Citation information: DOI: 10.1109/TAC.2026.3703305, IEEE Transactions on Automatic Control


# Distributed Recursive State Estimation Over Sensor Networks Under the Push-Pull-Based Gossip Protocol: A Locally Minimum Variance Approach

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***Abstract*—This paper addresses the problem of distributed recursive state estimation over sensor networks utilizing the gossip protocol, where communication resources are conserved by allowing each node to randomly select neighboring nodes for data exchange. An easily implementable and scalable mathematical model is developed for the push-pull-based gossip protocol, facilitating bidirectional data exchange. Random variables are introduced to characterize node selection, with their probability distributions considered as customizable protocol parameters. Distributed state estimators integrated with the gossip protocol are subsequently formulated within the framework of recursive estimation. The relationship between estimation error and gossip protocol parameters is analyzed, leading to the derivation of computationally efficient upper bounds for estimation error covariance matrices. Estimator gains are then designed to minimize the traces of these upper bounds, thereby enhancing estimation accuracy. Furthermore, performance analysis is conducted to establish the monotonicity of estimation precision with respect to noise intensity. Finally, numerical simulations are presented to validate the effectiveness of the proposed method.**

***Index Terms*—Sensor network, distributed state estimation, recursive filtering, push-pull-based gossip protocol.**

## I. Introduction

State estimation, as a fundamental issue in control engineering, has been widely studied in both academia and industry in recent years due primarily to its extensive applications in the process industry, aerospace, navigation, power grids, and other domains [10], [30], [40]. Its core objective is to obtain precise estimates of the unknown system states based on the available measurement information collected over time. The state information is essential for several tasks such as control, decision-making, and fault diagnosis [4], [5], [11], [25], [39], [44], [48]. As a result, increasing attention has been drawn to state estimation problems, leading to significant advancements in various methodologies and applications, e.g., [6], [32], [33], [38], [45], [46], [49].

With the rapid advancement of network technologies, the development of networked state estimation has been presented with both opportunities and challenges. Sensor networks, as a key application scenario of network technologies, are composed of large quantities of spatially distributed sensor nodes that are interconnected according to specific topologies. The utilization of sensor networks offers several advantages, including ease of implementation, structural flexibility, and reduced maintenance costs [21], [34]. Consequently, distributed state estimation over sensor networks has been established as a prominent research topic, benefiting from efficient measurement collection and data exchange in sensor networks [1], [2], [7], [27], [41].

In practice, the application of sensor networks is inevitably constrained by various limitations. A major challenge is posed by the restricted availability of communication resources, such as bandwidth, energy, and computational power. In a sensor network comprising a large number of nodes, the resources allocated to each node are particularly limited. The insufficiency of communication resources typically results in network-induced phenomena, such as packet losses, quantization errors, and delays [35], [43], [47]. These issues contribute to the degradation of data transmission reliability and, consequently, impair the performance of state estimation. As a result, effectively addressing the limitations of communication resources has been recognized as a significant challenge and a focal point in recent research on sensor networks, e.g., [8], [19], [20], [36].

To address the issue of limited communication resources in sensor networks, an effective approach is the implementation of communication protocols to schedule data transmission between nodes, thereby reducing communication resource consumption. In recent years, extensive research has been conducted on various protocols, including the round-robin protocol, random access protocol, try-once-discard protocol, and so on [9], [14], [18], [37], [42]. In engineering practice, the *gossip protocol* has been widely adopted in peer-to-peer communication, primarily due to its ease of implementation and high scalability. This protocol reduces the frequency of data exchanges between neighboring nodes in a stochastic manner, achieving a balance between conserving communication resources and maintaining data-sharing efficiency. Due to its compatibility with sensor networks, the gossip protocol has attracted significant research interest in various studies, such as [12], [28], [31].

The fundamental concept of the gossip protocol is to allow each node in a sensor network to randomly select a subset of its neighbors for data exchange, rather than broadcasting information to all neighbors. This protocol reduces the consumption of communication resources in a straightforward manner and primarily employs three different methods for data exchange between nodes: the pull-based, push-based, and push-pull-based schemes [29]. In the pull-based or push-based scheme, information is either requested from or forwarded to neighboring nodes [24]. In contrast, the push-pull-based scheme enables bidirectional data exchange, allowing a node and its neighbors to share information simultaneously [15]. Although slightly more complex than the other two schemes, the push-pull-based scheme can significantly enhance the efficiency of data dissemination, and has been widely adopted in practice due to this advantage [16], [22].

