

On k -clusters of high-intensity random geometric graphs

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ABSTRACT

Let k, d be positive integers. We determine a sequence of constants that are asymptotic to the probability that the cluster at the origin in a d -dimensional Poisson Boolean model with balls of fixed radius is of order k , as the intensity becomes large. Using this, we determine the asymptotics of the mean of the number of components of order k , denoted $S_{n,k}$ in a random geometric graph on n uniformly distributed vertices in a smoothly bounded compact region of d -dimensional Euclidean space, with distance parameter $r(n)$ chosen so that the expected degree grows slowly as n becomes large (the so-called mildly dense limiting regime). We also show that the variance of $S_{n,k}$ is asymptotic to its mean, and prove Poisson and normal approximation results for $S_{n,k}$ in this limiting regime. We provide analogous results for the corresponding Poisson process (i.e. with a Poisson number of points).

We also give similar results in the so-called mildly sparse limiting regime where $r(n)$ is chosen so the expected degree decays slowly to zero as n becomes large.

1. Overview

1.1. Introduction and background

The *Poisson blob model* (PBM) of overlapping spheres centred on random points in space, also known as the *Gilbert graph*, is perhaps the simplest model of random clustering in a continuous space. Cluster formation is governed entirely by spatial proximity of particles, and their locations are governed by complete spatial randomness with no interactions, i.e. a homogeneous spatial Poisson process. The *random geometric graph* (RGG) is obtained by restricting the PBM to a finite window, or in an alternate version, by specifying the number of particles in the window.

In the present paper we investigate these models at high intensity, i.e. with a high density of particles relative to the range at which connections form. For high densities, one expects most of the particles to lie in a single giant cluster, but the spatial randomness means that from time to time smaller clusters may also be seen. We consider here clusters of fixed order k . We provide precise asymptotics on the frequency of these clusters as the intensity increases, and moreover describe the fluctuations of the number of clusters in a window that grows in size simultaneously with the growth of the intensity; the window needs to be large enough to observe clusters of order k . We also describe the limiting internal spatial distribution of the k -clusters (suitably rescaled) at high intensity, and show it is governed by an interesting energy functional defined on configurations of k points.

Formally, the RGG based on a random sample \mathcal{X} of points in \mathbb{R}^d (with $d \in \mathbb{N}$) is the graph $G(\mathcal{X}, r)$ with vertex set \mathcal{X} and with an edge between each pair of points distant at most r apart, in the Euclidean metric, for a specified distance parameter $r > 0$. Such

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graphs are important in a variety of applications (see [1]), such as wireless communications (see [2]) and topological data analysis (TDA) (see [3,4]).

The PBM is given by a union of balls of equal radius, say $\frac{1}{2}$, centred a homogeneous Poisson process \mathcal{H}_λ of intensity λ in \mathbb{R}^d . The clusters of the PBM correspond to those of the graph $G(\mathcal{H}_\lambda, 1)$, which is often called the *Gilbert graph*. The terminology ‘Poisson blob model’ is used in e.g. [5,6]. The PBM is a special case of the *Poisson Boolean model* with balls of fixed radius. This is a fundamental model of spatial clustering and continuum percolation in stochastic geometry; see [7–12].

In this paper we consider, for fixed $k \in \mathbb{N}$, the *number of components of order k* (i.e., having k vertices) of the graph $G(\mathcal{X}, r)$, here denoted $K_{k,r}(\mathcal{X})$, where \mathcal{X} is either a random sample of n points, denoted \mathcal{X}_n , uniformly distributed over a compact set A in \mathbb{R}^d with a smooth (in fact, $C^{1,1}$) boundary or the corresponding Poisson process, denoted \mathcal{P}_n . In particular, we investigate asymptotic properties of $K_{k,r}(\mathcal{X}_n)$ and $K_{k,r}(\mathcal{P}_n)$ for large n with $r = r(n)$ specified and decaying to zero according to a certain limiting regime (see (1.1) and (1.2) below). In the special case $k = 1$, $K_{1,r}(\mathcal{X})$ is the number of isolated vertices in $G(\mathcal{X}, r)$.

Some limiting regimes have been considered already. In the *thermodynamic* limiting regime with nr^d held constant, $K_{k,r}(\mathcal{X}_n)$ is known to grow proportionately to n , with a strong law of large numbers (LLN) (see [1, Theorem 3.15]) and there are central limit theorems (CLTs) both for $K_{k,r}(\mathcal{X}_n)$ (see [1, Theorem 3.14]) and for $K_{k,r}(\mathcal{P}_n)$ (see [1, Theorem 3.11]). There is also a strong LLN for $K_{k,r}(\mathcal{X}_n)$ in the *sparse* regime with $nr^d \rightarrow 0$, subject to some further conditions on $r(n)$; see [1, Theorem 3.19]. Within the sparse regime, when $n(nr^d)^{(k-1)} \rightarrow \infty$ but $n(nr^d)^k \rightarrow 0$, a CLT for $K_{k,r}(\mathcal{X}_n)$ can be derived from [1, Theorem 3.5] and the fact that the second condition implies that the number of components of order larger than k vanishes, in probability. In fact the results in [1] are stated for the number of components isomorphic to some specified connected graph Γ , but results for $K_{k,r}(\mathcal{X}_n)$ or $K_{k,r}(\mathcal{P}_n)$ can then be obtained by summing over all possible Γ with k vertices; note that [1, Theorem 3.5] gives asymptotic independence, and hence a multivariate CLT, for the Γ -component counts indexed by all different feasible Γ with k vertices (as discussed in [1], only Γ which are feasible contribute to the k -component count). More recently, [13] gives a large deviations principle for the k -component count in the sparse regime when also $n(nr^d)^{k-1} \rightarrow \infty$.

In the *logarithmic regime* we take $n\theta r^d = b \log n$ for some constant b , where θ denotes the volume of the unit radius ball in \mathbb{R}^d . Provided $d \geq 2$ and b exceeds the critical value $b_c = (2 - 2/d)\text{Vol}(A)$ (where Vol denotes Lebesgue measure), it is known that with probability tending to 1 as $n \rightarrow \infty$, $G(\mathcal{X}_n, r)$ is fully connected so that $K_{k,r}(\mathcal{X}_n) = 0$ (see [1, Theorem 13.7]).

The main limiting regime for $r = r(n)$ that we consider here is to make the assumptions

$$\lim_{n \rightarrow \infty} (nr^d) = \infty; \tag{1.1}$$

$$\limsup_{n \rightarrow \infty} ((\theta nr^d) / \log n) < \min((2/d), 1)\text{Vol}(A). \tag{1.2}$$

We call this the *intermediate* or *mildly dense* regime because the average vertex degree is of order $\Theta(nr^d)$ and therefore grows to infinity as n becomes large, but only slowly in this regime. Our intermediate regime includes the logarithmic regime for small values of the constant b ; we discuss the significance of the upper bound in (1.2) later on.

While asymptotics for the number of k -clusters in continuum clustering models of this sort are a classical object of study [1,6,8,12,14,15], there has also been renewed interest more recently, e.g. [13], often motivated by TDA, where topological properties of a simplicial complex based on the sample of points are used to try to understand those of the underlying space. The sum $\sum_k K_{k,r}(\mathcal{X})$ is the *total number of components* in $G(\mathcal{X}, r)$, and this is of interest in TDA as one of the Betti numbers [3,4,16]

In TDA the number of components (and other Betti numbers) are of interest in the whole range of values of r , not just in the sparse or thermodynamic limiting regime. One can also imagine other situations where the dense limit is of interest: for example the structure of impurities in a gas that gets more and more compressed, or that of isolated communities in a society that becomes more and more interconnected. However, most of the existing work on the limit theory is concerned with the sparse or thermodynamic limit; in the TDA context, this is noted in the last paragraph of [4, Section 2.4.1]. It seems well worth building our understanding of the mildly dense limit as well.

In a companion paper [17] we investigate the limiting behaviour of the total number of components in the mildly dense limiting regime. It turns out that this quantity is dominated by the first term in the sum, namely $K_{1,r}(\mathcal{X})$ (which is also the case in the sparse limit, as observed long ago in [15]). The results in [17] rely on those in the present paper, in particular the variance asymptotics in Propositions 2 and 3 and the Poisson approximation results in Theorem 6. While it is the special case $k = 1$ of our results here that are most relevant to [17], one can imagine some algorithm whereby all singletons are removed from the sample, and one is interested in the number of remaining clusters. In this case we would expect the 2-clusters to dominate, and if these too were removed, the 3-clusters would dominate, and so on. So it seems natural to investigate the behaviour of k -clusters for general k .

1.2. Summary of results

Regarding the Poisson blob model, the number of components of order k in $G(\mathcal{H}_\lambda, 1)$ will be infinite. Instead, we consider the probability $p_k(\lambda)$ that an inserted point at the origin o lies in a component of order k of $G(\mathcal{H}_\lambda \cup \{o\}, 1)$. This can be interpreted, loosely speaking, as the proportion of vertices in $G(\mathcal{H}_\lambda, 1)$ that lie in components of order k .

Our first result, Theorem 1, determines the asymptotic behaviour of $p_k(\lambda)$ for large λ and fixed $k \in \mathbb{N}$. It says that $p_k(\lambda) \sim \alpha_k \lambda^{(1-k)(d-1)} e^{-\theta \lambda}$ as $\lambda \rightarrow \infty$, where $\alpha_1 := 1$ and for $k \in \mathbb{N}$, $\alpha_{k+1} := (1/k!) \int_{(\mathbb{R}^d)^k} e^{-g(\mathbf{z})} d\mathbf{z}$, where

$$g(\mathbf{z}) := \int_{i=1}^k \int_{B(\frac{1}{2}z_i, \|z_i\|/2)} \|x\|^{1-d} dx, \quad \mathbf{z} = (z_1, \dots, z_k) \in (\mathbb{R}^d)^k.$$

Here and elsewhere, for $x \in \mathbb{R}^d$, $r > 0$ we write $B(x, r)$ or $B_r(x)$ for $\{y \in \mathbb{R}^d : \|y - x\| \leq r\}$, and $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d . This result is both of interest in its own right, and important for understanding the asymptotics for number of components of order k in RGGs in the mildly dense regime.

In **Theorem 2**, we shall provide a result on the large- λ limiting distribution of the (rescaled) points of the component containing the origin, given that it is of order $k + 1$. Namely, their joint density is given by $e^{-g(x)}$, normalized to a probability density function. We have not seen the ‘energy functional’ $g(\cdot)$ in the literature before.

We now describe our main results on finite RGGs. Let $A \subset \mathbb{R}^d$ be compact with smooth boundary (in the $C^{1,1}$ sense that we define later), and let X_1, X_2, \dots be independent identically distributed random d -vectors, uniformly distributed over A . For $n \in \mathbb{N}$ set $\mathcal{X}_n := \{X_1, \dots, X_n\}$, which is a binomial point process (see e.g. [9]). Also, for $n \in (0, \infty)$ let Z_n be a Poisson random variable with mean n , independent of (X_1, X_2, \dots) , and set $\mathcal{P}_n := \mathcal{X}_{Z_n}$. Then (see [9, Proposition 3.5]) \mathcal{P}_n is a Poisson point process in A with intensity measure $(n/\text{Vol}(A))dx$ (in this case n does not need to be an integer).

Our results are concerned with the variables $S_{n,k} := K_{k,r}(\mathcal{X}_n)$ and $S'_{n,k} := K_{k,r}(\mathcal{P}_n)$, with $r = r(n)$ given, satisfying (1.1) and (1.2) unless stated otherwise. For now we present them as asymptotic results as $n \rightarrow \infty$ with k fixed, but the precise statements of these results later on come with bounds on the rates of convergence.

As first-order results we shall give the asymptotic behaviour of $\mathbb{E}[S_{n,k}]$ and $\mathbb{E}[S'_{n,k}]$. Setting $I_{n,k} := \mathbb{E}[S'_{n,k}]$, we show in **Theorem 3** that (with α_k given above)

$$I_{n,k} \sim k^{-1} \alpha_k n (nr^d / \text{Vol}(A))^{(1-k)(d-1)} \exp(-(\theta / \text{Vol}(A))nr^d) \tag{1.3}$$

as $n \rightarrow \infty$. We show in **Proposition 1** that also $\mathbb{E}[S_{n,k}] \sim I_{n,k}$ as $n \rightarrow \infty$.

Turning to second-order results, we shall show in **Propositions 2** and **3** that

$$\text{Var}[S_{n,k}] \sim \text{Var}[S'_{n,k}] \sim I_{n,k} \quad \text{as } n \rightarrow \infty. \tag{1.4}$$

Note that (1.3), (1.1) and (1.2) together imply that $I_{n,k} \rightarrow \infty$ as $n \rightarrow \infty$. It is immediate from the asymptotics already described, and Chebyshev’s inequality, that we have the weak LLNs $(S_{n,k}/I_{n,k}) \xrightarrow{\mathbb{P}} 1$ and $(S'_{n,k}/I_{n,k}) \xrightarrow{\mathbb{P}} 1$ as $n \rightarrow \infty$, and hence using (1.3) we have $S_{n,k}/(k^{-1} \alpha_k n (nr^d / \text{Vol}(A))^{(1-k)(d-1)} e^{-\theta nr^d / \text{Vol}(A)}) \xrightarrow{\mathbb{P}} 1$ and likewise for $S'_{n,k}$. In **Theorem 5**, under the extra condition $\limsup((\theta nr^d) / \log n) < \text{Vol}(A)/2$, we improve the weak LLNs to strong LLNs, i.e. we prove almost sure convergence

$$S_{n,k}/(k^{-1} \alpha_k n (nr^d / \text{Vol}(A))^{(1-k)(d-1)} e^{-\theta nr^d / \text{Vol}(A)}) \xrightarrow{a.s.} 1 \quad \text{as } n \rightarrow \infty,$$

and likewise for $S'_{n,k}$ as $n \rightarrow \infty$ through \mathbb{N} for any coupling of $(Z_n)_{n \geq 1}$. We do this by giving concentration of measure results for $S_{n,k}$ and for $S'_{n,k}$.

Turning to CLTs, in **Theorem 6** we show that as $n \rightarrow \infty$,

$$I_{n,k}^{-1/2}(S'_{n,k} - I_{n,k}) \xrightarrow{D} N(0, 1); \quad I_{n,k}^{-1/2}(S_{n,k} - \mathbb{E}[S_{n,k}]) \xrightarrow{D} N(0, 1). \tag{1.5}$$

We shall give two approaches to proving (1.5); one via approximating $S_{n,k}$ by a Poisson distribution with mean $I_{n,k}$ (of interest in itself), and the other via more direct normal approximation. Both methods provide bounds on rates of convergence in various metrics on probability distributions.

Finally, in **Section 6** we shall present some new results in the *sparse* limiting regime, where instead of (1.1) we assume $nr^d \rightarrow 0$ and $n(nr^d)^{k-1} \rightarrow \infty$ as $n \rightarrow \infty$. In this case, instead of (1.3) we have $I_{n,k} \sim cn(nr^d)^{k-1}$ for appropriate c . Again in this regime we can show that $\mathbb{E}[S_{n,k}] \sim I_{n,k}$, and that (1.4) and (1.5) still hold, by similar arguments to those we use in the mildly dense regime.

It is natural to ask whether our results on $S_{n,k}$ and $S'_{n,k}$ generalize to non-uniform distributions. To answer this, we shall present these results in greater generality. We shall assume X_1, X_2, \dots are independent random d -vectors having a common probability distribution ν with density f , supported by A (so that $\mathbb{P}[X_i \in dx] = \nu(dx) = f(x)dx$ for $x \in A, i \in \mathbb{N}$). We then define \mathcal{X}_n and \mathcal{P}_n as already described.

For all of our results on the mildly dense limiting regime, we shall assume that f is continuous on A , and that $f_0 > 0$, where we set

$$f_0 := \inf_{x \in A} f(x); \quad f_1 := \inf_{x \in \partial A} f(x); \quad f_{\max} := \sup_{x \in A} f(x),$$

and ∂A denotes the boundary of A . By compactness of A and continuity of f , we have $f_{\max} < \infty$; moreover $f_1 \geq f_0$. In the special case where f is constant on A , we have $f_0 = f_1 = f_{\max} = 1/\text{Vol}(A)$; we call this the *uniform case*.

In the general (non-uniform) case, instead of (1.2) we assume

$$\limsup_{n \rightarrow \infty}((\theta nr^d) / \log n) < \frac{1}{\max(f_0, d(f_0 - f_1/2))}, \tag{1.6}$$

which reduces to (1.2) in the uniform case. This condition may be understood as follows. Suppose $\theta nr^d = b \log n$ for some constant b . A ‘back-of-the-envelope’ calculation suggests that the mean number of components of order k (which we shall call *k-components* or *k-clusters* for short) in the interior of A is roughly of the order $n(nr^d)^{k-1} e^{-f_0 \theta nr^d} \approx n^{1-bf_0}$ (up to logarithmic factors), which tends to infinity if $b < 1/f_0$. Similarly the mean number of k -clusters within distance kr of the boundary of A is of order $nr(nr^d)^{k-1} e^{-f_1 \theta nr^d / 2} \approx n^{1-(1/d)-(bf_1/2)}$. If b is less than the right hand side of (1.6), then $bf_0 < (1/d) + bf_1/2$, so there are more k -clusters in the interior of A than near the boundary, suggesting we can ignore boundary effects when analysing the asymptotic behaviour of $S_{n,k}$ and $S'_{n,k}$. If b

were larger than this, we would have to take boundary effects into account more carefully, for example to determine the analogue to (1.3) in the uniform case, and we leave this as possible future research.

All of the results for $S_{n,k}$ and $S'_{n,k}$ described earlier for the uniform case remain valid in the general (non-uniform) case under assumptions (1.1) and (1.6), except for the asymptotic (1.3). In general, $I_{n,k} = \mathbb{E}[S'_{n,k}]$ is equal to a multiple integral, given at (3.3). In the non-uniform case we are not able to describe the limiting behaviour of $I_{n,k}$ so explicitly, but we do show that $I_{n,k} \rightarrow \infty$ and $(nr^d)^{-1} \log(I_{n,k}/n) \rightarrow -\theta f_0$ as $n \rightarrow \infty$; see Lemma 2 and Theorem 4.

Remark 1.

1. As alluded to earlier, in the thermodynamic or sparse limiting regime, it is of interest to further distinguish between different components of order k , according to which of the possible graphs Γ they are isomorphic to. In the mildly dense limiting regime considered here, this is perhaps of less interest, because of the *compression* phenomenon already noted in [5]; for fixed k and large n , nearly all of the components of order k are likely to be compressed into balls of diameter less than r (in fact, less than δr for any fixed $\delta > 0$), and therefore fully connected. In other words, nearly all of the components of order k are likely to be cliques.
2. It seems plausible that one could derive a result on the limiting within-cluster spatial distribution of points in k -clusters of the RGG in the mildly dense limit, suitably rescaled, using Theorem 2, but we do not do so here.
 In [6] the mean volume of k -clusters of the PBM at finite intensities is investigated, in particular providing an expansion in λ for λ small. We would expect Theorem 2 to be useful for understanding the limiting behaviour of the mean volume of k -clusters for λ large.
3. To prove our Gaussian approximation results, we shall use the local dependence methods of Chen and Shao [18] to deal with $S'_{n,k}$ and those of Chatterjee [19] to deal with $S_{n,k}$. For the Poisson approximation, we shall use the coupling bounds from e.g. Lindvall [20] for $S_{n,k}$ and from Penrose [21] for $S'_{n,k}$. All of these approximation bounds are obtained by a blend of Stein’s method and stochastic analysis of general Poisson or binomial point process functionals. For a systematic account of the subject, we refer the reader to [9,22–25], and to [26] for some recent developments about the methodology.
4. Our results hold for all $d \geq 2$, and also for $d = 1$ with the sole exception of our result on Poisson approximation for $S_{n,k}$ (Proposition 5); we conjecture that even this result could be proved by other means for $d = 1$. Many of our results seem to be new even for the special case with $k = 1$ (i.e., the isolated vertex count) and/or $d = 1$.
5. In TDA one is interested in interactions between topological invariants of $G(\mathcal{X}, r)$ for different values r (and the same \mathcal{X}). With this in mind, it could be of interest to investigate processes of the form $(K_{k,r_n(t)}(\mathcal{X}_n))_{t \geq 0}$ (suitably scaled and centred) for some suitable sequence of increasing functions $(r_n(t), t \geq 0)_{n \in \mathbb{N}}$ such that for each fixed t , $r(n) := r_n(t)$ satisfies (1.1) and (1.2). In view of (1.5), one might hope for suitable $r_n(t)$ to be able to find a Gaussian process limit. The choice of $r_n(t)$ might require a stretching or squeezing as well as scaling of the time-scale to get a limiting Gaussian process with non-trivial covariances. In particular, finding $(r_n(t))_{t \geq 0}$ such that we can apply the same scaling constant to $(K_{k,r_n(t)}(\mathcal{X}_n))$ for different t would seem to require some work. This topic is beyond the scope of the present paper.

