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FLUCTUATIONS OF THE CONNECTIVITY THRESHOLD AND LARGEST NEAREST-NEIGHBOUR LINK

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Consider a random uniform sample $\mathcal{X}_n$ of n points in a compact region A of Euclidean d-space, $d \geq 2$, with a smooth or (when $d = 2$) polygonal boundary. Fix $k \in \mathbb{N}$. Let $M_k(\mathcal{X}_n)$ be the threshold r at which the geometric graph on these n vertices with distance parameter r becomes k-connected. We show that if $d = 2$ then $n(\pi/|A|)M_1(\mathcal{X}_n)^2 - \log n$ is asymptotically standard Gumbel. For $(d, k) \neq (2, 1)$, it is

$$n(\theta_d/|A|)M_k(\mathcal{X}_n)^d - (2 - 2/d)\log n - (4 - 2k - 2/d)\log\log n$$

that converges in distribution to a nondegenerate limit, where θ_d is the volume of the unit ball. The limit is Gumbel with scale parameter 2 except when $(d, k) = (2, 2)$ where the limit is two component extreme value distributed. The different cases reflect the fact that boundary effects are more important in some cases than others. We also give similar results for the largest k-nearest neighbour link $L_k(\mathcal{X}_n)$ in the sample, and show $M_k(\mathcal{X}_n) = L_k(\mathcal{X}_n)$ with high probability. We provide estimates on rates of convergence and give similar results for Poisson samples in A. Finally, we give similar results even for nonuniform samples, with a less explicit sequence of centring constants.

1. Introduction.

1.1. *Overview and motivation.* This paper is concerned with the threshold at which the random geometric graph becomes connected. This graph is defined as follows. Let $d \in \mathbb{N}$. Given a finite set $\mathcal{X} \subset \mathbb{R}^d$, and $r > 0$, the *geometric graph* $G(\mathcal{X}, r)$ has vertex set $\mathcal{X}$, with an edge drawn between any two vertices at Euclidean distance at most r from each other. We say $G(\mathcal{X}, r)$ is *k-connected* if it is not possible to disconnect the graph by removing $k - 1$ or fewer vertices. (in particular, 1-connectivity is the same as connectivity). The *k-connectivity threshold* of $\mathcal{X}$ is the number

$$M_k(\mathcal{X}) := \inf\{r > 0 : G(\mathcal{X}, r) \text{ is } k\text{-connected}\}.$$

An alternative characterisation of $M_1(\mathcal{X})$ is in terms of *minimal spanning tree* (MST). A MST on $\mathcal{X}$ is a tree spanning $\mathcal{X}$ that minimises the total (Euclidean) length of the edges. It is not hard to see that $M_1(\mathcal{X})$ equals the longest edge length of a MST on $\mathcal{X}$.

For the *random* geometric graph, the vertex set $\mathcal{X}$ is given by the set of points of a Poisson point process $\mathcal{P}_n$ in $\mathbb{R}^d$ with intensity measure $n\nu$, where ν is a probability measure on $\mathbb{R}^d$ with probability density function $f : \mathbb{R}^d \to [0, \infty)$, and $n \in (0, \infty)$ is the mean number of vertices. Alternatively, for $n \in \mathbb{N}$ we can take $\mathcal{X} = \mathcal{X}_n$, where $\mathcal{X}_n$ denotes a binomial point process whose points are n independent random d-vectors with common density f. Since the vertices are placed randomly in $\mathbb{R}^d$, the thresholds $M_k(\mathcal{X}_n)$ and

$$M_{n,k} := M_k(\mathcal{P}_n) = \inf\{r > 0 : G(\mathcal{P}_n, r) \text{ is } k\text{-connected}\} \tag{1.1}$$

are random variables.

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In this paper we investigate the limiting behaviour of the connectivity threshold $M_{n,k}$ and $M_k(\mathcal{X}_n)$ for large n and fixed $k \in \mathbb{N}$. We assume throughout that $d \geq 2$. We consider a broad class of measures ν, subject to the *working assumption* that ν has compact support $A \subset \mathbb{R}^d$, and its density f is continuous and bounded away from zero on A. As n grows, the spacing between vertices becomes smaller, so one expects to have $M_{n,k} \to 0$ as $n \to \infty$. A simple consideration by computing typical spacing of vertices leads to the belief that $M_{n,k}$ should decay more slowly than $n^{-1/d}$, in the sense that $nM_{n,k}^d$ should tend to infinity in probability. In the special case where ν is the uniform distribution on $[0, 1]^d$, it is known [11, 12, 14], that there is an explicit sequence of centring constants $(a_n)_{n\geq 1}$ satisfying $a_n \to \infty$ as $n \to \infty$ such that

$$nM_{n,k}^d - a_n \xrightarrow{d} X; \qquad nM_k(\mathcal{X}_n)^d - a_n \xrightarrow{d} X, \tag{1.2}$$

where X is an explicit nondegenerate random variable. Clearly (1.2) is what is needed to determine $\lim_{n\to\infty} \mathbb{P}[M_{n,k} \leq r_n]$ for any sequence $(r_n)_{n\geq 1}$ such that the limit exists.

In the present paper, we show that (1.2) holds for suitable a_n and X, for a broad class of measures ν. While one might perhaps anticipate that the limiting behaviour of the form (1.2) would carry over from the uniform distribution on $[0, 1]^d$ to more general ν satisfying our working assumption, proving this seems to be considerably harder than one might naively expect, and in the last 20 years or so there has been limited progress in proving such results. It turns out that even in the uniform case where f is constant on A, boundary effects are important because the "most isolated" vertex is likely to lie near the boundary when $d \geq 3$. Thus the formula for a_n, even when ν is uniform on $[0, 1]^d$, is quite complicated (see (1.10) below) due to having to consider all of the different kinds of faces making up the boundary, and does not necessarily provide much insight into the appropriate choice of centring constants for other A. In the nonuniform case, determining appropriate constants a_n is even harder because they depend in a delicate way on how f approaches its minimum, both in the interior and on the boundary of A.

In this work we chiefly consider the case where ∂A is smooth. In the uniform case we determine an explicit sequence of centring constants a_n such that (1.2) holds for suitable X. In the nonuniform case we are still (in most cases) able to derive (1.2) on taking a_n to be the median of the distribution of $nM_{n,k}^d$. Part of our proof involves approximating A with a polyhedral set A_n with the spacing between vertices of A_n decreasing slowly as n becomes large. This technique was developed recently for certain random coverage problems in [6, 18], and its availability is one reason why this problem is more tractable now than it was before.

We also address the case where $d = 2$ and A is polygonal; this case could be of importance for some applications, and it turns out that the effects of corners are asymptotically negligible for $d = 2$. We hope to deal with the case of polytopal A in dimension $d \geq 3$ in future work.

Understanding the connectivity threshold is important for a variety of applications. In telecommunications, the vertices could represent mobile transceivers and one might be interested in whether the network of transceivers is connected; see, for example, [5]. In topological data analysis (TDA), detecting connectivity is a fundamental step for inspecting all other higher dimensional topological features (here the dimension of the ambient space may be very high). See also applications in machine learning (clustering), statistical tests (e.g., for outliers), spatial epidemics or forest fires (see the description in [12]).

To go into more detail on the first of these applications, we follow the setup of [5], namely "a group of mobile nodes which communicate with each other over a wireless channel and without any centralized control". Suppose the number of nodes n and the region A in which they are scattered are known, but their locations are not known and are varying due to mobility. In the absence of further information one might assume that at any fixed instant the nodes are independently uniformly distributed over A (possibly a crude model, but one has to start somewhere, and it is used in [5] which is heavily cited). One can achieve connectivity by choosing the power of the transmitters (represented by the distance parameter r) to be sufficiently large; however taking the power too large can lead to problems with interference. One can imagine a protocol whereby the network is required to be fully connected 95% of the time, say, so that r is chosen to be just big enough to achieve a 95% probability of connectivity at any fixed instant. For this we need to know the distribution of $M_1(\mathcal{X}_n)$. For extra robustness of the network the protocol might instead require it to be k-connected 95% of the time for some fixed $k > 1$, and then one would need the distribution of $M_k(\mathcal{X}_n)$.

For another application in more detail, consider clustering. Suppose one has a uniform random sample of size n from some unknown region A in Euclidean space, and wishes to know whether or not A is connected; this might be of value in deciding whether the data can be considered as a single sample or as coming from two or more separate clusters. A possible test statistic would be the 1-connectivity threshold $M_1(\mathcal{X}_n)$ of the sample, which one would expect to be smaller if A is connected than otherwise. To develop hypothesis tests and p-values using this statistic, one would like to know the distribution of $M_1(\mathcal{X}_n)$.

Note that (1.2) implies the weaker statement that the sequence $(nM_{n,k}^d - a_n)_{n\geq 1}$ is tight as $n \to \infty$, and even in the (rather exceptional) cases where we cannot prove (1.2), we shall prove this weaker statement. Tightness in turn implies $nM_{n,k}^d/a_n' \to 1$ in probability as $n \to \infty$, for any sequence $(a_n')_{n\geq 1}$ satisfying $a_n'/a_n \to 1$ as $n \to \infty$. Another direction of research (not followed in the present paper) is to improve this to almost sure convergence, that is, a strong law of large numbers (SLLN), under the natural coupling of $(\mathcal{X}_n, n \geq 1)$:

$$\frac{nM_k(\mathcal{X}_n)^d}{a_n'} \xrightarrow{a.s.} 1, \tag{1.3}$$

or to establish (1.3) for some a_n' even in cases when (1.2) is not known; see (1.7), (1.8) below.

1.2. *The largest k-nearest-neighbour link.* Given $x \in \mathbb{R}^d$ and $r > 0$, we denote the closed Euclidean ball of radius r, centred at x, by $B_r(x)$. Given finite $\mathcal{X} \subset \mathbb{R}^d$ we define the *largest k-nearest-neighbour link* $L_k(\mathcal{X})$ of $\mathcal{X}$, by

$$L_k(\mathcal{X}) := \begin{cases} \max_{x\in\mathcal{X}}\big(\inf\{r > 0 : \mathcal{X}\big(B_r(x)\big) > k\}\big) & \text{if } |\mathcal{X}| \geq k+1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

where $|\mathcal{X}|$ denotes the number of elements of $\mathcal{X}$ and $\mathcal{X}(\cdot) := |\mathcal{X} \cap \cdot|$ denotes the counting measure associated with $\mathcal{X}$. Note that if $|\mathcal{X}| \geq k+1$ and $x \in \mathcal{X}$, then $\inf\{r > 0 : \mathcal{X}(B_r(x)) > k\}$ is the distance from x to its k-nearest neighbour in $\mathcal{X}$. Note also that $L_k(\mathcal{X}) \leq M_k(\mathcal{X})$ since if $r < L_k(\mathcal{X})$ and $|\mathcal{X}| \geq k+1$, then $G(\mathcal{X}, r)$ has at least one vertex with degree less than k and therefore is not k-connected, so $r \leq M_k(\mathcal{X})$.

Our analysis of $M_{n,k}$ and $M_k(\mathcal{X}_n)$ will involve first investigating $L_{n,k} := L_k(\mathcal{P}_n)$ and $L_k(\mathcal{X}_n)$. A priori, it is not obvious that $L_k(\mathcal{X}_n)$ is a sharp lower bound for $M_k(\mathcal{X}_n)$; nevertheless, it is known in some cases (see Section 1.3) that $M_k(\mathcal{X}_n)$ enjoys the same limiting behaviour as $L_k(\mathcal{X}_n)$, and even sometimes that

$$\lim_{n\to\infty} \mathbb{P}\big[L_k(\mathcal{X}_n) = M_k(\mathcal{X}_n)\big] = 1. \tag{1.5}$$

Equation (1.5), when true, says that with probability tending to 1 as $n \to \infty$ the point set $\mathcal{X}_n$ has the following property: If we start with no edges between the vertices of $\mathcal{X}_n$, and then add edges one by one in order of increasing Euclidean length until we arrive at a k-connected graph, then just before the addition of the last edge we still have a vertex of degree less than k; if $k = 1$ then just before the addition of the last edge we have exactly two components, one of which is a singleton.

The largest k-nearest neighbour link $L_k(\mathcal{X}_n)$ (or $L_k(\mathcal{P}_n)$) is of interest in its own right. To quote [3], it "comes up in almost all discussions of computational complexity involving nearest neighbours". See, for example, [12] for further motivation. As with $M_{n,k}$, its limiting behaviour has previously been studied on the torus, and the unit cube, and only at the level of a SLLN for regions with smooth or polytopal boundary. By providing a limiting distribution for $L_k(\mathcal{X}_n)$ for regions with smooth or polygonal boundary, we here add significantly to this body of work, too.

1.3. *Literature review*. Before stating our main results, let us give a literature review on this topic. Under our working assumption (WA), we use throughout the notation

$$f_0 := \inf_{x\in A} f(x); \qquad f_1 := \inf_{x\in \partial A} f(x); \qquad f_{\max} := \sup_{x\in A} f(x). \tag{1.6}$$

Note that $0 < f_0 \le f_1 \le f_{\max} < \infty$ under our WA. Let θ_d denote the volume of a d-dimensional unit ball, that is, $\theta_d := \pi^{d/2}/\Gamma(1+d/2)$,

In the case where A is the flat torus $\mathbb{T}^d$ of any dimension, it is known ([11], Theorem 13.6) that under the natural coupling of $(\mathcal{X}_n, n \ge 1)$ we have

$$\lim_{n\to\infty}\left(\frac{\theta_d n(M_k(\mathcal{X}_n))^d}{\log n}\right) = \frac{1}{f_0} \quad a.s.$$

The dimensionality and the density play a crucial role when one considers compact sets with boundaries. More precisely, if A has a smooth boundary, it is proved for $k = 1$ in [15, 16], and for general k in [11], that

$$\lim_{n\to\infty}\left(\frac{\theta_d n(M_k(\mathcal{X}_n))^d}{\log n}\right) = \lim_{n\to\infty}\left(\frac{\theta_d n(L_k(\mathcal{X}_n))^d}{\log n}\right) = \max\left(\frac{1}{f_0}, \frac{2-2/d}{f_1}\right) \quad a.s. \tag{1.7}$$

When A is a convex polytope, it is proved in [19] that

$$\lim_{n\to\infty}\left(\frac{n(M_k(\mathcal{X}_n))^d}{\log n}\right) = \lim_{n\to\infty}\left(\frac{n(L_k(\mathcal{X}_n))^d}{\log n}\right) = \max_{\varphi\in\Phi^*(A)}\left(\frac{D(\varphi)}{f_\varphi \rho_\varphi d}\right) \quad a.s., \tag{1.8}$$

where Φ^* denotes the collection of all faces of φ of all dimensions (including A itself, considered as a face of dimension d), and where $D(\varphi)$ is the dimension of face φ, and where f_φ denotes the infimum of f over face φ and ρ_φ is the angular volume of face φ.

Less is known about the fluctuations of $nM_k(\mathcal{X}_n)^d - a_n$. Weak limit results of the type (1.2) are proved for two cases in [12, 14]. The first case is when f is uniform on $\mathbb{T}^d$ for any $d \ge 2$. In this case, by [11], Theorem 8.3 and Corollary 13.20, one has

$$\theta_d n\big(L_k(\mathcal{X}_n)\big)^d - \log n - (k-1)\log\log n + \log\big((k-1)!\big) \xrightarrow{d} \mathsf{Gu}, \tag{1.9}$$

and likewise for $M_k(\mathcal{X}_n)$, where Gu denotes a standard Gumbel random variable, that is, one with cumulative distribution function $F(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$. (For $a \in \mathbb{R}$, $b > 0$ the random variable $b\mathsf{Gu} + a$ is said to be Gumbel distributed with scale parameter b and location parameter a.)

The second case is when f is uniform on $[0, 1]^d$. For this case, we describe only the results for $k = 1$ from [11] but the case of general k is also treated there. When f is uniform on $[0, 1]^d$, one has by [11], Corollary 13.21, that

$$\begin{aligned} &2^{2-d}\theta_d n\big(M_1(\mathcal{X}_n)\big)^d - (2/d)\log n + (d-3+2/d)\log\log n \\ &\quad + \log\left(\left(\frac{2^{2-2/d}}{d(d-1)}\right)(\theta_d d)^{3-d-2/d}\theta_{d-1}^{d-2}\right) \xrightarrow{d} \mathsf{Gu}. \end{aligned} \tag{1.10}$$

Similar results hold for $L_k(\mathcal{X}_n)$; see [11], Theorems 8.3 and 8.4. Moreover it is known that (1.5) holds. Also these results hold for $\mathcal{P}_n$ instead of $\mathcal{X}_n$.

So far as we know, there is no weak limit result for other shaped boundaries or for nonuniform distributions (until now). In the special case where $d = 2$ and f is uniformly distributed in a disk, [5] gives a partial result in the direction of a weak limit.

The main results of this paper considerably generalise previous findings and deepen the understanding of the connectivity threshold in terms of the geometry of A. In the uniform case, we also provide a bound on the rate of convergence that is new for all shapes under the WA.

We end this subsection by mentioning some related results. It is natural to ask what happens if we drop the working assumption and take the support of f to be unbounded. Penrose [13] found that the scaling is completely different in the case of standard Gaussian density in $\mathbb{R}^d$; see also Hsing and Rootzén [7] for a significant extension in dimension two, where a class of elliptically contoured distributions are included, for example, Gaussian densities with correlated coordinates. Gupta and Iyer [4] give a limiting distribution and SLLN for $L_{n,1}$ for a class of radially symmetric densities with unbounded support, including cases where $L_{n,1}$ (and hence also $M_{n,1}$) does not even tend to zero.

Regarding $L_k(\mathcal{X}_n)$ for general k, Chenavier, Henze and Otto [2] consider a statistic resembling $L_k(\mathcal{X}_n)$ but where for each point one considers the ν-measure of the k-nearest neighbour ball rather than its radius. Let us denote this quantity by $\tilde{L}_k(\mathcal{X}_n)$. One can obtain from [2], Theorem 2.2, a limiting distribution for $\tilde{L}_k(\mathcal{X}_n)$ when our working assumption holds and $A = [0, 1]^d$ (but their method also applies for general convex A). If ν is uniform on the torus, then $\tilde{L}_k(\mathcal{X}_n) = \theta_d L_k(\mathcal{X}_n)^d$ and one readily recovers (1.9) from [2], Theorem 2.2. In other cases, there is no obvious equivalence between $L_k(\mathcal{X}_n)$ and $\tilde{L}_k(\mathcal{X}_n)$ since boundary effects and nonuniformity of ν are treated differently in the definitions of $L_k(\mathcal{X}_n)$ and $\tilde{L}_k(\mathcal{X}_n)$.

In a similar vein, Otto [10], Theorem 4.2, and Bobrowski, Schulte and Yogeshwaran [1], Theorems 6.4 and 6.5, derive point process approximation results for point processes associated with $\tilde{L}_k(\mathcal{X}_n)$ or $\tilde{L}_k(\mathcal{P}_n)$, namely the point process of (upper) extreme values (and associated spatial locations) of ν-measures of k-nearest neighbour balls in $\mathcal{X}_n$ or in $\mathcal{P}_n$.

1.4. *Main results.* Throughout this paper, c and c' denote positive finite constants whose values may vary from line to line and do not depend on n. Also if $n_0 \in (0, \infty)$ and $f(n)$, $g(n)$ are two functions, defined for all $n \geq n_0$ with $g(n) > 0$ for all $n \geq n_0$, the notation $f(n) = O(g(n))$ as $n \to \infty$ means that $\limsup_{n\to\infty}(|f(n)|/g(n)) < \infty$. If also $f(n) > 0$ for all $n \geq n_0$, we use notation $f(n) = \Theta(g(n))$ to mean that both $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

Given $d, k \in \mathbb{N}$, define the constant

$$c_{d,k} := \theta_{d-1}^{-1}\theta_d^{1-1/d}(2 - 2/d)^{k-2+1/d}2^{1-k}/(k-1)! \tag{1.11}$$

In this paper, given $A \subset \mathbb{R}^d$, we say that A has C^2 *boundary* (or for short: $\partial A \in C^2$) if for each $x \in \partial A$, the topological boundary of A, there exists a rigid motion ρ_x of $\mathbb{R}^d$, an open set $U_x \subset \mathbb{R}^d$ and a C^2 function $f_x : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\rho_x(A \cap U_x) = \rho_x(U_x) \cap \mathrm{epi}(f_x)$, where $\mathrm{epi}(f_x) := \{(u, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : z \geq f_x(u)\}$, the closed epigraph of f_x. If, for each x, we can take the function f_x to be Lipschitz-continuously differentiable, then we say that A has $C^{1,1}$ *boundary* (for short, $\partial A \in C^{1,1}$). Clearly $\partial A \in C^2$ implies $\partial A \in C^{1,1}$.