Although the gossip protocol directly alleviates the burden on communication channels, sharing data with only a subset of neighbors inevitably slows down information dissemination. In distributed state

This work was supported in part by the National Natural Science Foundation of China under Grants 62525308, 62473223, 52172323, and 62473285, in part by the Beijing Natural Science Foundation under Grant L241016, in part by the Fundamental Research Funds for the Central Universities, in part by the China Scholarship Council under Grant 202206210302, in part by the Royal Society of the U.K., and in part by the Alexander von Humboldt Foundation of Germany. *(Corresponding author: Xiao He)*

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estimation over sensor networks, delayed data leads to a degradation in estimation performance to some extent. Furthermore, in the gossip protocol, neighbors for data exchange are selected by each node based on predefined probabilities, which serve as customizable protocol parameters. Understanding how these parameters influence specific estimation performance remains a crucial challenge. In summary, the design of effective distributed state estimators requires a balance between communication resource conservation and estimation precision. Achieving this balance necessitates a comprehensive analysis of the impact of the push-pull-based gossip protocol, which continues to pose a significant challenge.

To the best of our knowledge, several pioneering studies have explored related research questions, such as the problem of communication reduction in distributed filtering [3], as well as the application of the gossip protocol in average consensus [22], [28], [31] and distributed optimization [26]. Nevertheless, within the framework of distributed state estimation, the push-pull-based gossip protocol remains insufficiently studied. In particular, the effects of protocol parameters, specifically the probabilities of each node selecting its neighbors, have not yet been thoroughly investigated. This study aims to bridge these gaps by systematically examining the incorporation of the push-pull-based gossip protocol into distributed state estimation. The key challenges addressed in this paper include: 1) how to describe the push-pull-based gossip protocol in a detailed and mathematically implementable manner? 2) how to integrate the gossip protocol into distributed state estimators and analyze the quantitative relationship between protocol parameters and estimation performance? and 3) how to design estimator gains to optimize estimation accuracy, analyze the impact of noise, and derive conditions for the boundedness of estimation error covariances?

To tackle the challenges discussed above, this paper primarily focuses on the integration of the push-pull-based gossip protocol into distributed state estimation. The main contributions can be summarized as follows.

1) A mathematical model of the gossip protocol that is well-suited for state estimation is established, providing a detailed explanation and formulation of the push-pull-based protocol parameters.
2) With the assistance of random variable definitions, an explicit formulation of the workflow of the push-pull-based gossip protocol is incorporated into the design of the distributed state estimator. Considering the protocol parameters, the impact of the gossip protocol on estimation performance is comprehensively analyzed.
3) Estimator gains are designed to enhance the overall estimation accuracy, and performance analysis is conducted to describe the influence of noise characteristics.

## II. Preliminaries and Problem Formulation

### A. System Description

Consider the following linear time-varying system with both additive and multiplicative noises:

$$x_{k+1} = A_k x_k + \sum_{j=1}^{p} \alpha_{j,k} \bar{A}_{j,k} x_k + F_k u_k + B_k w_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ represents the system state vector, $\alpha_{j,k} \in \mathbb{R}$ denotes the multiplicative noise, and $w_k \in \mathbb{R}^{n_w}$ is the additive process noise. $n_x, n_w, p \in \mathbb{N}^+$ are known integers, and the matrices $A_k, \bar{A}_{j,k}, B_k, F_k$ are given with compatible dimensions. Here, $\mathbb{N}^+$ represents the set of positive integers. In this model, we consider the case with a total of $p$ multiplicative noise terms, corresponding to disturbances on system characteristics in practice.

In this paper, we study the case where a sensor network with $N$ sensor nodes is deployed to monitor the state of the system (1). The topology of the sensor network is represented by an undirected graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E}, \mathcal{L})$, where $\mathcal{V} \triangleq \{1, 2, \cdots, N\}$ stands for the set of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents the set of all edges, and $\mathcal{L} \triangleq [l_{ij}]_{N\times N}$ is the adjacency matrix with $l_{ij} \in \{0, 1\}$. $l_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}$, which means node $i$ is connected with node $j$, and vice versa. With the above definitions, the set of neighbors of node $i$ is defined as $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Furthermore, we assume that no node is connected to itself, i.e., $l_{ii} = 0$ and $i \notin \mathcal{N}_i$ for all $i \in \mathcal{V}$.