1.3. Notation

We use the following notation for asymptotics. Given $g : (0, \infty) \rightarrow \mathbb{R}$, and $h : (0, \infty) \rightarrow (0, \infty)$, we write $g(x) = O(h(x))$ if $\limsup |g(x)|/h(x) < \infty$, and $g(x) = o(h(x))$ if $\limsup |g(x)|/h(x) = 0$, and $g(x) = \Omega(h(x))$ if $\liminf |g(x)|/h(x) > 0$, and $g(x) = \Theta(h(x))$ if both $g(x) = O(h(x))$ and $g(x) = \Omega(h(x))$, and $g(x) \sim h(x)$ if $g(x)/h(x)$ tends to 1. Here, the limit is taken either as $x \rightarrow 0$ or $x \rightarrow \infty$, to be specified in each appearance.

We denote the origin in \mathbb{R}^d by o , and write just B_r for $B_r(o)$. Also for finite $\mathcal{X} \subset \mathbb{R}^d$ and $r > 0$, we define the set $B_r(\mathcal{X}) := \cup_{x \in \mathcal{X}} B_r(x)$, and the function $h_r(\mathcal{X})$ to be the indicator of the event that $G(\mathcal{X}, r)$ is connected. Also for $m \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_m) \in (\mathbb{R}^d)^m$ we define $B_r(\mathbf{x})$ and $h_r(\mathbf{x})$ similarly, identifying \mathbf{x} with $\{x_1, \dots, x_m\}$; that is

$$B_r(\mathbf{x}) := \cup_{i=1}^m B_r(x_i); \quad h_r(\mathbf{x}) = \mathbf{1}\{G(\{x_1, \dots, x_m\}, r) \text{ is connected}\}. \tag{1.7}$$

Also we define $h_r^*(\mathbf{x})$ to be the product of $h_r(\mathbf{x})$ and the indicator of the event that x_1 precedes all of x_2, \dots, x_m in the lexicographic ordering (that is, x_1 is the left-most point of $\{x_1, \dots, x_m\}$).

For $A \subset \mathbb{R}^d$, we set $A^{(r)} := \{x \in A : B_r(x) \subset A\}$, and $A^\circ := \cup_{r>0} A^{(r)}$, the interior of A . If $\mathcal{X} \subset \mathbb{R}^d$, and $a > 0$ let $a\mathcal{X} := \{ax : x \in \mathcal{X}\}$. If also $\mathcal{Y} \subset \mathbb{R}^d$ and \mathcal{X} and \mathcal{Y} are locally finite and non-empty, let $\text{dist}(\mathcal{X}, \mathcal{Y}) := \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} \|y - x\|$.

For $n \in \mathbb{N}$, set $[n] := \{1, \dots, n\}$. If $\ell, m \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_\ell) \in (\mathbb{R}^d)^\ell$, $\mathbf{y} = (y_1, \dots, y_m) \in (\mathbb{R}^d)^m$, then we set $\text{dist}(\mathbf{x}, \mathbf{y}) := \min_{i \in [\ell], j \in [m]} \|x_i - y_j\|$.

Given a random vector \mathbf{X} and an event \mathcal{E} on the same probability space, let $\mathcal{L}(\mathbf{X})$ denote the probability distribution (also called the law) of \mathbf{X} , and $\mathcal{L}(\mathbf{X}|\mathcal{E})$ the conditional probability distribution of \mathbf{X} given that event \mathcal{E} occurs.

Given $r > 0$, given locally finite $\mathcal{X} \subset \mathbb{R}^d$ and given $x \in \mathcal{X}$, let $C_r(x, \mathcal{X})$ denote the vertex set of the component of the graph $G(\mathcal{X}, r)$ containing x , and $|C_r(x, \mathcal{X})|$ the number of elements of this set.

Given $m \in \mathbb{N}$ and $p \in [0, 1]$, we write $\text{Bin}(m, p)$ for a Binomial random variable with parameters m, p ,

2. High-intensity boolean model

Given $\lambda > 0$, let \mathcal{H}_λ be a homogeneous Poisson point process in \mathbb{R}^d of intensity λ (viewed as a random set of points), and let $\mathcal{H}_{\lambda,0} := \mathcal{H}_\lambda \cup \{o\}$. Then for $r > 0$ the graph $G(\mathcal{H}_\lambda, r)$ is the intersection graph of a PBM, i.e. a Boolean model of continuum percolation with balls of fixed radius $r/2$. See [9,10] for background information on models of this type.

Let $C(\lambda) := C_1(o, \mathcal{H}_{\lambda,0})$. For $k \in \mathbb{N}$ let $p_k(\lambda) := \mathbb{P}[|C(\lambda)| = k]$. This section is concerned with the large- λ asymptotics of $p_k(\lambda)$ for fixed k , and the asymptotic conditional distribution of the point process $C(\lambda)$ given $|C(\lambda)| = k$. These are relevant to the RGG model described in the Section 1, as we shall show later on.

In [5, Theorem 2.2] Alexander proved (among other things) that as $\lambda \rightarrow \infty$ we have

$$p_{k+1}(\lambda) = \Theta(\lambda^{-k(d-1)} e^{-\theta\lambda}). \tag{2.1}$$

In other words, $\lambda^{k(d-1)} e^{\theta\lambda} p_{k+1}(\lambda)$ remains bounded away from 0 and ∞ as $\lambda \rightarrow \infty$. This suggests that this quantity might tend to a positive finite limit as $\lambda \rightarrow \infty$, but this is not proved in [5]. Our next result shows that this is indeed the case, and provides a rate of convergence. To describe the limit, we define for each $\mathbf{z} = (z_1, \dots, z_k) \in (\mathbb{R}^d)^k$ the set

$$D(\mathbf{z}) := \cup_{i=1}^k B((1/2)z_i, \|z_i\|/2), \tag{2.2}$$

that is, the union of balls B_1^*, \dots, B_k^* where for each i the ball B_i^* has opposite poles at o and z_i (Fig. 1 shows an example with $d = 2$). Define the function $g : (\mathbb{R}^d)^k \rightarrow [0, \infty)$ by

$$g(\mathbf{z}) = \int_{D(\mathbf{z})} \|x\|^{1-d} dx, \quad \mathbf{z} \in (\mathbb{R}^d)^k. \tag{2.3}$$

When $d = 2$, this can be interpreted as the *gravitational energy* of a flat object with the shape of $D(\mathbf{z})$ and mass equal to the area of $D(\mathbf{z})$, and with its mass evenly spread, with respect to a large point mass at the origin; that is, the energy required to remove the object from the gravitational field of the point mass. We are not aware of any physical interpretation of $g(\cdot)$ in other dimensions, but nevertheless refer to $g(\mathbf{z})$ as the *quasi-gravitational energy* of the set $D(\mathbf{z})$. Set $\alpha_1 := \alpha_1(d) := 1$, and for $k \in \mathbb{N}$ define the constant

$$\alpha_{k+1} := \alpha_{k+1}(d) := (1/k!) \int_{(\mathbb{R}^d)^k} \exp(-g(\mathbf{z})) d\mathbf{z}. \tag{2.4}$$

Theorem 1. *Let $k \in \mathbb{N} \cup \{0\}$. Then as $\lambda \rightarrow \infty$,*

$$\lambda^{k(d-1)} e^{\theta\lambda} p_{k+1}(\lambda) = \alpha_{k+1} + O(\lambda^{-1}). \tag{2.5}$$

The existence of the limit in (2.5) is crucial for our other results later on, in which we shall establish the existence of limiting constants in the limit theory for the mean and variance of the number of k -components of the finite RGG in the mildly dense limit.

Let $k \in \mathbb{N}$. Another part of [5, Theorem 2.2] says, in effect, that conditionally on $|C(\lambda)| = k + 1$, the random variable $\lambda \text{Diam}(C(\lambda))$ is bounded away from zero and infinity in probability (here Diam is Euclidean diameter). In other words, $\text{Diam}(C(\lambda)) = \Theta(\lambda^{-1})$ in probability as $\lambda \rightarrow \infty$, given $|C(\lambda)| = k + 1$; that is, given $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $\mathbb{P}[\text{Diam}(C(\lambda)) \in (M^{-1}\lambda^{-1}, M\lambda^{-1}) \mid |C(\lambda)| = k + 1] > 1 - \epsilon$ for all large enough λ . This phenomenon is known as *compression* [5,10]: the density of points in the cluster $C(\lambda)$ is of the order of λ^d , compared to an ambient density of λ .

We give a stronger result here, namely convergence in distribution of the point process $\lambda C(\lambda)$. To describe the limit, let (U_1, \dots, U_k) be a random vector in $(\mathbb{R}^d)^k$ with joint density with respect to (dk) -dimensional Lebesgue measure, given by $(k! \alpha_{k+1})^{-1} \exp(-g(\cdot))$, i.e. with

$$\mathbb{P}[(U_1, \dots, U_k) \in d\mathbf{z}] = (k! \alpha_{k+1})^{-1} \exp(-g(\mathbf{z})) d\mathbf{z}, \quad \mathbf{z} \in (\mathbb{R}^d)^k, \tag{2.6}$$

with the energy function g given at (2.3), and α_{k+1} given at (2.4). Informally, our result says that the point process $\lambda C(\lambda) \setminus \{o\}$ converges in distribution to the point process $\{U_1, \dots, U_k\}$.

To make this precise while avoiding technicalities on point process convergence, we list the points of our point processes in increasing lexicographic order to get random vectors in $(\mathbb{R}^d)^k$ (alternatively we could use distance from the origin for the ordering). Let $Y_{1,\lambda}, \dots, Y_{k,\lambda}$ be the points of $C(\lambda) \setminus \{o\}$ taken in increasing lexicographic order, and let V_1, \dots, V_k be the points of $\{U_1, \dots, U_k\}$ taken in increasing lexicographic order.

Theorem 2. *The conditional distribution of the random vector $(\lambda Y_{1,\lambda}, \dots, \lambda Y_{k,\lambda})$, given $|C(\lambda)| = k + 1$, converges weakly, as $\lambda \rightarrow \infty$, to the distribution of (V_1, \dots, V_k) .*

Remark 2.

1. The result (2.1) (but not (2.5)) also appears in [10, Section 5.3]. The results in [5,10] are also presented for components of order $k + 1$ in $G(\mathcal{H}_{\lambda,0}, s)$ for general fixed $s > 0$ (rather than just for $s = 1$). It is easy to generalize (2.5) to this setting, since by Poisson scaling (see e.g. [10], or the Mapping Theorem in [9])

$$\mathbb{P}[|C_s(o, \mathcal{H}_{\lambda,0})| = k] = \mathbb{P}[|C_1(o, s^{-1}\mathcal{H}_{\lambda,0})| = k] = p_k(\lambda s^d).$$

2. When $d = 1$ we can compute α_{k+1} exactly. In this case we have $g(\mathbf{z}) = \max_{i \in [k]} z_i^+ + \max_{i \in [k]} z_i^-$, and considering separately the case where z_1, \dots, z_k all have the same sign and the case where they do not, we obtain that

$$\begin{aligned} k! \alpha_{k+1} &= 2k \int_0^\infty z^{k-1} e^{-z} dz + k(k-1) \int_0^\infty du \int_0^\infty dv (u+v)^{k-2} e^{-(u+v)} \\ &= 2k! + k(k-1) \int_0^\infty w^{k-2} e^{-w} w dw \\ &= 2k! + k!(k-1), \end{aligned}$$

so $\alpha_{k+1}(1) = k + 1$.

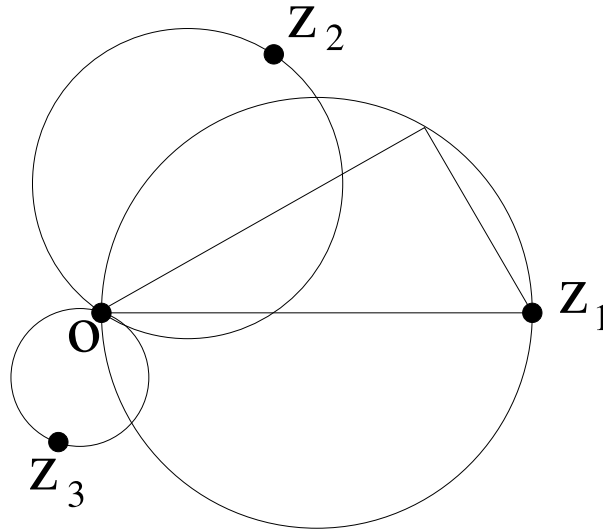


Fig. 1. The union of the disks is the set $D(z_1, z_2, z_3)$.

3. We can also compute $\alpha_2(d)$ more explicitly for general d . Indeed, when $k = 1$ it can be seen directly from (2.8) below that $g(z) = \lim_{r \downarrow 0} g_r(z) = \theta_{d-1} \|z\|$ for all $z \in \mathbb{R}^d$, where θ_{d-1} denotes the volume of a unit radius ball in $d - 1$ dimensions. Then by (2.4) we have

$$\begin{aligned} \alpha_2(d) &= \int_{\mathbb{R}^d} \exp(-\theta_{d-1} \|z\|) dz = \theta_{d-1}^{-d} \int_{\mathbb{R}^d} e^{-\|u\|} du = \theta_{d-1}^{-d} d \theta \int_0^\infty e^{-r} r^{d-1} dr \\ &= d! \theta / \theta_{d-1}^d. \end{aligned}$$

4. It may be possible to improve the right hand side of (2.5) to an infinite power series in λ^{-1} and to compute the first few coefficients. This series expansion would provide a high-intensity complement to the low-intensity expansions in λ which have been considered previously in the literature dating back to [14]; see [6], [12, p.242] and references therein.

For the proof of Theorem 1, we introduce the following notation. Given $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{R}^d)^k$, recalling $B_r(\mathbf{x}) := \cup_{i=1}^k B_r(x_i)$, we define

$$V(\mathbf{x}) := \text{Vol}(B_1(\mathbf{x})); \quad V'(\mathbf{x}) := \text{Vol}(B_1(\mathbf{x}) \setminus B_1(o)); \quad \|\mathbf{x}\| := \max_{1 \leq i \leq k} \|x_i\|. \tag{2.7}$$

The proof of Theorem 1 will use the following geometrical lemma, which begins to show the relevance of $g(\mathbf{z})$, defined above at (2.3) as the quasi-gravitational energy of $D(\mathbf{z})$.

Lemma 1. For $\mathbf{z} \in (\mathbb{R}^d)^k$, and $r > 0$, set $g_r(\mathbf{z}) := r^{-1} V'(r\mathbf{z})$. There exists a constant $c = c(d) \in (0, \infty)$ such that for all $r \in (0, 1]$ and $\mathbf{z} \in (\mathbb{R}^d)^k$ with $r\|\mathbf{z}\| \leq 1$,

$$|g_r(\mathbf{z}) - g(\mathbf{z})| \leq cr\|\mathbf{z}\|^2. \tag{2.8}$$

Proof. When $d = 1$, it is easy to see directly that $g_r(\mathbf{z}) = \max_i z_i^+ + \max_i z_i^- = g(\mathbf{z})$ whenever $r\|\mathbf{z}\| \leq 1$, and hence that (2.8) holds in this case. Therefore we can and do assume from now on that $d \geq 2$.

Let $r \in (0, 1]$ and $\mathbf{z} = (z_1, \dots, z_k) \in (\mathbb{R}^d)^k$ with $r\|\mathbf{z}\| \leq 1$. Using polar coordinates, write

$$g_r(\mathbf{z}) = r^{-1} \int_{\partial B_1} \int_0^\infty \mathbf{1}\{(1+s)x \in \cup_{i=1}^k B_1(rz_i)\} (1+s)^{d-1} ds \sigma(dx),$$

where the outer integral is a surface integral with the surface measure denoted σ .

Given $x \in \partial B_1$, $y \in B_1$, let $s(x, y) := (\max\{s : (1+s)x \in B_1(y)\})^+$. Then

$$\begin{aligned} g_r(\mathbf{z}) &= r^{-1} \int_{\partial B_1} \int_0^{\max_i s(x, rz_i)} (1+s)^{d-1} ds \sigma(dx) \\ &= \int_{\partial B_1} (rd)^{-1} ((1 + \max_i s(x, rz_i))^d - 1) \sigma(dx). \end{aligned} \tag{2.9}$$

By the triangle inequality, $s(x, rz_i) \leq r\|z_i\|$, so the integrand in (2.9) is bounded by a constant independent of x . We shall prove that the limit $\lim_{r \downarrow 0} r^{-1} s(x, rz_i)$ exists, for each $i \in [k]$ and $x \in \partial S$, with a rate of convergence.

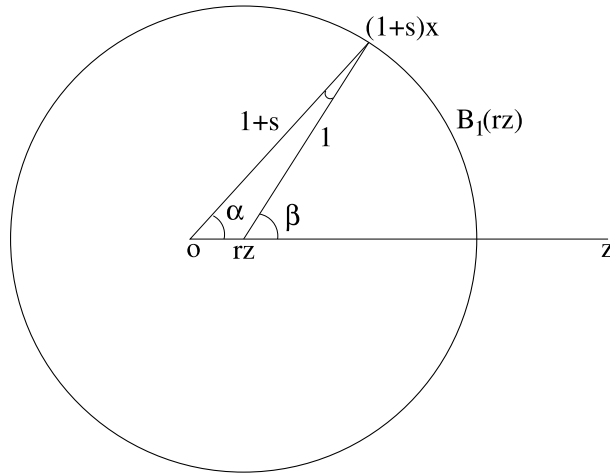


Fig. 2. Illustration of the proof of Lemma 1.

The reader is invited to refer to Fig. 2 now. Let $i \in [k]$ and set $z = z_i$. If $z = o$ then $s(x, rz) = 0$. Given $r \in (0, 1]$, $z \in B_{1/r} \setminus \{o\}$ and $x \in \partial B_1$, let $\alpha = \alpha(x, z)$ be the angle between x and z , i.e the angle of (x, o, z) , and assume $\alpha < \pi/2$ (otherwise, it is not hard to see that $s(x, rz) = 0$). Set $s = s(x, rz)$. Let $\beta = \beta(r, x, z)$ be the angle of $((1 + s)x, rz, z)$, i.e. the angle between $(1 + s)x - rz$ and z . Then $\|(1 + s)x - rz\| = 1$, and by the cosine rule,

$$(1 + s)^2 = \|rz + ((1 + s)x - rz)\| = (r\|z\|)^2 + 1 + 2r\|z\| \cos \beta.$$

Also $\beta > \alpha$, and $\beta - \alpha$ is the most acute angle shown in Fig. 2; in other words, $\beta - \alpha = \arcsin(r\|z\| \sin \alpha)$.

If $t = \sin \omega$ with $\omega \in [0, \pi/2]$ then $t \geq (2/\pi)\omega$, so that $\arcsin(t) = \omega \leq (\pi/2)t$. Hence for all $r \in (0, 1]$, $z \in B_{1/r} \setminus \{o\}$ and all $x \in \partial B_1$ such that $\alpha(x, z) \in [0, \pi/2]$, we have $0 < \beta(r, x, z) - \alpha(x, z) \leq (\pi/2)r\|z\|$.

There is a constant c_1 such that for all $t \geq -1$ we have $|(1 + t)^{1/2} - 1 - t/2| \leq c_1 t^2$. Hence there is another constant c_2 such that for all $r \in (0, 1]$, $z \in B_{1/r} \setminus \{o\}$ and $x \in \partial B_1$ with $\alpha(x, z) \in (0, \pi/2)$ we have

$$\begin{aligned} |s(x, rz) - r\|z\| \cos \alpha(x, z)| &= |(1 + 2r\|z\| \cos \beta(r, x, z) + r^2\|z\|^2)^{1/2} - 1 - r\|z\| \cos \alpha(x, z)| \\ &\leq |r\|z\|(\cos \beta(r, x, z) - \cos \alpha(x, z))| + \frac{r^2}{2}\|z\|^2 + c_1(r\|z\| \cos \beta(r, x, z) + r^2\|z\|^2)^2 \\ &\leq c_2 r^2 \|z\|^2. \end{aligned}$$

There is a further constant c_3 such that for $t, u \in [0, 1]$ we have $|(1 + u)^d - 1 - dt| \leq c_3(|u|^2 + |u - t|)$. Therefore

$$(1 + \max_i s(x, rz_i))^d - 1 - rd \max_i \{ \|z_i\| (\cos \alpha(x, z_i))^+ \} \leq c_3 (\max_i s(x, rz_i)^2 + c_2 r^2 \|z\|^2).$$

Hence by (2.9), and the inequality $s(x, rz_i) \leq r\|z_i\|$, there is a constant c_4 such that

$$\left| g_r(z) - \int_{\partial B_1} \max_{i \in [k]} \{ \|z_i\| (\cos \alpha(x, z_i))^+ \} \sigma(dx) \right| \leq c_4 r \|z\|^2. \tag{2.10}$$

Using Thales' theorem, one can show that $\max_i \|z_i\| (\cos \alpha(x, z_i))^+$ is the distance from the origin to the furthest point from the origin in the direction of x lying in the set $D(z)$, defined at (2.2) (this is illustrated in Fig. 1; the triangle shown there is right angled.). Therefore using polar coordinates again, we can verify that $g(z)$, defined at (2.3), equals the integral in (2.10). \square

Proof. (Proof of Theorem 1) Since $\mathbb{P}[|C(\lambda)| = 1] = e^{-\lambda\theta}$, and we set $\alpha_1 = 1$, (2.5) is immediate for $k = 0$. Therefore we can and do assume from now on that $k \geq 1$.