For compact $A \subset \mathbb{R}^d$ with $C^{1,1}$ or polytopal boundary, let $|A|$ denote the volume (Lebesgue measure) of A, and $|\partial A|$ the perimeter of A, that is, the $(d-1)$-dimensional Hausdorff measure of ∂A. Also define

$$\sigma_A := \frac{|\partial A|}{|A|^{1-1/d}}. \tag{1.12}$$

Note that σ_A^d is sometimes called the *isoperimetric ratio* of A, and is at least $d^d\theta_d$ by the isoperimetric inequality.

THEOREM 1.1 (Weak limits in the uniform case). *Suppose either that $d \geq 2$ and A a compact subset of $\mathbb{R}^d$ with C^2 boundary, or that $d = 2$ and A is a convex polygon. Let $f \equiv f_0 \mathbf{1}_A$ with $f_0 = |A|^{-1}$. Let $\beta \in \mathbb{R}$. Then, if $d = 2$, we have as $n \to \infty$ that*

$$\mathbb{P}\big[nf_0\pi M_1(\mathcal{X}_n)^2 - \log n \leq \beta\big] = \exp\biggl(-\frac{\sigma_A \pi^{1/2} e^{-\beta/2}}{2(\log n)^{1/2}}\biggr) e^{-e^{-\beta}} + O\big((\log n)^{-1}\big); \tag{1.13}$$

$$\mathbb{P}\big[nf_0\pi M_{n,1}^2 - \log n \leq \beta\big] = \exp\biggl(-\frac{\sigma_A \pi^{1/2} e^{-\beta/2}}{2(\log n)^{1/2}}\biggr) e^{-e^{-\beta}} + O\big((\log n)^{-1}\big). \tag{1.14}$$

Also, given $k \in \mathbb{N}$, set

$$u_{n,k} := \mathbb{P}\big[n\theta_d f_0 M_{n,k}^d - (2 - 2/d)\log n + (4 - 2k - 2/d)\log\log n \leq \beta\big];$$
$$u'_{n,k} := \mathbb{P}\big[n\theta_d f_0 M_k(\mathcal{X}_n)^d - (2 - 2/d)\log n + (4 - 2k - 2/d)\log\log n \leq \beta\big].$$

If $d = 2$, then as $n \to \infty$,

$$\begin{aligned} u_{n,2} = {}& \exp\biggl(-\frac{\pi^{1/2}\sigma_A e^{-\beta/2}\log\log n}{8\log n} - \frac{e^{-\beta}\log\log n}{\log n}\biggr)\exp\biggl(-e^{-\beta} - \frac{\pi^{1/2}\sigma_A e^{-\beta/2}}{4}\biggr) \\ & + O\biggl(\frac{1}{\log n}\biggr), \end{aligned} \tag{1.15}$$

and likewise, for $u'_{n,2}$. If $d \geq 3$, or if $d = 2$, $k \geq 3$ we have as $n \to \infty$ that

$$\begin{aligned} u_{n,k} = {}& \exp\biggl(-\frac{c_{d,k}\sigma_A e^{-\beta/2}(k - 2 + 1/d)^2 \log\log n}{(1 - 1/d)\log n}\biggr)\exp\big(-c_{d,k}\sigma_A e^{-\beta/2}\big) \\ & + O\biggl(\frac{1}{\log n}\biggr), \end{aligned} \tag{1.16}$$

and likewise for $u'_{n,k}$. Also (1.5) holds, and all of the above results hold with $L_{n,k}$ (resp. $L_k(\mathcal{X}_n)$) instead of $M_{n,k}$ (resp. $M_k(\mathcal{X}_n)$).

REMARK 1.2.

(i) The statements about $L_{n,k}$ and $L_k(\mathcal{X}_n)$ in this result are spelt out in Corollaries 6.3 and 6.4.

(ii) These results imply certain convergence in distribution results. Namely, $n\theta_d f_0 M_{n,k}^d$, suitably centred, is asymptotically Gumbel distributed with scale parameter 1 when $d = 2$, $k = 1$ but with scale parameter 2 when $d \geq 3$ or $d = 2$, $k \geq 3$. When $d = 2$, $k = 2$ the limiting distribution is a so-called *two-component extreme value distribution* (TCEV), that is, a probability distribution with cumulative distribution function (cdf) given by the product of two Gumbel cdfs with different scale parameters, in this case 1 and 2. The terminology TCEV was introduced in the hydrology literature [20].

(iii) We have included a multiplicative correction factor in each of (1.13)–(1.16), namely the first factor in the right hand side in each case, because this factor tends to 1 very slowly, especially in (1.13) and (1.14) where the correction factor is $1 + O((\log n)^{-1/2})$ so for moderately large values of n the limiting Gumbel distribution with cdf $e^{-e^{-\beta}}$ is not very close to the centred distribution of $nf_0\theta_d M_{n,1}^d$. If $d \geq 3$ it is possible to give some extra terms in the correction and improve the error bound to $O((\log n)^{\varepsilon-2})$.

(iv) The error bounds above are for fixed β but we would need to make them uniform in β for an error bound in the Kolmogorov distance between probability distributions. We do not address this in this paper.

(v) It seems likely that our results carry over to the case where $d = 2$ and A is a nonconvex polygon. Our main reason for restricting attention to polygons that are convex its that some of our arguments in Section 4 are based on results from [19] that are stated there only for convex polytopes.

(vi) The statements about $L_{n,k}$ and $L_k(\mathcal{X}_n)$ in Theorem 1.1, and all of our results, still hold if we replace the condition $\partial A \in C^2$ with the weaker condition $\partial A \in C^{1,1}$. Our proof of the results for $M_{n,k}$ and $M_k(\mathcal{X}_n)$ (see Lemma 4.5) relies on [12], Lemma 3.5, which is proved there under the C^2 condition. We suspect that that result still holds under the weaker $C^{1,1}$ condition, but its proof is quite long and we do not attempt to adapt it here.

If A is a convex polytope, then by (1.8), in the uniform case both $nL_k(\mathcal{X}_n)^d/\log n$ and $nM_k(\mathcal{X}_n)^d/\log n$ converge almost surely to a constant times $\max_{\varphi\in\Phi^*(A)} D(\varphi)/\rho_\varphi$. For $d \ge 3$ this maximum is not attained at $\varphi = A$, so boundary effects dominate. However, different dimensions of faces may dominate, depending on their geometry and on k. For example, if $A = [0, 1]^d$ with $d \ge 3$, the $(d-1)$- or $(d-2)$-dimensional faces may be the most important, depending on the value of k; see [11], Theorem 8.4 and Corollary 13.21. A full treatment of all convex polytopes in $d \ge 3$ is beyond the scope of this paper.

We now give a result for the general nonuniform case; that is, we still use our WA on f, but drop the stronger assumption that f is constant on A. Recall f_0, f_1 defined at (1.6). In this more general case, subject to the condition $f_1 \neq f_0(2-2/d)$, we still provide a result along the lines of (1.2), but now, instead of using the explicit centring constants $a_n = (2 - 2/d)\log n - (4 - 2k - 2/d)\mathbf{1}\{d \ge 3 \text{ or } k \ge 2\}\log\log n$ as in Theorem 1.1, we take a_n to be the median of the distribution $nM_{n,k}^d$. In the case $f_1 = f_0(2-2/d)$ we prove only the weaker result that our sequence of centred random variables is tight.

Given a random variable X, let $\mu(X) := \inf\{x \in \mathbb{R} : \mathbb{P}[X \le x] \ge 1/2\}$, the median of the distribution of X. Note that $\mu(\mathsf{Gu}) = -\log(\log 2)$, so for $\alpha > 0$, the random variable $\alpha(\mathsf{Gu} + \log(\log 2))$ has a Gumbel distribution with median 0 and with scale parameter α.

THEOREM 1.3 (Weak limit in the nonuniform case). *Suppose our working assumption applies, either with $d \ge 2$ and A a compact subset of $\mathbb{R}^d$ with C^2 boundary, or with $d = 2$ and A a convex polygon. Let $k \in \mathbb{N}$.*

(i) *If $f_1 > f_0(2-2/d)$, then as $n \to \infty$,*

$$nM_k(\mathcal{X}_n)^d - n\mu\big(M_k(\mathcal{X}_n)\big)^d \xrightarrow{d} (\theta_d f_0)^{-1}(\mathsf{Gu} + \log\log 2); \tag{1.17}$$

$$nL_k(\mathcal{X}_n)^d - n\mu\big(L_k(\mathcal{X}_n)\big)^d \xrightarrow{d} (\theta_d f_0)^{-1}(\mathsf{Gu} + \log\log 2); \tag{1.18}$$

$$nM_{n,k}^d - n\mu(M_{n,k})^d \xrightarrow{d} (\theta_d f_0)^{-1}(\mathsf{Gu} + \log\log 2); \tag{1.19}$$

$$nL_{n,k}^d - n\mu(L_{n,k})^d \xrightarrow{d} (\theta_d f_0)^{-1}(\mathsf{Gu} + \log\log 2). \tag{1.20}$$

(ii) *If $f_1 < f_0(2-2/d)$, then as $n \to \infty$,*

$$nM_k(\mathcal{X}_n)^d - n\mu\big(M_k(\mathcal{X}_n)\big)^d \xrightarrow{d} \big(2/(\theta_d f_1)\big)(\mathsf{Gu} + \log\log 2); \tag{1.21}$$

$$nL_k(\mathcal{X}_n)^d - n\mu\big(L_k(\mathcal{X}_n)\big)^d \xrightarrow{d} \big(2/(\theta_d f_1)\big)(\mathsf{Gu} + \log\log 2). \tag{1.22}$$

$$nM_{n,k}^d - n\mu(M_{n,k})^d \xrightarrow{d} \big(2/(\theta_d f_1)\big)(\mathsf{Gu} + \log\log 2); \tag{1.23}$$

$$nL_{n,k}^d - n\mu(L_{n,k})^d \xrightarrow{d} \big(2/(\theta_d f_1)\big)(\mathsf{Gu} + \log\log 2); \tag{1.24}$$

(iii) *In all cases, including when* $f_1 = f_0(2-2/d)$, (1.5) *holds, and also the family of random variables* $(n(M_{n,k}^d - \mu(M_{n,k})^d))_{n\geq 1}$ *is tight. Likewise the collection of random variables* $(n(L_{n,k}^d - \mu(L_{n,k})^d))_{n\geq 1}$ *is tight, as are the sequences* $(n(M_k(\mathcal{X}_n)^d - \mu(M_k(\mathcal{X}_n))^d))_{n\geq 1}$ *and* $(n(L_k(\mathcal{X}_n)^d - \mu(L_k(\mathcal{X}_n))^d))_{n\geq 1}$.

REMARK 1.4. For any sequence (X_n) of random variables, if X_n converges almost surely to a constant x then $\mu(X_n) \to x$ as $n \to \infty$. Therefore, $\mu(M_k(\mathcal{X}_n))$ and the other medians mentioned in Theorem 1.3 enjoy the same limiting behaviour described in (1.7) and (1.8) for $M_k(\mathcal{X}_n)$ and $L_k(\mathcal{X}_n)$. We are not aware of any more detailed results on limiting behaviour of these medians in the nonuniform case.

Before proceeding to proofs, we give a rough calculation indicating why, in the uniform case with $f \equiv f_0 \mathbf{1}_A$, we might expect to see qualitative differences between the cases with $d = 2$ and $k = 1$ or $k = 2$, and other cases, as seen in Theorem 1.1. Suppose we take a sequence of distance parameters r_n with $nf_0\theta_d r_n^d = \log n + (k-1)\log\log n + c$ for some constant c. Given r_n, let F_n^o, F_n^∂ be the number of vertices in $G(\mathcal{P}_n, r_n)$ of degree less than k in the interior of A, respectively near the boundary of A. We give a rough calculation suggesting that for $d \geq 3$ we have $\mathbb{E}[F_n^\partial] \gg \mathbb{E}[F_n^o]$ so the boundary region dominates, while for $d = 2$, it depends on the value of k whether the interior or boundary region dominates. First, by the Mecke formula (see (3.2) below) and the assumption $nr_n^d \to \infty$,

$$\mathbb{E}[F_n^o] \approx n\big((nf_0\theta_d r_n^d)^{k-1}/(k-1)!\big)e^{-nf_0\theta_d r_n^d} \sim (e^{-c})/(k-1)!.$$

For F_n^∂, note that for small positive s the volume of the intersection of A with a ball of radius r_n centred at distance sr_n from ∂A is about $(\theta_d/2)r_n^d + \theta_{d-1}sr_n^d$, suggesting

$$\mathbb{E}[F_n^\partial] \approx nr_n\big(f_0\theta_d nr_n^d/2\big)^{k-1}\big(|\partial A|/(k-1)!\big)e^{-nf_0\theta_d r_n^d/2}\int_0^\infty e^{-nf_0\theta_{d-1}sr_n^d}\,ds.$$

The integral on the right is asymptotic to a constant times $(nr_n^d)^{-1}$. Also by the choice of r_n the exponential factor is $n^{-1/2}(\log n)^{-(k-1)/2}e^{-c/2}$, so that

$$\begin{aligned}\mathbb{E}[F_n^\partial] &\approx \text{const.} \times n^{(1/2)-(1/d)}\big(nr_n^d\big)^{(1/d)+k-2}(\log n)^{(1-k)/2}\\ &\approx \text{const.} \times n^{(1/2)-(1/d)}(\log n)^{(1/d)+((k-3)/2)}.\end{aligned}$$

If $d \geq 3$ this tends to infinity (regardless of k). Thus the boundary effects dominate in this case; we should choose a slightly bigger r_n to make $\mathbb{E}[F_n^\partial]$ tend to a constant, and then $\mathbb{E}[F_n^o]$ will tend to zero.

If $d = 2$, the last expression for $\mathbb{E}[F_n^\partial]$ tends to zero if $k = 1$ and to infinity if $k \geq 3$, so the interior contribution dominates when $k = 1$ but the boundary contribution dominates when $k \geq 3$. When $k = 2$ the interior and boundary effects are of comparable size.

Having chosen r_n so that $\mathbb{E}[F_n^o + F_n^\partial]$ tends to a constant, we shall use Poisson approximation to show that $\mathbb{P}[L_n^o \leq r_n]$ tends to a nontrivial constant, and then some percolation arguments to show the same limit holds for $\mathbb{P}[M_{n,1} \leq r_n]$.

The rest of the paper is organised as follows. After the preparation of geometrical ingredients in Section 2, we prove Poisson approximation for the number of k-isolated vertices in Section 3, asymptotic equivalence of $L_{n,k}$ and $M_{n,k}$ in Section 4, the weak law in the nonuniform case (Theorem 1.3) in Section 5 and finally the weak law in the uniform case (Theorem 1.1) in Section 6.

2. Geometrical preliminaries. In this section, we prepare some geometrical ingredients for later use. Let A be a compact subset of $\mathbb{R}^d$ with $d \geq 2$.

Given $B, C \subset \mathbb{R}^d$, set $B \oplus C := \{x + y : x \in B, y \in C\}$. Let o denote the origin in $\mathbb{R}^d$. Given $x \in \mathbb{R}^d$, $a \in \mathbb{R}$, we write $B + x$ for $B \oplus \{x\}$ and aB for $\{ay : y \in B\}$.

Given $s > 0$, and $\Gamma \subset A$, we write $\Gamma^{(s)}$ for $(\Gamma \oplus B_s(o)) \cap A$, the set of points in A distant at most s from Γ. Also we set $\operatorname{diam}(\Gamma) := \sup_{x,y\in\Gamma} \|y - x\|$, or zero if $\Gamma = \varnothing$.

Given $x \in A$, set $\operatorname{dist}(x, \partial A) := \inf_{z\in\partial A}\{\|z - x\|\}$. We write $A^{(-s)}$ for $A \setminus (\partial A)^{(s)}$, the set of points in A distant more than s from the boundary ∂A of A.

When A is polygonal, we denote by Cor the set of corners of A.

DEFINITION 2.1 (Sphere condition). For $z \in \partial A$ let $\hat{n}_z$ be the unit normal to ∂A at z pointing inside A. Given $\tau \geq 0$, let us say τ satisfies the *sphere condition* for A if, for all $x \in \partial A$, we have $B_\tau(x + \tau\hat{n}_x) \subset A$ and $B_\tau(x - \tau\hat{n}_x) \cap A = \{x\}$. Let $\tau(A)$ denote the supremum of the set of all τ satisfying the sphere condition for A.

LEMMA 2.2 (Lemma 7 of [9]). *Suppose $\partial A \in C^{1,1}$. Then $\tau(A) > 0$; that is, there exists a constant $\tau > 0$ such that τ satisfies the sphere condition for A.*

Note that if $0 < \tau < \tau'$ and τ' satisfies the sphere condition for A, then so does τ. Note moreover that if $x \in \mathbb{R}^d$ with $\operatorname{dist}(x, \partial A) < \tau(A)$, then x has a unique closest point in ∂A.

LEMMA 2.3. *Suppose A has a $C^{1,1}$ boundary. Let $\varepsilon \in (0, 1]$. Then:*

(i) *For all small enough $r > 0$ we have*

$$|B_r(x) \cap A| \geq ((\theta_d/2) + (\theta_d\varepsilon/4))r^d, \quad \forall\, x \in A^{(-\varepsilon r)}. \tag{2.1}$$

(ii) *There exists $\delta > 0$ and $r_0 > 0$ such that if $0 < r < s < 2r < r_0$, then*

$$|A \cap B_s(x) \setminus B_r(x)| \leq ((\theta_d/2) + \varepsilon)(s^d - r^d), \quad \forall x \in (\partial A)^{(\delta s)}. \tag{2.2}$$

PROOF. For $a > 0$ set $g(a) := |B_1(o) \cap (\mathbb{R}^{d-1} \times [0, a])|$. Then $g(a)/a$ is decreasing in a, so for $0 \leq a \leq 1$ we have $g(a)/a \geq g(1) = \theta_d/2$.

Suppose $0 < r < \theta_d\tau(A)\varepsilon/(8\theta_{d-1})$, and $x \in A^{(-\varepsilon r)} \setminus A^{(-r)}$. Then by [6], Lemma 3.4,

$$\begin{aligned}|B_r(x) \cap A| &\geq (\theta_d/2)r^d + g(\varepsilon)r^d - (2\theta_{d-1}/\tau(A))r^{d+1}\\ &\geq (\theta_d/2)r^d + (\varepsilon\theta_d/4)r^d.\end{aligned}$$

Also, (2.1) clearly holds for $x \in A^{(-r)}$, and hence we have part (i).

For part (ii), fix $\tau \in (0, \tau(A))$. Let $\delta \in (0, 1)$ be a constant to be chosen later. Let $s \in (0, \delta\tau)$ and let $x \in (\partial A)^{(\delta s)}$. Without loss of generality (after a translation and rotation), we can assume that the closest point of ∂A to x lies at the origin, and $x = he_d$ for some $h \in (0, \delta s]$, where $e_d := (0, \dots, 0, 1)$ is the dth coordinate unit vector.

Define the half-spaces $\mathbb{H} := \{y \in \mathbb{R}^d : y \cdot e_d \geq 0\}$ and $\mathbb{H}' := \{y \in \mathbb{R}^d : y \cdot e_d \geq h\}$. Then $A \cap B_s(x) \setminus \mathbb{H} \subset A \cap B_s(o) \setminus \mathbb{H}$, and by the proof of [6], Lemma 3.4, the set $A \cap B_s(o) \setminus \mathbb{H}$ is contained in a cylinder of radius s and height s^2/τ with upper face part of $\partial\mathbb{H}$ centred on o. Also $B_s(x) \cap \mathbb{H} \setminus \mathbb{H}'$ is contained in a cylinder of radius s and height h. Taking the union of these two cylinders we see that $A \cap B_s(x) \setminus \mathbb{H}'$ is contained in a cylinder of radius s and height $h + s^2/\tau$, and hence of height less than $2\delta s$.

