha (2005), is a framework in which the conditional expectations of range-based volatility metrics follow VARMA-type dynamic structures.
- The multivariate heterogeneous autoregressive realized volatility (MHAR-RV) model, introduced by Hwang and Hong (2021), is specifically designed to capture interdependencies in realized financial volatilities.

Explicitly, the proposed matrix inequalities should be enforced on VARMA-type nonnegative processes, which are widely used in financial engineering and various business operations, including multivariate time-series forecasting for customer service queue counts and demand prediction in marketing analytics.

Beyond their essential value, our findings gain additional significance from the new questions they raise, which we believe can inspire future research. In this regard, we propose three recommendations.

Future research should focus on implementing our techniques to derive matrix inequalities in multivariate systems of order higher than $(1, q)$. Such advancements would enhance the applicability of our results to multivariate generalizations of the following:

i) Nonnegative ARMA process of Tsai and Chan (2007).
ii) Tukey nonnegative type autoregression model, as introduced by Eriksson, Preve, and Yu (2019).
iii) Several nonnegative versions of the generalized ARMA model, proposed by Zheng, Xiao, and Chen (2015).

The second proposal involves a methodological strategy for explicitly formulating the second-moment structure of higher order models. While He and Teräsvirta (2004) derived only recursive

complexity of the estimation process increases accordingly. In high-dimensional scenarios, this complexity introduces additional concerns, including overfitting and numerical instability (see Bauwens, Laurent, and Rombouts 2006, for an in-depth analysis).

[19] For a recent application using the MGARCH model, see Yfanti et al. (2023). The acronym HEAVY (High frEquency bAsed VolatilitY) was introduced by Shephard and Sheppard (2010).

solutions for the MGARCH system of order (2,2), our approach—based on the ARMA representation of an MEM (or a GARCH model)—provides a more analytically tractable alternative.

As a third consideration, we propose relaxing the assumption of constant parameters. Establishing necessary and sufficient non-negativity conditions for N-dimensional systems in a time-varying setting would significantly enhance the theoretical understanding of their behavior. Although this task presents a considerable challenge it underscores the necessity of our methodology.

Since "time-varying" multivariate models do not allow for the derivation of "univariate representations," the approach proposed by Conrad and Karanasos (2010) is inapplicable in this context. In stark contrast, the multivariate "one-sided representation" introduced in this paper, combined with the innovative methodology proposed by Karanasos et al. (2025a) for addressing time-varying models, provides a viable solution to this issue.

Last but not least, this research is dedicated to the memory of Daniel Nelson (1959–1995), whose pioneering contributions continue to inspire this work. Had he not passed away on May 4, 1995, at the age of 36, he might have derived results on nonnegativity constraints for MGARCH models by the late 1990s.[20]

Supplemental material

Supplemental material is available at *Journal of Financial Econometrics* online.

Funding

None declared.

Conflicts of interest

None declared.

APPENDIX A

In what follows, we will prove the symmetric case of Theorem 1. The proof for the other extreme case (Conditions (C2b) and (C3b)), that is when $\mathbf{S}_t = \mathbf{I}$ for all t is omitted, as it is similar to the one presented for the symmetric case (see also footnote 21). First, we will present some definitions and two propositions.

A.1 Definitions

In this appendix, we will introduce the following definitions.

Definition A2. *Let μ_i be the ith element of the following vector:*

$$\boldsymbol{\mu} = adj[\mathbf{I} - \mathbf{B}]\boldsymbol{\omega}. \tag{A.1}$$

[20] Nelson and Cao (1992) stated that

> Presumably, such sufficient (but not necessary) conditions should not be imposed in estimation. In practice, however, it is usually necessary to impose positivity on in-sample fitted values of σ_t^2 to keep nonlinear maximization routines from encountering overflows. For the ARCH(p), GARCH($1, q$), and GARCH($2, q$), the inequality constraints of Sections 2.1 and 2.2 should suffice. For higher-order GARCH models and *multivariate GARCH*, some other tactic is required.

Let $Y_{ij}(L)$ be a polynomial (i.e., of order N). Then $\mathbf{Y}(L) = [Y_{ij}(L)]$ indicates a matrix polynomial in the lag operator.