Recall the definitions of $h_r(\cdot)$ and $B_r(\cdot)$ at (1.7), and $V(\cdot)$ at (2.7). It is known (see e.g. [27]) that

$$p_{k+1}(\lambda) = \frac{\lambda^k}{k!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h_1((o, x_1, \dots, x_k)) \exp(-\lambda V((o, x_1, \dots, x_k))) dx_k \cdots dx_1. \tag{2.11}$$

This formula is a consequence of the multivariate Mecke formula (see e.g. [9]); a similar formula to (2.11) is derived later on at (3.3).

Since $V((o, x_1, \dots, x_k)) = V'((x_1, \dots, x_k)) + \theta$ by (2.7), we obtain from (2.11) that

$$\begin{aligned} p_{k+1}(\lambda) &= \frac{\lambda^k e^{-\lambda\theta}}{k!} \int_{(\mathbb{R}^d)^k} h_1(o, \mathbf{x}) \exp(-\lambda V'(\mathbf{x})) d\mathbf{x} \\ &= \frac{\lambda^{k-kd} e^{-\lambda\theta}}{k!} \int_{(\mathbb{R}^d)^k} h_1((o, \lambda^{-1}\mathbf{z})) \exp(-\lambda V'(\lambda^{-1}\mathbf{z})) d\mathbf{z}. \end{aligned} \tag{2.12}$$

If $\|z\| \leq \lambda/2$ then $h_1((o, \lambda^{-1}z)) = h_\lambda((o, z)) = 1$, and if $\|z\| > k\lambda$ then $h_\lambda((o, z)) = 0$. Thus provided $\lambda \geq 4$ so that $\lambda^{1/2} \leq \lambda/2$, recalling the definition $g_r(z) := r^{-1}V'(rz)$, by (2.12) and (2.4) we have

$$k!|\lambda^{k(d-1)}e^{\lambda\theta}p_{k+1}(\lambda) - \alpha_{k+1}| \leq \int_{(B_{\sqrt{\lambda}})^k} |e^{-g_{1/\lambda}(z)} - e^{-g(z)}| dz + \int_{(B_{\lambda k})^k \setminus (B_{\sqrt{\lambda}})^k} \exp(-g_{1/\lambda}(z)) dz + \int_{(\mathbb{R}^d)^k \setminus (B_{\sqrt{\lambda}})^k} e^{-g(z)} dz. \tag{2.13}$$

There is a constant $c = c(d) > 0$ such that $g_r(z) \geq c\|z\|$, for all $r > 0$ and $z \in (\mathbb{R}^d)^k$ with $\|rz\| \leq k$. Hence $g_{1/\lambda}(z) \geq c\|z\|$ whenever $\|z\| \leq k\lambda$, and when $\|z\| > k\lambda$ we have $h_1((o, \lambda^{-1}z)) = 0$. Moreover by (2.8) we also have $g(z) = \lim_{r \downarrow 0} g_r(z) \geq c\|z\|$ for all $z \in (\mathbb{R}^d)^k$. Therefore the second integral in (2.13) is bounded by $\int_{(B_{\lambda k})^k \setminus (B_{\sqrt{\lambda}})^k} \exp(-c\|z\|) dz$, and hence by $\theta^k(\lambda k)^{dk} \exp(-c\lambda^{1/2})$.

The third integral in (2.13) is bounded by $\int_{(\mathbb{R}^d)^k \setminus (B_{\lambda^{1/2}(o)})^k} e^{-c\|z\|} dz$, which is $O(\exp(-c'\lambda^{1/2}))$ as $\lambda \rightarrow \infty$, for some $c' > 0$.

The integrand in the first integral in (2.13) can be written as $e^{-g(z)}|e^{g(z)-g_{1/\lambda}(z)} - 1|$. By Lemma 1, there are constants c'', c''' such that for all large λ and all $z \in (B_{\sqrt{\lambda}})^k$ we have $|g(z) - g_{1/\lambda}(z)| \leq c''\lambda^{-1}\|z\|^2$ and thus $|e^{g(z)-g_{1/\lambda}(z)} - 1| \leq c'''\lambda^{-1}\|z\|^2$, so that the first integral in (2.13) is bounded by

$$c'''\lambda^{-1} \int_{(\mathbb{R}^d)^k} e^{-g(z)} \|z\|^2 dz,$$

and since $g(z) \geq c\|z\|$ for all z , the last integral is finite and the result is proved. \square

Proof. (Proof of Theorem 2) Let $f : (\mathbb{R}^d)^k \rightarrow \mathbb{R}$ be bounded and continuous. Write $Y_\lambda := (Y_{1,\lambda}, \dots, Y_{k,\lambda})$. We need to show that $\mathbb{E}[f(\lambda Y_\lambda) \mathbf{1}\{|C(\lambda)| = k + 1\}] \rightarrow \mathbb{E}[f(V_1, \dots, V_k)]$. For $(x_1, \dots, x_k) \in (\mathbb{R}^d)^k$ let $f^*((x_1, \dots, x_k)) := f((x_{(1)}, \dots, x_{(k)}))$, where $x_{(1)}, \dots, x_{(k)}$ are the elements of $\{x_1, \dots, x_k\}$ taken in increasing lexicographic order. Then f^* is a symmetric function of (x_1, \dots, x_k) . By the multivariate Mecke formula

$$\begin{aligned} \mathbb{E}[f(\lambda Y_\lambda) \mathbf{1}\{|C(\lambda)| = k + 1\}] &= \frac{\lambda^k}{k!} \int_{(\mathbb{R}^d)^k} f^*(\lambda \mathbf{x}) h_1((o, \mathbf{x})) \exp(-\lambda V((o, \mathbf{x}))) d\mathbf{x} \\ &= \frac{e^{-\lambda\theta} \lambda^{k(1-d)}}{k!} \int_{(\mathbb{R}^d)^k} f^*(z) h_1((o, \lambda^{-1}z)) \exp(-g_{1/\lambda}(z)) dz. \end{aligned}$$

We divide by $\mathbb{P}\{|C(\lambda)| = k + 1\}$ and take the $\lambda \rightarrow \infty$ limit. By Lemma 1 the integrand tends to $f^*(z)e^{-g(z)}$, and it is dominated by an integrable function of z , as in the proof of Theorem 1. Hence, using (2.5) and the dominated convergence theorem we obtain that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \mathbb{E}[f(\lambda Y_\lambda) \mathbf{1}\{|C(\lambda)| = k + 1\}] &= (k! \alpha_{k+1})^{-1} \int_{(\mathbb{R}^d)^k} f^*(z) \exp(-g(z)) dz \\ &= \mathbb{E}[f^*((U_1, \dots, U_k))], \end{aligned}$$

the last line coming from (2.6) and the law of the unconscious statistician. Since $f((V_1, \dots, V_k)) = f^*((U_1, \dots, U_k))$, the result is proved. \square

3. First order asymptotics

Now we return to the model of finite random geometric graphs, described in Section 1. To recall, we let $A \subset \mathbb{R}^d$ be compact with $\text{Vol}(A) > 0$. If $d \geq 2$ then assume, for the rest of this paper, that A has a $C^{1,1}$ boundary in the sense that for each $x \in \partial A$ there exists a neighbourhood U of x and a real-valued function ϕ that is defined on an open set in \mathbb{R}^{d-1} and differentiable with Lipschitz continuous partial derivatives, such that $\partial A \cap U$, after a rotation, is the graph of the function ϕ . If $d = 1$ we assume A is an interval. This is our smoothness assumption.

Recall that (X_1, X_2, \dots) is a sequence of independent random d -vectors, each of which has distribution, denoted ν , with density function f with support A and with $0 < f_0 \leq f_{\max} < \infty$. Recall the point processes $\mathcal{X}_n := \{X_1, \dots, X_n\}$ and $\mathcal{P}_n := \{X_1, \dots, X_{Z_n}\}$, where Z_n is Poisson with parameter n (in the second case n need not be an integer). Then \mathcal{X}_n is a binomial point process, and \mathcal{P}_n is a Poisson point process in A with $\mathbb{E}[\mathcal{P}_n(dx)] = nf(x)dx$. In the uniform case we take $f = f_0 \mathbf{1}_A$, with $f_0 = 1/\text{Vol}(A)$. For $n > 0$ let $r = r_n = r(n) > 0$ be given, satisfying (1.1) and (1.6), which we re-state here:

$$\lim_{n \rightarrow \infty} nr_n^d = \infty; \quad \limsup_{n \rightarrow \infty} n\theta r_n^d / (\log n) < \frac{1}{\max(f_0, d(f_0 - f_1/2))}. \tag{3.1}$$

We shall continue to assume (3.1) throughout the rest of this paper, except for Section 6.

Let $k \in \mathbb{N}$, and recall that $S'_{n,k}$ denotes the number of components of $G(\mathcal{P}_n, r)$ of order k (again, n need not be an integer here).

We give a formula for $I_{n,k} := \mathbb{E}[S'_{n,k}]$. With $h_r(\cdot)$ and $B_r(\cdot)$ defined at (1.7),

$$S'_{n,k} = \sum_{\varphi \in \mathcal{P}_n, |\varphi|=k} h_r(\varphi) \mathbf{1}\{(\mathcal{P}_n \setminus \varphi) \cap B_r(\varphi) = \emptyset\}. \tag{3.2}$$

Hence by the multivariate Mecke formula,

$$I_{n,k} = \mathbb{E}[S'_{n,k}] = \frac{n^k}{k!} \int_{A^k} h_r(\mathbf{x}) \exp(-n\nu(B_r(\mathbf{x}))) \nu^k(d\mathbf{x}). \tag{3.3}$$

It will be useful to have a lower bound on this expectation.

Lemma 2. *Let f_0^+ be any constant with $f_0^+ > f_0$. Then*

$$n \exp(-\theta f_0^+ nr^d) = o(I_{n,k}) \quad \text{as } n \rightarrow \infty. \tag{3.4}$$

In particular, $I_{n,k} \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. Let $f_0 < \alpha < \alpha' < f_0^+$. By the definition of f_0 and the assumption that f is continuous on A , we can (and do) choose $x_0 \in A^o$ with $f(x_0) < \alpha$ and f continuous at x_0 . Choose $r_0 > 0$ such that $B(x_0, 2r_0) \subset A$ and $f(y) \leq \alpha$ for all $y \in B(x_0, 2r_0)$. Let $\varepsilon \in (0, 1/2)$ with $\alpha(1 + \varepsilon)^d < \alpha'$. If $2r \leq r_0$, and $\mathbf{x} = (x_1, \dots, x_k) \in A^k$ with $x_1 \in B(x_0, r_0)$, and $\{x_1, \dots, x_k\} \subset B(x_1, \varepsilon r)$ then $B_r(\mathbf{x}) \subset B_{r(1+\varepsilon)}(x_1)$ and $v(B_r(\mathbf{x})) \leq \alpha\theta(1 + \varepsilon)^d r^d < \alpha'\theta r^d$; also $h_r(\mathbf{x}) = 1$. Hence by (3.3), for all large enough n ,

$$\begin{aligned} I_{n,k} &\geq \frac{n^k}{k!} \int_{B(x_0, r_0)} \int_{B(x_1, \varepsilon r)^{k-1}} \exp(-n\alpha'\theta r^d) v^{k-1}(d(x_2, \dots, x_k)) v(dx_1) \\ &\geq n(f_0^k \theta^k r_0^d / k!) (n\varepsilon r)^d)^{k-1} \exp(-n\alpha'\theta r^d), \end{aligned}$$

so that $(ne^{-n\theta f_0^+ r^d})/I_{n,k}$ tends to zero by the first part of (3.1), and hence (3.4).

Since the assumption (3.1) implies $\limsup_{n \rightarrow \infty} ((\theta f_0 nr^d)/\log n) < 1$, (3.4) implies $I_{n,k} \rightarrow \infty$ as $n \rightarrow \infty$. \square

We next determine the asymptotics for $\mathbb{E}[S_{n,k}]$.

Proposition 1 (Mean k -cluster count in binomial process). *There exists $c > 0$ such that*

$$\mathbb{E}[S_{n,k}] = I_{n,k}(1 + O(e^{-c nr^d})) \quad \text{as } n \rightarrow \infty.$$

Proof. Observe that

$$\mathbb{E}[S_{n,k}] = \binom{n}{k} \int_{A^k} h_r(\mathbf{x})(1 - v(B_r(\mathbf{x})))^{n-k} v^k(d\mathbf{x}). \tag{3.5}$$

We compare this with (3.3). Let $\mathbf{x} \in A^k$ and set $p = v(B_r(\mathbf{x}))$. Then

$$|e^{-np} - (1 - p)^{n-k}| \leq e^{-np}|1 - (e^p(1 - p))^n| + (1 - p)^{n-k}|(1 - p)^k - 1|.$$

Now $e^p(1 - p) \leq e^p e^{-p} = 1$, and moreover $e^p(1 - p) \geq (1 + p)(1 - p) = 1 - p^2$. Therefore

$$0 \leq e^{-np}(1 - (e^p(1 - p))^n) \leq e^{-np}(1 - (1 - p^2)^n) \leq np^2 e^{-np}.$$

Moreover $1 - (1 - p)^k = O(r^d)$. Thus $|e^{-np} - (1 - p)^{n-k}| = O(nr^{2d})$, uniformly over $\mathbf{x} \in A^k$. Also $\int_{A^k} h_r(\mathbf{x}) v^k(d\mathbf{x}) = O(r^{d(k-1)})$. Therefore

$$\binom{n}{k} \int_{A^k} h_r(\mathbf{x}) |e^{-nv(B_r(\mathbf{x}))} - (1 - v(B_r(\mathbf{x})))^{n-k}| v^k(d\mathbf{x}) = O(n^k r^{d(k-1)} (nr^{2d})),$$

which is $O((\log n)^{k+1})$ by (3.1). By Lemma 2 and (3.1) there exists $\delta > 0$ such that $I_{n,k} = \Omega(n^{2\delta})$. Hence by (3.1) again there exists $c > 0$ such that $(\log n)^{k+1}/I_{n,k} = O(n^{-\delta}) = O(e^{-c nr^d})$. Thus the last display is $O(e^{-c nr^d} I_{n,k})$ for some $c > 0$. Moreover

$$\left| \frac{n^k}{k!} - \binom{n}{k} \right| \int_{A^k} h_r(\mathbf{x}) e^{-nv(B_r(\mathbf{x}))} v^k(d\mathbf{x}) = O(n^{-1} I_{n,k}),$$

which is $O(e^{-c nr^d} I_{n,k})$ for some other $c > 0$. Combining these estimates, and using (3.5) and (3.3), we obtain that $|\mathbb{E}[S_{n,k}] - I_{n,k}| = O(e^{-c nr^d} I_{n,k})$ for some $c > 0$, as required. \square

We now verify the asymptotic expression (1.3) for $I_{n,k}$ in the uniform case, with a bound on the rate of convergence.

Theorem 3 (Asymptotic for $I_{n,k}$ in the uniform case). *Let $k \in \mathbb{N}$. In the uniform case,*

$$n^{-1} (f_0 nr^d)^{(k-1)(d-1)} e^{f_0 \theta nr^d} I_{n,k} = k^{-1} \alpha_k + O((nr^d)^{-1}) \quad \text{as } n \rightarrow \infty. \tag{3.6}$$

Before proving this, we give two geometrical lemmas that we shall use repeatedly later on to deal with boundary effects.

Definition 1 (Sphere condition). For $z \in \partial A$ let \hat{n}_z be the unit normal to ∂A at z pointing inside A .

Given $\tau \geq 0$, let us say τ satisfies the *sphere condition* for A if, for all $x \in \partial A$, we have $B(x + \tau \hat{n}_x, \tau) \subset A$ and $B(x - \tau \hat{n}_x, \tau) \cap A = \{x\}$.

Let $\tau(A)$ denote the supremum of the set of all τ satisfying the sphere condition for A .

Lemma 3 (Sphere condition lemma). $\tau(A) > 0$; that is, there exists a constant $\tau > 0$ such that τ satisfies the sphere condition for A .

Proof. See [28, Lemma 7]. \square

Remark 3. (i) If $0 < \tau < \tau'$ and τ' satisfies the sphere condition for A , then so does τ .

(ii) If $x \in \mathbb{R}^d$ with $\text{dist}(x, \partial A) < \tau(A)$, then x has a unique closest point in ∂A .

Lemma 4. Writing $\kappa(\partial A, r)$ for the number of balls of radius r required to cover ∂A , we have

$$\limsup_{s \downarrow 0} s^{d-1} \kappa(\partial A, s) < \infty. \tag{3.7}$$

Moreover

$$\liminf_{s \downarrow 0} (s^{-d} \inf_{x \in A} \text{Vol}(A \cap B_s(x))) \geq \theta/2. \tag{3.8}$$

Finally, as $s \downarrow 0$ we have $\limsup(s^{-1} \text{Vol}(A \setminus A^{(s)})) < \infty$.

Proof. If $d = 1$ we are assuming A is a compact interval, and all the assertions of the lemma are clear.

Suppose $d \geq 2$, and let $x \in \partial A$. By our smoothness assumption, there is a rotation about x and a neighbourhood V of x such that after rotation the set $(\partial A) \cap V$ equals $\{(u, \phi(u)) : u \in U\}$ where U is an open set in \mathbb{R}^{d-1} and $\phi : U \rightarrow \mathbb{R}$ is continuously differentiable with Lipschitz derivatives. By a compactness argument, for (3.7) it suffices to prove that $\limsup_{s \downarrow 0} s^{d-1} \kappa((\partial A) \cap V, s) < \infty$.

Without loss of generality we can assume that $x = o$ and our rotation is the identity map, and that $\nabla \phi(x) = 0$. Also since ϕ is continuously differentiable we can also assume (by taking a smaller neighbourhood V if needed) that $U = B(o, \delta)$ for some $\delta > 0$ and that $|\langle \nabla \phi(u), e \rangle| \leq 1$ for all $u \in U$ and all unit vectors $e \in \mathbb{R}^{d-1}$. Then by the Intermediate Value theorem, for all $u, v \in U$ we have for some $w \in [u, v]$ that

$$|\phi(v) - \phi(u)| = |\langle v - u, \nabla \phi(w) \rangle| \leq \|v - u\|.$$

Given $s > 0$, we can and do cover U by $(d - 1)$ -dimensional balls $B_1, \dots, B_{m(s)}$ of radius $s/2$ centred in U , with the centre of B_i denoted u_i for each i , and with $m(s) = O(s^{1-d})$ as $s \downarrow 0$. Then for $i \leq m(s)$ and $v \in B_i$,

$$\|(v, \phi(v)) - (u_i, \phi(u_i))\| \leq \|v - u_i\| + \|\phi(v) - \phi(u_i)\| \leq (s/2) + (s/2) = s.$$

Therefore the balls $B((u_i, \phi(u_i)), s)$, $1 \leq i \leq m(i)$, cover $\partial A \cap V$, and this gives us (3.7).

For (3.8), we use [29, Lemma 3.4]. By that result, for $0 < s < \tau(A)$ and $x \in A \setminus A^{(s)}$,

$$\left| \text{Vol}(A \cap B_s(x)) - ((\theta_d/2) + h(\text{dist}(x, \partial A)/s))s^d \right| \leq \frac{2\theta_{d-1}s^{d+1}}{\tau(A)},$$

where $h(a) := \text{Vol}(B_1(o) \cap ([0, a] \times \mathbb{R}^{d-1}))$. In particular $h(a) \geq 0$ so that

$$s^{-d} \text{Vol}(A \cap B_s(x)) \geq (\theta_d/2) - 2\theta_{d-1}s/\tau(A),$$

and (3.8) follows.