The measurement model of the sensor node $i$ is formulated as follows:

$$y_{i,k} = C_{i,k} x_k + \sum_{j=1}^{p} \beta_{i,j,k} \bar{C}_{i,j,k} x_k + v_{i,k}, \tag{2}$$

where $y_{i,k} \in \mathbb{R}^{n_{i,y}}$ denotes the measurement vector produced by the sensor node $i$, $\beta_{i,j,k} \in \mathbb{R}$ represents the multiplicative noise, and $v_{i,k} \in \mathbb{R}^{n_{i,y}}$ is the additive measurement noise. Constants $n_{i,y} \in \mathbb{N}^+$ and matrices $C_{i,k}, \bar{C}_{i,j,k}$ are known, and we also consider the case with $p$ multiplicative noise terms.

Regarding the noises and network topology, the following assumptions are made.

*Assumption 1:* The additive noises $w_k$ and $v_{i,k}$ are zero-mean white Gaussian noises with known covariance matrices $Q_k$ and $R_{i,k}$, respectively. The multiplicative noises $\alpha_{i,k}$ and $\beta_{i,j,k}$ are zero-mean white noise sequences with unity variances, i.e., $\mathbb{E}\left\{\alpha_{i,k}^2\right\} = \mathbb{E}\left\{\beta_{i,j,k}^2\right\} = 1$. The initial system state $x_0$ follows a Gaussian distribution with a known mean $\mu_0$ and covariance $P_0 > 0$.

*Assumption 2:* The random variables $x_0$, $w_k$, $v_{i,k}$, $\alpha_{j,k}$, $\beta_{i,j,k}$ with $i \in \mathcal{V}$ and $j \in \{1, 2, \cdots, p\}$ are mutually independent.

### B. Push-Pull-Based Gossip Protocol

In this study, we consider the case where each sensor node is equipped with a local estimator and the push-pull-based gossip protocol is applied for data exchange between different local estimators. According to the design of the push-pull-based gossip protocol, at each time instant, each node randomly selects one neighbor for data exchange, i.e., sharing its own local estimate and requesting the neighbor's local estimate. Let $\xi_{i,k}$ denote the index of the neighbor selected by node $i$ at time instant $k$. In light of the random data exchange in the gossip protocol, $\xi_{i,k}$ is sampled from a certain probability distribution and then utilized in the estimator as a known integer. Furthermore, the relation $\xi_{i,k} \in \mathcal{N}_i,\ \forall i \in \mathcal{V}, k \in \mathbb{N}^+$ must hold. Furthermore, we make the following assumption.

*Assumption 3:* Each node has at least one neighbor. For $i \in \mathcal{V}$, $\xi_{i,k}$ is i.i.d. for each $k$, and is independent of system states and measurements. For $i \neq j$, $\xi_{i,k}$ and $\xi_{j,k}$ are independent of each other.

Let $\mathrm{Prob}\{\cdot\}$ denote the probability of an event. Then, the probability distribution of $\xi_{i,k}$ is described by

$$\mathrm{Prob}\{\xi_{i,k} = j\} = \pi_{i,j},\ \forall i, j \in \mathcal{V}, \tag{3}$$

where $\pi_{i,j}$ is a pre-defined parameter of the gossip protocol. The constraint

$$\pi_{i,j} \in [0, 1],\ \ \sum_{j=1}^{N} \pi_{i,j} = 1,\ \ \pi_{i,j} = 0,\ \forall j \notin \mathcal{N}_i \tag{4}$$

should also be satisfied since $\pi_{i,j}$ represents a probability distribution and must not violate the constraint arising from the topology.

*Remark 1:* Since $\xi_{i,k} \in \mathcal{N}_i \subseteq \mathcal{V}$ is a discrete-valued random variable, we use $\{\pi_{i,j}\}_{i=1}^{N}$ to describe the probability distribution of $\xi_{i,k}$, which has a direct influence on the state estimation error.

Thus, a proper design of $\pi_{i,j}$ can enhance the overall estimation performance. This design process is quite a meaningful problem and can be considered in future studies.