Definition A3. *We also define the square matrix polynomial $\boldsymbol{\alpha}(L)$, with ijth element denoted by $\alpha_{ij}(L)$, as*

$$\boldsymbol{\alpha}(L) = adj[\mathbf{I} - \mathbf{B}L] \sum_{l=1}^{q} \mathbf{A}^{(l)} L^{l}. \tag{A.2}$$

Similarly to $\beta(L)$ in Equation (3), since under Assumption A1: $a_{ij}^{(N-1+q)} \neq 0$ for all i, j, the scalar polynomials $\alpha_{ij}(L) = \sum_{n=1}^{N-1+q} a_{ij}^{(n)} L^n$ are of order $N-1+q$.

Definition A4. *Let the invertibility condition (Assumption A2) hold, and let $\Psi_{ij}(L) = \sum_{k=1}^{\infty} \psi_{ij}^{(k)} L^k$ be the ijth element of $\boldsymbol{\Psi}(L)$ where*

$$\boldsymbol{\Psi}(L) = \boldsymbol{\alpha}(L)/\beta(L), \tag{A.3}$$

that is

$$\Psi_{ij}(L) = \alpha_{ij}(L)/\beta(L). \tag{A.4}$$

Equivalently, $\boldsymbol{\Psi}(L)$ can be written as

$$\boldsymbol{\Psi}(L) = \sum_{k=1}^{\infty} \boldsymbol{\Psi}^{(k)} L^k, \tag{A.5}$$

where the ijth element of $\boldsymbol{\Psi}^{(k)}$ in Equation (A.5) is $\psi_{ij}^{(k)}$.

A.2 Univariate representations

The following proposition gives the "one-sided" representation of the symmetric version of the vMEM of order $(1, q)$. The proof is trivial (the result is obtained from Equation (2)—ignoring the asymmetries—by inverting the matrix polynomial $\mathbf{I} - \mathbf{B}L$ and using the notation in Equations (A.1), (3), (A.2) and (A.4)).

Proposition A1. *Let Assumptions A1 and A2 be satisfied. Then the symmetric version of the vMEM $(1, q)$ in Equation (2) admits the multivariate "one-sided" representation:*

$$\boldsymbol{\mu}_t = \frac{\boldsymbol{\mu}}{\beta(1)} + \boldsymbol{\Psi}(L)\mathbf{y}_t,$$

with the corresponding univariate "one-sided" representations given by

$$\mu_{it} = \frac{\mu_i}{\beta(1)} + \sum_{j=1}^{\infty} \Psi_{ij}(L) y_{jt} \tag{A.6}$$

(recall that $\boldsymbol{\Psi}(L)$ and $\Psi_{ij}(L)$ have been defined in Equations (A.3) and (A.4), respectively).

Remark A1. *Each $\Psi_{ij}(L)$ can be thought of as an infinite-order kernel of a univariate unrestricted symmetric $(N, N-1+q)$ specification. Clearly, for the symmetric version of the N-dimensional process in Equation (2) to be well defined and the N conditional means to be positive almost surely for all t, all the constants μ_i (see Equation (A.1)) must be positive and all the $\psi_{ij}^{(k)}$ coefficients in the univariate "one-sided" representations, that is Equation (A.6), must be nonnegative: $\psi_{ij}^{(k)} \geq 0$, $i, j = 1, \ldots, N$ for $k = 1, 2, \ldots$.*

In other words, the nonnegativity of the conditional means is guaranteed if and only if all the kernels are nonnegative, that is, if the *infinite* number of parameters in the "one-sided" expansions of the N^2 kernels are nonnegative. For this, one should express these parameters as functions of the parameters of the original process. It can then be shown that checking a finite number of inequality constraints on these parameters ensures the nonnegativity of all MEM kernels of the system (see also Conrad and Karanasos 2010, who, in the GARCH context, paid special attention only to the bivariate case of order $(1, 1)$).

A.3 Results in Conrad and Karanasos (2010)

In this part of the appendix, we will present the main result of Conrad and Karanasos (2010). As in their paper, we consider here the symmetric case, see the symmetric version of Equation (2), that is when $\mathbf{S}_t = \mathbf{0}$, for all t.

First, we recall that:

i) the intercept in each of the N univariate representations of the symmetric system of order $(1, q)$ is a fraction with μ_i as the scaling coefficient in the numerator; see Equations (A.1) in Definition A2 and (A.6) in Proposition A1,
ii) $\beta(L) = \prod_{i=1}^{N}(1 - \phi_i L)$, is the common "autoregressive" polynomial of the univariate representations; see Equation (3) in Definition 1,
iii) $\alpha_{ij}(L) = \sum_{n=1}^{N-1+q} a_{ij}^{(n)} L^n$ in Definition A3, for $i, j = 1, \ldots, N$, are the N^2 polynomials associated with the "univariate" representations, and
iv) the corresponding N^2 infinite-order kernels, that is, $\Psi_{ij}(L) = \sum_{k=1}^{\infty} \psi_{ij}^{(k)} L^k$, are given in Equation (A.4) in Definition A4.