For the last assertion, given $s > 0$, set $\kappa_s := \kappa(\partial A, s)$. By definition of κ_s , we can find $x_1, \dots, x_{\kappa_s} \in \mathbb{R}^d$ such that

$$\partial A \subset \bigcup_{i=1}^{\kappa_s} B(x_i, s).$$

Then $A \setminus A^{(s)} \subset \bigcup_{i=1}^{\kappa_s} B(x_i, 2s)$. Also $\kappa_s = O(s^{1-d})$ as $s \downarrow 0$, by (3.7). Hence by the union bound,

$$\text{Vol}(A \setminus A^{(s)}) \leq \sum_{i=1}^{\kappa_s} \text{Vol}(B(x_i, 2s)) = O(s),$$

which is the last assertion. \square

Proof. (Proof of Theorem 3) Assume ν is uniform on A . By definition $I_{n,k} = \mathbb{E}[S'_{n,k}]$. Hence, by the (univariate) Mecke formula, writing \mathcal{P}_n^x for $\mathcal{P}_n \cup \{x\}$, we have

$$I_{n,k} = \frac{n}{k} \int_A \mathbb{P}[|C_r(x, \mathcal{P}_n^x)| = k] f_0 dx, \tag{3.9}$$

By the multivariate Mecke formula, setting $V_r(\mathbf{x}) := \text{Vol}(B_r(\mathbf{x}))$ for $\mathbf{x} \in (\mathbb{R}^d)^k$, for $x \in A^{(kr)}$ we have

$$\mathbb{P}[|C_r(x, \mathcal{P}_n^x)| = k] = \frac{(nf_0)^{k-1}}{(k-1)!} \int_{(\mathbb{R}^d)^{k-1}} h_r((x, \mathbf{x})) \exp(-nf_0 V_r((x, \mathbf{x}))) d\mathbf{x},$$

where we used the fact that if $x \in A^{(kr)}$ and $h_r((x, \mathbf{x})) = 1$ then $B_r((x, \mathbf{x})) \subset A$. Hence by translation invariance, for $x \in A^{(kr)}$, writing $h_r(o, \mathbf{x})$ for $h_r((o, \mathbf{x}))$ and $V_r(o, \mathbf{x})$ for $V_r((o, \mathbf{x}))$ we have

$$\begin{aligned} \mathbb{P}[|C_r(x, \mathcal{P}_n^x)| = k] &= \frac{(nf_0)^{k-1}}{(k-1)!} \int_{(\mathbb{R}^d)^{k-1}} h_r(o, \mathbf{x}) \exp(-nf_0 V_r(o, \mathbf{x})) d\mathbf{x} \\ &= \frac{(nf_0 r^d)^{k-1}}{(k-1)!} \int_{(\mathbb{R}^d)^{k-1}} h_r(o, r\mathbf{y}) \exp(-nf_0 V_r(o, r\mathbf{y})) d\mathbf{y} \\ &= \frac{(nf_0 r^d)^{k-1}}{(k-1)!} \int_{(\mathbb{R}^d)^{k-1}} h_1(o, \mathbf{y}) \exp(-nf_0 r^d V_1(o, \mathbf{y})) d\mathbf{y} \end{aligned}$$

and by (2.11), this equals $p_k(f_0 nr^d)$. Therefore using (2.5) and the last part of Lemma 4 we obtain that as $n \rightarrow \infty$,

$$\int_{A^{(kr)}} \mathbb{P}[|C_r(x, \mathcal{P}_n^x)| = k] f_0 dx = (1 + O(r)) \alpha_k (f_0 nr^d)^{(1-k)(d-1)} e^{-\theta f_0 nr^d} (1 + O((nr^d)^{-1})). \tag{3.10}$$

To deal with $x \in A \setminus A^{(kr)}$, let $f_0^- \in (0, f_0)$. By Lemma 4 we have for all large enough n and all $x \in A$ that $v(B_r(x)) \geq \theta f_0^- r^d / 2$. Therefore using the multivariate Mecke formula we have for all large enough n and all $x \in A$ that

$$\mathbb{P}[|C_r(x, \mathcal{P}_n^x)| = k] \leq \frac{(nf_0)^{k-1}}{(k-1)!} \int_{A^{k-1}} h_r(x, \mathbf{x}) \exp(-n\theta f_0^- r^d / 2) d\mathbf{x},$$

which is $O((nr^d)^{k-1} \exp(-n\theta f_0^- r^d / 2))$, uniformly over $x \in A$. Hence by the last part of Lemma 4 we obtain that

$$\int_{A \setminus A^{(kr)}} \mathbb{P}[|C_r(x, \mathcal{P}_n^x)| = k] f_0 dx = O(r(nr^d)^{k-1} \exp(-n\theta f_0^- r^d / 2)),$$

which is negligible compared to the right hand side of (3.10) by (3.1), provided we take f_0^- close enough to f_0 (here $f_0 = f_1$ so (3.1) implies $\limsup_{n \rightarrow \infty} (n\theta f_0 r_n^d / \log n) < 2/d$). Thus using (3.9) we obtain (3.6). \square

In the non-uniform case, we do not have such precise asymptotics for $I_{n,k}$. However we do have the following, which formalizes back-of-the-envelope calculations suggesting $I_{n,k} \approx n \exp(-n\theta f_0 r_n^d)$.

Theorem 4 (Limiting behaviour of $I_{n,k}$ in the non-uniform case). *Let $k \in \mathbb{N}$. Then*

$$\lim_{n \rightarrow \infty} ((nr^d)^{-1} \log(I_{n,k}/n)) = -\theta f_0. \tag{3.11}$$

Proof. By Lemma 2, given $f_0^+ > f_0$ we have for large n that $I_{n,k} \geq n \exp(-n\theta f_0^+ r^d)$, so that $\log(I_{n,k}/n) \geq -n\theta f_0^+ r^d$, and hence

$$\liminf_{n \rightarrow \infty} ((nr^d)^{-1} \log(I_{n,k}/n)) \geq -\theta f_0.$$

For a bound the other way, note that $v(B_r(x)) \geq \theta f_0 r^d$ for $x \in A^{(r)}$. Also using Lemma 4, given $f_1^- < f_1$ we have for n large enough that $v(B_r(x)) \geq (\theta/2) f_1^- r^d$ for all $x \in A$. Therefore using the last part of Lemma 4, we have that

$$\begin{aligned} I_{n,k} &\leq n v(A^{(r)}) (n f_{\max} \theta ((k-1)r)^d)^{k-1} \exp(-n\theta f_0 r^d) \\ &\quad + O(nr(nr^d)^{k-1} \exp(-n(\theta/2) f_1^- r^d)). \end{aligned} \tag{3.12}$$

The second term in the right-hand side of (3.12), divided by the first term, is bounded by a constant times $r \exp(n\theta r^d (f_0 - f_1^- / 2))$, and using (3.1) we can show this tends to zero, provided f_1^- is chosen close enough to f_1 . Hence by (3.12) we can deduce that

$$\limsup_{n \rightarrow \infty} ((nr^d)^{-1} \log(I_{n,k}/n)) \leq -\theta f_0,$$

and hence (3.11). \square

4. Variance asymptotics and laws of large numbers

In this section, we show that the variance of the k -cluster count $S_{n,k}$ and its expectation are asymptotically equivalent, with explicit rates, and likewise for $S'_{n,k}$. As mentioned in Section 1, this is enough to yield weak laws of large numbers for $S_{n,k}$ and $S'_{n,k}$ but we shall also derive concentration results which yield strong laws of large numbers.

4.1. Variance asymptotics in the poisson setting

Proposition 2. *Let $k \in \mathbb{N}$. Then there exists $c > 0$ such that as $n \rightarrow \infty$,*

$$\text{Var}[S'_{n,k}] = \mathbb{E}[S'_{n,k}] (1 + O(e^{-c nr^d})) = I_{n,k} (1 + O(e^{-c nr^d})).$$

Proof. Since $S'_{n,k} (S'_{n,k} - 1)$ is the number of ordered pairs of distinct k -clusters in $G(\mathcal{P}_n, r)$, with $h(\cdot) := h_r(\cdot)$ as given at (1.7), we can write $S'_{n,k} (S'_{n,k} - 1)$ as follows

$$\sum_{\varphi, \psi \subset \mathcal{P}_n, |\varphi|=|\psi|=k} h(\varphi) h(\psi) \mathbf{1}\{(\mathcal{P}_n \setminus \varphi \setminus \psi) \cap B_r(\varphi \cup \psi) = \emptyset, \text{dist}(\varphi, \psi) > r\}.$$

Indeed, for distinct k -subsets ψ, φ of \mathcal{P}_n , their distance must be larger than r in order that they are both connected components of order k , for otherwise ψ and φ are connected with $|\psi \cup \varphi| > k$. Thus by the multivariate Mecke equation $\mathbb{E}[(S'_{n,k})^2] - \mathbb{E}[S'_{n,k}]^2$ equals

$$\frac{n^{2k}}{k!k!} \int_{A^k} \int_{A^k} h(\mathbf{x}) h(\mathbf{y}) \exp(-n v(B_r(\mathbf{x}, \mathbf{y}))) \mathbf{1}\{\text{dist}(\mathbf{x}, \mathbf{y}) > r\} v^k(d\mathbf{y}) v^k(d\mathbf{x}),$$

where $B_r(\mathbf{x}, \mathbf{y}) := B_r(\mathbf{x}) \cup B_r(\mathbf{y})$. We compare this integral with

$$\mathbb{E}[S'_{n,k}]^2 = \frac{n^{2k}}{k!k!} \int_{A^k} \int_{A^k} h(\mathbf{x}) h(\mathbf{y}) \exp(-n[v(B_r(\mathbf{x})) + v(B_r(\mathbf{y}))]) v^k(d\mathbf{y}) v^k(d\mathbf{x}),$$

which comes from (3.3). Observe that $v(B_r(\mathbf{x}, \mathbf{y})) = v(B_r(\mathbf{x})) + v(B_r(\mathbf{y}))$ whenever $\text{dist}(\mathbf{x}, \mathbf{y}) > 2r$. Therefore $|\text{Var}[S'_{n,k}] - \mathbb{E}[S'_{n,k}]^2| \leq J_{1,n} + J_{2,n}$, where

$$J_{1,n} := \frac{n^{2k}}{k!k!} \int_{A^k} \int_{A^k} h(\mathbf{x}) h(\mathbf{y}) \exp(-n v(B_r(\mathbf{x}, \mathbf{y}))) \mathbf{1}\{\text{dist}(\mathbf{x}, \mathbf{y}) \in (r, 2r]\} v^k(d\mathbf{y}) v^k(d\mathbf{x}); \tag{4.1}$$

$$J_{2,n} := \frac{n^{2k}}{k!k!} \int_{A^{2k}} h(\mathbf{x})h(\mathbf{y}) \exp(-n[v(B_r(\mathbf{x})) + v(B_r(\mathbf{y}))]) \mathbf{1}\{\text{dist}(\mathbf{x}, \mathbf{y}) \leq 2r\} v^{2k}(d(\mathbf{x}, \mathbf{y})). \tag{4.2}$$

We estimate $J_{1,n}$ in Lemma 5 below. For $J_{2,n}$, note that by Lemma 4 there exists $n_0 \in (0, \infty)$ such that for all $n \geq n_0$ and all $\mathbf{y} \in A^k$ we have $v(B_r(\mathbf{y})) \geq (\theta/3)f_0r^d$. Hence for $\mathbf{x} \in A^k$ we have

$$\frac{n^k}{k!} \int_{A^k} h(\mathbf{y}) \exp(-nv(B_r(\mathbf{y}))) \mathbf{1}\{\text{dist}(\mathbf{x}, \mathbf{y}) \leq 2r\} v^k(d\mathbf{y}) = O((nr^d)^k \exp(-n(\theta/3)f_0r^d)),$$

and hence $J_{2,n} = O(I_{n,k}(nr^d)^k \exp(-nf_0(\theta/3)r^d))$, which is $O(I_{n,k} \exp(-nf_0(\theta/4)r^d))$. Combined with Lemma 5, this completes the proof. \square

Lemma 5. *Let $J_{1,n}$ be given by (4.1). There exists $c > 0$ such that $J_{1,n} = O(e^{-c nr^d} I_{n,k})$ as $n \rightarrow \infty$.*

Proof. Let us write $\mathbf{x} < \mathbf{y}$ if there exists $y \in \{y_1, \dots, y_k\}$ such that all points of $\{x_1, \dots, x_k\}$ precede y in the lexicographic ordering (or in other words, the rightmost entry of \mathbf{x} lies to the left of the rightmost entry of \mathbf{y}). Since the integrand in J_1 is symmetric in \mathbf{x} and \mathbf{y} , J_1 is twice the same expression with the inner integral restricted to \mathbf{y} satisfying $\mathbf{x} < \mathbf{y}$.

Suppose $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^k$ with $h(\mathbf{x}) = h(\mathbf{y}) = 1$ and $\text{dist}(\mathbf{y}, \mathbf{x}) \in (r, 2r]$. These conditions imply that $\mathbf{y} \in (B_{2kr}(x_1))^k$, where we write $\mathbf{x} = (x_1, \dots, x_k)$. Thus, if also $\mathbf{x} \in (A^{(9kr)})^k$ then $\mathbf{y} \in (A^{(7kr)})^k$, and if moreover $\mathbf{x} < \mathbf{y}$, by considering the right half of the ball of radius r centred on the right-most entry of \mathbf{y} , we see that $v(B_r(\mathbf{y}) \setminus B_r(\mathbf{x})) \geq (\theta f_0/2)r^d$. Therefore since $B_r(\mathbf{x}, \mathbf{y}) = B_r(\mathbf{x}) \cup (B_r(\mathbf{y}) \setminus B_r(\mathbf{x}))$ and this is a disjoint union,

$$J_{1,n} \leq \frac{2n^k}{k!} \int_{(A^{(9kr)})^k} h(\mathbf{x}) \exp(-nv(B_r(\mathbf{x}))) a(\mathbf{x}) v^k(d\mathbf{x}) + z_n,$$

where

$$a(\mathbf{x}) := \frac{n^k}{k!} \int_{A^k} e^{-(\theta f_0/2)nr^d} h(\mathbf{y}) \mathbf{1}\{\text{dist}(\mathbf{x}, \mathbf{y}) \in (r, 2r], \mathbf{x} < \mathbf{y}\} v^k(d\mathbf{y}), \tag{4.3}$$

and z_n is a remainder term from \mathbf{x} near ∂A (to be dealt with later).

As already mentioned, if $\mathbf{x}, \mathbf{y} \in A^k$ with $h(\mathbf{x}) = h(\mathbf{y}) = 1$ and $r < \text{dist}(\mathbf{x}, \mathbf{y}) \leq 2r$, then $\mathbf{y} \in (B_{2kr}(x_1))^k$, and consequently

$$a(\mathbf{x}) \leq \frac{(nf_{\max})^k}{k!} e^{-(\theta f_0/2)nr^d} \theta^k (2kr)^{kd} = O((nr^d)^k e^{-(\theta f_0/2)nr^d}) = O(e^{-(\theta f_0/3)nr^d}),$$

uniformly over $\mathbf{x} \in A^k$. Therefore by virtue of (3.3), we have $J_{1,n} - z_n = O(e^{-(\theta f_0/3)nr^d} I_{n,k})$.

For the remainder term, note that given $f_1^- < f_1$, by Lemma 4 and the assumed continuity of $f|_A$, there exists n_1 such that for all $n \geq n_1$ and all $\mathbf{x} \in A^k \setminus (A^{(9r)})^k$, we have $v(B_r(\mathbf{x})) \geq n(\theta f_1^-/2)r^d$. Also if $\mathbf{x} = (x_1, \dots, x_k) \in A^k \setminus (A^{(9kr)})^k$ then at least one of x_1, \dots, x_k lies in $A \setminus A^{(9kr)}$. Introducing a factor of k we can assume this is x_1 , and hence deduce that

$$|z_n| \leq \frac{2(nf_{\max})^{2k}}{(k-1)!} \int_{A \setminus A^{(9kr)}} (\theta(2kr)^d)^{2k-1} dx_1 e^{-n\theta f_1^- r^d/2} = O(nr(nr^d)^{2k-1} e^{-n\theta f_1^- r^d/2}).$$

Combined with (3.4) this shows that $z_n/I_{n,k} = O(r \exp(n\theta r^d(f_0^+ - f_1^-/2)))$, for any $f_0^+ > f_0$. Using (3.1), we can choose $\epsilon > 0$, $f_0^+ > f_0$ and $f_1^- < f_1$ in such a way that this upper bound implies $z_n/I_{n,k} = O(rn^{(1/d)-\epsilon})$, and using (3.1) again we see that this is $O(e^{-c nr^d})$ for some $c > 0$. Thus $J_{1,n} = O(e^{-c nr^d} I_{n,k})$ for some $c > 0$. \square

4.2. Variance asymptotics in the binomial case

Proposition 3. *There exists $c > 0$ such that as $n \rightarrow \infty$,*

$$\text{Var}[S_{n,k}] = \mathbb{E}[S_{n,k}](1 + O(e^{-c nr^d})). \tag{4.4}$$

Proof. Let \prec denote the lexicographic ordering on \mathbb{R}^d . Write $S_{n,k} = \sum_{i=1}^n \xi_i$, here taking

$$\xi_i := \mathbf{1}\{|C_r(X_i, \mathcal{X}_n)| = k, X_i < \mathbf{y} \forall \mathbf{y} \in C_r(X_i, \mathcal{X}_n) \setminus \{X_i\}\}. \tag{4.5}$$

Then $\mathbb{E}[S_{n,k}] = n\mathbb{E}[\xi_1]$, and $\mathbb{E}[S_{n,k}^2] = n\mathbb{E}[\xi_1^2] + n(n-1)\mathbb{E}[\xi_1 \xi_2]$, so that $\mathbb{E}[S_{n,k}^2] = \mathbb{E}[S_{n,k}] + n(n-1)\mathbb{E}[\xi_1 \xi_2]$. Hence,

$$\text{Var}[S_{n,k}] - \mathbb{E}[S_{n,k}] = -n\mathbb{E}[\xi_1 \xi_2] + n^2(\mathbb{E}[\xi_1 \xi_2] - \mathbb{E}[\xi_1]\mathbb{E}[\xi_2]). \tag{4.6}$$

In view of Proposition 1 we need to show the right hand side of (4.6) is $O(e^{-c nr^d} I_{n,k})$.

For $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{R}^d)^k$, recall from (1.7) that $h_r^*(\mathbf{x})$ is the indicator variable of the event that $G(\{x_1, \dots, x_k\}, r)$ is connected and moreover $x_1 < x_i$ for $i = 2, \dots, k$. Then

$$\mathbb{E}[\xi_1] = \binom{n-1}{k-1} \int_{A^k} h_r^*(\mathbf{x})(1 - v(B_r(\mathbf{x})))^{n-k} v^k(d\mathbf{x}). \tag{4.7}$$

Consider the first term in the right hand side of (4.6). Observe that if $\xi_1 = \xi_2 = 1$ then $C_r(X_2, \mathcal{X}_n) \cap B_r(C_r(X_1, \mathcal{X}_n)) = \emptyset$. Hence, using the bound $1 - t \leq e^{-t}$ for all $t \geq 0$ and writing $B_r(\mathbf{x}, \mathbf{y}) := B_r(\mathbf{x}) \cup B_r(\mathbf{y})$, we have that $n\mathbb{E}[\xi_1 \xi_2]$ equals

$$n \binom{n-2}{k-1} \binom{n-1-k}{k-1} \int_{A^k} \int_{(A \setminus B_r(\mathbf{x}))^k} h_r^*(\mathbf{x})h_r^*(\mathbf{y})(1 - v(B_r(\mathbf{x}, \mathbf{y})))^{n-2k} v^k(d\mathbf{y})v^k(d\mathbf{x}) \tag{4.8}$$

$$\leq \frac{2n^{2k-1}}{(k-1)!^2} \int_{A^k} \int_{(A \setminus B_r(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) \exp(-n\nu(B_r(\mathbf{x}, \mathbf{y}))) \nu^k(d\mathbf{y}) \nu^k(d\mathbf{x}), \tag{4.9}$$

provided n is large enough so that $e^{2k\nu(B_r(\mathbf{x}, \mathbf{y}))} \leq 2$ for all $\mathbf{x}, \mathbf{y} \in A^k$.