Let $r \in (s/2, s)$. By the argument above,

$$A \cap B_s(x) \setminus B_r(x) \subset (B_s(x) \setminus B_r(x)) \cap (\mathbb{R}^{d-1} \times (h - 2\delta s, \infty)).$$

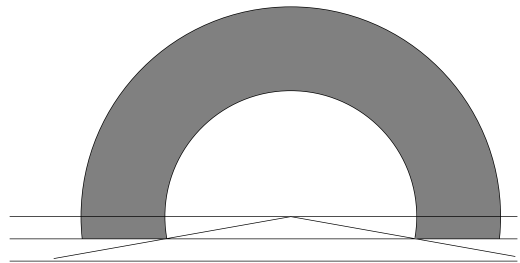

FIG. 1. *The horizontal lines are at height* 0, $-2\delta s$ *and* $-4\delta s$. *The circles are of radius* r, s *centred on the origin.*

Define the set $\Lambda_{-4\delta} := \{z \in B_1(o) : z \cdot e_d > -4\delta \|z\|\}$. Since $s \le 2r$, as shown in Figure 1,

$$\big|A \cap B_s(x) \setminus B_r(x)\big| \le \big(s^d - r^d\big)|\Lambda_{-4\delta}|.$$

On taking δ small enough so that $|\Lambda_{-4\delta}| \le (\theta_d/2) + \varepsilon$, we obtain (2.2), completing the proof of part (ii). □

LEMMA 2.4. *Suppose* $A \subset \mathbb{R}^d$ *is compact with* $\partial A \in C^{1,1}$.

(i) *Given* $\varepsilon > 0$, *there exists* $r_0 = r_0(d, A, \varepsilon) > 0$ *such that*

$$\big|B_r(x) \cap A\big| \ge \big((\theta_d/2) - \varepsilon\big)r^d, \quad \forall x \in A, r \in (0, r_0). \tag{2.3}$$

(ii) *There is a constant* $r_1 = r_1(d, A)$, *such that if* $r \in (0, r_1)$ *and* $x, y \in A$ *with* $\|y - x\| \le 3r$ *and* $\mathrm{dist}(x, \partial A) \le \mathrm{dist}(y, \partial A)$, *then*

$$\big|A \cap B_r(y) \setminus B_r(x)\big| \ge 8^{-d}\theta_{d-1}r^{d-1}\|y - x\|. \tag{2.4}$$

(iii) *Given* $\varepsilon > 0$, *there exists* $r_2 = r_2(d, A, \varepsilon) > 0$ *such that*

$$\big|A \cap B_s(x) \setminus B_r(x)\big| \ge \big((\theta_d/2) - \varepsilon\big)\big(s^d - r^d\big), \quad \forall x \in A, s \in (0, r_2), r \in (0, s).$$

PROOF. Part (i) follows from [6], Lemma 3.4. For (ii), see [6], Lemma 3.6. For (iii), as in the proof of Lemma 2.3, it suffices to consider $x \in A$ such that the closest point of ∂A to x is at o with $x = he_d$ for some $h \in (0, s]$.

Given $a > 0$, define the set $\Lambda_a \subset \mathbb{R}^d$ by

$$\Lambda_a := \big\{z \in B_1(o) : z \cdot e_d > a\|z\|\big\}.$$

Then $|\Lambda_a| \uparrow \theta_d/2$ as $a \downarrow 0$ so we can and do choose $\delta \in (0, \frac{1}{2})$ such that $|\Lambda_\delta| \ge (\theta_d/2) - \varepsilon$.

Fix $\tau \in (0, \tau(A))$ and suppose $s \in (0, \delta\tau)$. Let $\mathbb{H}' := \{y \in \mathbb{R}^d : y \cdot e_d \ge x \cdot e_d\}$. Set $C_s := \{z \in \mathbb{R}^d : 0 \le z \cdot e_d \le \tau, \|z - (z \cdot e_d)e_d\| \le s\}$, a cylinder of radius s and height τ. Then as in the proof of [6], Lemma 3.4, the set $C_s \setminus A$ is contained in a cylinder of radius s and height s^2/τ. Then the set $B_s(x) \cap \mathbb{H}' \setminus A$ is contained in the cylinder $C_{x,s}$ given by

$$C_{x,s} := \big\{z \in \mathbb{R}^d : 0 \le (z - x) \cdot e_d \le \delta s, \big\|z - (z \cdot e_d)e_d\big\| \le s\big\}.$$

Now suppose $y \in B_s(x)$ with $(y - x) \cdot e_d > \delta\|y - x\|$. Then $y \in \mathbb{H} \setminus C_{x,\|y-x\|}$ so $y \in A$. Therefore, $s\Lambda_\delta + x \subset A$.

For $0 \le r < s$, given x as above define the set $S := ((s\Lambda_\delta) \setminus (r\Lambda_\delta)) + x$. Then

$$|S| = \big|(s\Lambda_\delta) \setminus (r\Lambda_\delta)\big| = \big(s^d - r^d\big)|\Lambda_\delta| \ge \big((\theta_d/2) - \varepsilon\big)\big(s^d - r^d\big).$$

Since $S \subset A \cap B_s(x) \setminus B_r(x)$, this gives us the result, with $r_2 = \delta\tau$. □

Recall that Cor denotes the set of corners of A when A is polygonal.

LEMMA 2.5. *Assume $d = 2$ and A is polygonal, then there exist $K > 0$ and $r_1 > 0$ depending on A such that for all $r \in (0, r_1)$, $x, y \in A \setminus \mathsf{Cor}^{(Kr)}$ with* $\operatorname{dist}(x, \partial A) \le \operatorname{dist}(y, \partial A)$ *and $\|x - y\| \le 3r$, the lower bound (2.4) holds.*

PROOF. Let r_1 be small enough such that nonoverlapping edges of A are distant at least $8r_1$ from each other. Consider $x \in A \setminus \mathsf{Cor}^{(Kr)}$ with $r < r_1$ where K is made explicit later. We can assume that the corner of A closest to x is formed by edges e, e' meeting at the origin with angle $\alpha \in (0, 2\pi) \setminus \{\pi\}$. We claim that, provided $K > 4 + 8/|\sin\alpha|$, the disk $B_{4r}(x)$ intersects at most one of the two edges. Indeed, if it intersects both edges, then taking $w \in B_{4r}(x) \cap e$, $w' \in B_{4r}(x) \cap e'$ we have $\|w - w'\| \le 8r$; hence $\operatorname{dist}(w, e') \le 8r$. Then, $\|w\| \le \operatorname{dist}(w, e')/|\sin\alpha| \le 8r/|\sin\alpha|$. However, $\|w\| \ge (K - 4)r$ by the triangle inequality, so we arrive at a contradiction. We have thus shown that any ball of radius $4r$ with centre distant at least Kr from the corners of A cannot intersect two edges at the same time, where $K = 5 + (8/\min_i |\sin\alpha_i|)$ and $\{\alpha_i\}$ are the angles of the corners of A.

We have $B_r(x) \cup B_r(y) \subset B_{4r}(x)$; hence, the argument leading to Lemma 2.4-(ii), namely [6], Lemma 3.6, gives the estimate (2.4) in this case too. □

LEMMA 2.6. *Let $\varepsilon \in (0, 1]$. Then for all $r > 0$ and all compact $B \subset \mathbb{R}^d$ with* $\operatorname{diam} B \ge \varepsilon r$ *we have $|B \oplus B_r(o)| \ge |B| + \theta_d(1 + 2^{-d-1}d^{-d}\varepsilon^d)r^d$.*

PROOF. By scaling, it suffices to show that for all compact $B \subset \mathbb{R}^d$ with $\operatorname{diam} B \ge \varepsilon$, we have $|B \oplus B_1(o)| \ge |B| + \theta_d(1 + 2^{-d-1}d^{-d}\varepsilon^d)$.

Let $B \subset \mathbb{R}^d$ with $\varepsilon \le \operatorname{diam} B < \infty$. Without loss of generality we can assume $\operatorname{diam}(\pi_1(B)) \ge \varepsilon/d$, where π_1 denotes projection onto the first coordinate.

Let x be a left-most point of B, y a right-most point of B and u a top-most point of B. Here "left" and "right" refer to ordering using the first coordinate and "top" refers to ordering using the last coordinate. Let H^+ be the right half of $B_1(y)$ and H^- the left-half of $B_1(x)$. Let $D := B_{\varepsilon/(2d)}(u + (0, \dots, 0, \varepsilon/(2d)))$, and let D^+ and D^- be the left half and right half of D, respectively. Then the interiors of H^+ and of H^- are disjoint from B and from each other, and the interior of either D^+ or D^- (or both) is disjoint from all of B, H^+ and H^-. Therefore, since H^+, H^- and D are all contained in $B \oplus B_1(o)$, we obtain that

$$|B \oplus B_1(o)| \ge |B| + \theta_d + (\theta_d 2^{-d-1}d^{-d})\varepsilon^d,$$

as required. □

LEMMA 2.7. *Suppose $\partial A \in C^{1,1}$. Let $\rho, \varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \rho$. Then there exist $\delta = \delta(d, \rho, \varepsilon) > 0$, and $r_0 = r_0(d, \rho, \varepsilon, A)$, such that for all $r \in (0, r_0)$ and all compact $B \subset A$ with $\varepsilon r \le$* $\operatorname{diam} B$ *$\le \rho r$ we have*

$$|(B \oplus B_r(o)) \cap A| \ge |B| + ((\theta_d/2) + \delta)r^d, \tag{2.5}$$

and also, letting x_0 denote a closest point of B to ∂A, we have

$$|(B \oplus B_r(o)) \cap A| \ge |B| + |B_r(x_0) \cap A| + 2\delta r^d. \tag{2.6}$$

PROOF. It suffices to prove (2.6). Indeed, if (2.6) holds for some δ and r_0, then using (2.6) and Lemma 2.4-(i) readily yields (2.5) for some new, possibly smaller, choice of r_0.

Without loss of generality we may assume $\varepsilon < 1 < \rho$. Let $r > 0$, and let $B \subset A$ be compact with $\varepsilon r \le \operatorname{diam}(B) \le \rho r$. If $B \subset A^{(-r)}$ we can use Lemma 2.6 so it suffices to consider the case where $B \cap (\partial A)^{(r)} \ne \varnothing$.

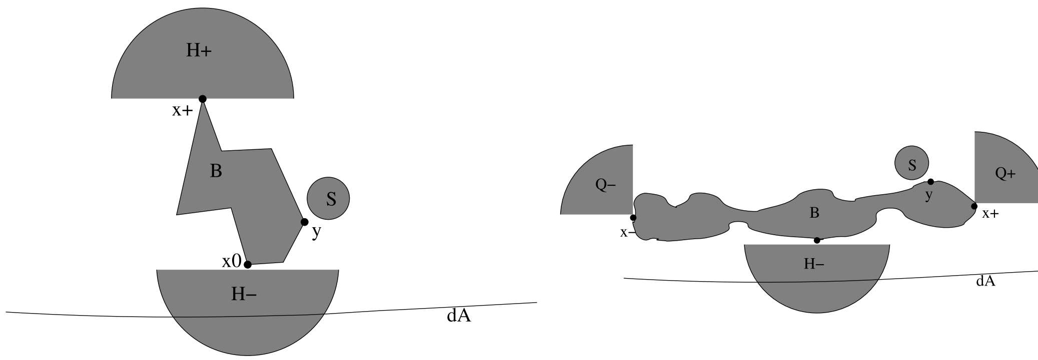

FIG. 2. *Illustration of the proof of Lemma 2.7. Left: when* $\mathrm{diam}(\pi_d(B)) \geq \varepsilon r/d$, *the sets* B, H^+, H^-, S *are disjoint. The point* x^- *(not indicated) could be the same as* x_0; *if not, it is only slightly lower than* x_0. *Right: when* $\mathrm{diam}(\pi_1(B)) \geq \varepsilon r/d$, *the sets* B, H^-, Q^+, Q^-, S *are disjoint.*

Fix $\tau \in (0, \tau(A))$. Let x_0 be a closest point of B to ∂A. As in the proof of Lemma 2.3, we can assume without loss of generality that the closest point of ∂A to x_0 lies at the origin, and $x_0 = he_d$ for some $h \in [0, r]$.

For $u \in \mathbb{R}^{d-1}$ with $\|u\| \leq \tau$, define $\phi(u) := \sup\{a \in [-\tau, \tau] : (u, a) \notin A\}$. Set $K = 4/\tau$. Then as a consequence of the sphere condition (see [6], equation (3.4)), provided $2\rho r < \tau$,

$$|\phi(u)| \leq \tau^{-1}\|u\|^2 \leq K\rho^2 r^2, \quad \forall u \in \mathbb{R}^{d-1} \text{ with } \|u\| \leq 2\rho r. \tag{2.7}$$

Assume $r \leq \varepsilon/(144dK\rho^2)$. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ denote projection onto the first $d-1$ coordinates, and for $1 \leq i \leq d$, let $\pi_i : \mathbb{R}^d \to \mathbb{R}$ denote projection onto the ith coordinate. Define the set H^- (slightly less than half a ball of radius r: see Figure 2) by

$$H^- := \{z \in B_r(x_0) : \pi_d(z) < \pi_d(x_0) - K\rho^2 r^2\}. \tag{2.8}$$

For all $w \in B$ we have $\|\pi(w)\| \leq \|w - x_0\| \leq \rho r$, so by (2.7),

$$\pi_d(x_0) = \mathrm{dist}(x_0, \partial A) \leq \mathrm{dist}(w, \partial A) \leq \pi_d(w) + |\phi(\pi(w))| \leq \pi_d(w) + K\rho^2 r^2. \tag{2.9}$$

Therefore, any $z \in H^-$, $w \in B$ satisfy $\pi_d(z) < \pi_d(w)$, so that $H^- \subset (B \oplus B_r(o)) \setminus B$.

We can bound above the volume difference of a half-ball and H^- by the volume of a cylinder of thickness $K\rho^2 r^2$ with base of radius r. Using this and the union bound, we obtain that

$$|B_r(x_0) \cap A| \leq (\theta_d/2)r^d + |H^- \cap A| + \theta_{d-1}r^{d-1}K\rho^2 r^2. \tag{2.10}$$

For at least one $i \in \{1, \dots, d\}$, we must have $\mathrm{diam}(\pi_i(B)) \geq \varepsilon r/d$. We distinguish the cases where this holds for $i = d$, and where it holds for some $i \leq d-1$.

First suppose $\mathrm{diam}(\pi_d(B)) \geq \varepsilon r/d$. Choose $x^+ \in B$ of maximal height (i.e., maximal d-coordinate), $x^- \in B$ of minimal height, and $y \in B$ of maximal 1-coordinate (see Figure 2 (Left)).

For all $z \in B \oplus B_r(o)$ we have $\|\pi(z)\| \leq \|z - x_0\| \leq 2\rho r$ so by (2.7) we have $|\phi(\pi(z))| \leq K\rho^2 r^2$. Applying this in the case $z = x^-$, and using the condition $r < \varepsilon/(144K\rho^2)$, we deduce that

$$\pi_d(x^+) \geq \pi_d(x^-) + \varepsilon r/d \geq (\varepsilon r/d) - K\rho^2 r^2 \geq (8/9)\varepsilon r/d,$$

and thus

$$\pi_d(x^+) - \phi(\pi(z)) \geq (8/9)\varepsilon r/d - K\rho^2 r^2 \geq (7/9)\varepsilon r/d, \quad \forall z \in B \oplus B_r(o).$$

Let $H^+ := \{z \in B_r(x^+) : \pi_d(z) > \pi_d(x^+)\}$ (see Figure 2). Then $H^+ \cap B = \varnothing$, since x^+ is a point of maximal height in B. Also $H^+ \subset B \oplus B_r(o)$, so for all $z \in H^+$ we have

$$\phi(\pi(z)) \le K\rho^2 r^2 \le (\varepsilon/(9d))r \le \pi_d(x^+) < \pi_d(z),$$

so $z \in A$. Also, since $\pi_d(z) > \pi_d(x^+) \ge \pi_d(x_0)$ we have from (2.8) that $z \notin H^-$. Hence $H^+ \subset A \setminus H^-$.

Now consider y. For $1 \le i \le d$ let e_i denote the ith unit coordinate vector. Define a point $\tilde{y}$ slightly to the right of y by

$$\tilde{y} := \begin{cases} y + (\varepsilon r/(8d))e_1 - (\varepsilon r/(8d))e_d & \text{if } \pi_d(y) \ge (\pi_d(x^+) + \pi_d(x^-))/2, \\ y + (\varepsilon r/(8d))e_1 + (\varepsilon r/(8d))e_d & \text{if } \pi_d(y) < (\pi_d(x^+) + \pi_d(x^-))/2, \end{cases}$$

and define the small ball $S := B_{\varepsilon r/(9d)}(\tilde{y})$. Then $|S| = \delta_1 r^d$, where we set $\delta_1 := \theta_d(\varepsilon/(9d))^d$.

Suppose $\pi_d(y) \ge (\pi_d(x^+) + \pi_d(x^-))/2$ (as well as $\operatorname{diam}(\pi_d(B)) \ge \varepsilon r/d$).

Then for all $z \in S$ we have $\pi_d(z) \le \pi_d(y) \le \pi_d(x^+)$ so $z \notin H^+$. Moreover

$$\pi_d(z) \ge \pi_d(y) - \varepsilon r/(4d) \ge \pi_d(x^-) + \varepsilon r/(4d) \ge \pi_d(x_0) + \varepsilon r/(8d)$$

by (2.9), applied to $w = x^-$. Therefore, $z \notin H^-$ by (2.8), and also (by (2.7)) $\pi_d(z) \ge \phi(z)$ so $z \in A$. Thus $S \subset A \setminus (H^+ \cup H^-)$ in this case.

Now suppose $\pi_d(y) < (\pi_d(x^+) + \pi_d(x^-))/2$ (as well as $\operatorname{diam}(\pi_d(B)) \ge \varepsilon r/d$).

Then for all $z \in S$ we have $\pi_d(z) \le \pi_d(y) + \varepsilon r/(4d) \le \pi_d(x^+)$, so $z \notin H^+$. Also, since $\pi_d(y) \ge \phi(\pi(y)) \ge -K\rho^2 r^2$, using the condition $r < \varepsilon/(144K\rho^2)$, we have

$$\pi_d(z) \ge \pi_d(y) + \varepsilon r/(72d) \ge K\rho^2 r^2 \ge \phi(\pi(z)),$$

so $z \in A$, and also by (2.9) applied to $w = y$, we have $\pi_d(z) \ge \pi_d(x_0)$, so $z \notin H^-$. Therefore, $S \subset A \setminus (H^+ \cup H^-)$ in this case too. Thus, whenever $\operatorname{diam}(\pi_d(B)) \ge \varepsilon r/d$, we have

$$|(B \oplus B_r(o)) \cap A| \ge |B| + |H^+| + |S| + |H^- \cap A|. \tag{2.11}$$

Combining (2.11) and (2.10), provided $r \le \delta_1/(2K\rho^2\theta_{d-1})$ we have

$$|(B \oplus B_r(o)) \cap A| \ge |B| + |B_r(x_0) \cap A| + (\delta_1/2)r^d, \quad \text{if } \operatorname{diam}(\pi_d(B)) \ge \varepsilon r/d. \tag{2.12}$$

Now suppose $\operatorname{diam} \pi_i(B) \ge \varepsilon r/d$ for some $i \in \{1, \dots, d-1\}$. We shall consider here the case where this holds for $i = 1$; the other cases may be treated similarly.

Let x^-, x^+, y be points in B of minimal 1-coordinate, maximal 1-coordinate, and maximum height respectively. Let $\delta_2 := \delta_1/(2\theta_{d-1})$. Define the sets Q^- and Q^+ (slightly less than quarter-balls of radius r: see Figure 2 (Right)) by

$$Q^- := \{z \in B_r(x^-) : \pi_d(z) \ge \pi_d(x^-) + \delta_2 r, \pi_1(z) < \pi_1(x^-)\};$$
$$Q^+ := \{z \in B_r(x^+) : \pi_d(z) \ge \pi_d(x^+) + \delta_2 r, \pi_1(z) > \pi_1(x^+)\}.$$

By (2.7), for $z \in Q^-$ we have $|\phi(\pi(z))| \le K\rho^2 r^2$, so provided $r < \delta_2/(2K\rho^2)$, for all $z \in Q^-$ we have

$$\pi_d(z) \ge \pi_d(x^-) + \delta_2 r \ge \delta_2 r - K\rho^2 r^2 \ge K\rho^2 r^2 \ge \phi(\pi(z)),$$

so that $z \in A$. Also by (2.9) applied to $w = x^-$ we have $\pi_d(z) \ge \pi_d(x^-) \ge \pi_d(x_0) - K\rho^2 r^2$, so $z \notin H^-$ by (2.8). Thus $Q^- \subset A \setminus H^-$, and similarly $Q^+ \subset A \setminus H^-$. Also $(Q^- \cup Q^+) \cap B = \varnothing$, and $|Q^- \cup Q^+| \ge (\theta_d - 2\delta_2\theta_{d-1})r^d/2 = (\theta_d - \delta_1)r^d/2$.