### C. State Estimation Problem

In this study, a recursive architecture for local estimators is considered. At time instant $k$, node $i$ ($i \in \mathcal{V}$) first computes a one-step state prediction as follows:

$$\hat{x}_{i,k|k-1} = A_{k-1}\hat{x}_{i,k-1|k-1} + F_{k-1}u_{k-1} \tag{5}$$

Then, the push-pull-based gossip protocol is enforced so that all nodes exchange one-step state predictions with each other. Node $i$ also acquires measurement information (i.e., $y_{i,k}$) from the corresponding sensor. Utilizing all the available information, the measurement update for each local state estimator is performed as

$$\begin{aligned}\hat{x}_{i,k|k} &= \hat{x}_{i,k|k-1} + K_{i,k}\left(y_{i,k} - C_{i,k}\hat{x}_{i,k|k-1}\right)\\ &\quad + L_{i,k}\left(\hat{x}_{\xi_{i,k},k|k-1} - \hat{x}_{i,k|k-1}\right)\\ &\quad + L_{i,k}\sum_{\substack{j\in\mathcal{N}_i\\ \xi_{j,k}=i}}\left(\hat{x}_{j,k|k-1} - \hat{x}_{i,k|k-1}\right),\end{aligned} \tag{6}$$

where $K_{i,k}, L_{i,k}$ are estimator gains to be designed. The initial local estimate is set as $\hat{x}_{i,0|0} = \mu_0$ for any $i \in \mathcal{V}$.

*Remark 2:* In the estimator model proposed in (6), the term $K_{i,k}\left(y_{i,k} - C_{i,k}\hat{x}_{i,k|k-1}\right)$ indicates the measurement update, which is the same as a regular Kalman filter. In light of the definition of $\xi_{i,k}$, the term $\hat{x}_{\xi_{i,k},k|k-1}$ represents the information *pulled* from the neighbor selected by the $i$-th local estimator, while the term $\sum_{\substack{j\in\mathcal{N}_i\\ \xi_{j,k}=i}}\left(\hat{x}_{\xi_{j,k},k|k-1} - \hat{x}_{i,k|k-1}\right)$ stands for information *pushed* by other neighbors.

The primary objective of the conventional optimal state estimation problem is to design estimator gains such that the estimates are unbiased and the trace of the estimation error covariance matrix is minimized. Unfortunately, as will be illustrated later, obtaining the exact expression for estimation error covariances is computationally intensive. As an alternative, we pursue a trade-off between computational cost and design conservativeness by calculating and minimizing the trace of upper bounds of estimation error covariances.

## III. Main Results

### A. Analysis of the Estimation Errors

Denote $e_{i,k|k-1} \triangleq x_k - \hat{x}_{i,k|k-1}$ and $e_{i,k|k} \triangleq x_k - \hat{x}_{i,k|k}$ as the one-step prediction error and the estimation error of node $i$ at time instant $k$, respectively. Their covariances are defined as $P_{i,k|k-1} \triangleq \mathbb{E}\left\{e_{i,k|k-1}e_{i,k|k-1}^T\right\}$ and $P_{i,k|k} \triangleq \mathbb{E}\left\{e_{i,k|k}e_{i,k|k}^T\right\}$. To facilitate the subsequent derivation, the Kronecker delta function is introduced as

$$\delta(i,j) = \begin{cases}1, & \text{if } i = j\\ 0, & \text{if } i \neq j.\end{cases} \tag{7}$$

Using the above notations, (5)–(6) can be rewritten as

$$\hat{x}_{i,k|k-1} = A_{k-1}\hat{x}_{i,k-1|k-1} + F_{k-1}u_{k-1} \tag{8}$$

$$\begin{aligned}\hat{x}_{i,k|k} &= \hat{x}_{i,k|k-1} + K_{i,k}\left(y_{i,k} - C_{i,k}\hat{x}_{i,k|k-1}\right)\\ &\quad + L_{i,k}\sum_{j=1}^{N}\delta\left(\xi_{i,k},j\right)\left(\hat{x}_{j,k|k-1} - \hat{x}_{i,k|k-1}\right)\\ &\quad + L_{i,k}\sum_{j=1}^{N}\delta\left(\xi_{j,k},i\right)\left(\hat{x}_{j,k|k-1} - \hat{x}_{i,k|k-1}\right).\end{aligned} \tag{9}$$

It follows from (1)–(2) that

$$e_{i,k|k-1} = A_{k-1}e_{i,k-1|k-1} + \sum_{j=1}^{p}\alpha_{j,k-1}\bar{A}_{j,k-1}x_{k-1} + B_{k-1}w_{k-1}. \tag{10}$$

Since $\xi_{i,k} \in \mathcal{N}_i \subset \{1, 2, \cdots, N\}$, we have $\sum_{j=1}^{N}\delta\left(\xi_{i,k}, j\right) = 1$, and thus

$$\begin{aligned}e_{i,k|k} &= \left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\delta\left(\xi_{j,k},i\right)L_{i,k}\right)e_{i,k|k-1}\\ &\quad - K_{i,k}\left(\sum_{j=1}^{p}\beta_{i,j,k}\bar{C}_{i,j,k}x_k + v_{i,k}\right)\\ &\quad + L_{i,k}\sum_{j=1}^{N}\left(\delta\left(\xi_{i,k},j\right) + \delta\left(\xi_{j,k},i\right)\right)e_{j,k|k-1}.\end{aligned} \tag{11}$$

In light of the above formulations, the unbiasedness of the designed estimators is analyzed in the following theorem.