Let $\alpha < \min(f_0, f_1/2)$. Using Lemma 4, we have for all large enough n and all $\mathbf{x} \in A^k, \mathbf{y} \in (A \setminus B_{2r}(\mathbf{x}))^k$ that $\nu(B_r(\mathbf{x}, \mathbf{y})) \geq \nu(B_r(\mathbf{x})) + \theta\alpha r^d$. Hence for such (\mathbf{x}, \mathbf{y}) the integrand in (4.9) is bounded by $h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) e^{-n\nu(B_r(\mathbf{x}))} e^{-\alpha\theta nr^d}$. Since $\int_{A^k} h_r^*(\mathbf{y}) \nu^k(d\mathbf{y}) = O(r^{d(k-1)})$, and $I_{n,k} = \frac{n^k}{(k-1)!} \int_{A^k} h_r^*(\mathbf{x}) e^{-n\nu(B_r(\mathbf{x}))} \nu^k(d\mathbf{x})$ by the multivariate Mecke formula,

$$n^{2k-1} \int_{A^k} \int_{(A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) e^{-n\nu(B_r(\mathbf{x}, \mathbf{y}))} \nu^k(d\mathbf{y}) \nu^k(d\mathbf{x}) = O((nr^d)^{k-1} e^{-n\theta\alpha r^d} I_{n,k}).$$

On the other hand, if $\mathbf{y} \in A^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k$, and $h_r^*(\mathbf{y}) = 1$, then $\mathbf{y} \in (B_{2kr}(x_1))^k$, and hence $\int_{A^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{y}) \nu^k(d\mathbf{y}) = O(r^{dk})$, and $\nu(B_r(\mathbf{x}, \mathbf{y})) \geq \nu(B_r(\mathbf{x}))$, so that

$$n^{2k-1} \int_{A^k} \int_{A^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) e^{-n\nu(B_r(\mathbf{x}, \mathbf{y}))} \nu^k(d\mathbf{y}) \nu^k(d\mathbf{x}) = O(r^d (nr^d)^{k-1} I_{n,k}),$$

which is $O(e^{-c nr^d} I_{n,k})$ for some $c > 0$ by (3.1). Thus the first term in the right hand side of (4.6) is $O(e^{-c nr^d} I_{n,k})$ for some $c > 0$.

Now consider the second term in the right hand side of (4.6). Given n , for $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^k$ let $p_x = \nu(B_r(\mathbf{x}))$ and $p_{\mathbf{x}, \mathbf{y}} := \nu(B_r(\mathbf{x}, \mathbf{y}))$. By (4.7) and (4.8),

$$\begin{aligned} \mathbb{E}[\xi_1 \xi_2] - \mathbb{E}[\xi_1] \mathbb{E}[\xi_2] &= \int_{A^k} \int_{(A \setminus B_{2r}(\mathbf{x}))^k} \left[\binom{n-2}{k-1} \binom{n-1-k}{k-1} (1-p_x - p_y)^{n-2k} \right. \\ &\quad \left. - \binom{n-1}{k-1}^2 (1-p_x)^{n-k} (1-p_y)^{n-k} \right] h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) \nu^k(d\mathbf{y}) \nu^k(d\mathbf{x}) \\ &+ \binom{n-2}{k-1} \binom{n-1-k}{k-1} \int_{A^k} \int_{(A \setminus B_r(\mathbf{x}))^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) (1-p_{\mathbf{x}, \mathbf{y}})^{n-2k} \nu^k(d\mathbf{y}) \nu^k(d\mathbf{x}) \\ &\quad - \binom{n-1}{k-1}^2 \int_{A^k} \int_{A^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) (1-p_x)^{n-k} (1-p_y)^{n-k} \nu^k(d\mathbf{y}) \nu^k(d\mathbf{x}). \end{aligned} \tag{4.10}$$

Note that $p_x \leq k f_{\max} \theta r^d$ for all $\mathbf{x} \in A^k$. Writing p for p_x and q for p_y , and using the fact that as $n \rightarrow \infty$ with k fixed,

$$\binom{n-1}{k-1}^2 \div \left[\binom{n-2}{k-1} \binom{n-1-k}{k-1} \right] = 1 + O(n^{-1}),$$

we estimate the integrand in the first term in the right hand side of (4.10) as follows:

$$\begin{aligned} &\binom{n-2}{k-1} \binom{n-1-k}{k-1} (1-p-q)^{n-2k} - \binom{n-1}{k-1}^2 (1-p-q+pq)^{n-k} \\ &= \binom{n-2}{k-1} \binom{n-1-k}{k-1} (1-p-q)^{n-k} \left[(1-p-q)^{-k} - (1+O(n^{-1})) \left(1 + \frac{pq}{1-p-q} \right)^{n-k} \right] \\ &\leq \frac{n^{2k-2}}{(k-1)!^2} \exp((n-k) \log(1-p-q)) \times [(n-k)pq + O(r^d) + O(n^{-1})] \\ &\leq \frac{n^{2k-1}}{(k-1)!^2} e^{-np} (k f_{\max} \theta)^2 r^{2d} e^{-nq} (1+o(1)). \end{aligned} \tag{4.11}$$

By Lemma 4, for all large enough n and all $\mathbf{y} \in A^k$ we have $q = p_y \geq (f_0 \theta / 3) r^d$. Multiplying the expression at (4.11) by $h_r^*(\mathbf{x}) h_r^*(\mathbf{y})$ and integrating over $\mathbf{y} \in (A \setminus B_{2r}(\mathbf{x}))^k$ yields an expression bounded by

$$c' n^k (nr^d)^{k-1} \exp(-(f_0 \theta / 3) nr^d) \exp(-np_x) r^{2d} h_r^*(\mathbf{x}),$$

for some constant c' independent of n . Recalling $I_{n,k} = \frac{n^k}{(k-1)!} \int e^{-np_x} h_r^*(\mathbf{x}) \nu^k(d\mathbf{x})$, we find that the absolute value of the first term in the right hand side of (4.10), multiplied by n^2 , is bounded above by

$$c' (k-1)! (nr^d)^{k+1} I_{n,k} \exp(-(f_0 \theta / 3) nr^d) = O(I_{n,k} \exp(-(f_0 \theta / 4) nr^d)).$$

We turn to the second term in the right hand side of (4.10). By the bound $1-t \leq e^{-t}$, for large n this term, multiplied by n^2 , is bounded by

$$\frac{2n^{2k}}{(k-1)!^2} \int_{A^k} \int_{(A \setminus B_r(\mathbf{x}))^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) \exp(-n\nu(B_r(\mathbf{x}, \mathbf{y}))) \nu^k(d\mathbf{x}) \nu^k(d\mathbf{y}).$$

By the multivariate Mecke formula, the last displayed expression equals twice the expected number of ordered pairs (φ, ψ) of distinct k -clusters of $G(\mathcal{P}_n, r)$ with $\text{dist}(\varphi, \psi) < 2r$. Hence by a further application of the multivariate Mecke formula, it is precisely equal to $2J_{1,n}$, where $J_{1,n}$ was defined at (4.1). Therefore by Lemma 5, the second term in the right hand side of (4.10), multiplied by n^2 , is $O(e^{-c nr^d} I_{n,k})$ for some $c > 0$.

We turn to the last term in the right hand side of (4.10). For large enough n , by Lemma 4 we have $p_y \geq f_0(\theta/3)r^d$ and also $(1 - p_x)^{-k}(1 - p_y)^{-k} \leq 2$ for all $\mathbf{x}, \mathbf{y} \in A$. Then

$$\begin{aligned} & n^{2+2(k-1)} \int_A \int_{A^k \setminus (A \setminus B_{2r}(\mathbf{x}))^k} h_r^*(\mathbf{x}) h_r^*(\mathbf{y}) (1 - p_x)^{n-k} (1 - p_y)^{n-k} v^k(d\mathbf{y}) v^k(d\mathbf{x}) \\ & \leq c n^{2k} r^{dk} \int_A h_r^*(\mathbf{x}) e^{-n p_x} e^{-(f_0 \theta/3) n r^d} v^k(d\mathbf{x}) \\ & = I_{n,k} \times O((n r^d)^k e^{-f_0(\theta/3) n r^d}), \end{aligned}$$

which is $O(I_{n,k} \exp(-(f_0 \theta/4) n r^d))$. Thus there exists $c > 0$ such that all terms in the right hand side of (4.10), multiplied by n^2 , are $O(e^{-c n r^d} I_{n,k})$. Therefore the right hand side of (4.6) is $O(e^{-c n r^d} I_{n,k})$, as required. \square

4.3. LLNs And concentration results for $S_{n,k}$ and $S'_{n,k}$

A sequence of random variables $(\xi_n)_{n \in \mathbb{N}}$ is said to converge completely to a constant C (written $\xi_n \xrightarrow{c.c.} C$) as $n \rightarrow \infty$ if $\mathbb{P}[|\xi_n - C| > \varepsilon]$ is summable in n for any $\varepsilon > 0$. In particular, complete convergence implies almost sure convergence for any sequence of random variables defined on the same probability space. We now prove $S_{n,k}/I_{n,k} \xrightarrow{c.c.} 1$ and $S'_{n,k}/I_{n,k} \xrightarrow{c.c.} 1$ under an extra condition on $r = r_n$, namely

$$\limsup_{n \rightarrow \infty} (\theta n r_n^d / (\log n)) < 1 / (2 f_0). \tag{4.12}$$

Using our results about $I_{n,k}$, we can then give a sequence of constants that are almost surely asymptotic to $S_{n,k}$ (in the uniform case) or to $\log(S_{n,k}/n)$ (in the general case).

Theorem 5 (Concentration results for $S_{n,k}$ and $S'_{n,k}$). *Suppose that $r = r_n$ satisfies (4.12) as well as (3.1). Given $\varepsilon > 0$, there exist $\delta, n_2 \in (0, \infty)$ such that for all $n \geq n_2$,*

$$\mathbb{P}[|(S_{n,k}/I_{n,k}) - 1| > \varepsilon] \leq \exp(-n^\delta); \tag{4.13}$$

$$\mathbb{P}[|(S'_{n,k}/I_{n,k}) - 1| > \varepsilon] \leq \exp(-n^\delta). \tag{4.14}$$

In particular, as $n \rightarrow \infty$ we have the complete convergence $(S_{n,k}/I_{n,k}) \xrightarrow{c.c.} 1$, and also

$$(n r^d)^{-1} \log(S_{n,k}/n) \xrightarrow{c.c.} -\theta f_0, \tag{4.15}$$

and likewise for $S'_{n,k}$ (as $n \rightarrow \infty$ through \mathbb{N}). Finally, in the uniform case,

$$n^{-1} e^{f_0 \theta n r^d} (n r^d)^{(k-1)(d-1)} S_{n,k} \xrightarrow{c.c.} k^{-1} f_0^{(1-k)(d-1)} \alpha_k, \tag{4.16}$$

and likewise for $S'_{n,k}$.

Remark 4. If (3.1) holds but (4.12) does not, then as mentioned in Section 1, we still have $S_{n,k}/I_{n,k} \xrightarrow{\mathbb{P}} 1$, and (4.15) and (4.16) with convergence in probability.

When $f_0 = f_1$, the upper bound in (3.1) is $f_0^{-1} \min(2/d, 1)$, so (4.12) is a stronger condition than the upper bound in (3.1) when $d \leq 3$ (but not when $d \geq 4$).

Proof. (Proof of Theorem 5.) By definition $\mathbb{E}[S'_{n,k}] = I_{n,k}$. Hence by Lemma 2, given any $\varepsilon > 0$ and $f_0^+ > f_0$, for n large enough,

$$\mathbb{P}[|S'_{n,k} - I_{n,k}| \geq \varepsilon I_{n,k}] \leq \mathbb{P}[|S'_{n,k} - \mathbb{E}[S'_{n,k}]| \geq n e^{-f_0^+ \theta n r^d}].$$

Partition \mathbb{R}^d into cubes of side length r . Then the number of k -clusters intersecting a fixed cube in the partition is bounded by a constant c' depending only on d . Therefore, removing all the Poisson points in any fixed cube of the partition reduces or increases the total number of k -clusters by at most a constant depending only on d . Enumerate the cubes in the partition that intersect A as C_1, \dots, C_{m_n} , where $m_n = O(r^{-d})$ as $n \rightarrow \infty$. For each $\ell \in [m_n]$, let \mathcal{F}_ℓ be the sigma-algebra generated by $\mathcal{P}_n \cap (\cup_{i=1}^\ell C_i)$. Set $D_i = \mathbb{E}[S'_{n,k} | \mathcal{F}_i] - \mathbb{E}[S'_{n,k} | \mathcal{F}_{i-1}]$ for $i \in [m_n]$. Then $S'_{n,k}$ is written as the sum of martingale differences $S'_{n,k} - \mathbb{E}[S'_{n,k}] = \sum_{i=1}^{m_n} D_i$. To see why $|D_i|$ is bounded from above by a finite constant, we re-sample the Poisson points in C_i from an independent copy \mathcal{P}'_n defined on the same probability space. By the superposition theorem [9], recalling that $K_{k,r}(\mathcal{X})$ is the number of k -clusters of $G(\mathcal{X}, r)$, we have that

$$D_i = \mathbb{E}[K_{k,r}(\mathcal{P}_n) - K_{k,r}(\mathcal{P}'_n \cap C_i) \cup (\mathcal{P}_n \setminus C_i) | \mathcal{F}_i].$$

Hence $|D_i| \leq 2c'$ uniformly in $i \in [m_n]$, as discussed in the last paragraph. From here, it follows easily by an application of Azuma's inequality [1, page 33] that for all n large

$$\mathbb{P}[|S'_{n,k} - \mathbb{E}[S'_{n,k}]| \geq n e^{-n f_0^+ \theta r^d}] \leq 2 \exp\left(-\frac{n^2 e^{-2 n f_0^+ \theta r^d}}{8(c')^2 m_n}\right).$$

By the assumption (4.12), we may assume f_0^+ is chosen so that $\limsup_{n \rightarrow \infty} n\theta f_0^+ r^d / (\log n) < 1/2$. Using this we can find $\delta > 0$ and $c > 0$ such that for n large, the last bound is at most $2 \exp(-c''n^\delta)$. This gives us the result (4.14).

Now consider $S_{n,k}$ (i.e. the binomial case). By Proposition 1 we have for some $c > 0$ that $\mathbb{E}[S_{n,k}] = I_{n,k}(1 + O(e^{-cnr^d}))$, and hence for n large,

$$\begin{aligned} \mathbb{P}[|S_{n,k} - I_{n,k}| \geq \varepsilon I_{n,k}] &\leq \mathbb{P}[|S_{n,k} - \mathbb{E}[S_{n,k}]| \geq \varepsilon I_{n,k}/2] \\ &\leq \mathbb{P}[|S_{n,k} - \mathbb{E}[S_{n,k}]| \geq ne^{-nf_0^+ \theta r^d}]. \end{aligned}$$

Then (4.13) can be proved in the same manner as (4.14) with the filtration now given by $\mathcal{F}_\ell = \sigma(X_i : i \leq \ell)$ and $D_i = \mathbb{E}[K_{k,r}(\mathcal{X}_n) - K_{k,r}(\mathcal{X}_{n+1} \setminus \{X_i\}) | \mathcal{F}_i]$. We omit the details.

It is immediate from (4.13) and (4.14) that $(S_{n,k}/I_{n,k}) \xrightarrow{c.c.} 1$, and $(S'_{n,k}/I_{n,k}) \xrightarrow{c.c.} 1$.

For (4.15), note that since $(S_{n,k}/I_{n,k}) \xrightarrow{c.c.} 1$ and $nr^d \rightarrow \infty$, $(nr^d)^{-1} \log(S_{n,k}/I_{n,k}) \xrightarrow{c.c.} 0$, and thus by Theorem 4,

$$(nr^d)^{-1} \log(S_{n,k}/n) = (nr^d)^{-1} \log(S_{n,k}/I_{n,k}) + (nr^d)^{-1} \log(I_{n,k}/n) \xrightarrow{c.c.} -\theta f_0,$$

which is (4.15). We obtain a similar result for $S'_{n,k}$ using the fact that $(S'_{n,k}/I_{n,k}) \xrightarrow{c.c.} 1$.

In the uniform case, we obtain (4.16) using (3.6). \square

5. Asymptotic distribution of the k -cluster count

To present Poisson and normal approximation results for $S_{n,k}$ and $S'_{n,k}$, we recall three notions of distance which measure the proximity between the distributions of real-valued random variables X, Y . The Kolmogorov distance is defined by

$$d_K(X, Y) := \sup_{z \in \mathbb{R}} |\mathbb{P}[X \leq z] - \mathbb{P}[Y \leq z]|.$$

The Wasserstein-1 distance is defined by

$$d_W(X, Y) := \sup_{h \in \text{Lip}_1} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,$$

where Lip_1 denotes the family of Lipschitz continuous functions with Lipschitz constant at most one. The total variation distance is defined by

$$d_{TV}(X, Y) := \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]|,$$

where the supremum is taken over all Borel measurable subsets of \mathbb{R} . If X, Y both take values in \mathbb{Z} , then this is the same as $(1/2) \sum_{n \in \mathbb{Z}} |\mathbb{P}[X = n] - \mathbb{P}[Y = n]|$. Convergence in any of the three aforementioned distances implies convergence in distribution.

In this section we prove the following result, concerning the asymptotic distributions of $S_{n,k}$ (the number of k -clusters in $G(\mathcal{X}_n, r)$) and $S'_{n,k}$ (the number of k -clusters in $G(\mathcal{P}_n, r)$), in the mildly dense regime. Recall from (3.3) that $\mathbb{E}[S'_{n,k}] = I_{n,k}$. Recall that Z_t denotes a Poisson variable with mean t , and let \mathcal{N} denote a standard normal random variable.

Theorem 6 (Distributional results on the k -cluster count). *Suppose that $r = r(n)$ satisfies (3.1), and set $b := \limsup_{n \rightarrow \infty} (n\theta r^d / \log n)$. Let $k \in \mathbb{N}$. There exist finite $c > 0, n_3 > 0$ such that*

$$d_{TV}(S'_{n,k}, Z_{I_{n,k}}) \leq e^{-cnr^d}, \quad \forall n \geq n_3, \tag{5.1}$$

and if $d \geq 2$ then

$$d_{TV}(S_{n,k}, Z_{\mathbb{E}[S_{n,k}]}) \leq e^{-cnr^d}, \quad \forall n \geq n_3. \tag{5.2}$$

Moreover (for all d), given $\varepsilon > 0$ we have as $n \rightarrow \infty$ that

$$d_K(\text{Var}[S'_{n,k}]^{-1/2}(S'_{n,k} - I_{n,k}), \mathcal{N}) = O(n^{\varepsilon + (bf_0 - 1)/2}) \tag{5.3}$$

and

$$d_W(\text{Var}[S_{n,k}]^{-1/2}(S_{n,k} - \mathbb{E}[S_{n,k}]), \mathcal{N}) = O(n^{\varepsilon + (3bf_0 - 1)/2}). \tag{5.4}$$

Also if $d \geq 2$ then there exist finite $c > 0, n_4 > 0$ such that

$$d_K(\text{Var}[S_{n,k}]^{-1/2}(S_{n,k} - \mathbb{E}[S_{n,k}]), \mathcal{N}) \leq e^{-cnr^d}, \quad \forall n \geq n_4, \tag{5.5}$$

and a similar result holds for $S'_{n,k}$ for all $d \geq 1$.

Remark 5.

1. The normal approximation (5.5) (and the corresponding result for $S'_{n,k}$) will be proved via Poisson approximation, while (5.3) and (5.4) will be obtained by direct normal approximation of $S_{n,k}$ or $S'_{n,k}$. It is of interest to compare the rates of normal convergence in these results with one another.

When $b = 0$, the rates are better in (5.3) and (5.4) than in (5.5) since e^{-cnr^d} decays more slowly than $n^{-\epsilon}$ for any $\epsilon > 0$. To fully compare when $b > 0$, we would need to optimize the value of c in (5.5), which is beyond the scope of this paper.

2. We conjecture that the rates in (5.3), and in (5.4) for $b = 0$, are optimal up to a factor of n^ϵ . When $b > 0$ the rate of convergence given for $S_{n,k}$ in (5.4) is worse than the rate given for $S'_{n,k}$ in (5.3). We expect that in this case the rate in (5.4) is sub-optimal.
3. When $bf_0 \geq 1/3$, (5.4) does not give a CLT for $S_{n,k}$ at all. However, provided $d \geq 2$ and (3.1) holds (which amounts to $bf_0 < 2/d$ when $f_0 = f_1$), we do still have a CLT for $S_{n,k}$ (possibly with a sub-optimal rate of convergence) by (5.5).
4. In (5.4) we found it more convenient to use d_W rather than d_K . The proof of (5.4) is based on a general result on normal approximation (in d_W) for stabilizing functionals of binomial point processes (Lemma 12) which is itself based on a result in [19]. It might be possible to obtain a similar result with d_K instead of d_W by utilizing [30] instead of [19].
5. A result along the lines of (5.3) (without any rate of convergence) is proved in [21] for a class of soft RGGs where the probability of two vertices being connected given their locations at x, y say, is some function of x and y (called the connection function), rather than being $\mathbf{1}\{\|y - x\| \leq r\}$ as in the graphs considered here. However, the result in [21] requires the connection function to be bounded away from 1, so the result there does not cover the RGGs that we consider here.