We define a point $\tilde{y}$ slightly above y by

$$\tilde{y} := \begin{cases} y + (\varepsilon r/(8d))e_d + (\varepsilon r/(8d))e_1 & \text{if } \pi_1(y) \le (\pi_1(x^-) + \pi_1(x^+))/2, \\ y + (\varepsilon r/(8d))e_d - (\varepsilon r/(8d))e_1 & \text{if } \pi_1(y) > (\pi_1(x^-) + \pi_1(x^+))/2, \end{cases}$$

and set $S := B_{\varepsilon r/(9d)}(\tilde{y})$: see Figure 2 (Right). Then $|S| = \delta_1 r^d$ as before.

Then for all $z \in S$ we have $\pi_d(z) > \pi_d(y) \geq \pi_d(x_0)$, so that $z \notin B$ and $z \notin H^-$. Also $\phi(\pi(z)) \leq K\rho^2 r^2 < \varepsilon r/(72d) \leq \pi_d(z)$, so $z \in A$. Moreover, if $\pi_1(y) > (\pi_1(x^-) + \pi_1(x^+))/2$ then

$$\pi_1(z) - \pi_1(x^-) = \pi_1(y) - \pi_1(x^-) + (\pi_1(z) - \pi_1(y)) \geq (\varepsilon r/(2d)) - (\varepsilon r/(4d)) > 0,$$

while if $\pi_1(y) \leq (\pi_1(x^-) + \pi_1(x^+))/2$ then $\pi_1(z) - \pi_1(x^-) > \pi_1(y) - \pi_1(x^-) \geq 0$ so in both cases $z \notin Q^-$. Similarly $z \notin Q^+$. Thus $S \subset A \setminus (Q^+ \cup Q^- \cup B \cup H^-)$. Combining all of this and using (2.10) in the third line below yields

$$\begin{aligned}
|(B \oplus B_r(o)) \cap A| &\geq |B| + |Q^- \cup Q^+| + |S| + |H^- \cap A| \\
&\geq |B| + ((\theta_d + \delta_1)/2)r^d + |H^- \cap A| \\
&\geq |B| + |B_r(x_0) \cap A| - \theta_{d-1} r^{d-1} K\rho^2 r^2 + (\delta_1/2)r^d \\
&\geq |B| + |B_r(x_0) \cap A| + (\delta_1/4)r^d \quad \text{if } \operatorname{diam}(\pi_1(B)) \geq \varepsilon r/d,
\end{aligned}$$

provided $r \leq \delta_1/(4K\rho^2\theta_{d-1})$. Combined with (2.12), this shows that (2.6) holds for r small if we take $\delta = \delta_1/8$. □

3. Poisson approximation for the *k*-isolated vertices. Fix $k \in \mathbb{N}$. We say a vertex is k-isolated if its degree is at most $k-1$. Given $n, r > 0$ let $\xi_{n,r}$ denote the number of k-isolated vertices in $G(\mathcal{P}_n, r)$:

$$\xi_{n,r} := \sum_{x \in \mathcal{P}_n} \mathbf{1}\{\mathcal{P}_n(B_r(x)) \leq k\}. \tag{3.1}$$

The goal of this section is to prove (in Proposition 3.1 below) Poisson approximation for $\xi_{n,r}$ when n is large and r is small.

Throughout this section we adopt our working assumption on ν. Moreover we assume either that $d \geq 2$ and the support A of ν is compact with $C^{1,1}$ boundary, or that $d = 2$ and A is a polygon. We do *not* assume in this section that ν is necessarily uniform on A. Recall that $\mathcal{P}_n$ is the Poisson process in $\mathbb{R}^d$ with intensity measure $n\nu$.

A fundamental identity used throughout this paper is the Mecke equation which basically says that the law of a Poisson process $\mathcal{P}$ conditioned on having a point mass at x is that of $\mathcal{P} \cup \{x\}$. More precisely, let $\mathcal{P}$ be a Poisson process on $\mathbb{R}^d$ with diffuse intensity measure λ (i.e., λ does not charge atoms). The Mecke equation says that

$$\mathbb{E}\Big[\sum_{x \in \mathcal{P}} f(x, \mathcal{P})\Big] = \int \mathbb{E}[f(x, \mathcal{P} \cup \{x\})]\lambda(dx), \tag{3.2}$$

for all $f : \mathbb{R}^d \times \mathbf{N}(\mathbb{R}^d) \to \mathbb{R}$ such that both sides of the identity are finite, where $\mathbf{N}(\mathbb{R}^d)$ denotes the space of all locally finite subsets of $\mathbb{R}^d$—see [8], Chapter 4, for a more general statement.

By the Mecke equation, given $n, r > 0$ we have

$$\mathbb{E}[\xi_{n,r}] = n \int_A p_{n,r}(x)\nu(dx), \tag{3.3}$$

where for each $x \in A$ we set

$$p_{n,r}(x) := \mathbb{P}[\mathcal{P}_n(B_r(x)) \leq k-1] = \sum_{j=0}^{k-1} (n(\nu(B_r(x)))^j/j!) \exp(-n\nu(B_r(x))). \tag{3.4}$$

Given random variables X, Z taking values in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, define the total variation distance

$$d_{\mathrm{TV}}(X, Z) := \sup_{B \subset \mathbb{N}_0} \big|\mathbb{P}[X \in B] - \mathbb{P}[Z \in B]\big|.$$

Given $\alpha > 0$, let Po_α be Poisson distributed with mean α.

PROPOSITION 3.1 (Poisson approximation). *Let $\beta' > 0$, and suppose $(r_n)_{n\geq 1}$ are such that $r_n \geq 0$ for all n and*

$$\lim_{n\to\infty} \mathbb{E}[\xi_{n,r_n}] = \beta'. \tag{3.5}$$

Then we have

$$d_{\mathrm{TV}}(\xi_{n,r_n}, \mathsf{Po}_{\mathbb{E}[\xi_{n,r_n}]}) = O\big((\log n)^{1-d}\big) \quad \text{as } n \to \infty. \tag{3.6}$$

In particular, with $L_{n,k} = L_k(\mathcal{P}_n)$ defined at (1.4),

$$\mathbb{P}[L_{n,k} \leq r_n] - \exp\big(-\mathbb{E}[\xi_{n,r_n}]\big) = O\big((\log n)^{1-d}\big) \quad \text{as } n \to \infty. \tag{3.7}$$

If for some β', no $(r_n)_{n\geq 1}$ satisfying $r_n \geq 0$ and (3.5) existed, we could think of Proposition 3.1 as being vacuously true for that β'. In fact we shall show later that for any choice of β', such an $(r_n)_{n\geq 1}$ does exist.

We prepare for proving Proposition 3.1 with three lemmas, the first of which is used repeatedly later on.

LEMMA 3.2. *Under the WA, assuming either that $\partial A \in C^{1,1}$ or $d = 2$ and A is polygonal, there exists a constant $\delta_0 > 0$ (depending on A and f) such that*

$$2\delta_0 r^d \leq \nu\big(B_r(x)\big) \leq \theta_d r^d f_{\max}, \quad \forall\, x \in A, r \in (0, 1]. \tag{3.8}$$

PROOF. The second inequality is clear. The first inequality follows from Lemma 2.4-(i) in the case where $\partial A \in C^{1,1}$, and can be seen directly when A is polygonal. □

LEMMA 3.3. *Let $\beta' \in (0, \infty)$ and suppose that $(r_n)_{n\geq 1}$ satisfies (3.5). Then we have that* $\liminf_{n\to\infty}(nr_n^d/\log n) \geq 1/(f_{\max}\theta_d)$, *and* $\limsup_{n\to\infty}(nr_n^d/\log n) < \infty$.

PROOF. Let $\alpha \in (0, 1/(f_{\max}\theta_d))$. If $nr_n^d < \alpha \log n$, then for all $x \in A$ we have $n\nu(B_{r_n}(x)) \leq n\theta_d f_{\max} r_n^d \leq \alpha\theta_d f_{\max} \log n$. Therefore, by (3.3), $\mathbb{E}[\xi_{n,r}] \geq n \int e^{-n\nu(B_{r_n}(x))} \times \nu(dx) \geq n^{1-\alpha\theta_d f_{\max}}$, so the condition (3.5) implies that $nr_n^d \geq \alpha \log n$ for all large enough n. The first claim follows.

For the second claim, let $\delta_0 > 0$ be as in (3.8). Take $s_n > 0$ so that $ns_n^d = \delta_0^{-1} \log n$. Using (3.8), for some constant c we have

$$n \int_A p_{n,s_n}(x)\nu(dx) \leq cn(\log n)^{k-1} \exp\big(-2n\delta_0 s_n^d\big) = c(\log n)^{k-1} n^{-1},$$

which tends to zero. Hence by (3.5) we have $r_n \leq s_n$ for n large, and hence the second claim. □

Given $x, y \in \mathbb{R}^d$ and $n, r > 0$, setting $\mathcal{P}_n^x := \mathcal{P}_n \cup \{x\}$, we define the quantity

$$q_{n,r}(x, y) := \mathbb{P}\big[\mathcal{P}_n^y\big(B_r(x)\big) \leq k - 1, \mathcal{P}_n^x\big(B_r(y)\big) \leq k - 1\big].$$

Our proof of Proposition 3.1 is based on the following estimate which was proved in [11] by the local dependence approach of Stein's method.

LEMMA 3.4 ([11], Theorem 6.7). *Let $n, r > 0$. Then*

$$d_{\mathrm{TV}}(\xi_{n,r}, \mathsf{Po}_{\mathbb{E}[\xi_{n,r}]}) \le 3\big(I_1(n,r) + I_2(n,r)\big),$$

where

$$I_1(n,r) = n^2 \int \mathbf{1}\{\|x-y\| \le 3r\} p_{n,r}(x) p_{n,r}(y) \nu^2\big(d(x,y)\big),$$

$$I_2(n,r) = n^2 \int \mathbf{1}\{\|x-y\| \le 3r\} q_{n,r}(x,y) \nu^2\big(d(x,y)\big).$$

REMARK 3.5. One could try various other approaches to proving Proposition 3.1. Indeed, when $k = 1$ one can alternatively use [17], Theorem 3.1, which is proved using Stein's method via coupling; however it might not be so easy to use this approach for general k. One might also try using the generator approach to Stein's method, as was done in [1] for a variant of $\xi_{n,r}$ where one considers balls of given ν-measure rather than balls of given radius (as discussed earlier).

PROOF OF PROPOSITION 3.1. Observe first that whenever $|\mathcal{P}_n| \ge k+1$, the statement $L_{n,k} \le r_n$ is equivalent to $\xi_{n,r_n} = 0$, so that $|\mathbb{P}[L_{n,k} \le r_n] - \mathbb{P}[\xi_{n,r_n} = 0]| \le \mathbb{P}[|\mathcal{P}_n| \le k] = O(n^k e^{-n})$. Therefore, (3.6) will imply (3.7), so it suffices to prove (3.6).

By (3.8), provided n is large enough, for all $y \in A$ we have

$$p_{n,r_n}(y) \le k\big(n f_{\max}\theta_d r_n^d\big)^{k-1} \exp\big(-n\nu(B_{r_n}(y))\big) \le \exp\big(-\delta_0 n r_n^d\big).$$

Therefore, using (3.3) in the second line below we have

$$\begin{aligned} I_1(n,r_n) &\le n\big(3^d f_{\max}\theta_d r_n^d\big)\exp\big(-\delta_0 n r_n^d\big) n \int p_{n,r_n}(x)\nu(dx) \\ &\le \exp\big(-(\delta_0/2) n r_n^d\big)\mathbb{E}[\xi_{n,r_n}]. \end{aligned} \tag{3.9}$$

Now we estimate $I_2 := I_2(n,r_n)$. Since the integrand of I_2 is symmetric in x and y,

$$I_2 \le 2n^2 \int \mathbf{1}\{\|x-y\| \le 3r_n, \operatorname{dist}(x,\partial A) \le \operatorname{dist}(y,\partial A)\} q_{n,r_n}(x,y)\nu^2\big(d(x,y)\big).$$

To further simplify the integral, writing $B_x = B_{r_n}(x)$ and likewise for $B_{r_n}(y)$, we have

$$\begin{aligned} q_{n,r_n}(x,y) &\le \mathbb{P}\big[\mathcal{P}_n(B_x) \le k-1, \mathcal{P}_n(B_y \setminus B_x) \le k-1\big] \\ &= p_{n,r_n}(x)\mathbb{P}\big[\mathcal{P}_n(B_y \setminus B_x) \le k-1\big]. \end{aligned}$$

Consider first the case where A has a $C^{1,1}$ boundary. If $\operatorname{dist}(x,\partial A) \le \operatorname{dist}(y,\partial A)$, setting $\kappa_d := 2^{-3d-1}\theta_{d-1}$ and using Lemma 2.4-(ii) for the lower bound and Fubini's theorem for the upper bound below, we have

$$f_0\kappa_d\|y-x\| r_n^{d-1} \le \nu(B_y \setminus B_x) \le f_{\max}\theta_{d-1} r_n^{d-1}\|y-x\|,$$

and hence

$$q_{n,r_n}(x,y) \le p_{n,r_n}(x)\sum_{j=0}^{k-1}\big(n f_{\max}\theta_{d-1} r_n^{d-1}\|y-x\|\big)^j \exp\big(-\kappa_d f_0\|y-x\| n r_n^{d-1}\big).$$

Therefore, we have

$$\begin{aligned} I_2 &\le 2\max(f_{\max}\theta_{d-1},1)^{k-1} n^2 \\ &\quad\times \int_A \left(\int_{B_{3r_n}(x)} \sum_{j=0}^{k-1}\big(n r_n^{d-1}\|y-x\|\big)^j \exp\big(-\kappa_d f_0\|y-x\| n r_n^{d-1}\big)\nu(dy)\right) p_{n,r_n}(x)\nu(dx). \end{aligned}$$

A change of variables $z = nr_n^{d-1}(y-x)$ shows that the inner integral is bounded by $c' r_n^d (nr_n^d)^{-d}$ for some finite constant c'. Together with (3.3), this yields for some further constant c'' that

$$I_2 \le 2c''(nr_n^d)^{1-d}\mathbb{E}[\xi_{n,r_n}]. \tag{3.10}$$

This, together with (3.9) and (3.5), shows that $I_1 + I_2 = O((nr_n^d)^{1-d})$; applying Lemmas 3.4 and 3.3 proves (3.6) as required for this case.

Now consider the other case, where $d = 2$ and A is polygonal. Let $x, y \in A$ with $\|y - x\| \le 3r_n$ and $\mathrm{dist}(x, \partial A) \le \mathrm{dist}(y, \partial A)$. By Lemma 2.5, there exists $\delta_1 > 0$ such that $\nu(B_{r_n}(y) \setminus B_{r_n}(x)) \ge \delta_1 \|x - y\| r_n$. Using this, we can estimate the contribution to I_2 from x, y not too close to the corners similar to how we estimated I_2 at (3.10) in the previous case.

Suppose instead that x is close to a corner of A and $\|x - y\| \le 3r_n$. By (3.8) the contribution to I_2 from such pairs (x, y) is at most $c''' n^2 r_n^4 \exp(-\delta_2 n r_n^2)$ for suitable constants $c''' < \infty$, $\delta_2 > 0$. Hence by Lemma 3.3, this contribution is $O(n^{-\delta_3})$ for some $\delta_3 > 0$. The proof is now complete. □

4. Relating $L_{n,k}$ to $M_{n,k}$.

Throughout this section we assume that $\partial A \in C^2$ or that A is a convex polygon (though for some of the results in this section we could relax the C^2 condition to a $C^{1,1}$ condition). We adopt our WA but do not assume f is necessarily constant on A.

Fix $k \in \mathbb{N}$. Recall that $L_{n,k}$ and $M_{n,k}$ were defined at (1.4) and (1.1). While Proposition 3.1 provides an understanding of $\mathbb{P}[L_{n,k} \le r_n]$ for suitable r_n, for Theorems 1.1 and 1.3 we also need to understand the limiting behaviour of $\mathbb{P}[M_{n,k} \le r_n]$ and $\mathbb{P}[M_k(\mathcal{X}_n) \le r_n]$. In this section, we work towards this by showing (in Proposition 4.6) that $\mathbb{P}[L_{n,k} \le r < M_{n,k}]$ and $\mathbb{P}[L_k(\mathcal{X}_n) \le r < M_k(\mathcal{X}_n)]$ are small for n large and r small.

Suppose $\mathcal{X} \subset \mathbb{R}^d$ is finite, and $r > 0$. We adapt terminology from [11], page 282. For $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ a *j-separating pair* for the geometric graph $G(\mathcal{X}, r)$ means a pair of disjoint nonempty subsets $\mathcal{Y}, \mathcal{Y}'$ of $\mathcal{X}$ such that $G(\mathcal{Y}, r)$ and $G(\mathcal{Y}', r)$ are both connected, $G(\mathcal{Y} \cup \mathcal{Y}', r)$ is not, and $\mathcal{X} \setminus (\mathcal{Y} \cup \mathcal{Y}')$ contains at most j points within distance r of $\mathcal{Y} \cup \mathcal{Y}'$.

When we need to refer to an individual set in a separating pair, we use the terminology *separating set*. That is, for $j \in \mathbb{N}_0$ a j-separating set for the graph $G(\mathcal{X}, r)$ is a set $\mathcal{Y} \subset \mathcal{X}$ such that $G(\mathcal{Y}, r)$ is connected, and with $\Delta\mathcal{Y}$ denoting the set of sites in $\mathcal{X} \setminus \mathcal{Y}$ adjacent to $\mathcal{Y}$, we have $|\Delta\mathcal{Y}| \le j$ and $\mathcal{X} \setminus (\mathcal{Y} \cup \Delta\mathcal{Y}) \ne \varnothing$.

LEMMA 4.1. *Suppose $\mathcal{X} \subset \mathbb{R}^d$ is finite with $|\mathcal{X}| \ge k+3$. Let $r > 0$, and suppose $L_k(\mathcal{X}) \le r < M_k(\mathcal{X})$. Then there exists a $(k-1)$-separating pair $(\mathcal{Y}, \mathcal{Y}')$ for $G(\mathcal{X}, r)$ such that neither $\mathcal{Y}$ nor $\mathcal{Y}'$ is a singleton.*

PROOF. Since $M_k(\mathcal{X}) > r$, the graph $G(\mathcal{X}, r)$ is not k-connected. Therefore by [11], Lemma 13.1, it has a $(k-1)$-separating pair $\mathcal{Y}, \mathcal{Y}' \subset \mathcal{X}$. Since also $L_k(\mathcal{X}) \le r$, every vertex $x \in \mathcal{X}$ has degree at least k, which implies that neither $\mathcal{Y}$ nor $\mathcal{Y}'$ can be a singleton. □

Our strategy in this section is to estimate the probability that there exists a pair of nonsingleton separating sets for $G(\mathcal{P}_n, r)$ or $G(\mathcal{X}_n, r)$. We do this in stages, according to the size of the separating sets.

For $x, y \in A$, we write $x < y$ if x precedes y in the lexicographic ordering. We define the following ordering $\prec$ on A, that we shall use repeatedly:

$$x \prec y \Leftrightarrow \big(\mathrm{dist}(x, \partial A) < \mathrm{dist}(y, \partial A)\big) \quad \text{or} \quad \big(\mathrm{dist}(x, \partial A) = \mathrm{dist}(y, \partial A) \text{ and } x < y\big). \tag{4.1}$$

4.1. *Small separating sets.* The goal of this section is to prove that for any fixed vertex $x \in A$, the probability that x belongs to a nonsingleton $(k-1)$-separating set of "small" diameter in $G(\mathcal{P}_n \cup \{x\}, r)$ is negligible compared to the probability that it has degree at most $k-1$, provided that $x \prec y$ for all other y in the separating set containing x, where the ordering $\prec$ was defined at (4.1).

We introduce further notation. With k fixed, for $r > 0$ and finite $\mathcal{X} \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, let $\mathcal{C}_r(x, \mathcal{X})$ denote the collection of $(k-1)$-separating sets $\mathcal{Y}$ for $G(\mathcal{X} \cup \{x\}, r)$ containing x such that moreover $x \prec y$ for all $y \in \mathcal{Y} \setminus \{x\}$. Given also $\rho > 0$, we are interested in the event

$$E_{x,\rho,r}(\mathcal{X}) := \{\exists \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{X}), 0 < \operatorname{diam}(\mathcal{Y}) \le \rho r\}. \tag{4.2}$$

LEMMA 4.2.