A.3.1 Non-negativity constraints

As pointed out by Conrad and Karanasos (2010), in practice, given a particular set of parameters, checking the nonnegativity of $\{\psi_{ij}^{(k)}\}_{k=1,\ldots,\infty}$, $i, j = 1, \ldots, N$ may be a numerically unfeasible task. However, in their Theorem 1, they show that under some conditions, the nonnegativity of $\{\psi_{ij}^{(k)}\}_{k=1,\ldots,\kappa_{ij}}$ for some tractable integers κ_{ij} (see Notation 2), is necessary and sufficient for the nonnegativity of $\{\psi_{ij}^{(k)}\}_{k=1,\ldots,\infty}$. We state Theorem 1 in Conrad and Karanasos (2010) as Proposition A2 below.

Proposition A2. *Consider the symmetric version of the N-dimensional vMEM$(1, q)$ model in Equation (2), and let Assumptions A1 and A2 be satisfied for the symmetric case. Then, the following conditions are necessary and sufficient for $\mu_{it} > 0$, $i = 1, \ldots, N$, for all t:*

(A) μ_i are positive for all $i = 1, \ldots, N$.

$$(B)\begin{cases} \phi_1 \textit{ is real, and } \phi_1 > 0 & (C1) \\ a_{ij}(\phi_1^{-1}) > 0, \textit{ for } i,j = 1,\ldots,N & (C2) \\ \psi_{ij}^{(k_{ij})} \geq 0, \textit{ for } i,j = 1,\ldots,N \textit{ and } k_{ij} = 1,\ldots,\kappa_{ij}, & (C3) \end{cases}$$

where κ_{ij} *is the smaller integer greater than or equal to* $\max\{0,\varphi_{ij}\}$ *with*

$$\begin{aligned} \varphi_{ij} &= [\log \eta_{ij}^{(1)} - \log(N-1)\eta_{ij}]/[\log(|\phi_2|) - \log(|\phi_1|)], \\ \eta_{ij} &= \max_{2\leq n\leq N} |\eta_{ij}^{(n)}|, \text{ and } \eta_{ij}^{(n)} = -\frac{a_{ij}(\phi_n^{-1})}{\beta'(\phi_n^{-1})}, \ 1\leq n\leq N, \end{aligned}$$

and $\beta'(z)$ *denotes the first derivative of* $\beta(z)$.

As pointed out by Conrad and Karanasos (2010), the proof of Proposition A2 relies on the observation that the results of Tsai and Chan (2008) can be applied separately to each of the N^2 kernels $\Psi_{ij}(L)$. Hence, an infinite number of inequality constraints on the coefficients of the "one-sided" decomposition of the model are reduced to a finite number of conditions on the parameters of the process.

A.4 Proof of Theorem 1

Proof.

The proof of Theorem 1 (the symmetric case)[21] is obtained by observing that:

(i) Condition (A) in Proposition A2, in view of Definition A2, can be expressed in a matrix form as

$$adj[\mathbf{I}-\mathbf{B}]\boldsymbol{\omega} > \mathbf{0},$$

(ii) Condition (C2) in Proposition A2, taking into account Definition A3, can be written in a matrix form as $\boldsymbol{\alpha}(\phi^{-1}) > \mathbf{0}$ or equivalently as

$$\phi_1^{q+1}\boldsymbol{\alpha}(\phi_1^{-1}) = adj[\mathbf{I}\phi_1 - \mathbf{B}]\sum_{l=1}^{q}\mathbf{A}^{(l)}\phi_1^{q-l} > \mathbf{0}.$$

(iii) The matrix expression of Condition (C3) in Proposition A2, in light of Proposition 1 (presented in Section 2; see Equation (4)), is

$$\sum_{s=1}^{\min(q,k)} \mathbf{B}^{k-s}\mathbf{A}^{(s)} \geq \mathbf{0}.$$

For every element indexed by (i,j) the above matrix inequalities must be satisfied for all $k\in\{1,\ldots,\kappa_{ij}\}$, where κ_{ij} are given in Notation 2. The above results, in conjunction with Claim 1, complete the proof of the Theorem. ■

[21] The proof of Conditions (C2b) and (C3b), that is when all the signed variables, x_{it}, are negative for all t, is similar to the proof of the symmetric case and, therefore, is omitted. The only changes needed are: to replace $\mathbf{A}^{(l)}$ in eq. (A.2) and in the two equations below by $\mathbf{A}^{(l)} + \Gamma^{(l)}$.