5.1. Poisson approximation for $S'_{n,k}$

Let $\mathbf{N}(\mathbb{R}^d)$ be the space of all finite subsets of \mathbb{R}^d , equipped with the smallest σ -algebra $\mathcal{S}(\mathbb{R}^d)$ containing the sets $\{\mathcal{X} \in \mathbf{N}(\mathbb{R}^d) : |\mathcal{X} \cap B| = m\}$ for all Borel $B \subset \mathbb{R}^d$ and all $m \in \mathbb{N} \cup \{0\}$. Given $m \in \mathbb{N}$, let $\mathbf{N}_m(\mathbb{R}^d) := \{\mathcal{X} \in \mathbf{N}(\mathbb{R}^d) : |\mathcal{X}| = m\}$.

Our main tool for proving the Poisson approximation result (5.1) in Theorem 6 is the following coupling bound in [21] adapted to our situation (i.e. without marking). The function g in the next result has nothing to do with the g in Section 2.

Lemma 6 ([21, Theorem 3.1]). *Let $g : \mathbf{N}_k(\mathbb{R}^d) \times \mathbf{N}(\mathbb{R}^d) \rightarrow \{0, 1\}$ be measurable. Define*

$$W := F(\mathcal{P}_n) := \sum_{\varphi \subset \mathcal{P}_n, |\varphi|=k} g(\varphi, \mathcal{P}_n \setminus \varphi).$$

Let $n > 0, k \in \mathbb{N}$. For $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{R}^d)^k$ with distinct entries, set $p(\mathbf{x}) := \mathbb{E}[g(\{x_1, \dots, x_k\}, \mathcal{P}_n)]$ and set $\mu = nv$. Assume that for μ^k -a.e. \mathbf{x} with $p(\mathbf{x}) > 0$, we can find coupled random variables $U_{\mathbf{x}}, V_{\mathbf{x}}$ such that

- $\mathcal{L}(U_{\mathbf{x}}) = \mathcal{L}(W)$;
- $\mathcal{L}(1 + V_{\mathbf{x}}) = \mathcal{L}(F(\mathcal{P}_n \cup \{x_1, \dots, x_k\}) | g(\{x_1, \dots, x_k\}, \mathcal{P}_n) = 1)$.
- $\mathbb{E}[|U_{\mathbf{x}} - V_{\mathbf{x}}|] \leq w(\mathbf{x})$, where $w : (\mathbb{R}^d)^k \rightarrow [0, \infty)$ is measurable.

Then

$$d_{TV}(W, Z_{\mathbb{E}[W]}) \leq \frac{\min(1, \mathbb{E}[W]^{-1})}{k!} \int w(\mathbf{x}) p(\mathbf{x}) \mu^k(d\mathbf{x}). \tag{5.6}$$

We restate the first Poisson approximation result (5.1) from Theorem 6 in the following proposition. Recall that Z_t denotes a Poisson variable with mean t .

Proposition 4. *Let $k \in \mathbb{N}$. Then there exist finite constants $c, n_5 > 0$ such that*

$$d_{TV}(S'_{n,k}, Z_{I_{n,k}}) \leq e^{-cnr^d}, \quad \forall n \geq n_5.$$

Proof. In view of (3.2), we apply Lemma 6 with

$$g(\varphi, \psi) := h_r(\varphi) \mathbf{1}\{\psi \cap B_r(\varphi) = \emptyset\}.$$

For $\mathbf{x} \in (\mathbb{R}^d)^k$, with $h_r(\mathbf{x}) = 1$, we construct coupled random variables $(U_{\mathbf{x}}, V_{\mathbf{x}})$ as follows. Define $U_{\mathbf{x}} := \sum_{\varphi \subset \mathcal{P}_n, |\varphi|=k} g(\varphi, \mathcal{P}_n \setminus \varphi)$, and

$$V_{\mathbf{x}} := \sum_{\varphi \subset \mathcal{P}_n \setminus B_r(\mathbf{x}), |\varphi|=k} g(\varphi, \mathcal{P}_n \setminus B_r(\mathbf{x}) \setminus \varphi).$$

This coupling satisfies the distributional requirement because the conditional distribution of \mathcal{P}_n given the event $\{g(\mathbf{x}, \mathcal{P}_n) = 1\}$ is the same as the distribution of $\mathcal{P}_n \setminus B_r(\mathbf{x})$.

There are two sources of contribution to the change $U_{\mathbf{x}} - V_{\mathbf{x}}$ of k -cluster counts after removing all the Poisson points in $B_r(\mathbf{x})$. First, after removal, all k -clusters of $G(\mathcal{P}_n, r)$ that were originally intersecting $B_r(\mathbf{x})$ are destroyed, therefore reducing the k -cluster count. Second, every k -set $\varphi \subset \mathcal{P}_n \setminus B_r(\mathbf{x})$ satisfying the two properties

- (a) $g(\varphi, \mathcal{P}_n \setminus B_r(\mathbf{x}) \setminus \varphi) = 1$;
- (b) $\mathcal{P}_n(B_r(\mathbf{x}) \cap B_r(\varphi)) \geq 1$;

becomes a k -cluster *only* after removing all the Poisson points in $B_r(\mathbf{x})$, thereby increasing the number of k -clusters. Let $\xi_1(\mathbf{x})$ denote the number of k -clusters of $G(\mathcal{P}_n, r)$ that intersect $B_r(\mathbf{x})$ and let $\xi_2(\mathbf{x})$ denote the number of k -subsets φ of $\mathcal{P}_n \setminus B_r(\mathbf{x})$ satisfying property (a) and property (c) $B_r(\varphi) \cap B_r(\mathbf{x}) \neq \emptyset$. It is clear that (b) implies (c) and

$$|U_{\mathbf{x}} - V_{\mathbf{x}}| \leq \xi_1(\mathbf{x}) + \xi_2(\mathbf{x}). \tag{5.7}$$

We estimate $\mathbb{E}[\xi_1(\mathbf{x})]$ and $\mathbb{E}[\xi_2(\mathbf{x})]$ separately. Since

$$\xi_1(\mathbf{x}) = \sum_{\varphi \subset \mathcal{P}_n: |\varphi|=k, \varphi \cap B_r(\mathbf{x}) \neq \emptyset} h_r(\varphi) \mathbf{1}\{(\mathcal{P}_n \setminus \varphi) \cap B_r(\varphi) = \emptyset\},$$

applying the multivariate Mecke equation leads to

$$\mathbb{E}[\xi_1(\mathbf{x})] = \frac{1}{k!} \int_{A^k \setminus (A \setminus B_r(\mathbf{x}))^k} h_r(\mathbf{y}) e^{-nv(B_r(\mathbf{y}))} (nv)^k (d\mathbf{y}).$$

If $\mathbf{y} \in A^k \setminus (A \setminus B_r(\mathbf{x}))^k$ and $h_r(\mathbf{y}) = 1$ then $\mathbf{y} \subset B_{kr}(\mathbf{x})^k$. Moreover by Lemma 4, if n is large enough then $v(B_r(\mathbf{y})) \geq f_0(\theta/3)r^d$ for any $\mathbf{y} \in A^k$. Hence for all n large enough and all \mathbf{x} ,

$$\mathbb{E}[\xi_1(\mathbf{x})] \leq \frac{n^k}{k!} (k\theta(kr)^d f_{\max})^k e^{-f_0(\theta/3)nr^d}.$$

Therefore setting $p(x_1, \dots, x_k) = \mathbb{E}[g(\{x_1, \dots, x_k\}, \mathcal{P}_n)]$, and using (3.3), we have

$$\begin{aligned} \int_{A^k} \mathbb{E}[\xi_1(\mathbf{x})] p(\mathbf{x}) (nv)^k (d\mathbf{x}) &\leq (f_{\max} \theta k^{d+1})^k (nr^d)^k e^{-(\theta/3)f_0 nr^d} \left(\frac{n^k}{k!}\right) \int_{A^k} p(\mathbf{x}) v^k (d\mathbf{x}) \\ &= (f_{\max} \theta k^{d+1})^k (nr^d)^k e^{-(\theta/3)f_0 nr^d} I_{n,k}. \end{aligned} \tag{5.8}$$

Set $\gamma(\mathbf{x}, \mathbf{y}) = \mathbf{1}\{r < \text{dist}(\mathbf{x}, \mathbf{y}) \leq 2r\}$. By the multivariate Mecke equation

$$\mathbb{E}[\xi_2(\mathbf{x})] = \frac{n^k}{k!} \int_{A^k} h_r(\mathbf{y}) \gamma(\mathbf{x}, \mathbf{y}) e^{-nv(B_r(\mathbf{y}) \setminus B_r(\mathbf{x}))} v^k (d\mathbf{y}),$$

and therefore writing $B_r(\mathbf{x}, \mathbf{y})$ for $B_r(\mathbf{x}) \cup B_r(\mathbf{y})$, we have that

$$\int_{A^k} \mathbb{E}[\xi_2(\mathbf{x})] p(\mathbf{x}) (nv)^k (d\mathbf{x}) = \frac{n^{2k}}{k!} \int_{A^k} \int_{A^k} h_r(\mathbf{x}) h_r(\mathbf{y}) \gamma(\mathbf{x}, \mathbf{y}) e^{-nv(B_r(\mathbf{x}, \mathbf{y}))} v^k (d\mathbf{y}) v^k (d\mathbf{x}).$$

By (4.1) this expression is equal to $k! J_{1,n}$, and therefore by Lemma 5 it is $O(e^{-c'nr^d} I_{n,k})$ for some $c' > 0$.

Combining this with (5.8), and using (5.7), we obtain for suitable $c > 0$ that

$$\int_{A^k} |U_{\mathbf{x}} - V_{\mathbf{x}}| p(\mathbf{x}) (nv)^k (d\mathbf{x}) = O(e^{-c nr^d} I_{n,k}).$$

Applying Lemma 6 with the present choice of g (so that the W of that result is $S'_{n,k}$) gives the desired bound in Poisson approximation, completing the proof. \square

5.2. Poisson approximation for $S_{n,k}$

For Poisson approximation in the binomial setting, i.e. for $S_{n,k}$, we use the following result from [20, Theorem II.24.3] or [22, Theorem 1.B].

Lemma 7. *Let $n \in \mathbb{N}$. Suppose Y_1, \dots, Y_n are Bernoulli random variables on a common probability space. Set $W := \sum_{i=1}^n Y_i$, and $p_i := \mathbb{E}[Y_i]$ for $i \in [n]$. Suppose for each $i \in [n]$ that there exist coupled random variables U_i, V_i such that $\mathcal{L}(U_i) = \mathcal{L}(W)$ and $\mathcal{L}(1 + V_i) = \mathcal{L}(W | Y_i = 1)$. Then*

$$d_{\text{TV}}(W, Z_{\mathbb{E}[W]}) \leq (\min(1, 1/\mathbb{E}[W])) \sum_{i=1}^n p_i \mathbb{E}[|U_i - V_i|].$$

We use this to obtain the analogue of Proposition 4 in the binomial setting, i.e. the second Poisson approximation result (5.2) in Theorem 6:

Proposition 5. *Suppose $d \geq 2$. Let $k \in \mathbb{N}$. There exist $c > 0$, $n_6 \in (0, \infty)$ such that*

$$d_{\text{TV}}(S_{n,k}, Z_{\mathbb{E}[S_{n,k}]}) \leq e^{-c nr^d}, \quad \forall n \geq n_6.$$

We shall prove this in stages. To apply Lemma 7 to $S_{n,k}$, we let Y_i be the indicator of the event that $|C_r(X_i, \mathcal{X}_n)| = k$ and X_i is the left-most point of $C_r(X_i, \mathcal{X}_n)$ (we called this ξ_i at (4.5)). Then $S_{n,k} = \sum_{i=1}^n Y_i$.

We need to define U_i, V_i for each $i \in [n]$ so that $\mathcal{L}(U_i) = \mathcal{L}(S_{n,k})$, and $\mathcal{L}(1 + V_i) = \mathcal{L}(S_{n,k} | Y_i = 1)$, and so that we can find a good bound for $\mathbb{E}[|U_i - V_i|]$. We do this for $i = 1$ as follows. First define the event

$$\mathcal{E} := \{Y_1 = 1\} \cap \{C_r(X_1, \mathcal{X}_n) = \{X_1, \dots, X_k\}\}.$$

Let $(\tilde{X}_1, \dots, \tilde{X}_k)$ be a random vector in $(\mathbb{R}^d)^k$ with $\mathcal{L}(\tilde{X}_1, \dots, \tilde{X}_k) = \mathcal{L}((X_1, \dots, X_k) | \mathcal{E})$. Also let $(X_{i,j}, i \in [n], j \in \mathbb{N})$ be an array of independent ν -distributed random variables, independent of $(\tilde{X}_1, \dots, \tilde{X}_k)$. Set $\mathcal{X}_{n,1} := \{X_{1,1}, \dots, X_{n,1}\}$.

For $k < i \leq n$ set $J_i := \min\{j : X_{i,j} \notin \cup_{\ell=1}^k B_r(\tilde{X}_\ell)\}$ and set $\tilde{X}_i := X_{i,J_i}$. Then set $\mathcal{X}_{n,2} := \{\tilde{X}_1, \dots, \tilde{X}_n\}$.

In other words, we sample the random vector $(\tilde{X}_1, \dots, \tilde{X}_k)$ from the conditional distribution of (X_1, \dots, X_k) given that \mathcal{E} occurs, independently of $\mathcal{X}_{n,1}$. Given the outcome of $\tilde{X}_1, \dots, \tilde{X}_k$, for $i \in [n] \setminus [k]$, if $X_{i,1} \notin \cup_{\ell=1}^k B_r(\tilde{X}_\ell)$ we take $\tilde{X}_i := X_{i,1}$. Otherwise we resample a random vector with distribution ν repeatedly until we get a value that is not in $\cup_{\ell=1}^k B_r(\tilde{X}_\ell)$, and call this \tilde{X}_i . Thus, given the value of $(\tilde{X}_1, \dots, \tilde{X}_k)$, the distribution of \tilde{X}_i is given by the measure ν restricted to $A \setminus \cup_{\ell=1}^k B_r(\tilde{X}_\ell)$, normalized to a probability measure.

Set $U_1 := K_{k,r}(\mathcal{X}_{n,1})$, and $V_1 := K_{k,r}(\mathcal{X}_{n,2}) - 1$. Note that in this coupling we make no attempt to make a ‘good’ coupling of $(\tilde{X}_1, \dots, \tilde{X}_k)$ to $(X_{1,1}, \dots, X_{k,1})$. However, for $k + 1 \leq i \leq n$ the variables \tilde{X}_i , are the same as $X_{i,1}$ except in those rare cases where $X_{i,1}$ lies in $B_r((\tilde{X}_1, \dots, \tilde{X}_k))$. This is what makes the coupling effective. Clearly $\mathcal{L}(U_1) = \mathcal{L}(S_{n,k})$.

Lemma 8. *With V_1 as just defined, $\mathcal{L}(1 + V_1) = \mathcal{L}(S_{n,k} | Y_1 = 1)$.*

Proof. Given $Y_1 = 1$, we have that $C_r(X_1, \mathcal{X}_n)$ has k vertices with X_1 the left-most of these, and by exchangeability we can assume the other vertices of $C_r(X_1, \mathcal{X}_n)$ are X_2, \dots, X_k without affecting the distribution of the point process \mathcal{X}_n . In other words, we have $\mathcal{L}(\mathcal{X}_n | Y_1 = 1) = \mathcal{L}(\mathcal{X}_n | \mathcal{E})$. Moreover, we claim that

$$\mathcal{L}((\tilde{X}_1, \dots, \tilde{X}_n)) = \mathcal{L}((X_1, \dots, X_n) | \mathcal{E}). \tag{5.9}$$

This implies that $\mathcal{L}(\{\tilde{X}_1, \dots, \tilde{X}_n\}) = \mathcal{L}(\mathcal{X}_n | \mathcal{E}) = \mathcal{L}(\mathcal{X}_n | Y_1 = 1)$, and hence $\mathcal{L}(1 + V_1) = \mathcal{L}(K_{k,r}(\{\tilde{X}_1, \dots, \tilde{X}_n\})) = \mathcal{L}(S_{n,k} | Y_1 = 1)$, as required.

It remains to confirm the claim (5.9). The reader may think this is obvious; we provide a sketch proof. For $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, let us set $\mathbf{x}_1^k := (x_1, \dots, x_k)$, $\mathbf{x}_{k+1}^n := (x_{k+1}, \dots, x_n)$ and $g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) := \prod_{\ell=k+1}^n \mathbf{1}_{\{x_\ell \notin B_r(\mathbf{x}_1^k)\}}$. Then with h_r^* defined just after (1.7),

$$\mathbb{P}[(X_1, \dots, X_n) \in d\mathbf{x}; \mathcal{E}] = h_r^*(\mathbf{x}_1^k) g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^n(d\mathbf{x}), \tag{5.10}$$

so that $\mathbb{P}(\mathcal{E}) = \int_{A^n} h_r^*(\mathbf{x}_1^k) g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^n(d\mathbf{x})$. For $\mathbf{x}_1^k \in (\mathbb{R}^d)^k$, let us set $I(\mathbf{x}_1^k) := \int_{(\mathbb{R}^d)^{n-k}} g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^{n-k}(d\mathbf{x}_{k+1}^n)$. Then by (5.10),

$$\mathbb{P}((\tilde{X}_1, \dots, \tilde{X}_k) \in d\mathbf{x}_1^k) = \mathbb{P}[(X_1, \dots, X_k) \in d\mathbf{x}_1^k | \mathcal{E}] = \frac{h_r^*(\mathbf{x}_1^k) I(\mathbf{x}_1^k) \nu^k(d\mathbf{x}_1^k)}{\int_{A^n} h_r^*(\mathbf{x}_1^k) g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^n(d\mathbf{x})}.$$

Then

$$\begin{aligned} \mathbb{P}((\tilde{X}_1, \dots, \tilde{X}_n) \in d\mathbf{x}) &= \frac{g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^{n-k}(d\mathbf{x}_{k+1}^n)}{I(\mathbf{x}_1^k)} \times \mathbb{P}[(\tilde{X}_1, \dots, \tilde{X}_k) \in d\mathbf{x}_1^k] \\ &= \frac{h_r^*(\mathbf{x}_1^k) g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^n(d(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n))}{\int_{A^n} h_r^*(\mathbf{x}_1^k) g_r(\mathbf{x}_1^k, \mathbf{x}_{k+1}^n) \nu^n(d\mathbf{x})} \\ &= \mathbb{P}[(X_1, \dots, X_n) \in d\mathbf{x} | \mathcal{E}], \end{aligned}$$

where the last line comes from (5.10). Thus we have (5.9). \square

Proof. (Proof of Proposition 5) We need to find a useful bound on $\mathbb{E}[|U_1 - V_1|]$. Note that the ‘new’ point process $\mathcal{X}_{n,2} = \{\tilde{X}_1, \dots, \tilde{X}_n\}$ is obtained from the ‘old’ point process $\mathcal{X}_{n,1}$ by replacing the points $X_{1,1}, \dots, X_{k,1}$ with $\tilde{X}_1, \dots, \tilde{X}_k$ and also, for those $i \in [n] \setminus [k]$ such that $J_i > 1$, replacing the point $X_{i,1}$ with \tilde{X}_i , leaving the other points unchanged.

We refer to vertices and components (here also called *clusters*) of $G(\mathcal{X}_{n,1}, r)$ as being *old* while vertices and clusters of $G(\mathcal{X}_{n,2}, r)$ are *new*. We write \tilde{C} for $\{\tilde{X}_1, \dots, \tilde{X}_k\}$. By construction \tilde{C} is a cluster of $G(\mathcal{X}_{n,2}, r)$.

The value of $|U_1 - V_1|$ is bounded by the sum of the following variables $N_i, 1 \leq i \leq 6$.

N_1 is the number of old k -clusters involving $X_{1,1}, \dots, X_{k,1}$, and N_2 is the number of new k -clusters within distance r of one of $X_{1,1}, \dots, X_{k,1}$ but not using any of the new vertices \tilde{X}_i with $i > k, J_i > 1$, and with no vertex in $B_{2r}(\tilde{C})$. These will be affected by removing $X_{i,1}$ for $1 \leq i \leq k$.

N_3 is the number of old k -clusters intersecting $B_r(\tilde{X}_i)$ for some $i \in [n] \setminus [k]$ with $J_i > 1$, and not using $X_{i,1}$ (if such a cluster does use $X_{i,1}$ it is included in N_5 below). N_4 is the number of new k -clusters involving \tilde{X}_i for some $i > k$ with $J_i > 1$, and having no vertex in $B_{2r}(\tilde{C})$. These are affected by the creation of new vertices at \tilde{X}_i with $i > k, J_i > 1$.