(i) *Suppose $d \ge 2$ and A has C^2 boundary. Then there exist $\delta, r_0 \in (0,1)$ and $c < \infty$ such that for all $n \ge k+2$, any $x \in A$ and any $r \in (0, r_0)$ we have*

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{P}_n)] \le c p_{n,r}(x) (nr^d)^{1-d}; \tag{4.3}$$

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{X}_{n-1})] \le c p_{n,r}(x) (nr^d)^{1-d}, \tag{4.4}$$

where $p_{n,r}(x)$ was defined at (3.4)

(ii) *Suppose $d = 2$ and A is polygonal. Then there exist $K \in (0, \infty)$, and $\delta, r_0 \in (0,1)$ and $c < \infty$ such that for all $n \ge 3$, $x \in A \setminus \mathsf{Cor}^{(Kr)}$, and $r \in (0, r_0)$,* (4.3) *and* (4.4) *hold.*

PROOF. (i) Let $\delta \in (0,1)$. Suppose that $E_{x,\delta,r}(\mathcal{P}_n)$ occurs with some $\mathcal{Y}$. Then by considering the vertex furthest from x in $\mathcal{Y}$, we see that there exists $y \in \mathcal{P}_n$ such that $\mathcal{Y} \subset B_{\|y-x\|}(x)$ and $\|y - x\| \le \delta r$. Moreover, setting $D_{x,y} := (B_r(x) \cup B_r(y)) \setminus B_{\|y-x\|}(x)$ we have that $\mathcal{P}_n(D_{x,y}) \le k-1$. By Markov's inequality, $\mathbb{P}[E_{x,\delta,r}(\mathcal{P}_n)]$ is bounded above by the expected number of $y \in \mathcal{P}_n \cap B_{\delta r}(x)$ satisfying $\mathcal{P}_n(D_{x,y}) \le k-1$, and hence by the Mecke formula

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{P}_n)] \le n \int_{B_{\delta r}(x)} \mathbb{P}[\mathcal{P}_n(D_{x,y}) \le k-1] \nu(dy). \tag{4.5}$$

To proceed, we need to bound the volume of $D_{x,y}$ from below. By Lemma 2.4-(ii), there exists $r_0 > 0$ such that for all $r \in (0, r_0)$ and $x, y \in A$ with $\|x - y\| \le r$ and $x \prec y$, setting $\kappa_d := 2^{-3d-1}\theta_{d-1}$ we have

$$\nu(B_r(x) \cup B_r(y)) \ge \nu(B_r(x)) + 2\kappa_d f_0 r^{d-1} \|y - x\|.$$

Hence, for $r < r_0$, for $x, y \in A$ with $\|y - x\| \le \delta r$ and $x \prec y$,

$$\nu(D_{x,y}) \ge \nu(B_r(x)) + 2\kappa_d f_0 r^{d-1} \|y - x\| - f_{\max}\theta_d \|y - x\|^d.$$

Now provided $\delta \le (\kappa_d f_0/(f_{\max}\theta_d))^{1/(d-1)}$, we have $f_{\max}\theta_d \|y - x\|^d \le \kappa_d f_0 r^{d-1} \|y - x\|$, yielding

$$\nu(D_{x,y}) \ge \nu(B_r(x)) + \kappa_d f_0 r^{d-1} \|y - x\|. \tag{4.6}$$

By (3.8), there is also a bound the other way, namely $\nu(D_{x,y}) \le \nu(B_r(x) \cup B_r(y)) \le c_0 \nu(B_r(x))$ for some constant $c_0 \in [1, \infty)$. Using (4.5) and the preceding upper and lower bounds on $\nu(D_{x,y})$, we have

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{P}_n)] \le c_0^{k-1} n \int_{B_{\delta r}(x)} \sum_{j=0}^{k-1} ((n\nu(B_r(x)))^j / j!) e^{-n\nu(B_r(x)) - nr^{d-1} f_0 \kappa_d \|y-x\|} \nu(dy). \tag{4.7}$$

Recall that $p_{n,r}(x) = \sum_{j=0}^{k-1}((n\nu(B_r(x)))^j/j!)e^{-n\nu(B_r(x))}$. Changing variable to $y' = y - x$, and then to $z = nr^{d-1}y'$ leads to

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{P}_n)] \le c_0^{k-1} f_{\max} n p_{n,r}(x) \int_{\|y'\|\le\delta r} e^{-nf_0\kappa_d r^{d-1}\|y'\|}\,dy'$$

$$\le c' p_{n,r}(x) n (nr^{d-1})^{-d} \int_{\mathbb{R}^d} e^{-f_0\kappa_d\|z\|}\,dz,$$

for a suitable positive constant c', not depending on r or n. This proves (4.3).

To prove (4.4), we use similar reasoning to before, now using the union bound (instead of the Mecke formula) and the binomial distribution, to obtain that

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{X}_{n-1})] \le (n-1)\int_{B_{\delta r}(x)} \sum_{j=0}^{k-1} \binom{n-2}{j} \nu(D_{x,y})^j (1-\nu(D_{x,y}))^{n-2-j}\nu(dy).$$

As before we bound $\nu(D_{x,y})^j$ from above by $c_0^{k-1}\nu(B_r(x))^{j-1}$. Provided r is sufficiently small, we have for all x, y and all $j \le k-1$ that $(1-\nu(D_{x,y}))^{-2-j} \le 2$. Also we can bound the binomial coefficient from above by $n^j/j!$. Combining all of these and also using the bound $(1-t) \le e^{-t}$ we obtain that

$$\mathbb{P}[E_{x,\delta,r}(\mathcal{X}_{n-1})] \le 2c_0^{k-1} n \int_{B_{\delta r}(x)} \sum_{j=0}^{k-1} \frac{(n\nu(B_r(x)))^j}{j!}\exp(-n\nu(D_{x,y}))\nu(dy);$$

then using (4.6) and arguing similar to the Poisson case, we obtain (4.4).

(ii) Suppose $d = 2$ and A is polygonal. We use Lemma 2.5 in place of Lemma 2.4 to get the lower bound (4.6). This together with the simple upper bound $\nu(D_{x,y}) \le c_0\nu(B_r(x))$ and the same reasoning as in part (i) leads to (4.3) and (4.4) in this case too. □

4.2. *Medium sized separating sets.* Recall the definition of $\mathcal{C}_r(x, \mathcal{X})$ before the previous lemma, and $p_{n,r}(x)$ at (3.4). Given ε, ρ with $0 < \varepsilon < \rho < \infty$, define the event

$$F_{x,\varepsilon,\rho,r}(\mathcal{P}_n) := \{\exists \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n),\ \varepsilon r < \mathrm{diam}(\mathcal{Y}) < \rho r\}. \tag{4.8}$$

The next lemma helps us bound the probability of having a medium-sized separating set.

LEMMA 4.3.

(i) *Suppose $\partial A \in C^2$. Given $\rho, \varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \rho$, there exist $\delta, r_0, c > 0$ such that for all $n \ge 2 + k$, $r \in (0, r_0)$ and all $x \in A$, we have*

$$\mathbb{P}[F_{x,\varepsilon,\rho,r}(\mathcal{P}_n)] \le c p_{n,r}(x) e^{-\delta n r^d}; \tag{4.9}$$

$$\mathbb{P}[F_{x,\varepsilon,\rho,r}(\mathcal{X}_{n-1})] \le c p_{n,r}(x) e^{-\delta n r^d}. \tag{4.10}$$

(ii) *Suppose $d = 2$ and A is polygonal. Given $0 < \varepsilon < \rho < \infty$, there exists $K \in (0, \infty)$ and $\delta, r_0 > 0$ such that for all $n \ge 3$, $r \in (0, r_0)$ and all $x \in A \setminus \mathrm{Cor}^{(Kr)}$, we have* (4.9) *and* (4.10).

PROOF. Later in the proof we shall use the fact that since we assume A is compact and f is continuous on A with $f_0 > 0$,

$$\lim_{s\downarrow 0}(\sup\{f(y)/f(x) : x, y \in A, \|y - x\| \le s\}) = 1. \tag{4.11}$$

(i) Suppose $\partial A \in C^2$. Without loss of generality, we can and do assume $\varepsilon < 1$. Let $\delta_1 := \delta(d, \rho, \varepsilon)$ be as in Lemma 2.7. With e_1 denoting an arbitrary unit vector in $\mathbb{R}^d$, choose $\delta_2 \in (0, 1/(99\sqrt{d}))$ such that

$$\big|B_1(o) \setminus B_{1-\sqrt{d}\delta_2}(o)\big| \le \delta_1. \tag{4.12}$$

Partition $\mathbb{R}^d$ into cubes of side length $\delta_2 r$. Given $\mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n)$, denote by $\mathsf{A}_{\delta_2}(\mathcal{Y})$ the closure of the union of all the cubes in the partition that intersect $\mathcal{Y}$. Here A stands for "animal" and is unrelated to our underlying domain A. If $\operatorname{diam} \mathcal{Y} \in (\varepsilon r, \rho r]$, then $\mathsf{A}_{\delta_2}(\mathcal{Y}) \subset B_{\rho r + \delta_2 d^{1/2} r}(x)$ and $\mathsf{A}_{\delta_2}(\mathcal{Y})$ can take at most $c := 2^{(2\lceil(\rho/\delta_2)+\sqrt{d}\rceil)^d}$ different possible shapes.

If the event $F_{x,\varepsilon,\rho,r}(\mathcal{P}_n)$ occurs there is at least one set $\mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n)$ with $\varepsilon r < \operatorname{diam} \mathcal{Y} \le \rho r$. If there are several such sets $\mathcal{Y}$, choose one of these according to some deterministic rule, and denote it by $\mathcal{Y}^*(\mathcal{P}_n)$.

Fix a possible shape σ that might arise as $\mathsf{A}_{\delta_2}(\mathcal{Y})$ for some $\mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n)$ with $\operatorname{diam} \mathcal{Y} \in (\varepsilon r, \rho r]$, and suppose the event $F_{x,\varepsilon,\rho,r}(\mathcal{P}_n) \cap \{\mathsf{A}_{\delta_2}(\mathcal{Y}^*(\mathcal{P}_n)) = \sigma\}$ occurs. Let $\sigma^* := \{z \in \sigma : x \prec z\} \cup \{x\}$. Set $H := H(\sigma) = (\sigma^* \oplus B_{(1-\sqrt{d}\delta_2)r}(o)) \setminus \sigma^*$. By the triangle inequality, $H \subset \mathcal{Y}^*(\mathcal{P}_n) \oplus B_r(o)$. We claim that $\mathcal{P}_n(H) \le k-1$. Indeed, if there are k or more points in $\mathcal{P}_n \cap H$, then since $\mathcal{Y}^*(\mathcal{P}_n)$ is $(k-1)$-separating, necessarily one of these points, denoted by y, belongs to $\mathcal{Y}^*(\mathcal{P}_n)$. Hence $y \in \mathcal{P}_n \cap H \cap \mathcal{Y}^*(\mathcal{P}_n)$, implying $y \in \sigma$ and therefore $y \in \sigma \setminus \sigma^*$ (since $y \in H$), but this would contradict the assumption that $x \prec y$ for all $y \in \mathcal{Y}^*(\mathcal{P}_n) \setminus \{x\}$.

Now we estimate from below the volume of $H \cap A$. Recall that $\delta_1 = \delta(d, \rho, \varepsilon)$ is as in Lemma 2.7. Applying (2.6) from there leads to

$$|H \cap A| \ge \big|B_{r(1-\sqrt{d}\delta_2)}(x) \cap A\big| + 2\delta_1 r^d.$$

By (4.12), $|(B_r(x) \setminus B_{r(1-\sqrt{d}\delta_2)}(x)) \cap A| \le \delta_1 r^d$ and hence

$$\big|B_{r(1-\sqrt{d}\delta_2)}(x) \cap A\big| \ge \big|B_r(x) \cap A\big| - \delta_1 r^d.$$

Let $\delta_3 \in (0, 1/2)$ be such that $\delta_4 := (1 - 2\delta_3)(1 + \delta_1/(f_{\max}\theta_d)) - 1 > 0$. By the preceding estimates, and (4.11), provided r is small we have that

$$\begin{aligned}\nu(H) &\ge (1-\delta_3) f(x)\big(\big|B_r(x) \cap A\big| + \delta_1 r^d\big) \\ &\ge (1-2\delta_3)\nu\big(B_r(x)\big)\left(1 + \frac{\delta_1 r^d}{f_{\max}\theta_d r^d}\right) = (1+\delta_4)\nu\big(B_r(x)\big).\end{aligned}$$

Let $\delta_5 = \delta_0\delta_4$, with δ_0 given at (3.8). Then

$$\nu(H) \ge \nu\big(B_r(x)\big) + \delta_5 r^d. \tag{4.13}$$

Also, because of the upper bound on diameters and (3.8), there is a constant $c_1 \in [1, \infty)$ such that $\nu(H) \le c_1 \nu(B_r(x))$ uniformly over all possible x, all small r, and all possible σ.

Using these upper and lower bounds on $\nu(H)$, we can deduce that

$$\begin{aligned}\mathbb{P}\big[F_{x,\varepsilon,\rho,r}(\mathcal{P}_n) \cap \big\{\mathsf{A}_{\delta_2}\big(\mathcal{Y}^*(\mathcal{P}_n)\big) = \sigma\big\}\big] &\le \mathbb{P}\big[\mathcal{P}_n(H) \le k-1\big] \\ &\le c_1^{k-1} \sum_{j=0}^{k-1} \big(\big(n\nu\big(B_r(x)\big)\big)^j / j!\big) e^{-n\nu(B_r(x)) - n\delta_5 r^d} \\ &= c_1^{k-1} p_{n,r}(x) e^{-n\delta_5 r^d}.\end{aligned}$$

This, together with the union bound over the choice of possible shapes σ, gives us (4.9).

To prove the result (4.10) for the binomial case we use the volume estimates (4.13) and $\nu(H) \leq c_1 \nu(B_r(x))$ once more. For n large, we have

$$\begin{aligned}
\mathbb{P}\big[F_{x,\varepsilon,\rho,r}(\mathcal{X}_{n-1}) \cap \{\mathsf{A}_{\delta_2}(\mathcal{Y}^*(\mathcal{X}_{n-1})) = \sigma\}\big] &\leq \mathbb{P}[\mathcal{X}_{n-1}(H) \leq k-1] \\
&= \sum_{j=0}^{k-1} \binom{n-1}{j} \nu(H)^j (1-\nu(H))^{n-1-j} \\
&\leq 2c_1^{k-1} \sum_{j=0}^{k-1} (n^j/j!) \nu(B_r(x))^j \exp(-n\nu(H)) \\
&\leq 2c_1^{k-1} p_{n,r}(x) e^{-n\delta_5 r^d},
\end{aligned}$$

and hence (4.10).

(ii) Suppose $d = 2$ and A is polygonal. Let $0 < \varepsilon < \rho < \infty$. Choose K such that for all r and all $x \in A \setminus \mathsf{Cor}^{(Kr)}$, the ball $B_{(\rho+9)r}(x)$ intersects at most one edge of A. We can choose such a K by a similar argument to the proof of Lemma 2.5. Then for $x \in A \setminus \mathsf{Cor}^{(Kr)}$ we can deduce (4.9) and (4.10) in the same manner as in the proof of part (i). □

Next we consider the probability of having a small or medium-sized separating set near the corner of a polygon in dimension 2. We do not attempt to compare this probability with $p_{n,k-1}(x)$ because the corners have negligible area and we can get by with a less precise estimate.

LEMMA 4.4. *Suppose that $d = 2$ and A is a convex polygon. Given $\rho, K \in (0, \infty)$, there exist constants $c, \delta, r_0 \in (0, \infty)$ depending only on A, f_0, ρ and K, such that if $n \geq k+2$ then*

$$\sup_{x \in \mathsf{Cor}^{(Kr)}, r \in (0, r_0)} \mathbb{P}[\exists \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n), \operatorname{diam}(\mathcal{Y}) \leq \rho r] \leq c \exp(-\delta n r^2); \tag{4.14}$$

$$\sup_{x \in \mathsf{Cor}^{(Kr)}, r \in (0, r_0)} \mathbb{P}[\exists \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{X}_{n-1}), \operatorname{diam}(\mathcal{Y}) \leq \rho r] \leq c \exp(-\delta n r^2). \tag{4.15}$$

PROOF. Fix $\rho, K \in (0, \infty)$. Let $\alpha_{\min}$ be the smallest angle of the corners of A. Assume without loss of generality that one of the corners of A lies at the origin, and moreover one of the edges of A incident to the origin is in the direction of the positive x-axis, while the other edge is in the anti-clockwise direction from the positive x-axis, and therefore lies in the upper half-plane since A is assumed convex.

Let $\delta_2 := 1/4$. Define $\mathsf{A}_{\delta_2}(\mathcal{Y}^*(\mathcal{P}_n))$ as in the proof of Lemma 4.3. As argued there, if there exists $\mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n)$ with $\operatorname{diam}(\mathcal{Y}) \leq \rho r$, then $\mathsf{A}_{\delta_2}(\mathcal{Y}^*(\mathcal{P}_n))$ can take at most c different possible shapes for some finite c not depending on r.

Suppose $x \in B_{Kr}(o)$. Fix a possible shape σ that might arise when there exists $\mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n)$ with $\operatorname{diam}(\mathcal{Y}) \leq \rho r$, and suppose the event $\{\mathsf{A}_{\delta_2}(\mathcal{Y}^*(\mathcal{P}_n)) = \sigma\}$ occurs. Let $y_{\max}$ be the largest point of σ in the lexicographic ordering (i.e., the highest rightmost point). Let S be a sector centred on $y_{\max}$ of radius $r/2$ and with one straight edge from y in the direction of the positive x-axis, while the other edge is in the anti-clockwise direction with angle $\min(\alpha_{\min}, \pi/2)$ from the first edge.

Then provided r is small enough, $S \subset A$ and the interior of S is disjoint from σ. Also the squares making up σ have diameter less than $r/2$, so S is contained in $\mathcal{P}_n \oplus B_r(o)$; hence $\mathcal{P}_n(S) \leq k-1$. Also $f_0 \min(\alpha_{\min}, \pi/2) r^2/2 \leq \nu(S) \leq (\pi/4) f_{\max} r^2$. Therefore

$$\begin{aligned}
&\mathbb{P}\big[\{\exists \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{P}_n), \operatorname{diam}(\mathcal{Y}) \leq \rho r\} \cap \{\mathsf{A}_{\delta_2}(\mathcal{Y}^*(\mathcal{P}_n)) = \sigma\}\big] \\
&\quad \leq k\big(n f_{\max}(\pi/4) r^2\big)^{k-1} \times \exp\big(-n f_0 \min(\alpha_{\min}, \pi/2) r^2/2\big).
\end{aligned}$$

Summing over all possible σ and treating other corners similarly, we obtain (4.14). Also

$$\begin{aligned}\mathbb{P}\big[\{\exists \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{X}_{n-1}), \operatorname{diam}(\mathcal{Y}) \le \rho r\} \cap \{\mathsf{A}_\delta(\mathcal{Y}^*(\mathcal{X}_{n-1})) = \sigma\}\big] \\ \le \sum_{j=0}^{k-1} \binom{n-1}{j} \nu(S)^j (1-\nu(S))^{n-1-j} \\ \le \sum_{j=0}^{n} 2\big((n\nu(S))^j/j!\big) \exp(-n\nu(S)),\end{aligned}$$

and using the same upper and lower bounds on $\nu(S)$ as before gives us (4.15). □

4.3. *Large separating pairs*. Given $r > 0$, $\rho > 0$, recall that if $L_{n,k} \le r < M_{n,k}$ then there is a $(k-1)$-separating pair for $G(\mathcal{P}_n, r)$, and each individual set in the pair is nonsingleton. Then, either there exists a nonsingleton $(k-1)$-separating set with diameter at most ρr, or both sets in the pair have diameter greater than ρr. Our next lemma deals with the latter possibility. Given $\rho > 0$, $r \ge 0$, define the event

$$H_{r,\rho}(\mathcal{X}) = \{\exists \text{ a } (k-1)\text{-separating pair } \mathcal{Y}, \mathcal{Y}' \text{ for } G(\mathcal{X}, r), \min(\operatorname{diam}(\mathcal{Y}), \operatorname{diam}(\mathcal{Y}')) > \rho r\}.$$

LEMMA 4.5. *Suppose $(r_n)_{n>0}$ satisfies (3.5) for some $\beta' \in (0,\infty)$. Then there exists $\rho \in (0,\infty)$ such that $\mathbb{P}[H_{r_n,\rho}(\mathcal{P}_n)] = O(n^{-2})$ and $\mathbb{P}[H_{r_n,\rho}(\mathcal{X}_n)] = O(n^{-2})$ as $n \to \infty$.*

PROOF. *Case 1: $\partial A \in C^2$.* See [16], equation (3.14), (it is here that we rely on [16], Lemma 3.5, which requires the C^2 boundary condition). That result is formulated only for $\mathcal{X}_n$, not for $\mathcal{P}_n$, and also only for the case $k = 0$. However, it uses only the probability bound that if X is binomial with mean μ then $\mathbb{P}[X = 0] \le e^{-\mu}$. Using a standard Chernoff bound, for example, [11], Lemmas 1.1 and 1.2, we have that if X is either binomial or Poisson distributed with mean μ for μ sufficiently large, we have $\mathbb{P}[X < k] \le e^{-\mu/2}$, and using this we can readily adapt the argument in [16] to the generality required here.