*Theorem 1:* The state estimators described by (5)–(6) are unbiased, i.e., $\mathbb{E}\left\{e_{i,k|k}\right\} = 0$ for any $i \in \mathcal{V}$ and $k \in \mathbb{N}^+$.

*Proof:* We prove this theorem by using mathematical induction. First, based on Assumption 1 and the initial estimates, it follows that $\mathbb{E}\left\{e_{i,0|0}\right\} = \mathbb{E}\left\{x_0\right\} - \mu_0 = 0$.

Assuming that $\mathbb{E}\left\{e_{i,k|k}\right\} = 0$ for all $i \in \mathcal{V}$, it can be derived from Assumptions 1–2 and (11) that

$$\begin{aligned}\mathbb{E}\left\{e_{i,k+1|k}\right\} &= A_k\mathbb{E}\left\{e_{i,k|k}\right\} + B_k\mathbb{E}\left\{w_k\right\}\\ &\quad + \sum_{j=1}^{p}\mathbb{E}\left\{\alpha_{j,k}\right\}\bar{A}_{j,k}\mathbb{E}\left\{x_k\right\} = 0 \quad (\forall i \in \mathcal{V}).\end{aligned}$$

Then, we immediately arrive at

$$\begin{aligned}&\mathbb{E}\left\{e_{i,k+1|k+1}\right\}\\ &= \left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\delta\left(\xi_{j,k},i\right)L_{i,k}\right)\mathbb{E}\left\{e_{i,k|k-1}\right\}\\ &\quad - K_{i,k}\left(\mathbb{E}\left\{v_{i,k}\right\} + \sum_{k=1}^{p}\mathbb{E}\left\{\beta_{i,j,k}\right\}\bar{C}_{i,j,k}\mathbb{E}\left\{x_k\right\}\right)\\ &\quad - L_{i,k}\sum_{j=1}^{N}\mathbb{E}\left\{\delta\left(\xi_{i,k},j\right) + \delta\left(\xi_{j,k},i\right)\right\}\mathbb{E}\left\{e_{j,k|k-1}\right\} = 0,\end{aligned} \tag{12}$$

which completes the proof. ■

To compute the precise expression of $P_{i,k|k}$, up to $N^2 - N$ cross-terms such as $\mathbb{E}\left\{e_{i,k|k-1}e_{j,k|k-1}\right\}$ with $i, j \in \mathcal{V}$ must be considered, which imposes a significant computational burden, particularly in sensor networks with limited computational resources. To cope with this issue, we derive and minimize upper bounds of $P_{i,k|k}$ as a computationally tractable alternative. Before proceeding further, we present the following lemmas as fundamental tools.

*Lemma 1 ( [17]):* For vectors $x, y$ with the same dimension and any positive scalar $a > 0$, the following inequality holds:

$$xy^T + yx^T \preceq axx^T + a^{-1}yy^T.$$

As a direct extension of Lemma 1, we also have the following lemma.

*Lemma 2:* For $N$ vectors $x_1, x_2, \cdots, x_N$ and positive scalars $a_1, a_2, \cdots, a_{N-1} > 0$, the following relation holds:

$$\left(\sum_{i=1}^{N}x_i\right)\left(\sum_{i=1}^{N}x_i\right)^T \preceq \sum_{i=1}^{N}\eta_i x_i x_i^T,$$

where

$$\eta_1 \triangleq 1 + a_1 \tag{13}$$

$$\eta_j \triangleq (1+a_j)\prod_{i=1}^{j-1}\left(1+a_i^{-1}\right),\ j\in\{2,3,\cdots,N-1\} \tag{14}$$

$$\eta_N \triangleq \prod_{i=1}^{N-1}\left(1+a_i^{-1}\right). \tag{15}$$

Additionally, based on (1), the sequence $X_k \triangleq \mathbb{E}\left\{x_k x_k^T\right\}$ can be computed using the following recursion:

$$\begin{aligned} X_{k+1} = {} & A_k X_k A_k^T + \sum_{j=1}^{p}\bar{A}_{j,k}X_k\bar{A}_{j,k}^T + B_k Q_k B_k^T \\ & + A_k\mu_k u_k^T F_k^T + F_k u_k \mu_k^T A_k^T, \end{aligned} \tag{16}$$

where $X_0 = P_0 + \mu_0\mu_0^T,\ \ \mu_{k+1} = A_k\mu_k + F_k u_k$.