N_5 is the number of old k -clusters having at least one vertex in $B_r(\tilde{C})$. These clusters are affected by the removal of old vertices in $B_r(\tilde{C})$.

N_6 is the number of new k -clusters having at least one vertex in $B_{2r}(\tilde{C})$, other than \tilde{C} itself. These could be created from previously larger clusters due to the removal of old vertices in $B_r(\tilde{C})$.

We estimate $\mathbb{E}[N_i]$ for each $i \in [6]$, repeatedly using the fact that $\nu(B_s(x)) \geq (\theta/3) f_0 s^d$ for all small enough $s > 0$ and all $x \in A$ by Lemma 4. We have for large enough n that

$$\begin{aligned} \mathbb{E}[N_1] &\leq k \binom{n-1}{k-1} \int_{A^k} h_r(\mathbf{x})(1 - \nu(B_r(\mathbf{x})))^{n-k} \nu^k(d\mathbf{x}) \\ &\leq 2kn^{k-1} (f_{\max} \theta((k-1)r)^d)^{k-1} \exp(-f_0(\theta/3)nr^d), \end{aligned}$$

and writing $h_r(x, \mathbf{x})$ for $h_r((x, \mathbf{x}))$, we have

$$\begin{aligned} \mathbb{E}[N_2] &\leq k \binom{n-k}{k} \int_A \int_{A^k} h_r(x, \mathbf{x}) (1 - \nu(B_r(\mathbf{x})))^{n-2k} \nu^k(dx) \nu(d\mathbf{x}) \\ &\leq 2n^k (f_{\max} \theta (kr)^d)^k \exp(-f_0(\theta/3)nr^d). \end{aligned}$$

Using the fact that $\mathbb{P}[J_i > 1] \leq kf_{\max} \theta r^d$ for $i > k$, and the point process $\mathcal{X}_{n,1} \setminus \{X_{i,1}\}$ is independent of the event $\{J_i > 1\}$ and the random vector \tilde{X}_i , we obtain that

$$\begin{aligned} \mathbb{E}[N_3] &\leq (n-k)(kf_{\max} \theta r^d) \binom{n-1}{k} \sup_{x \in A} \int_{A^k} h_r(x, \mathbf{x}) (1 - \nu(B_r(\mathbf{x})))^{n-1-k} \nu^k(d\mathbf{x}) \\ &\leq c(nr^d)^{k+1} \exp(-f_0(\theta/3)nr^d). \end{aligned}$$

Using the same bound on $\mathbb{P}[J_i > 1]$ and the fact that for $i \in [n] \setminus [k]$ the distribution of $\mathcal{X}_{n,2} \setminus \{\tilde{X}_1, \dots, \tilde{X}_k, \tilde{X}_i\}$, given $(\tilde{X}_1, \dots, \tilde{X}_k, \tilde{X}_i)$ is that of a sample of size $n - k - 1$ from the restriction of ν to $A \setminus \cup_{\ell=1}^k \tilde{X}_\ell$ normalized to be a probability measure, we have that

$$\begin{aligned} \mathbb{E}[N_4] &\leq (n-k)(kf_{\max} \theta r^d) \binom{n-k-1}{k-1} \sup_{x \in A} \{(2f_{\max} \theta (kr)^d)^{k-1} (1 - \nu(B_r(x)))^{n-2k}\} \\ &\leq c(nr^d)^k \exp(-f_0(\theta/3)nr^d). \end{aligned}$$

Next,

$$\begin{aligned} \mathbb{E}[N_5] &\leq nk f_{\max} \theta r^d \binom{n-1}{k-1} \sup_{x \in A} \int_{A^{k-1}} h_r(x, \mathbf{x}) (1 - \nu(B_r(x, \mathbf{x})))^{n-k} \nu^{k-1}(d\mathbf{x}) \\ &\leq c(nr^d)^k \exp(-f_0(\theta/3)nr^d). \end{aligned}$$

We give an estimate for $\mathbb{E}[N_6]$ in Lemma 9 below. Combining this with the earlier estimates for N_1, \dots, N_5 , we obtain that there exists $\epsilon > 0$ that

$$\mathbb{E}[|U_1 - V_1|] = O(\exp(-\epsilon nr^d)).$$

By the exchangeability of X_1, \dots, X_n , we can construct U_i, V_i similarly for each $i \in [n]$, with the same bound for $\mathbb{E}[|U_i - V_i|]$. Also we know $\mathbb{E}[S_{n,k}] \rightarrow \infty$ as $n \rightarrow \infty$ by Proposition 1 and Lemma 2. Then by Lemma 7 (with the W of that result equal to our $S_{n,k}$) we obtain for n large enough that

$$d_{TV}(S_{n,k}, Z_{\mathbb{E}[S_{n,k}]}) \leq (\mathbb{E}[S_{n,k}])^{-1} \sum_{i=1}^n \mathbb{E}[|Y_i|] \times O(\exp(-n\epsilon r^d)) = O(\exp(-n\epsilon r^d)),$$

as required. \square

Lemma 9. *Suppose $d \geq 2$. Let N_6 be the number of k -clusters of $G(\mathcal{X}_{n,2}, r)$ within distance $2r$ of $\{\tilde{X}_1, \dots, \tilde{X}_k\}$, other than $\{\tilde{X}_1, \dots, \tilde{X}_k\}$ itself. Then there exists a constant $c > 0$ such that $\mathbb{E}[N_6] = O(\exp(-c nr^d))$ as $n \rightarrow \infty$.*

It is harder to find a good upper bound for $\mathbb{E}[N_6]$ than for $\mathbb{E}[N_i], 1 \leq i \leq 5$. The main problem is that for fairly large k (for example, $k = 12$ for $d = 2$) there can be a configuration of $\{\tilde{X}_1, \dots, \tilde{X}_k\}$ such that the union of balls of radius r centred on these points surrounds a very small (but non-empty) region. This causes a problem because the probability of having a second k -cluster inside this small region does not include the $\exp(-\Omega(nr^d))$ factor seen in the estimates for the other $\mathbb{E}[N_i]$, since all other points in the union of r -balls centred on $\{\tilde{X}_1, \dots, \tilde{X}_k\}$ are removed anyway.

To deal with this difficulty, we shall integrate over $\tilde{X}_1, \dots, \tilde{X}_k$ and verify that the aforementioned eventuality is very unlikely because of the compression phenomenon mentioned before: given small but fixed $\delta_1 > 0$, a k -cluster is very likely to be compressed within a small ball of radius $\delta_1 r$, and if this happens the kind of possibility just mentioned cannot happen. We can deal with the probability of non-compressed k -clusters by other means.

The proof of Lemma 9 uses the following geometrical lemma. The lemma fails when $d = 1$, and is the reason for the restriction to $d \geq 2$ in the statement of Proposition 5. It may be possible to prove Lemma 9, and hence Proposition 5, by other means when $d = 1$. Recall from the start of Section 3 that we are assuming $A \subset \mathbb{R}^d$ is compact with $\text{Vol}(A) > 0$ and with $C^{1,1}$ boundary.

Lemma 10. *Suppose $d \geq 2$. There exists a constant $\delta_1 > 0$ such that we have for all small enough $s > 0$ that*

$$\text{Vol}(B_s(y) \cap A \setminus B_{(1+\delta_1)s}(x)) \geq \delta_1 s^d, \quad \forall x \in A, y \in A \setminus B_s(x). \tag{5.11}$$

Proof. Suppose $\delta_1 < 1/4$. If $x \in A, y \in A^{(s)} \setminus B_s(x)$ then the set $B_s(y) \setminus B_{(1+\delta_1)s}(x)$ contains a ball of radius $s/4$, and is contained in A , so that $\text{Vol}(B_s(y) \cap A \setminus B_{(1+\delta_1)s}(x)) \geq \theta(s/4)^d$.

Let $\delta_2 = 0.01$, and let $\tau \in (0, \tau(A))$. We show first that for all $s \in (0, \delta_2 \tau)$ and all $y \in A \setminus A^{(s)}$ we can find a unit vector $e(y)$ such that

$$B^*(y; s, \delta_2, e(y)) \subset A, \tag{5.12}$$

where the set $B^*(y; s, \eta, e)$ is as defined at [1, p. 98], namely, it is the set of $z \in B(x, r)$ such that $\langle (z - y), e \rangle > \eta s$, where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product.

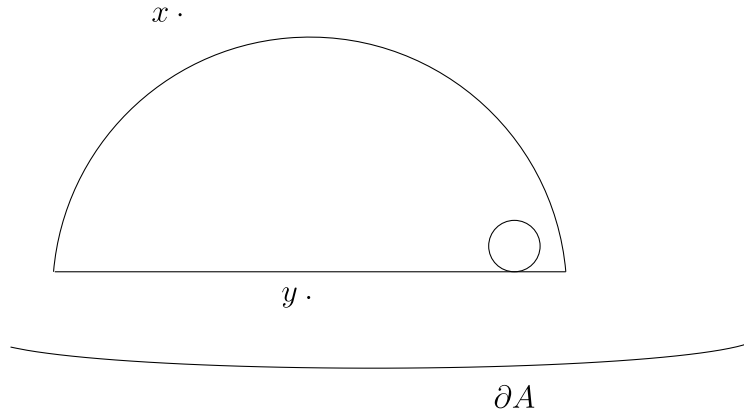


Fig. 3. Illustration for the proof of Lemma 10. The small circle represents $B_{\delta_2 s}(z)$. The larger segment represents $B^*(y, s, \delta_2, e(y))$.

Suppose $0 < s \leq \delta_2 \tau$. Given $y \in A \setminus A^{(s)}$, let w be the nearest point of ∂A to y . Let $e = \hat{n}_w$. By translating and rotating A we can assume without loss of generality that $y = o$ and $e = e_d := (0, \dots, 0, 1)$, the d th coordinate unit vector. For $u \in \mathbb{R}^{d-1}$ with $\|u\| < \tau$, define

$$\phi(u) := \sup\{a \in [-\tau, \tau] : (u, a) \notin A\}.$$

Then as in the proof of [29, Lemma 3.6], $\phi(u) \leq \|u\|^2/\tau$. But if $z \in B^*(y, s, \delta_2, e)$ then writing π for projection onto the first $d - 1$ coordinates, since $s \leq \delta_2 \tau$ we have

$$\langle z, e \rangle \geq \langle z - y, e \rangle \geq \delta_2 s \geq s^2/\tau \geq \phi(\pi(z))$$

so that $z \in A$. This gives us (5.12) for $s < \delta_2 \tau$.

Let $s \in (0, s_0)$. Pick $y \in A \setminus A^{(s)}$ and let $e = e(y)$ be as just described. Without loss of generality, now assume $y = o$ and $e = e_d$. The following argument is illustrated in Fig. 3.

Let $x \in A \setminus B_s(y)$. After a further rotation we can also assume without loss of generality that $x = -ue_1 + ve_d$ for some $u \geq 0$ and $v \in \mathbb{R}$ with $u^2 + v^2 \geq s^2$, where $e_1 := (1, 0, \dots, 0)$. It follows from this that $\|se_1 - x\| \geq \sqrt{2}s$.

Let $z = (1 - 7\delta_2)se_1 + 2\delta_2 se_d$. By the triangle inequality $\|z - x\| \geq (\sqrt{2} - 9\delta_2)s \geq (1 + 3\delta_2)s$. Hence $B_{\delta_2 s}(z) \subset \mathbb{R}^d \setminus B_{(1+\delta_2)s}(x)$. Moreover $B_{\delta_2 s}(z)^o \subset B^*(y; s, \delta_2, e_d) \subset A$. Therefore

$$\text{Vol}(B_s(y) \cap A \setminus B_{(1+\delta_2)s}(x)) \geq \text{Vol}(B_{\delta_2 s}(z)^o) = \theta(\delta_2 s)^d.$$

Thus taking $\delta_1 = \min(\delta_2, \theta\delta_2^d)$ we have (5.11). \square

Proof. (Proof of Lemma 9) Choose $\delta_1 \in (0, 1/2)$ with the property in Lemma 10. Let $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{R}^d)^k$ with $x_1 < x_i$ and $\|x_1 - x_i\| \leq \delta_1 r$ for $2 \leq i \leq k$ (here $<$ refers to the lexicographic ordering). For $i \in [n] \setminus [k]$, let $\mathcal{X}_{i,k,n} := \{\tilde{X}_{k+1}, \dots, \tilde{X}_n\} \setminus \{\tilde{X}_i\}$. Let

$$N_6(\mathbf{x}) := \sum_{i=k+1}^n \mathbf{1}\{\tilde{X}_i \in B_{2r}(\mathbf{x}) \setminus B_r(\mathbf{x}), \mathcal{X}_{i,k,n}(B_r(\tilde{X}_i) \setminus B_{(1+\delta_1)r}(x_1)) < k\}.$$

Then using the property from Lemma 10 (taking $y = \tilde{X}_i$ and $x = x_1$), and then a Chernoff-type bound (for example, [1, Lemma 1.1]), we have for large n that

$$\begin{aligned} \mathbb{E}[N_6(\mathbf{x})] &\leq (n - k)2kf_{\max}\theta(2r)^d \mathbb{P}[\text{Bin}(n - k - 1, f_0\delta_1 r^d) \leq k - 1] \\ &\leq \exp(-f_0(\delta_1/2)nr^d). \end{aligned}$$

Let \mathcal{M} be the event that $\{\tilde{X}_1, \dots, \tilde{X}_k\} \subset B_{\delta_1 r}(\tilde{X}_1)$, i.e. ‘compressed’. If \mathcal{M} occurs then $N_6 \leq N_6((\tilde{X}_1, \dots, \tilde{X}_k))$, and thus we have

$$\mathbb{E}[N_6 | \mathcal{M}] \leq \exp(-f_0(\delta_1/2)nr^d). \tag{5.13}$$

Also,

$$\begin{aligned} \mathbb{P}[\mathcal{M}^c] &= \mathbb{P}[\cup_{i=2}^k \{X_i \notin B_{\delta_1 r}(X_1)\} | \{h_r^*(X_1, \dots, X_k) = 1\} \cap \cap_{\ell=k+1}^n \{X_\ell \notin \cup_{i=1}^k B_r(X_i)\}] \\ &= \frac{\int_A \mathbb{P}[E_x]v(dx)}{\int_A \mathbb{P}[F_x]v(dx)} \end{aligned} \tag{5.14}$$

where we set

$$F_x := \{h_r^*(x, X_2, \dots, X_k) = 1\} \cap \cap_{\ell=k+1}^n \{X_\ell \notin B_r((x, X_2, \dots, X_k))\};$$

$$E_x := \{\{X_2, \dots, X_k\} \setminus B_{\delta_1 r}(x) \neq \emptyset\} \cap F_x.$$

Set $b = \limsup_{n \rightarrow \infty} (\theta nr^d / \log n)$ and note that $b(f_0 - f_1/2) < 1/d$ by (3.1). Take $\delta_2 > 0$ such that

$$b(f_0 - f_1/2) + bf_0\delta_2 < 1/d, \tag{5.15}$$

and such that moreover for all $y \in \mathbb{R}^d \setminus B_{\delta_1}(o)$ we have $\text{Vol}(B_1(y) \setminus B_1(o)) \geq \delta_2\theta$, and also $\delta_2 < 1/3$. Then for all small enough $s > 0$ and all $x \in A^{(3s)}, y \in A \setminus B_{\delta_1 s}(x)$ we have $\text{Vol}(A \cap B_s(y) \setminus B_s(x)) > \delta_2\theta s^d$. Moreover, $h_r^*(x, x_2, \dots, x_k) = 0$ unless x_2, \dots, x_k all lie in $B_{(k-1)r}(x)$, and hence for all large enough n and all $x \in A^{(3r)}$ we have

$$\begin{aligned} \mathbb{P}[E_x] &\leq \int_{(B_{kr}(x))^{k-1} \setminus (B_{\delta_1 r}(x))^{k-1}} (1 - v(B_r(x, \mathbf{x})))^{n-k} v^{k-1}(d\mathbf{x}) \\ &\leq 2 \int_{(B_{kr}(x))^{k-1} \setminus (B_{\delta_1 r}(x))^{k-1}} \exp(-nv(B_r(x, \mathbf{x}))) v^{k-1}(d\mathbf{x}) \\ &\leq 2 \int_{(B_{kr}(x))^{k-1} \setminus (B_{\delta_1 r}(x))^{k-1}} \exp(-nv(B_r(x)) - f_0\theta\delta_2 nr^d) v^{k-1}(d\mathbf{x}) \\ &\leq cr^{d(k-1)} \exp(-nv(B_r(x)) - f_0\theta\delta_2 nr^d), \end{aligned}$$

for some constant c (independent of x). Similarly using the simpler bound $v(B_r(x, \mathbf{x})) \geq v(B_r(\mathbf{x}))$ we obtain for $x \in A \setminus A^{(3r)}$ that $\mathbb{P}[E_x] \leq cr^{d(k-1)} e^{-nv(B_r(x))}$. Hence by splitting the integral into regions $A^{(3r)}$ and $A \setminus A^{(3r)}$, given $f_1^- < f_1$, and using Lemma 4 we obtain for some new constant c that for n large,

$$r^{d(1-k)} \int_A \mathbb{P}[E_x] v(dx) \leq c \exp(-\theta(f_0 + f_0\delta_2)nr^d) + cr \exp(-\theta(f_1^-/2)nr^d). \tag{5.16}$$

On the other hand, given $\delta_3 \in (0, \delta_1)$, and any $\varepsilon > 0$, we have for large enough n and all $x \in A^{(2r)}$, setting $B_s^+(x)$ to be the right half of $B_s(x)$, that

$$\begin{aligned} \mathbb{P}[F_x] &\geq \int_{(B_{\delta_3 r}^+(x))^{k-1}} (1 - v(B_{(1+\delta_3)r}(x)))^{n-k} v^{k-1}(d\mathbf{x}) \\ &\geq (f_0(\theta/2)\delta_3^d r^d)^{k-1} \exp(-(1 + \varepsilon)nv(B_{(1+\delta_3)r}(x))). \end{aligned}$$

Choose δ_3 so that $f_{\max}((1 + \delta_3)^d - 1) \leq f_0\delta_2/4$, and ε so that $2^d \varepsilon f_{\max} < f_0\delta_2/4$. Then

$$\begin{aligned} (1 + \varepsilon)v(B_{(1+\delta_3)r}(x)) &\leq v(B_r(x)) + f_{\max} \theta r^d ((1 + \delta_3)^d - 1) + \varepsilon f_{\max} \theta (2r)^d \\ &\leq v(B_r(x)) + \theta r^d f_0 \delta_2 / 2, \end{aligned}$$

so that given $f_0^+ > f_0$ there is a further constant $c' > 0$ such that for large n (using the continuity of f on A) we have

$$r^{d(1-k)} \int_A \mathbb{P}[F_x] v(dx) \geq c' \exp(-(f_0^+ + f_0\delta_2/2)\theta nr^d).$$

Therefore by (5.14) and (5.16) there is a further constant c'' such that

$$\begin{aligned} \mathbb{P}[\mathcal{M}^c] &\leq c'' \exp[(f_0^+ + f_0\delta_2/2 - f_0 - f_0\delta_2)\theta nr^d] + c'' r \exp[(f_0^+ + f_0\delta_2/2 - f_1^-/2)\theta nr^d] \\ &= c'' \exp[(f_0^+ - f_0 - f_0\delta_2/2)\theta nr^d] + c'' r \exp[(f_0^+ + f_0\delta_2/2 - f_1^-/2)\theta nr^d]. \end{aligned}$$

Taking f_0^+ sufficiently close to f_0 and f_1^- sufficiently close to f_1^- , we have that the first term is $O(\exp(-f_0(\delta_2/4)\theta nr^d))$, while taking $b := \limsup_{n \rightarrow \infty} (n\theta r^d) / \log n$, the second term is bounded by $rn^{b(f_0 - (f_1/2) + f_0\delta_2)}$, and by (5.15) and (3.1), this is $O(\exp(-c nr^d))$ for some $c > 0$. Thus using (5.13) and the fact that N_6 is bounded by a deterministic constant depending only on d and k , we have for some constant c' that

$$\begin{aligned} \mathbb{E}[N_6] &= \mathbb{E}[N_6 | \mathcal{M}] \mathbb{P}[\mathcal{M}] + \mathbb{E}[N_6 | \mathcal{M}^c] \mathbb{P}[\mathcal{M}^c] \\ &\leq \exp(-\delta_1(f_0/2)nr^d) + c' \exp(-c nr^d), \end{aligned}$$

which implies the result asserted. \square

5.3. Normal approximation

We shall prove our central limit theorem for $S'_{n,k}$ by expressing it as a sum of variables having the structure of an m -dependent random field, whose definition we now recall. Let $X = (X_\alpha, \alpha \in \mathcal{V})$ be a collection of random variables indexed by a set $\mathcal{V} \subset \mathbb{Z}^d$. Given $m > 0$, we say X is an m -dependent random field if for any two subsets A_1, A_2 of \mathbb{Z}^d with $\min_{\alpha \in A_1, \beta \in A_2} \|\alpha - \beta\|_\infty > m$, the sigma-algebras $\sigma\{X_\alpha, \alpha \in A_1\}$, and $\sigma\{X_\alpha, \alpha \in A_2\}$, are mutually independent.