Case 2: $d = 2$ and A is polygonal. In this case we use the proof of [19], Lemma 3.12. Our r_n is not quite the same as there, but the argument works for our r_n too; the properties of r_n given in Lemma 3.3 are sufficient. Again, the proof in [19] is only for $\mathcal{X}_n$, but it relies only on Chernoff probability bounds for a binomial random variable, which also apply for a Poisson random variable with the same mean and therefore the result holds for $\mathcal{P}_n$ as well as for $\mathcal{X}_n$. □

We can now bound $\mathbb{P}[L_{n,k} \le r_n < M_{n,k}]$, which is the result we have been leading up to in this whole section.

PROPOSITION 4.6. *Let $\beta' \in (0,\infty)$ and suppose $(r_n)_{n\ge1}$ satisfies (3.5). Then as $n \to \infty$,*

$$\mathbb{P}[L_{n,k} \le r_n < M_{n,k}] = O\big((\log n)^{1-d}\big); \tag{4.16}$$

$$\mathbb{P}\big[L_k(\mathcal{X}_n) \le r_n < M_k(\mathcal{X}_n)\big] = O\big((\log n)^{1-d}\big). \tag{4.17}$$

PROOF. Given $r, \rho \in (0,\infty)$, and finite $\mathcal{X} \subset \mathbb{R}^d$, define the event

$$J_{r,\rho}(\mathcal{X}) := \{\exists x \in \mathcal{X}, \mathcal{Y} \in \mathcal{C}_r(x, \mathcal{X}), 0 < \operatorname{diam}(\mathcal{Y}) \le \rho r\}.$$

By Lemma 4.1, if $L_k(\mathcal{X}) \le r_n < M_k(\mathcal{X})$, then either $J_{r_n,\rho}(\mathcal{X})$ or $H_{r_n,\rho}(\mathcal{X})$ occurs. Hence by Lemma 4.5, it suffices to prove that for any $\rho \in (0,\infty)$, the events $J_{r_n,\rho}(\mathcal{P}_n)$ and $J_{r_n,\rho}(\mathcal{X}_n)$ occur with probability $O((\log n)^{1-d})$ as $n \to \infty$.

Case 1: $\partial A \in C^2$. Fix $\rho \in (0,\infty)$. Let N_n denote the (random) number of $x \in \mathcal{P}_n$ such that there exists a $\mathcal{Y} \in \mathcal{C}_{r_n}(x,\mathcal{P}_n)$ with $0 < \mathrm{diam}(\mathcal{Y}) \le \rho r_n$. By Markov's inequality $\mathbb{P}[J_{\rho,r_n}(\mathcal{P}_n)] = \mathbb{P}[N_n \ge 1] \le \mathbb{E}[N_n]$.

Let δ be as in Lemma 4.2(i) and assume without loss of generality that $0 < \delta < \rho$. Then by the Mecke equation and the definitions of $E_{x,\rho,r}$ and $F_{x,\varepsilon,\rho,r}$ at (4.2) and (4.8), and the union bound,

$$\mathbb{E}[N_n] \le n \int_A \mathbb{P}[E_{x,\delta,r_n}(x,\mathcal{P}_n)]\nu(dx) + n\int_A \mathbb{P}[F_{x,\delta,\rho,r_n}(x,\mathcal{P}_n)]\nu(dx). \tag{4.18}$$

Using Lemma 4.2 for the first integral and Lemma 4.3 for the second integral, and (3.3), we can find $c, \delta_2 \in (0,\infty)$ such that for large enough n we have that

$$\mathbb{E}[N_n] \le c\big((nr_n^d)^{1-d} + e^{-\delta_2 n r_n^d}\big)\mathbb{E}[\xi_{n,r_n}],$$

where $\xi_{n,r}$ was defined at (3.1). From this, we obtain the claimed estimate (4.16) by (3.5) and Lemma 3.3.

Case 2: $d = 2$ and A is polygonal. Fix $\rho \in (0,\infty)$. Let δ be as in Lemma 4.2(ii); assume without loss of generality that $0 < \delta < \rho$. By a similar argument to (4.18) we have

$$\begin{aligned}\mathbb{P}[J_{\rho,r_n}(\mathcal{P}_n)] \le & \int_{A\setminus \mathsf{Cor}^{(Kr_n)}} \mathbb{P}[E_{x,\delta,r_n}(x,\mathcal{P}_n)]n\nu(dx).\\ &+ \int_{A\setminus \mathsf{Cor}^{(Kr_n)}} \mathbb{P}[F_{x,\delta,\rho,r_n}(x,\mathcal{P}_n)]n\nu(dx).\\ &+ \int_{\mathsf{Cor}^{(Kr_n)}} \mathbb{P}[\{\exists \mathcal{Y} \in \mathcal{C}_{r_n}(x,\mathcal{P}_n), \mathrm{diam}(\mathcal{Y}) \le \rho r_n\}]n\nu(dx).\end{aligned}$$

We can deal with the first two integrals just as we did in Case 1. By Lemma 4.4, there is a constant δ' such that the third integral is $O(nr_n^2 \exp(-\delta' n r_n^2))$, which completes the proof of (4.16) for this case.

The proof of (4.17) is identical, now relying on the binomial parts of Lemmas 4.2–4.5. We omit the details. □

5. Proof of Theorem 1.3.

Throughout this section we assume that $\partial A \in C^2$ or that A is a convex polygon. We adopt our WA but do not assume f is necessarily constant on A.

5.1. *Proof of parts* (i) *and* (ii). Given $\beta \in \mathbb{R}$, choose $n_0(\beta)$ such that $n_0(\beta) > e^{-\beta}$ and $e^{-n}\sum_{j=0}^{k-1}(n^{j+1})/j! < e^{-\beta}$ for all $n \in [n_0(\beta),\infty)$. Recall the definition of $p_{n,r}(x)$ at (3.4). Given $n \ge n_0(\beta)$, define $r_n(\beta) \in (0,\infty)$ by

$$r = r_n(\beta) \quad \Longleftrightarrow \quad n\int_A p_{n,r}(x)\nu(dx) = e^{-\beta}. \tag{5.1}$$

By the intermediate value theorem, such an $r_n(\beta)$ exists and is unique (note that the integrand is nonincreasing in r because the Poisson (λ) distribution is stochastically monotone in λ). Moreover, for $-\infty < \beta < \gamma < \infty$ and $n \ge \max(n_0(\beta), n_0(\gamma))$, we have $r_n(\beta) < r_n(\gamma)$.

We first determine the first-order limiting behaviour of $r_n(\beta)$.

LEMMA 5.1. *Let $\beta \in \mathbb{R}$ and let $r_n(\beta)$ satisfy (5.1) for all $n \ge n_0(\beta)$. Then*

$$\lim_{n\to\infty}\big(n\theta_d r_n(\beta)^d/\log n\big) = \max\big(1/f_0, (2-2/d)/f_1\big). \tag{5.2}$$

If also $\gamma \in \mathbb{R}$ with $\beta < \gamma$, then

$$\lim_{n\to\infty}\sup_{x\in A}\big(\nu(B_{r_n(\gamma)}(x))/\nu(B_{r_n(\beta)}(x))\big) = 1. \tag{5.3}$$

REMARK 5.2. It might seem that Lemma 5.1 is a stronger version of Lemma 3.3. Note, however, that Lemma 3.3 is under the weaker condition (3.5) rather than the equality in (5.1). Moreover, we used Lemma 3.3 to prove Proposition 3.1, which we shall use in turn to prove Lemma 5.1. Thus Lemma 3.3 is not redundant.

PROOF OF LEMMA 5.1. By Proposition 3.1, as $t \to \infty$ (through $\mathbb{R}$),

$$\mathbb{P}[L_{t,k} \le r_t(\beta)] \to \exp(-e^{-\beta}). \tag{5.4}$$

On the other hand, we claim that $t\theta_d L_{t,k}^d / \log t \xrightarrow{\mathbb{P}} \max(1/f_0, (2-2/d)/f_1)$ as $t \to \infty$. Indeed, writing N_t for the number of points of the Poisson process $\mathcal{P}_t$, we have

$$t\theta_d L_{t,k}^d / \log t = (N_t \theta_d L_{t,k}^d / \log N_t) \times (t/N_t) \times (\log N_t / \log t). \tag{5.5}$$

Since the conditional distribution of $L_{t,k}$, given $N_t = n$, is that of $L_k(\mathcal{X}_n)$, we have from (1.7) when A has a C^2 boundary, and from (1.8) when A is a convex polygon, that the first factor in the right hand side of (5.5) tends to $\max(1/f_0, (2-2/d)/f_1)$ in probability, and by Chebyshev's inequality the second factor also tends to 1 in probability, from which we can deduce the third factor also tends to 1 in probability. Combining these gives us the claim.

Let $\alpha < \max(1/f_0, (2-2/d)/f_1)$. By the preceding claim we have $\mathbb{P}[t\theta_d L_{t,k}^d / \log t < \alpha] \to 0$, and hence by (5.4), $t\theta_d r_t^d / \log t \ge \alpha$ for t large. Similarly, if $\alpha' > \max(1/f_0, (2-2/d)/f_1)$ then $\mathbb{P}[t\theta L_{t,k}^d / \log t \le \alpha] \to 1$, so $t\theta_d r_t^d / \log t \le \alpha'$ for t large. Combining these assertions gives us (5.2).

For (5.3), note that $\nu(B_s(x) \setminus B_r(x)) \le f_{\max}\theta_d(s^d - r^d)$ for $x \in A$ and $0 < r < s$. Therefore using (3.8), for all $x \in A$ and all large enough n we have

$$\frac{\nu(B_{r_n(\gamma)}(x) \setminus B_{r_n(\beta)}(x))}{\nu(B_{r_n(\beta)}(x))} \le \frac{f_{\max}\theta_d(r_n(\gamma)^d - r_n(\beta)^d)}{\delta_0 r_n(\beta)^d},$$

which tends to zero by (5.2), and (5.3) follows. □

Recall that $\mu(X)$ denotes the median of a random variable X. For nonuniform ν, it seems to be hard in general to find a formula for $r_n(\beta)$ satisfying (5.1) (even if the equality is replaced by convergence). However, if we can determine a limit for $nr_n(\gamma)^d - nr_n(\beta)^d$ for all $\beta < \gamma$, then we can still obtain a weak limiting distribution for $nM_{n,k}^d - n\mu(M_{n,k})^d$ without giving an explicit sequence for $\mu(M_{n,k})$. The next lemma spells out this argument.

LEMMA 5.3. *Suppose there exists $\alpha \in (0, \infty)$ such that for all $\beta, \gamma \in \mathbb{R}$ with $\beta < \gamma$, we have as $n \to \infty$ that*

$$\lim_{n\to\infty} n(r_n(\gamma)^d - r_n(\beta)^d) = \alpha(\gamma - \beta), \tag{5.6}$$

where $r_n(\beta)$ is defined by (5.1). Suppose that $(X_n)_{n>0}$ are random variables satisfying

$$\lim_{n\to\infty} \mathbb{P}[X_n \le r_n(\beta)] = \exp(-e^{-\beta}), \quad \forall \beta \in \mathbb{R}. \tag{5.7}$$

Then

$$nX_n^d - n\mu(X_n)^d \xrightarrow{d} \alpha(\mathsf{Gu} + \log\log 2) \quad \text{as } n \to \infty. \tag{5.8}$$

PROOF. Set $\beta_0 = -\log\log 2$. Let $r_n = r_n(\beta_0)$ and let $-\infty < y < x < y' < \infty$. Set $s_n := r_n(\beta_0 + y/\alpha)$ and $s_n' := r_n(\beta_0 + y'/\alpha)$. Then by (5.6), $n(s_n^d - r_n^d) \to y$ and $n((s_n')^d - r_n^d) \to y'$. Hence for n large we have $ns_n^d < x + nr_n^d$ and $n(s_n')^d > x + nr_n^d$, so that by (5.7), setting $F(x) := \exp(-e^{-x})$ we have

$$\mathbb{P}[nX_n^d - nr_n^d \le x] \ge \mathbb{P}[nX_n^d \le ns_n^d] \to F(\beta_0 + y/\alpha),$$

and similarly $\mathbb{P}[nX_n^d - nr_n^d \le x] \le \mathbb{P}[nX_n^d \le n(s_n')^d] \to F(\beta_0 + y'/\alpha)$. Since we can take y and y' arbitrarily close to x and $F(\cdot)$ is continuous, we can deduce that

$$\mathbb{P}[nX_n^d - nr_n^d \le x] \to F(\beta_0 + x/\alpha), \quad x \in \mathbb{R}.$$

Since $F(\beta_0 + z/\alpha) = \mathbb{P}[\alpha(\mathsf{Gu} + \log\log 2) \le z]$, we thus have

$$nX_n^d - nr_n^d \xrightarrow{d} \alpha(\mathsf{Gu} + \log\log 2). \tag{5.9}$$

Finally we need to check that $n\mu(X_n)^d - nr_n^d \to 0$. Let $\varepsilon > 0$. By (5.9), as $n \to \infty$ we have

$$\mathbb{P}[nX_n^d - nr_n^d \le \varepsilon] \to \mathbb{P}[\alpha(\mathsf{Gu} + \log\log 2) \le \varepsilon] > 1/2;$$
$$\mathbb{P}[nX_n^d - nr_n^d \le -\varepsilon] \to \mathbb{P}[\alpha(\mathsf{Gu} + \log\log 2) \le -\varepsilon] < 1/2,$$

so for large n we have

$$\mu(X_n^d) - nr_n^d = \mu(nX_n^d - nr_n^d) \in [-\varepsilon, \varepsilon],$$

so $\mu(X_n^d) - nr_n^d \to 0$ as $n \to \infty$, and then (5.8) follows from (5.9) and the continuity of the Gumbel cdf. □

To use Lemma 5.3, we need to show that (5.6) holds for some α. We do this first for the case where $f_1 > f_0(2 - 2/d)$.

LEMMA 5.4. *Let $\beta, \gamma \in \mathbb{R}$ with $\beta < \gamma$. Define $r_n(\beta)$ by (5.1). If $f_1 > f_0(2 - 2/d)$, then*

$$n(r_n(\gamma)^d - r_n(\beta)^d) \to (\gamma - \beta)/(\theta_d f_0) \quad \text{as } n \to \infty. \tag{5.10}$$

PROOF. Given n, set $r = r_n(\beta)$, $s = r_n(\gamma)$. For all $x \in A^{(-s)}$, $\nu(B_s(x) \setminus B_r(x)) \ge \theta_d f_0(s^d - r^d)$, and hence using (5.1), we have

$$\begin{aligned} e^{-\beta} &\ge \int_{A^{(-s)}} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_s(x))} e^{n\nu(B_s(x)\setminus B_r(x))} n\nu(dx) \\ &\ge e^{n\theta_d f_0(s^d - r^d)} \int_{A^{(-s)}} \sum_{j=1}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_s(x))} n\nu(dx). \end{aligned} \tag{5.11}$$

Suppose $\partial A \in C^2$. Using our assumption $f_1 > f_0(2 - 2/d)$, choose $\delta > 0$ such that $(2 - 2/d)(f_1 - 2\delta)^{-1} < f_0^{-1}$. Using Lemma 2.4-(i) and the continuity of $f|_A$ we find for all large enough n and all $x \in (\partial A)^{(s)}$ that $\nu(B_r(x)) \ge \theta_d(f_1 - \delta)r^d/2$. Then using Lemma 5.1 and our assumption $f_1 > f_0(2 - 2/d)$, we have that

$$\lim_{n\to\infty} (n\theta_d r^d/\log n) = f_0^{-1} > (2 - 2/d)(f_1 - 2\delta)^{-1}. \tag{5.12}$$

Hence there are constants c, c' such that for n large

$$\begin{aligned} &\int_{(\partial A)^{(s)}} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_r(x))} n\nu(dx) \le c(\log n)^{k-1} nse^{-n\theta_d(f_1-\delta)r^d/2} \\ &\quad \le c'(\log n)^{k-1+1/d} n^{1-1/d} \exp\big(-(f_1 - \delta)(1 - 1/d)(f_1 - 2\delta)^{-1} \log n\big), \end{aligned}$$

which tends to zero.

Suppose instead that $d = 2$ and A is polygonal. The preceding estimate shows that

$$\int_{(\partial A)^{(s)} \setminus \mathsf{Cor}^{(s)}} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) \exp(-n\nu(B_r(x))) n\nu(dx) \to 0.$$

Moreover, by (3.8) there exist $c, \delta_0 \in (0, \infty)$ such that

$$\int_{\mathsf{Cor}^{(s)}} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_r(x))} n\nu(dx) \le c(nr^2)^k e^{-\delta_0 nr^2}$$

which tends to zero since $nr^2 \to \infty$ by (5.2). Thus in both cases we have that

$$\int_{(\partial A)^{(s)}} \sum_{j=0}^{k-1} \big((n\nu(B_s(x)))^j/j!\big) \exp(-n\nu(B_r(x))) n\nu(dx) \to 0. \tag{5.13}$$

Therefore using (5.1) and (5.3) we obtain that

$$\begin{aligned} e^{-\gamma} &= \lim_{n\to\infty} \int_{A^{(-s)}} \sum_{j=0}^{k-1} \big((n\nu(B_s(x)))^j/j!\big) e^{-n\nu(B_s(x))} n\nu(dx), \\ &= \lim_{n\to\infty} \int_{A^{(-s)}} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_s(x))} n\nu(dx), \end{aligned}$$

and then taking $n \to \infty$ in (5.11) we obtain that $e^{-\beta} \ge e^{-\gamma} \limsup_{n\to\infty}(e^{n\theta_d f_0 (s^d - r^d)})$, so that

$$\limsup \big(n(s^d - r^d)\big) \le (\gamma - \beta)/(\theta_d f_0). \tag{5.14}$$

For an inequality the other way, let $\varepsilon > 0$ and let $A_\varepsilon := \{x \in A : f(x) \le f_0 + 4\varepsilon\}$. By the assumed continuity of f on A, for all n large enough, and all $x \in A_\varepsilon$, we have $\nu(B_s(x) \setminus B_r(x)) \le \theta_d (f_0 + 5\varepsilon)(s^d - r^d)$. Therefore, by (5.1),

$$\begin{aligned} e^{-\beta} \le{}& \int_{A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{n\theta_d(f_0+5\varepsilon)(s^d-r^d)} e^{-n\nu(B_s(x))} n\nu(dx) \\ &+ \int_{A^{(-r)} \setminus A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_r(x))} n\nu(dx) \\ &+ \int_{(\partial A)^{(r)} \setminus A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_r(x))} n\nu(dx). \end{aligned} \tag{5.15}$$

The third integral on the right tends to zero by (5.13). For n large enough, and all $x \in A^{(-r)} \setminus A_\varepsilon$, using (5.12) we have $n\nu(B_r(x)) \ge n(f_0 + 3\varepsilon)\theta_d r^d \ge (f_0 + 3\varepsilon)(\log n)/(f_0 + \varepsilon)$, and hence the second integral in (5.15) tends to zero. Therefore,

$$\liminf \left(\int_{A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{n\theta_d(f_0+5\varepsilon)(s^d-r^d)} e^{-n\nu(B_s(x))} n\nu(dx) \right) \ge e^{-\beta}.$$

Also, by (5.3) the second and third integrals in (5.15) still tend to zero if we change $B_r(x)$ to $B_s(x)$, so

$$e^{-\gamma} = \lim \int_{A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_s(x)))^j/j!\big) e^{-n\nu(B_s(x))} n\nu(dx).$$

Hence, using (5.3) again we obtain that

$$e^{-\gamma} = \lim \int_{A_\varepsilon} \sum_{j=0}^{k-1} \bigl((n\nu(B_r(x)))^j/j!\bigr) e^{-n\nu(B_s(x))} n\nu(dx).$$

Hence $\liminf(e^{n\theta_d(f_0+5\varepsilon)(s^d-r^d)}) \times e^{-\gamma} \geq e^{-\beta}$, so that

$$\liminf\bigl(n(s^d - r^d)\bigr) \geq (\gamma - \beta)/(\theta_d(f_0 + 5\varepsilon)).$$

Since $\varepsilon > 0$ is arbitrary, combining this with (5.14) yields (5.10). □

Next we show that (5.6) holds for a different α in the case where $f_1 < f_0(2 - 2/d)$.