References

Alizadeh, S., M. W. Brandt, and F. X. Diebold. 2002. Range-Based Estimation of Stochastic Volatility Models. *The Journal of Finance* 57: 1047–1091.

Bauwens, L., S. W. Laurent, and J. V. K. Rombouts. 2006. Multivariate GARCH Models: A Survey. *Journal of Applied Econometrics* 21: 79–109.

Bollerslev, T. 1990. Modelling the Coherence in Short-Run Nominal Exchange Rates: A Multivariate Generalised ARCH Model. *The Review of Economics and Statistics* 72: 498–505.

Bollerslev, T., R. F. Engle, and J. M. Wooldridge. 1988. A Capital Asset Pricing Model with Time-Varying Covariances. *Journal of Political Economy* 96: 116–131.

Brandt, M. W., and F. X. Diebold. 2006. A no-Arbitrage Approach to Range-Based Estimation of Return Covariances and Correlations. *Journal of Business* 79: 61–74.

Cattivelli, L., and G. M. Gallo. 2020. Adaptive Lasso for Vector Multiplicative Error Models. *Quantitative Finance* 20: 255–274.

Chou, R. Y. 2005. Forecasting Financial Volatilities with Extreme Values: The Conditional Autoregressive Range (CARR) Model. *Journal of Money, Credit and Banking* 37: 561–582.

Cipollini, F., R. F. Engle, and G. M. Gallo. 2013. Semiparametric Vector MEM. *Journal of Applied Econometrics* 28: 1067–1086.

Cipollini, F., R. F. Engle, and G. M. Gallo. 2017. Copula–Based vMEM Specifications versus Alternatives: The Case of Trading Activity. *Econometrics* 5: 16.

Cipollini, F., and G. M. Gallo. 2010. Automated Variable Selection in Vector Multiplicative Error Models. *Computational Statistics & Data Analysis* 54: 2470–2486.

Cipollini, F., and G. M. Gallo. 2025. Multiplicative Error Models: 20 Years on. *Econometrics and Statistics* 33: 209–229.

Conrad, C. 2010. Non-Negativity Conditions for the Hyperbolic GARCH Model. *Journal of Econometrics* 157: 441–457.

Conrad, C., and M. Karanasos. 2010. Negative Volatility Spillovers in the Unrestricted ECCC-GARCH Model. *Econometric Theory* 26: 838–862.

Ding, Z., C. W. J. Granger, and R. F. Engle. 1993. A Long Memory Property of Stock Market Returns and a New Model. *Journal of Empirical Finance* 1: 83–106.

Engle, R. F., and G. M. Gallo. 2006. A Multiple Indicators Model for Volatility Using Intra-Daily Data. *Journal of Econometrics* 131: 3–27.

Eriksson, A., D. P. A. Preve, and J. Yu. 2019. Forecasting Realized Volatility Using a Nonnegative Semiparametric Model. *Journal of Risk and Financial Management* 12: 139, 1–23.

Fernandes, M., B. de Sá Mota, and G. Rocha. 2005. A Multivariate Conditional Autoregressive Range Model. *Economics Letters* 86: 435–440.

Francq, C., M. D. Jiménez-Gamero, and S. G. Meintanis. 2017. Tests for Conditional Ellipticity in Multivariate GARCH Models. *Journal of Econometrics* 196: 305–319.

Francq, C., and G. Sucarrat. 2017. An Equation-by-Equation Estimator of a Multivariate log-GARCH-X Model of Financial Returns. *Journal of Multivariate Analysis* 153: 16–32.

Glosten, L. R., R. Jagannathan, and D. E. Runkle. 1993. On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance* 48: 1779–1801.

Gourieroux, C. 2007. Positivity Conditions for a Bivariate Autoregressive Volatility Specification. *Journal of Financial Econometrics* 5: 624–636.

Hautsch, N. 2008. Capturing Common Components in High-Frequency Financial Time Series: A Multivariate Stochastic Multiplicative Error Model. *Journal of Economic Dynamics and Control* 32: 3978–4015.