Lemma 11. (see [18, Theorem 2.6]) Let $2 < q \leq 3$. Let $\mathcal{V} \subset \mathbb{Z}^d$ and let $(W_i, i \in \mathcal{V})$ be an m -dependent random field with $\mathbb{E}[W_i] = 0$ for each $i \in \mathcal{V}$. Let $W = \sum_{i \in \mathcal{V}} W_i$. Assume that $\mathbb{E}[W^2] = 1$ and $\mathbb{E}[|W_i|^q] < \infty$ for all $i \in \mathcal{V}$. Then

$$d_K(W, N(0, 1)) \leq 75(10m + 1)^{(q-1)d} \sum_{i \in \mathcal{V}} \mathbb{E}[|W_i|^q]. \tag{5.17}$$

We shall apply Lemma 11 to $S'_{n,k}$ to prove (5.3) from Theorem 6. The claim is restated in the form of a proposition.

Proposition 6. Set $b = \limsup_{n \rightarrow \infty} n\theta r^d / (\log n)$. Set $\sigma = (\mathbb{V}\text{ar}[S'_{n,k}])^{1/2}$. Let $\varepsilon > 0$. Then as $n \rightarrow \infty$,

$$d_K(\sigma^{-1}(S'_{n,k} - I_{n,k}), N(0, 1)) = O(n^{\varepsilon + (bf_0 - 1)/2}). \tag{5.18}$$

Proof. Let $f_0^- < f_0$ and $f_0^+ > f_0$, and $f_1^- < f_1^* < f_1$. Fix $\delta \in (0, d^{-1/2})$, with $f_0(1 - 2d\delta)^d > f_0^-$ and also $f_1^*(1 - 2d\delta)^d > f_1^-$. Given n , partition \mathbb{R}^d into cubes of side length $\delta r = \delta r(n)$ centred on the points of $\delta r\mathbb{Z}^d$ and indexed by \mathbb{Z}^d ; for $z \in \mathbb{Z}^d$ let C_z be the cube in the partition that is centred on δrz . Let $\mathcal{V} = \{z \in \mathbb{Z}^d : C_z \cap A \neq \emptyset\}$. Then $|\mathcal{V}| = O(r^{-d})$ as $n \rightarrow \infty$.

For $z \in \mathcal{V}$, let U_z be the number of components of order k in $G(\mathcal{P}_n, r)$ having their left-most vertex in C_z , set $p_z := \mathbb{E}[U_z]$, and let $W_z = \sigma^{-1}(U_z - p_z)$. Then $\sum_{z \in \mathcal{V}} U_z = S'_{n,k}$ and $\sum_{z \in \mathcal{V}} W_z = \sigma^{-1}(S'_{n,k} - I_{n,k})$. Moreover, $(W_z, z \in \mathcal{V})$ is a centred m -dependent random field, where m is independent of n . Hence by Lemma 11 (taking $q = 3$), there is a constant c independent of n such that

$$d_K(\sigma^{-1}(S'_{n,k} - I_{n,k}), N(0, 1)) \leq c \sum_{z \in \mathcal{V}} \mathbb{E}[|W_z|^3] = c\sigma^{-3} \sum_{z \in \mathcal{V}} \mathbb{E}[|U_z - p_z|^3]. \tag{5.19}$$

Since $\delta < d^{-1/2}$, any two points of \mathcal{P}_n in the same cube are connected in $G(\mathcal{P}_n, r)$, and therefore U_z is a Bernoulli random variable with parameter p_z , for each $z \in \mathcal{V}$. Hence $\mathbb{E}[|U_z - p_z|^3] = p_z(1 - p_z)^3 + (1 - p_z)p_z^3 \leq p_z$.

Let $\mathcal{V}_1 := \{z \in \mathcal{V} : \delta rz \in A^{(r)}\}$ and $\mathcal{V}_2 := \mathcal{V} \setminus \mathcal{V}_1$.

Let $z \in \mathcal{V}$. If $\mathcal{P}_n(B_{r-d\delta r}(\delta rz)) \geq k + 1$, then $U_z = 0$. Therefore there exists $c > 0$ such that for $z \in \mathcal{V}_1$,

$$p_z \leq \mathbb{P}[\mathcal{P}_n(B_{r-d\delta r}(\delta rz)) \leq k] \leq c(nr^d)^k \exp(-f_0\theta(1 - d\delta)^d nr^d),$$

so that by Proposition 2, there exists c' such that for n large the contribution of \mathcal{V}_1 to the right hand side of (5.19) is at most

$$c' r^{-d} \exp(-f_0\theta(1 - 2d\delta)^d nr^d) I_{n,k}^{-3/2} \leq c' r^{-d} \exp(-f_0^- \theta nr^d) I_{n,k}^{-3/2},$$

and by Lemma 2, for n large this is at most

$$n^{-1/2} (nr^d)^{-1} \exp(((3/2)f_0^+ - f_0^-)\theta nr^d).$$

Set $b := \limsup_{n \rightarrow \infty} n\theta r^d / (\log n)$. Then $b < 1/f_0$ by (3.1), and for any $\varepsilon > 0$, provided f_0^+ and f_0^- are chosen close enough to f_0 the above is bounded by $n^{\varepsilon + (bf_0 - 1)/2}$.

Now consider $z \in \mathcal{V}_2$. For such z , using Lemma 4 we have

$$p_z \leq \mathbb{P}[\mathcal{P}_n(B_{r-d\delta r}(\delta rz)) \leq k] \leq c(nr^d)^k \exp(-f_1^*(1 - d\delta)^d (\theta/2)nr^d).$$

Also we claim that $|\mathcal{V}_2| = O(r^{1-d})$. Indeed, using the first part of Lemma 4, we can cover ∂A by $O(r^{1-d})$ balls of radius r . Taking balls of radius $3r$ with the same centres gives a collection of $O(r^{1-d})$ balls such that each point of \mathcal{V}_2 lies in at least one of these balls. Since the number of points of \mathcal{V}_2 contained in any one of these balls is uniformly bounded, the claim follows.

Therefore there exists a constant c'' such that the contribution of \mathcal{V}_2 to the right hand side of (5.19) is at most $c'' r^{1-d} \exp(-f_1^- (\theta/2)nr^d) I_{n,k}^{-3/2}$, and by Lemma 2, given $\varepsilon > 0$, for n large this is at most

$$n^{-1/2} r (nr^d)^{-1} \exp(((3/2)f_0^+ - f_1^-)\theta nr^d) < n^{\varepsilon - (1/2) - (1/d) + (b/2)(3f_0 - f_1)}.$$

Using (3.1) we have that $b(f_0 - f_1/2) < 1/d$ and therefore the exponent of n above is at most $\varepsilon - (1/2) + bf_0/2$, i.e. $\varepsilon + (bf_0 - 1)/2$.

Combining this with the contribution of \mathcal{V}_1 and applying (5.19) yields (5.18). \square

For a binomial counterpart of Proposition 6 using Wasserstein distance, we use the following lemma, which is based on a result of Chatterjee [19], which requires further notation.

Recall the definition of $\mathbb{N}(\mathbb{R}^d)$ from Section 5.1. Given $F : \mathbb{N}(\mathbb{R}^d) \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$, set $D_x F(\mathcal{X}) := F(\mathcal{X} \cup \{x\}) - F(\mathcal{X})$, and set $\|DF\| := \sup_{x \in \mathbb{R}^d, \mathcal{X} \in \mathbb{N}(\mathbb{R}^d)} |D_x F(\mathcal{X})|$. Given such F and given $s \in (0, \infty)$, we say that the constant s is a *radius of stabilization* for F if $D_x F(\mathcal{X})$ is determined by $\mathcal{X} \cap B_s(x)$ for all $x \in \mathbb{R}^d, \mathcal{X} \in \mathbb{N}(\mathbb{R}^d)$, i.e. $D_x F(\mathcal{X}) = D_x F(\mathcal{X} \cap B_s(x))$ for all x, \mathcal{X} .

Lemma 12. Let $n \in \mathbb{N}$, $s \in (0, \infty)$. Suppose $F : \mathbb{N}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is measurable with $\|DF\| < \infty$ and $\sigma := \sqrt{\mathbb{V}\text{ar}[F(\mathcal{X}_n)]} \in (0, \infty)$, and s is a radius of stabilization for F . Then

$$d_W\left(\frac{F(\mathcal{X}_n) - \mathbb{E}F(\mathcal{X}_n)}{\sigma}, \mathcal{N}\right) \leq \frac{Cn^{1/2}}{\sigma^2} \|DF\|^2 (\mathbb{E}[(\mathcal{X}_{n+4}(B_s(X_1)))^4])^{1/4} + \frac{4n}{\sigma^3} \|DF\|^3, \tag{5.20}$$

where C is a universal constant.

Proof. We apply [19, Theorem 2.5]. As explained below, the graph $G(\mathcal{X}_n, s)$ is a *symmetric interaction rule* (in the sense of [19]) for F , and we can take its symmetric extension G' to be $G(\mathcal{X}_{n+4}, s)$. The quantity denoted $\Delta_j f(X)$ in [19, Theorem 2.5] has its absolute value bounded by $2\|DF\|$.

Suppose $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ and $\mathbf{x}' = (x'_1, \dots, x'_n) \in (\mathbb{R}^d)^n$. For $i, j \in \{1, \dots, n\}$ with $i \neq j$, set $\mathbf{x}^i = (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$ and $\mathbf{x}^{ij} = (\mathbf{x}^i)^j$. As done elsewhere in this paper, we identify \mathbf{x} with the set $\{x_1, \dots, x_n\}$ (with any repeated entry in \mathbf{x} being included just once in $\{x_1, \dots, x_n\}$), and likewise for \mathbf{x}' and \mathbf{x}^{ij} .

By the stabilization condition $F(\mathbf{x}) - F(\mathbf{x}^j)$ is determined by x_j, x'_j and the sets $\mathbf{x} \cap B_s(x_j)$ and $\mathbf{x} \cap B_s(x'_j)$. Similarly, $F(\mathbf{x}^i) - F(\mathbf{x}^{ij})$ is determined by x_j, x'_j and the sets $\mathbf{x}^i \cap B_s(x_j)$ and $\mathbf{x}^i \cap B_s(x'_j)$.

If $\|x_i - x_j\|, \|x'_i - x'_j\|, \|x_i - x'_j\|$ and $\|x'_i - x'_j\|$ all exceed s then $\mathbf{x} \cap B_s(x_j) = \mathbf{x}^i \cap B_s(x_j)$ and $\mathbf{x} \cap B_s(x'_j) = \mathbf{x}^i \cap B_s(x'_j)$, so that $F(\mathbf{x}) - F(\mathbf{x}^j) = F(\mathbf{x}^i) - F(\mathbf{x}^{ij})$, and this shows that $G(\mathbf{x}, s)$ really is a symmetric interaction rule for F in the sense of [19], as claimed earlier. \square

We can now provide the binomial counterpart for Proposition 6 using Wasserstein distance. This gives us the penultimate assertion (5.4) of Theorem 6.

Proposition 7. Set $\sigma = \sqrt{\text{Var}[S_{n,k}]}$, and set $b := \limsup_{n \rightarrow \infty} n\theta r^d / (\log n)$. Then

$$d_W(\sigma^{-1}(S_{n,k} - \mathbb{E}[S_{n,k}]), N(0, 1)) = O\left(\frac{n}{\sigma^3}\right) = O(n^{\varepsilon+(3bf_0-1)/2}). \tag{5.21}$$

Proof. Given n , we apply Lemma 12, taking $F(\mathcal{X}) := K_{k,r}(\mathcal{X})$, the number of k -components in $G(\mathcal{X}, r)$ (with $r = r(n)$ as usual). Then $\|DF\|$ is bounded above by a constant independent of n , since for all $x \in \mathbb{R}^d$, $r > 0$ and finite $\mathcal{X} \subset \mathbb{R}^d$ the number of components of $G(\mathcal{X}, r)$ intersecting $B(x, r)$ is bounded by a constant depending only on d (and not on \mathcal{X}, x or r). Also $\|DF\| \geq 1$. By Propositions 1 and 2, $\mathbb{E}[F(\mathcal{X}_n)] \sim I_{n,k}$ and $\sigma^2 \sim I_{n,k}$ as $n \rightarrow \infty$.

Clearly $s = (k + 1)r$ is a radius of stabilization for F . Also $\mathcal{X}_{n+4}(B_{(k+1)r}(X_1))$ is stochastically dominated by $1 + \text{Bin}(n + 4, f_{\max}\theta((k + 1)r)^d)$ and hence $\mathbb{E}[\mathcal{X}_{n+4}(B_{(k+1)r}(X_1))^4] \leq c(nr^d)^4$, for some constant c . Therefore in the present instance, the first term in the right hand side of (5.20) divided by the second term is $O(n^{-1/2}\sigma(nr^d)) = O(n^{-1/2}(nr^d)I_n^{1/2})$, which tends to zero because $I_n^{1/2} = O(n^{1/2}(nr^d)^{(k-1)/2}e^{-cnr^d})$ for some $c > 0$, by (3.3) and (3.8). Then we obtain the first line of (5.21) from (5.20).

For the second line of (5.21) we use Proposition 3, Lemma 2 and (1.1). \square

Proof. (Proof of Theorem 6) We have already proved (5.1), (5.2), (5.3) and (5.4). We now prove the normal approximation result (5.5), using the Poisson approximation result (5.2). By the Berry-Esseen theorem $d_K(t^{-1/2}(Z_t - t), \mathcal{N}) = O(t^{-1/2})$ as $t \rightarrow \infty$. Hence, by the triangle inequality for d_K , and the fact that $d_K(X, Y) \leq d_{TV}(X, Y)$ for any X, Y , we have

$$d_K((\mathbb{E}S_{n,k})^{-1/2}(S_{n,k} - \mathbb{E}S_{n,k}), \mathcal{N}) \leq d_{TV}(S_{n,k}, Z_{\mathbb{E}S_{n,k}}) + O((\mathbb{E}S_{n,k})^{-1/2}). \tag{5.22}$$

By (5.2) the first term in the right hand side of (5.22) is $O(e^{-cnr^d})$ for some $c > 0$.

By Proposition 1 and Lemma 2, $\mathbb{E}S_{n,k} = \Omega(ne^{-nf_0^+ \theta r^d})$. Therefore $(\mathbb{E}S_{n,k})^{-1/2} = o(n^{\varepsilon+(bf_0-1)/2})$. Note that (3.1) implies $bf_0 < 1$. Moreover, using (3.1) and taking both ε and c to be sufficiently small, we see that the second term in the right hand side of (5.22) is also $O(e^{-cnr^d})$, and hence the left hand side of (5.22) is $O(e^{-cnr^d})$ for some $c > 0$. Thus

$$d_K((\text{Var}S_{n,k})^{-1/2}(S_{n,k} - \mathbb{E}S_{n,k}), (\mathbb{E}S_{n,k}/\text{Var}S_{n,k})^{1/2}\mathcal{N}) = d_K((\mathbb{E}S_{n,k})^{-1/2}(S_{n,k} - \mathbb{E}S_{n,k}), \mathcal{N}) = O(e^{-cnr^d}).$$

Now using the fact that $\sup_{t \in [-1/2, 1/2] \setminus \{0\}} t^{-1}d_K((1+t)\mathcal{N}, \mathcal{N}) < \infty$, and Proposition 3, and the triangle inequality for d_K , we can deduce (5.5).

The same argument works for $S'_{n,k}$, for all $d \geq 1$. \square

6. The sparse limiting regime

Fix $k \in \mathbb{N}$. In this section, instead of (3.1) we assume the ‘mildly sparse’ limiting regime

$$\lim_{n \rightarrow \infty} nr^d = 0; \quad \lim_{n \rightarrow \infty} n(nr^d)^{k-1} = \infty. \tag{6.1}$$

In Section 1.1, we mentioned how to obtain a CLT for $S_{n,k}$ from previously known results in this regime under the extra condition that $n(nr^d)^k \rightarrow 0$. We now describe, without giving full details, how we can adapt the methods of this paper to derive limiting expressions for means and variances, and CLTs both for $S_{n,k}$ and $S'_{n,k}$, assuming only (6.1).

We assume $f_{\max} < \infty$ but now we do not need to make any other assumptions on f or on the geometry of the support A of f . Defining $I_{n,k} := \mathbb{E}S'_{n,k}$ as before, we now have

$$I_{n,k} \sim k!^{-1}n(nr^d)^{k-1} \int_{\mathbb{R}^d} f(x)^k dx \int_{(\mathbb{R}^d)^{k-1}} h_1((o, \mathbf{x})) d\mathbf{x} \quad \text{as } n \rightarrow \infty. \tag{6.2}$$

Moreover $\mathbb{E}[S_{n,k}] \sim I_{n,k}$ as $n \rightarrow \infty$. These can be proved along the lines of [1, Propositions 3.1 and 3.2]. By (6.2), (6.1) implies $I_{n,k} \rightarrow \infty$ as $n \rightarrow \infty$. Under (6.1), any factors of the form $e^{-nv(B_r(\mathbf{x}))}$ arising in moment estimates no longer tend to zero, but remain bounded above by 1.

Next we have

$$\text{Var}[S'_{n,k}] = I_{n,k}(1 + O(nr^d)^k).$$

To see this, follow the proof of Proposition 2. In the sparse regime, it can be seen directly from (4.1) and (4.2) that both $J_{1,n}$ and $J_{2,n}$ are $O(n^{2k}r^{d(2k-1)})$, which is $O((nr^d)^k I_{n,k})$.

Next we have

$$d_{TV}(S'_{n,k}, Z_{I_{n,k}}) = O((nr^d)^k). \tag{6.3}$$

For this, we can follow the proof of Proposition 4. We now have $\mathbb{E}[\xi_1(\mathbf{x})] = O((nr^d)^k)$, uniformly over \mathbf{x} . Likewise $\mathbb{E}[\xi_2(\mathbf{x})] = O((nr^d)^k)$.

For $S_{n,k}$, we consider only the case $k \geq 2$ in the sparse limiting regime, since isolated vertices are not rare events. We claim that if $k \geq 2$ then

$$\text{Var}[S_{n,k}] = \mathbb{E}[S_{n,k}](1 + O(nr^d)^{k-1}). \quad (6.4)$$

This is proved by following the proof of Proposition 3. The expression at (4.9) is $O(n^{2k-1}r^{d(2k-2)})$ which is $O(I_{n,k}(nr^d)^{k-1})$. In the penultimate line of (4.11) the last factor (in square brackets) now simplifies to $O(n^{-1})$ so the expression in (4.11) is $O(n^{2k-3})$. Hence the first term in the right hand side of (4.10), multiplied by n^2 , is now $O(n^{2k-1}r^{d(2k-2)}) = O((nr^d)^{k-1}I_{n,k})$. It can be seen directly that the second and third terms in the right hand side of (4.10), multiplied by n^2 , are $O(n^{2k}r^{d(2k-1)})$ which is $O((nr^d)^kI_{n,k})$. Combining these estimates shows that $\text{Var}S_{n,k} - \mathbb{E}S_{n,k} = O((nr^d)^{k-1}I_{n,k}) = O((nr^d)^{k-1}\mathbb{E}S_{n,k})$, as claimed.

Next we claim that by following the proof of Proposition 5, one can show

$$d_{\text{TV}}(S_{n,k}, Z_{\mathbb{E}S_{n,k}}) = O((nr^d)^{k-1}). \quad (6.5)$$

Indeed, for each $i \in [6]$ we have $\mathbb{E}[N_i] = O((nr^d)^j)$ with $j = j(i) \in \{k-1, k, k+1\}$. For N_6 this can now be seen directly without using the more involved proof of Lemma 9. In particular, the restriction to $d \geq 2$ is not needed for (6.5).

By a similar argument to the proof of (5.5), using (6.5) and (6.4), we can obtain that

$$d_{\mathcal{K}}((\text{Var}S_{n,k})^{-1/2}(S_{n,k} - \mathbb{E}S_{n,k}), \mathcal{N}) = O(\max((nr^d)^{k-1}, (n(nr^d)^{k-1})^{-1/2})).$$

Using (6.3) instead of (6.5) we can similarly obtain that

$$d_{\mathcal{K}}((\text{Var}S'_{n,k})^{-1/2}(S'_{n,k} - \mathbb{E}S'_{n,k}), \mathcal{N}) = O(\max((nr^d)^k, (n(nr^d)^{k-1})^{-1/2})).$$

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