LEMMA 5.5. *Let $\beta, \gamma \in \mathbb{R}$ with $\beta < \gamma$. If $f_1 < f_0(2 - 2/d)$, then*

$$\lim_{n\to\infty} \bigl(n(r_n(\gamma)^d - r_n(\beta)^d)\bigr) = 2(\gamma - \beta)/(\theta_d f_1). \tag{5.16}$$

Also, if $f_1 = f_0(2 - 2/d)$, then

$$\limsup_{n\to\infty} \bigl(n(r_n(\gamma)^d - r_n(\beta)^d)\bigr) \leq 2(\gamma - \beta)/(\theta_d f_1). \tag{5.17}$$

REMARK 5.6.

(i) We do not know in general whether the limit in (5.16) exists when $f_1 = f_0(2 - 2/d)$.
(ii) We note that $\limsup_{n\to\infty}(n(r_n(\gamma)^d - r_n(\beta)^d))$, viewed as a functional of the underlying density f, is discontinuous in f_0 at $f_0 = f_1/(2 - 2/d)$. By Lemmas 5.4 and 5.5, the limit equals $(\gamma - \beta)/(\theta_d f_0)$ if $f_0 < f_1/(2 - 2/d)$, and is about $(\gamma - \beta)/(\theta_d f_0(1 - 1/d))$ if f_0 is slightly above $f_1/(2 - 2/d)$ (the second case can only happen if $d \geq 3$). This is due to the limit being determined by balls centred on an interior point of A in the first case, but centred on a boundary point of A in the second case.

PROOF OF LEMMA 5.5. Assume $f_1 \leq f_0(2-2/d)$. For each n set $r := r_n(\beta)$, $s := r_n(\gamma)$. Suppose $\partial A \in C^2$. By Lemma 2.4-(iii), given $\varepsilon > 0$, for n large enough and all $x \in (\partial A)^{(-s)}$, we have $\nu(B_s(x) \setminus B_r(x)) \geq (f_1 - \varepsilon)\theta_d(s^d - r^d)/2$. Also $\nu(B_s(x) \setminus B_r(x)) \geq f_0\theta_d(s^d - r^d)$ for $x \in A^{(-s)}$. Hence for n large enough

$$\begin{aligned} e^{-\beta} &\geq e^{n(f_1-\varepsilon)\theta_d(s^d-r^d)/2} \int_{(\partial A)^{(s)}} \sum_{j=0}^{k-1} \bigl((n\nu(B_r(x)))^j/j!\bigr) e^{-n\nu(B_s(x))} n\nu(dx) \\ &\quad + e^{nf_0\theta_d(s^d-r^d)} \int_{A^{(-s)}} \sum_{j=0}^{k-1} \bigl((n\nu(B_r(x)))^j/j!\bigr) e^{-n\nu(B_s(x))} n\nu(dx) \\ &\geq e^{n(f_1-\varepsilon)\theta_d(s^d-r^d)/2} \Biggl(\int_{(\partial A)^{(s)}} \sum_{j=0}^{k-1} \bigl((n\nu(B_r(x)))^j/j!\bigr) e^{-n\nu(B_s(x))} n\nu(dx) \\ &\quad + \int_{A^{(-s)}} \sum_{j=0}^{k-1} \bigl((n\nu(B_r(x)))^j/j!\bigr) e^{-n\nu(B_s(x))} n\nu(dx) \Biggr), \end{aligned} \tag{5.18}$$

since the assumption $f_1 \leq f_0(2-2/d)$ implies $f_0 \geq f_1/2$. Hence, for n large enough, setting $\psi_n := \inf_{x\in A}(\nu(B_r(x))/\nu(B_s(x)))$ we have

$$e^{-\beta} \geq e^{n(f_1-\varepsilon)\theta_d(s^d-r^d)/2} e^{-\gamma} \psi_n^{k-1}.$$

By (5.3) we have $\psi_n \to 1$ as $n \to \infty$, and thus

$$\limsup\big(n(s^d - r^d)\big) \le 2(\gamma - \beta)/\big(\theta_d(f_1 - \varepsilon)\big). \tag{5.19}$$

In the other case with $d = 2$ and A polygonal, on choosing a suitable large K (dependent on the smallest angle of A) we can obtain, similar to (5.18), that

$$\begin{aligned} e^{-\beta} &\ge e^{n(f_1-\varepsilon)\pi(s^2-r^2)/2}\psi_n^{k-1} \\ &\quad\times \int_{A\setminus \mathsf{Cor}^{(Kr)}} \sum_{j=0}^{k-1}\big((n\nu(B_s(x)))^j/j!\big)e^{-n\nu(B_s(x))}n\nu(dx) \\ &= e^{n(f_1-\varepsilon)\pi(s^2-r^2)/2}\psi_n^{k-1} \\ &\quad\times \bigg(e^{-\gamma} - \int_{\mathsf{Cor}^{(Kr)}} \sum_{j=0}^{k-1}\big((n\nu(B_s(x)))^j/j!\big)e^{-n\nu(B_s(x))}n\nu(dx)\bigg). \end{aligned} \tag{5.20}$$

Since the integral over $\mathsf{Cor}^{(Kr)}$ tends to zero by (3.8), we therefore obtain from (5.20) that (5.19) holds in this case too. Taking $\varepsilon \downarrow 0$, we deduce in both cases that (5.17) holds whenever $f_1 \le f_0(2 - 2/d)$.

Now assume the strict inequality $f_1 < f_0(2-2/d)$. Again set $r = r_n(\beta)$, $s = r_n(\gamma)$. Using (5.2) from Lemma 5.1, choose $\delta_2 > 0$ such that

$$\lim_{n\to\infty} \frac{n\theta_d r^d}{\log n} = \frac{2-2/d}{f_1} > \frac{1+2\delta_2}{f_0}. \tag{5.21}$$

Then there exists a constant c such that for n large,

$$\begin{aligned} \int_{A^{(-r)}} \sum_{j=0}^{k-1}\big((n\nu(B_r(x)))^j/j!\big)e^{-n\nu(B_r(x))}n\nu(dx) &\le cn(\log n)^{k-1}e^{-n\theta_d f_0 r^d} \\ &\le cn(\log n)^{k-1}e^{-(1+\delta_2)\log n}, \end{aligned} \tag{5.22}$$

which tends to zero.

Take a new $\varepsilon > 0$. Let $A_\varepsilon := \{x \in A : f(x) \le f_1 + 3\varepsilon\}$. Using Lemma 2.3-(ii) in the case $\partial A \in C^2$, take $\delta > 0$ such that for all large enough n and all $x \in (\partial A)^{(\delta r)}$, we have

$$\big|A \cap B_s(x) \setminus B_r(x)\big| \le \theta_d\big(s^d - r^d\big)(f_1 + 5\varepsilon)/\big(2(f_1 + 4\varepsilon)\big).$$

Such δ can also be found in the other case where A is a convex polygon.

Then for n large and $x \in (\partial A)^{(\delta r)} \cap A_\varepsilon$ we have $\sup_{B_s(x)\cap A} f \le f_1 + 4\varepsilon$, and hence

$$\nu\big(B_s(x) \setminus B_r(x)\big) \le \theta_d\big(s^d - r^d\big)(f_1 + 5\varepsilon)/2, \quad \forall\, x \in (\partial A)^{(\delta r)} \cap A_\varepsilon. \tag{5.23}$$

Using Lemma 2.4-(i), in the case where $\partial A \in C^2$ we have for large n, and all $x \in A \setminus A_\varepsilon$, that $\nu(B_r(x)) \ge (f_1 + 2\varepsilon)\theta_d r^d/2$. Hence for n large

$$\begin{aligned} &\int_{(\partial A)^{(r)}\setminus A_\varepsilon} \sum_{j=0}^{k-1}\big((n\nu(B_r(x)))^j/j!\big)e^{-n\nu(B_r(x))}n\nu(dx) \\ &\quad\le c(\log n)^k nr\exp\big(-n(f_1 + 2\varepsilon)\theta_d r^d/2\big) \\ &\quad\le cn^{1-1/d}(\log n)^{k+1/d}\exp\big(-(f_1+\varepsilon)(1-1/d)(\log n)/f_1\big) \to 0, \end{aligned}$$

where for the second inequality we have used the equality in (5.21).

By Lemma 2.3-(i), there is a constant $\delta_1 > 0$ such that for $x \in (\partial A)^{(r)} \setminus (\partial A)^{(\delta r)}$ (in the case where $\partial A \in C^2$) or for $x \in (\partial A)^{(r)} \setminus (\partial A)^{(\delta r)} \setminus \mathsf{Cor}^{(Kr)}$ (in the case where A is polygonal) we have $\nu(B_r(x)) \geq (f_1 + 2\delta_1)\theta_d r^d/2$. Thus if $\partial A \in C^2$ then

$$\int_{(\partial A)^{(r)} \setminus (\partial A)^{(\delta r)}} p_{n,r}(x) n\nu(dx) \leq c(\log n)^k nr \exp(-n(f_1 + 2\delta_1)\theta_d r^d/2),$$

which tends to zero by (5.21). In the polygonal case we get the same conclusion using also the fact that the integral over $\mathsf{Cor}^{(Kr)}$ tends to zero. Combining the last two estimates with (5.22) shows that

$$\int_{A \setminus ((\partial A)^{(\delta r)} \cap A_\varepsilon)} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_r(x))} n\nu(dx) \to 0, \tag{5.24}$$

and therefore, using (5.1) followed by (5.23), we have

$$\begin{aligned} e^{-\beta} &= \lim_{n\to\infty} \int_{(\partial A)^{(\delta r)} \cap A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_r(x))} n\nu(dx) \\ &\leq \liminf_{n\to\infty} \Bigg(e^{n\theta_d(s^d - r^d)(f_1+5\varepsilon)/2} \\ &\quad \times \int_{(\partial A)^{(\delta r)} \cap A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_s(x))} n\nu(dx) \Bigg). \end{aligned} \tag{5.25}$$

By (5.24) we have $\int_{A \setminus ((\partial A)^{(\delta r)} \cap A_\varepsilon)} \sum_{j=0}^{k-1} ((n\nu(B_r(x)))^j/j!) e^{-n\nu(B_s(x))} n\nu(dx) \to 0$, so using (5.3) from Lemma 5.1 we have

$$\int_{(\partial A)^{(\delta r)} \cap A_\varepsilon} \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) e^{-n\nu(B_s(x))} n\nu(dx) \to e^{-\gamma},$$

and hence by (5.25),

$$e^{-\beta} \leq \liminf_{n\to\infty} \big(e^{n\theta_d(s^d - r^d)(f_1+5\varepsilon)/2}\big) \times e^{-\gamma},$$

so that

$$\liminf_{n\to\infty} \big(n(s^d - r^d)\big) \geq 2(\gamma - \beta)/\big(\theta_d(f_1 + 5\varepsilon)\big).$$

Taking $\varepsilon \downarrow 0$ and combining with (5.17) shows that (5.16) holds. □

LEMMA 5.7 (De-Poissonization). *Let $\beta \in \mathbb{R}$ and suppose $r_n = r_n(\beta)$ is given by* (5.1) *for n sufficiently large. Then*

$$\lim_{n\to\infty} \mathbb{P}\big[L_k(\mathcal{X}_n) \leq r_n(\beta)\big] = \lim_{n\to\infty} \mathbb{P}\big[L_{n,k} \leq r_n(\beta)\big] = \exp\big(-e^{-\beta}\big). \tag{5.26}$$

PROOF. The statement about $L_{n,k}$ in (5.26) follows from Proposition 3.1. It remains to prove the statement about $L_k(\mathcal{X}_n)$.

Given $n > 0$, $r > 0$ define $\phi_{n,r} := \mathbb{E}[\xi_{n,r}]$, that is, by (3.3),

$$\phi_{n,r} := \int_A p_{n,r}(x) n\nu(dx) = \int_A \sum_{j=0}^{k-1} \big((n\nu(B_r(x)))^j/j!\big) \exp\big(-n\nu(B_r(x))\big) n\nu(dx), \tag{5.27}$$

which is decreasing in r. Set $n^- := n - n^{3/4}$, $n^+ := n + n^{3/4}$, and let $\beta \in \mathbb{R}$. Then

$$\frac{\phi_{n^-,r_n(\beta)}}{\phi_{n,r_n(\beta)}} \ge \left(\frac{n^-}{n}\right)^{k-1} = 1 + O(n^{-1/4}),$$

and

$$\frac{\phi_{n^-,r_n(\beta)}}{\phi_{n,r_n(\beta)}} \le \exp\bigl(n^{3/4} f_{\max}\theta_d r_n(\beta)^d\bigr) = 1 + O\bigl((\log n)n^{-1/4}\bigr),$$

so that $\phi_{n^-,r_n(\beta)}/\phi_{n,r_n(\beta)} \to 1$ as $n \to \infty$, and thus $\phi_{n^-,r_n(\beta)} \to e^{-\beta}$ as $n \to \infty$. Therefore, using Proposition 3.1, we have

$$\mathbb{P}\bigl[L_{n^-,k} \le r_n(\beta)\bigr] \to \exp\bigl(-e^{-\beta}\bigr). \tag{5.28}$$

Now, following the proof of [11], Theorem 8.1, we note that with $\mathcal{P}_{n^-}$, $\mathcal{P}_{n^+}$ and $\mathcal{X}_n$ coupled in the usual way (as described in [11]), we have

$$\bigl\{L_{n^-,k} \le r_n(\beta)\bigr\} \triangle \bigl\{L_k(\mathcal{X}_n) \le r_n(\beta)\bigr\} \subset E_n \cup F_n \cup G_n,$$

where, setting $N_t = \mathcal{P}_t(A)$ for all t, we set

$$\begin{aligned} E_n &:= \bigl\{\exists x \in \mathcal{P}_{n^+} \setminus \mathcal{P}_{n^-} : \mathcal{P}_{n^-}\bigl(B_{r_n(\beta)}(x)\bigr) \le k-1\bigr\}; \\ F_n &:= \bigl\{\exists x \in \mathcal{P}_{n^-}, y \in \mathcal{P}_{n^+} \setminus \mathcal{P}_{n^-} : \mathcal{P}_{n^-}\bigl(B_{r_n(\beta)}(x)\bigr) \le k, \|y - x\| \le r_n(\beta)\bigr\}; \\ G_n &:= \{N_{n^-} \le n \le N_{n^+}\}^c. \end{aligned}$$

By Chebyshev's inequality $\mathbb{P}[G_n] = O(n^{-1/2})$. By Markov's inequality,

$$\begin{aligned} \mathbb{P}[E_n] &\le 2n^{3/4} \int_A \mathbb{P}\bigl[\mathcal{P}_{n^-}\bigl(B_{r_n(\beta)}(x)\bigr) \le k-1\bigr]\nu(dx) \\ &= (2n^{3/4}/n^-)\phi_{n^-,r_n(\beta)}, \end{aligned}$$

which tends to zero. Finally, by the Mecke formula,

$$\begin{aligned} \mathbb{P}[F_n] &\le 2n^{3/4} f_{\max}\theta_d r_n(\beta)^d \int_A \mathbb{P}\bigl[\mathcal{P}_{n^-}\bigl(B_{r_n(\beta)}(x)\bigr) \le k-1\bigr]n^-\nu(dx) \\ &= 2 f_{\max}\theta_d n^{3/4} r_n(\beta)^d \phi_{n^-,r_n(\beta)} = O\bigl((\log n)n^{-1/4}\bigr). \end{aligned}$$

Therefore, using (5.28), we obtain (5.26). □

PROOF OF THEOREM 1.3 PARTS (I) AND (II). For part (i) we assume $f_1 > f_0(2 - 2/d)$; in this case set $\alpha = 1/(\theta_d f_0)$. For part (ii) we assume $f_1 < f_0(2 - 2/d)$; in this case set $\alpha = 2/(\theta_d f_1)$.

By Lemma 5.4 in the first case, or by Lemma 5.5 in the second case, for all $\beta, \gamma \in \mathbb{R}$ with $\beta < \gamma$ the condition (5.6) holds.

Let $\beta \in \mathbb{R}$ and suppose $r_n = r_n(\beta)$ is given by (5.1) for n sufficiently large. By Lemma 5.7, $\mathbb{P}[L_{n,k} \le r_n(\beta)] \to F(\beta)$ and $\mathbb{P}[L_k(\mathcal{X}_n) \le r_n(\beta)] \to F(\beta)$, where we set $F(x) := \exp(-e^{-x})$. By Proposition 4.6, $\mathbb{P}[M_{n,k} \le r_n(\beta)] \to F(\beta)$ as $n \to \infty$.

Then by Lemma 5.3 (taking $X_n = M_{n,k}$), we obtain that $nM_{n,k}^d - n\mu(M_{n,k})^d \xrightarrow{d} \alpha(\mathsf{Gu} + \log\log 2)$, that is, (1.19) holds if $f_1 > f_0(2 - 2/d)$, and (1.23) holds if $f_1 < f_0(2 - 2/d)$. Also by taking $X_n = L_{n,k}$ in Lemma 5.3 we obtain (1.20) if $f_1 > f_0(2 - 2/d)$, and (1.24) if $f_1 < f_0(2 - 2/d)$.

It remains to demonstrate the results for the binomial model, that is, (1.17), (1.18), (1.21) and (1.22). By Lemma 5.7, we can use Lemma 5.3 (now taking $X_n = L_k(\mathcal{X}_n)$) to deduce that (1.18) holds if $f_1 > f_0(2 - 2/d)$, and (1.22) holds if $f_1 < f_0(2 - 2/d)$.

Using (5.26), and (4.17) from Proposition 4.6, we obtain that

$$\mathbb{P}[M_k(\mathcal{X}_n) \le r_n(\beta)] \to \exp(-e^{-\beta}).$$

Then using Lemma 5.3 (now taking $X_n = M_k(\mathcal{X}_n)$) we can deduce that (1.17) holds if $f_1 > f_0(2-2/d)$, and (1.21) holds if $f_1 < f_0(2-2/d)$. □

5.2. *Proof of Theorem* 1.3: *Conclusion*. It remains to prove part (iii) of Theorem 1.3. We deal first with the assertions there concerning tightness. Again in the next proof, set $F(x) := \exp(-e^{-x})$, $x \in \mathbb{R}$.

LEMMA 5.8. *The collection of random variables* $\{nM_{n,k}^d - n\mu(M_{n,k})^d\}_{n\ge1}$ *is tight. So is the collection of random variables* $\{nL_{n,k}^d - n\mu(L_{n,k})^d\}_{n\ge1}$, *and also the sequence* $(nM_k(\mathcal{X}_n)^d - n\mu(M_k(\mathcal{X}_n))^d)_{n\in\mathbb{N}}$, *and the sequence* $(nL_k(\mathcal{X}_n)^d - n\mu(L_k(\mathcal{X}_n))^d)_{n\in\mathbb{N}}$.