He, C., and T. Teräsvirta. 1999. Properties of Moments of a Family of GARCH Processes. *Journal of Econometrics* 92: 173–192.

He, C., and T. Teräsvirta. 2004. An Extended Constant Conditional Correlation GARCH Model and Its Fourth-Moment Structure. *Econometric Theory* 20: 904–926.

Heinen, A., and E. Rengifo. 2007. Multivariate Autoregressive Modeling of Time Series Count Data Using Copulas. *Journal of Empirical Finance* 14: 564–583.

Hwang, E., and W.-T. Hong. 2021. A Multivariate HAR-RV Model with Heteroscedastic Errors and Its WLS Estimation. *Economics Letters* 203: 109855–109855.

Jeantheau, T. 1998. Strong Consistency of Estimators for Multivariate ARCH Models. *Econometric Theory* 14: 70–86.

Karanasos, M., A. G. Paraskevopoulos, A. Magdalinos, and A. Canepa. 2025a. A Unified Theory for ARMA Models with Varying Coefficients: One Solution Fits All. *Econometric Theory* 1–54. https://doi.org/10.1017/S0266466624000306.

Karanasos, M., Y. Xu, and S. Yfanti. 2025b. The MUFA GARCH Type of Model: Theory and an Application to Corporate Bond Markets. Unpublished Paper.

Ling, S., and M. McAleer. 2003. Asymptotic Theory for a Vector ARMA-GARCH Model. *Econometric Theory* 19: 280–310.

Manganelli, S. 2005. Duration, Volume and Volatility Impact of Trades. *Journal of Financial Markets* 8: 377–399.

Martens, M., and D. Van Dijk. 2007. Measuring Volatility with the Realized Range. *Journal of Econometrics* 138: 181–207.

Nakatani, T., and T. Teräsvirta. 2008. Positivity Constraints on the Conditional Variances in the Family of Conditional Correlation GARCH Models. *Finance Research Letters* 5: 88–95.

Nakatani, T., and T. Teräsvirta. 2009. Testing for Volatility Interactions in the Constant Conditional Correlation GARCH Model. *Econometrics Journal* 12: 147–163.

Nelson, D. B. 1991. Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica* 59: 347–370.

Nelson, D. B., and C. Q. Cao. 1992. Inequality Constraints in the Univariate GARCH Model. *Journal of Business & Economic Statistics* 10: 229–235.

Noureldin, D., N. Shephard, and K. Sheppard. 2012. Multivariate High-Frequency-Based Volatility (HEAVY) Models. *Journal of Applied Econometrics* 27: 907–933.

Otranto, E., and L. S. Domianello. 2026. Spillovers and Co-movements in Multivariate Volatility: A Vector Multiplicative Error Model. arXiv, arXiv:2601.16837, preprint: not peer reviewed.

Shephard, N., and K. Sheppard. 2010. Realizing the Future: Forecasting with High-Frequency-Based Volatility (HEAVY) Models. *Journal of Applied Econometrics* 25: 197–231.

Stavroyiannis, S. 2018. A Note on the Nelson Cao Inequality Constraints in the GJR-GARCH Model: Is There a Leverage Effect? *International Journal of Economics and Business Research* 16: 442–452.

Taylor, N., and Y. Xu. 2017. The Logarithmic Vector Multiplicative Error Model: An Application to High Frequency NYSE Stock Data. *Quantitative Finance* 17: 1021–1035.

Tsai, H., and K. S. Chan. 2007. A Note on Non-Negative ARMA Processes. *Journal of Time Series Analysis* 28: 350–360.

Tsai, H., and K. S. Chan. 2008. A Note on Inequality Constraints in the GARCH Model. *Econometric Theory* 24: 823–828.

Wang, J. 2007. Foreign Equity Trading and Emerging Market Volatility: Evidence from Indonesia and Thailand. *Journal of Development Economics* 84: 798–811.

Yfanti, S., M. Karanasos, C. Zopounidis, and A. Christopoulos. 2023. Corporate Credit Risk Counter-Cyclical Interdependence: A Systematic Analysis of Cross-Border and Cross-Sector Correlation Dynamics. *European Journal of Operational Research* 304: 813–831.

Zakoïan, J.-M. 1994. Threshold Heteroskedastic Models. *Journal of Economic Dynamics and Control* 18: 931–955.

Zheng, T., H. Xiao, and R. Chen. 2015. Generalized ARMA Models with Martingale Difference Errors. *Journal of Econometrics* 189: 492–506.