PROOF. Let $\varepsilon \in (0, 1/6)$. Choose $\beta < \beta'$ with $F(\beta) < \varepsilon/3$ and $F(\beta') > 1-\varepsilon/3$. Set $r_n = r_n(\beta)$, $s_n = r_n(\beta')$ as given by (5.1). By Lemma 5.5 there exists a constant K such that $n(s_n^d - r_n^d) \le K$ for all large enough n. By Proposition 3.1, $\mathbb{P}[L_{n,k} \le r_n(\beta)] \to F(\beta)$. By Proposition 4.6, $\mathbb{P}[M_{n,k} \le r_n] \to F(\beta)$ as $n \to \infty$. Similarly, $\mathbb{P}[L_{n,k} \le s_n] \to F(\beta')$ and $\mathbb{P}[M_{n,k} \le s_n] \to F(\beta')$ as $n \to \infty$. Therefore, since $F(\beta) < 1/2 < F(\beta')$, we have $r_n \le \mu(L_{n,k}) < s_n$ and $r_n \le \mu(M_{n,k}) < s_n$ for n large. Then for n large

$$\begin{aligned}\mathbb{P}[nM_{n,k}^d \le n\mu(M_{n,k})^d - K] &\le \mathbb{P}[nM_{n,k}^d \le ns_n^d - K]\\ &\le \mathbb{P}[M_{n,k} \le r_n] < \varepsilon/2,\end{aligned}$$

and likewise for $L_{n,k}$. Similarly for n large

$$\begin{aligned}\mathbb{P}[nM_{n,k}^d > n\mu(M_{n,k})^d + K] &\le \mathbb{P}[nM_{n,k}^d > nr_n^d + K]\\ &\le \mathbb{P}[M_{n,k} > s_n] < \varepsilon/2,\end{aligned}$$

and likewise for $L_{n,k}$. Thus $\mathbb{P}[|n(M_{n,k}^d - \mu(M_{n,k})^d)| > K] \le \varepsilon$ and $\mathbb{P}[|n(L_{n,k}^d - \mu(L_{n,k})^d)| > K] \le \varepsilon$ for all large enough n. Also $\{n(M_{n,k}^d - \mu(M_{n,k}^d))\}_{1\le n\le n_0}$ and $\{n(L_{n,k}^d - \mu(L_{n,k}^d))\}_{1\le n\le n_0}$ are uniformly bounded for any fixed $n_0 \in (0,\infty)$. This yields the asserted tightness of $(M_{n,k})_{n\ge1}$ and of $(L_{n,k})_{n\ge1}$.

The proof of tightness for $L_k(\mathcal{X}_n)$ and of $M_k(\mathcal{X}_n)$ is similar, except that instead of Proposition 3.1 we use (5.26). Proposition 4.6 still applies in the binomial setting. □

To prove (1.5) we shall adapt the "squeezing argument" from [12]. For $-\infty < \beta < \gamma < \infty$ we define the random variable

$$U_n(\beta,\gamma) := \sum_{x\in\mathcal{X}_n} \mathbf{1}\{\mathcal{X}_n(B_{r_n(\beta)}(x)) \le k, \mathcal{X}_n(B_{r_n(\gamma)}(x) \setminus B_{r_n(\beta)}(x)) \ge 2\}. \tag{5.29}$$

LEMMA 5.9. *Let* $K > 0$. *Then there is a constant* $c \in (0,\infty)$ *such that for all* $\beta,\gamma \in \mathbb{R}$ *with* $-K \le \beta < \gamma \le K$,

$$\limsup_{n\to\infty} \mathbb{P}[U_n(\beta,\gamma) \ge 1] \le c(\gamma-\beta)^2.$$

PROOF. Set $r_n := r_n(\beta)$, and $s_n := r_n(\gamma)$. By the union bound,

$$\mathbb{P}[U_n(\beta,\gamma) \ge 1] \le n\int_A \mathbb{P}[\mathcal{X}_{n-1}(B_{r_n}(x)) < k, \mathcal{X}_{n-1}(B_{s_n}(x) \setminus B_{r_n}(x)) \ge 2]\nu(dx). \tag{5.30}$$

Let $x \in A$ and set $Y := \mathcal{X}_{n-1}(B_{r_n}(x))$, $Z := \mathcal{X}_{n-1}(B_{s_n}(x) \setminus B_{r_n}(x))$. Also set $v_n(x) := \nu(B_{r_n}(x))$ and $w_n(x) := \nu(B_{s_n}(x))$. Then

$$\begin{aligned}\mathbb{P}[Y < k] &= \sum_{j=0}^{k-1} \binom{n-1}{j} v_n(x)^j \big(1 - v_n(x)\big)^{n-1-j} \\ &\le \big(1 - v_n(x)\big)^{-k} \sum_{j=0}^{k-1} \big((n v_n(x))^j / j!\big)\big(1 - v_n(x)\big)^n \\ &\le \big(1 - f_{\max}\theta_d r_n^d\big)^{-k} \sum_{j=0}^{k-1} \big((n v_n(x))^j / j!\big) e^{-n v_n(x)}.\end{aligned}$$

Also, using the fact that $\mathbb{P}[Z \ge 2 | Y = j]$ is nonincreasing in j, and the fact that for any binomial random variable W with mean α we have $\mathbb{E}[W(W-1)] \le \alpha^2$, we have

$$\begin{aligned}\mathbb{P}[Z \ge 2 | Y < k] &\le \mathbb{P}[Z \ge 2 | Y = 0] \\ &\le (1/2)\mathbb{E}\big[Z(Z-1) | Y = 0\big] \\ &\le (1/2) n^2 \big((w_n(x) - v_n(x))/(1 - v_n(x))\big)^2 \\ &\le (1/2) n^2 \big(1 - f_{\max}\theta_d r_n(\beta)^d\big)^{-2} \big(w_n(x) - v_n(x)\big)^2.\end{aligned}$$

If n is taken to be large enough we have $(1 - f_{\max}\theta_d r_n(\beta)^d)^{-k-2} \le 2$, and hence using (5.30) followed by (5.1), we have

$$\begin{aligned}\mathbb{P}\big[U_n(\beta,\gamma) \ge 1\big] &\le n^3 \sup_{y \in A} \big(w_n(y) - v_n(y)\big)^2 \int_A \sum_{j=0}^{k-1} \big((n v_n(x))^j / j!\big) e^{-n v_n(x)} \nu(dx) \\ &= \Big(n \sup_{y \in A} \big(w_n(y) - v_n(y)\big)\Big)^2 e^{-\beta}.\end{aligned} \tag{5.31}$$

By Lemmas 5.4 and 5.5,

$$\limsup_{n\to\infty} n\theta_d\big(s_n^d - r_n^d\big) \le \left(\frac{1}{f_0} \vee \frac{2}{f_1}\right)(\gamma - \beta), \tag{5.32}$$

where $\vee$ denotes maximum, and hence

$$\limsup_{n\to\infty} \sup_{x\in A} n\big(w_n(x) - v_n(x)\big) \le \limsup_{n\to\infty} n f_{\max}\theta_d\big(s_n^d - r_n^d\big) \le f_{\max}\left(\frac{1}{f_0} \vee \frac{2}{f_1}\right)(\gamma - \beta),$$

so by (5.31) we have $\limsup_{n\to\infty} \mathbb{P}[U_n(\beta,\gamma) \ge 1] \le e^K (f_{\max}(\frac{1}{f_0} \vee \frac{2}{f_1}))^2 (\gamma - \beta)^2$. $\square$

PROOF OF THEOREM 1.3 (CONCLUSION). It remains to prove part (iii), and by Lemma 5.8 it remains only to prove (1.5). Let $\varepsilon > 0$. Choose $K > 0$ such that $\exp(-e^{-K}) > 1 - \varepsilon$, and also $\exp(-e^K) < \varepsilon$. Then let c be as in Lemma 5.9. Choose $\beta_0 < \cdots < \beta_m$ with $\beta_0 = -K$ and $\beta_m = K$ such that $c\sum_{i=1}^m (\beta_i - \beta_{i-1})^2 < \varepsilon$. Write $L'_{n,k}$ for $L_k(\mathcal{X}_n)$ and $M'_{n,k}$ for $M_k(\mathcal{X}_n)$. Since $L'_{n,k} \le M'_{n,k}$, by the union bound

$$\begin{aligned}\mathbb{P}\big[L'_{n,k} \ne M'_{n,k}\big] &= \mathbb{P}\big[L'_{n,k} < M'_{n,k}\big] \\ &\le \mathbb{P}\big[L'_{n,k} \le r_n(\beta_0)\big] + \mathbb{P}\big[L'_{n,k} > r_n(\beta_m)\big] + \sum_{i=1}^m \mathbb{P}\big[L'_{n,k} \le r_n(\beta_i) < M'_{n,k}\big] \\ &\quad + \sum_{i=1}^m \mathbb{P}\big[r_n(\beta_{i-1}) < L'_{n,k} < M'_{n,k} \le r_n(\beta_i)\big].\end{aligned}$$

Using Lemma 5.7 and Proposition 4.6, we obtain that

$$\limsup_{n\to\infty} \mathbb{P}[L'_{n,k} \neq M'_{n,k}] \le 2\varepsilon + \sum_{i=1}^{m} \limsup_{n\to\infty} \mathbb{P}[r_n(\beta_{i-1}) < L'_{n,k} < M'_{n,k} \le r_n(\beta_i)].$$

Suppose $\beta < \gamma$, and suppose $r_n(\beta) < L'_{n,k} < M'_{n,k} \le r_n(\gamma)$ and all inter-point distances in $\mathcal{X}_n$ are distinct (the latter condition holds almost surely).

Then there exist $x, y \in \mathcal{X}_n$ with $\|x - y\| = M'_{n,k}$, and it is possible to remove k vertices from $G(\mathcal{X}_n, M'_{n,k})$ leaving the resulting graph connected, but disconnected if the edge $\{x, y\}$ is also removed. Removing the same set of vertices from $G(\mathcal{X}_n, r_n(\beta))$ leaves x and y in distinct components, and if also for some fixed $\rho > 0$, events $H_{r_n(\beta),\rho}(\mathcal{X}_n)$ (defined in Lemma 4.5) and $J_{r_n(\beta),\rho}(\mathcal{X}_n)$ (defined in the proof of Proposition 4.6), fail to occur, then x or y must have at most $k-1$ other points of $\mathcal{X}_n$ within distance $r_n(\beta)$. But $\mathcal{X}_n(B_{r_n(\gamma)}(x) \setminus \{x\}) \ge k+1$ since $L'_{n,k} < \|y-x\| \le r_n(\gamma)$, and similarly $\mathcal{X}_n(B_{r_n(\beta)}(y) \setminus \{y\}) \ge k+1$. Thus we have the event inclusion

$$\{r_n(\beta) < L'_{n,k} < M'_{n,k} \le r_n(\gamma)\} \subset H_{r_n(\beta),\rho}(\mathcal{X}_n) \cup J_{r_n(\beta),\rho}(\mathcal{X}_n) \cup U_n(\beta,\gamma),$$

where $U_n(\beta,\gamma)$ was defined at (5.29).

By Lemma 4.5, we can choose ρ so that $\mathbb{P}[H_{r_n(\beta),\rho}(\mathcal{X}_n)] \to 0$, and by the proof of Proposition 4.6 $\mathbb{P}[J_{r_n(\beta),\rho}(\mathcal{X}_n)] \to 0$. Therefore, using Lemma 5.9, we obtain that

$$\limsup_{n\to\infty} \mathbb{P}[r_n(\beta) < L'_{n,k} < M'_{n,k} \le r_n(\gamma)] \le \limsup_{n\to\infty} \mathbb{P}[U_n(\beta,\gamma) \ge 1] \le c(\gamma-\beta)^2.$$

Thus

$$\limsup_{n\to\infty} \mathbb{P}[L'_{n,k} \neq M'_{n,k}] \le 2\varepsilon + \sum_{i=1}^{m} c(\beta_i - \beta_{i-1})^2 < 3\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary this gives us (1.5). $\square$

6. Proof of Theorem 1.1. Now we specialise to the uniform case. We make the same assumptions on A as in the previous section, but now we take $f \equiv f_0 \mathbf{1}_A$ with $f_0 = |A|^{-1}$. Recall from (1.11) the definition

$$c_{d,k} := \theta_{d-1}^{-1} \theta_d^{1-1/d} (2-2/d)^{k-2+1/d} 2^{1-k}/(k-1)! \tag{6.1}$$

Given $k \in \mathbb{N}$ and $\beta \in \mathbb{R}$, let $r_n = r_n(\beta) \ge 0$ be defined for all $n > 0$ by

$$f_0 n \theta_d r_n^d = \max\big((2-2/d)\log n + (2k-4+2/d)\mathbf{1}_{\{d\ge 3 \text{ or } k\ge 2\}} \log\log n + \beta, 0\big). \tag{6.2}$$

We now show the convergence of $\mathbb{E}[\xi_{n,r_n}]$ (with $\xi_{n,r}$ defined at (3.1)). That is, we show that this choice of r_n satisfies (3.5) for appropriate β'. Recall the definition of the isoperimetric ratio σ_A at (1.12).

PROPOSITION 6.1 (Convergence of the expectation in the uniform case with $d = 2$). *Suppose $f \equiv f_0 \mathbf{1}_A$, with $d = 2$ and either A compact with $C^{1,1}$ boundary, or A polygonal. Fix $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, and let r_n, $\xi_{n,r}$ be as given in (6.2) and (3.1). Then as $n \to \infty$,*

$$\mathbb{E}[\xi_{n,r_n}] = \begin{cases} e^{-\beta} + \sigma_A e^{-\beta/2} \frac{\sqrt{\pi}}{2} (\log n)^{-1/2} + O\big((\log n)^{-3/2}\big) & \text{if } k = 1, \\ e^{-\beta} + \sigma_A e^{-\beta/2} \frac{\sqrt{\pi}}{4} \left(1 + \frac{\log\log n}{2\log n}\right) \\ \quad + \frac{e^{-\beta}\log\log n}{\log n} + O\big((\log n)^{-1}\big) & \text{if } k = 2, \\ \sigma_A e^{-\beta/2} \frac{\sqrt{\pi}}{(k-1)!2^k} \left(1 + \frac{(2k-3)^2 \log\log n}{2\log n}\right) + O\big((\log n)^{-1}\big) & \text{if } k \ge 3. \end{cases} \tag{6.3}$$

PROOF. Define the "k-vacant region" $V_{n,k} := \{x \in A : \mathcal{P}_n(B_{r_n}(x)) < k\}$. Recall the definition of $p_{n,r}(x)$ at (3.4). By (3.3), we have

$$\mathbb{E}[\xi_{n,r_n}] = nf_0 \int_A p_{n,r_n}(x)\,dx = n|A|^{-1}\mathbb{E}[|V_{n,k}|]. \tag{6.4}$$

Therefore, the result follows from [6], Proposition 5.3. □

PROPOSITION 6.2 (Convergence of the expectation in the uniform case with $d \ge 3$). *Suppose $f \equiv f_0\mathbf{1}_A$, with $d \ge 3$ and A compact with $\partial A \in C^{1,1}$. Fix $\beta \in \mathbb{R}$ and let $r_n(\beta)$, $\xi_{n,r}$, $c_{d,k}$, σ_A be as given in* (6.2), (3.1) *and* (6.1). *Let $\varepsilon > 0$. Then as $n \to \infty$,*

$$\begin{aligned}\mathbb{E}[\xi_{n,r_n}] = e^{-\beta/2}c_{d,k}\sigma_A\biggl(1 + \frac{(k-2+1/d)^2\log\log n}{(1-1/d)\log n}\\ + \frac{(k-2+1/d)\beta + 4k - 4}{(2-2/d)\log n}\biggr) + O\bigl((\log n)^{\varepsilon-2}\bigr).\end{aligned} \tag{6.5}$$

PROOF. Again using (6.4), we obtain this result from [6], Proposition 5.1. □

COROLLARY 6.3. *Let $d = 2$, $\beta \in \mathbb{R}$. Then* (1.14) *holds, and also*

$$\mathbb{P}\bigl[nf_0\pi L_{n,1}^2 - \log n \le \beta\bigr] = \exp\biggl(-\frac{\sigma_A\pi^{1/2}e^{-\beta/2}}{2(\log n)^{1/2}}\biggr)\exp\bigl(-e^{-\beta}\bigr) + O\bigl((\log n)^{-1}\bigr). \tag{6.6}$$

Moreover (1.15) *holds, and*

$$\begin{aligned}&\mathbb{P}\bigl[nf_0\pi L_{n,2}^2 - \log n - \log\log n \le \beta\bigr]\\ &\quad = \exp\biggl(-\frac{\sigma_A\pi^{1/2}e^{-\beta/2}\log\log n}{8\log n} - \frac{e^{-\beta}\log\log n}{\log n}\biggr)\\ &\qquad \times \exp\biggl(-e^{-\beta} - \frac{\pi^{1/2}\sigma_A e^{-\beta/2}}{4}\biggr) + O\biggl(\frac{1}{\log n}\biggr).\end{aligned} \tag{6.7}$$

PROOF. Let $r_n = r_n(\beta)$ be given by (6.2) with $d = 2$, $k = 1$; then $nf_0\pi r_n^2 - \log n = \beta$ for all large enough n.

Let $\xi_{n,r}$ be the number of isolated vertices of $G(\mathcal{P}_n, r)$ as defined at (3.1), taking $k = 1$. By Proposition 6.1, (3.5) holds on taking $\beta' = e^{-\beta}$. Hence by Proposition 3.1,

$$\mathbb{P}\bigl[nf_0\pi L_{n,1}^2 - \log n \le \beta\bigr] = \mathbb{P}[L_{n,1} \le r_n] = \exp\bigl(-\mathbb{E}[\xi_{n,r_n}]\bigr) + O\bigl(1/(\log n)\bigr).$$

Then using Proposition 6.1, and the fact that $|e^{-\lambda} - e^{-\lambda'}| \le |\lambda - \lambda'|$ for any $\lambda, \lambda' > 0$, we obtain (6.6). We can then deduce (1.14) using Proposition 4.6.

Next, let $r_n = r_n(\beta)$ be given by (6.2) again, but now with $d = 2$, $k = 2$. Then $nf_0\pi r_n^2 - \log n - \log\log n = \beta$ for n large. Repeating the previous argument gives us (6.7) and then (1.15). □

COROLLARY 6.4. *Suppose either $d \ge 3$, or $d = 2$, $k \ge 3$. Let $\beta \in \mathbb{R}$. Then* (1.16) *holds, and*

$$\begin{aligned}&\mathbb{P}\bigl[nf_0\theta_d L_{n,k}^d - (2-2/d)\log n + (4-2k-2/d)\log\log n \le \beta\bigr]\\ &\quad = \exp\biggl(-\frac{c_{d,k}\sigma_A e^{-\beta/2}(k-2+1/d)^2\log\log n}{(1-1/d)\log n}\biggr)\exp\bigl(-c_{d,k}\sigma_A e^{-\beta/2}\bigr) + O\biggl(\frac{1}{\log n}\biggr).\end{aligned}$$

PROOF. The proof is the same as for Corollary 6.3, using Proposition 6.2 in place of Proposition 6.1 when $d \geq 3$. □

We are now ready to finish the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We already showed (1.14), (1.15), (1.16) and the corresponding results for $L_{n,k}$, in Corollaries 6.3 and 6.4. We already proved (1.5) under weaker assumptions in Theorem 1.3. Therefore, it remains only to prove (1.13) and the binomial versions of (1.15) and (1.16), along with the corresponding results for $L_k(\mathcal{X}_n)$.

Let $\phi_{n,r}$ be as defined at (5.27). Set $n^- := n - n^{3/4}$. As shown in the proof of Lemma 5.7, given $\beta \in \mathbb{R}$ we have that

$$\phi_{n^-,r_n(\beta)} = \left(1 + O\left(\frac{\log n}{n^{1/4}}\right)\right)\phi_{n,r_n(\beta)}.$$

Then by Proposition 3.1,

$$\begin{aligned}\mathbb{P}[L_{n^-,k} \leq r_n(\beta)] &= \exp(-\phi_{n^-,r_n(\beta)}) + O((\log n)^{-1}) \\ &= \exp(-\phi_{n,r_n(\beta)}) + O((\log n)^{-1}).\end{aligned}$$

By the proof of Lemma 5.7,

$$\begin{aligned}\mathbb{P}[L_k(\mathcal{X}_n) \leq r_n(\beta)] &= \mathbb{P}[L_{n^-,k} \leq r_n(\beta)] + O((\log n)/n^{1/4}). \\ &= \exp(-\phi_{n,r_n(\beta)}) + O((\log n)^{-1}).\end{aligned}$$

Plugging in the expressions for $\phi_{n,r_n(\beta)} = \mathbb{E}[\xi_{n,r_n(\beta)}]$ in Lemmas 6.1 and 6.2 gives us the result (1.13) for $L_k(\mathcal{X}_n)$ and the binomial versions of (1.15) and (1.16) for $L_k(\mathcal{X}_n)$. Finally, applying Proposition 4.6 gives the same results for $M_k(\mathcal{X}_n)$. □

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