Using Lemma 2 and (16), an upper bound of $P_{i,k|k}$ is calculated through the following theorem.

*Theorem 2:* Given positive scalars $a_1, a_2, \cdots, a_{N-1}$, for any $i \in \mathcal{V}$ and $k \in \mathbb{N}^+$, upper bounds of $P_{i,k|k-1}$ and $P_{i,k|k}$ satisfying $P_{i,k|k-1} \preceq \Sigma_{i,k|k-1},\ \ P_{i,k|k} \preceq \Sigma_{i,k|k}$, denoted as $\Sigma_{i,k|k-1}$ and $\Sigma_{i,k|k}$, can be computed using the recursion

$$\begin{aligned} \Sigma_{i,k|k-1} = {} & A_{k-1}\Sigma_{i,k-1|k-1}A_{k-1}^T + \sum_{j=1}^{p}\bar{A}_{j,k-1}X_{k-1}\bar{A}_{j,k-1}^T \\ & + B_{k-1}Q_{k-1}B_{k-1}^T \\ \Sigma_{i,k|k} = {} & \eta_i\left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\pi_{j,i}L_{i,k}\right)\Sigma_{i,k|k-1} \\ & \times\left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\pi_{j,i}L_{i,k}\right)^T \\ & + \eta_i\sum_{j=1}^{N}\left(\pi_{j,i} - \pi_{j,i}^2\right)L_{i,k}\Sigma_{i,k|k-1}L_{i,k}^T \\ & + \sum_{\substack{j=1\\ j\neq i}}^{N}\eta_j\left(\pi_{i,j} + \pi_{j,i} + 2\pi_{i,j}\pi_{j,i}\right)L_{i,k}\Sigma_{j,k|k-1}L_{i,k}^T \\ & + K_{i,k}R_{i,k}K_{i,k}^T + \sum_{j=1}^{N}K_{i,k}\bar{C}_{i,j,k}X_k\bar{C}_{i,j,k}^T K_{i,k}^T \end{aligned}$$

with initial condition $\Sigma_{0|0} = P_0$ and $\eta_i$ given in Lemma 2.

*Proof:* The proof is conducted using mathematical induction. Clearly, the initial condition $P_{0|0} = \Sigma_{0|0} = P_0$ holds. Assuming that $P_{i,k-1|k-1} \preceq \Sigma_{i,k-1|k-1}$, we aim to prove that $P_{i,k|k-1} \preceq \Sigma_{i,k|k-1}$ and $P_{i,k|k} \preceq \Sigma_{i,k|k}$ hold.

Using Assumptions 1–2, (10), (16), and the relation $P_{i,k-1|k-1} \preceq \Sigma_{i,k-1|k-1}$, it follows that, for any $i \in \mathcal{V}$,

$$\begin{aligned} P_{i,k|k-1} = {} & A_{k-1}\Sigma_{i,k-1|k-1}A_{k-1}^T \\ & + A_{k-1}\left(P_{i,k-1|k-1} - \Sigma_{i,k-1|k-1}\right)A_{k-1}^T \\ & + \sum_{j=1}^{p}\bar{A}_{j,k-1}X_{k-1}\bar{A}_{j,k-1}^T + B_{k-1}Q_{k-1}B_{k-1}^T \\ \preceq {} & A_{k-1}\Sigma_{i,k-1|k-1}A_{k-1}^T + \sum_{j=1}^{p}\bar{A}_{j,k-1}X_{k-1}\bar{A}_{j,k-1}^T \\ & + B_{k-1}Q_{k-1}B_{k-1}^T = \Sigma_{i,k|k-1}. \end{aligned} \tag{17}$$

According to the topology, for $i = j$, we have $\delta(\xi_{i,k}, j) = \delta(\xi_{j,k}, i) = 0$. Then, (11) can be rewritten as

$$e_{i,k|k} = \xi_{i,k|k}^{(1)} - \xi_{i,k|k}^{(2)}, \tag{18}$$

where

$$\begin{aligned} \xi_{i,k|k}^{(1)} \triangleq {} & \left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\delta\left(\xi_{j,k}, i\right)L_{i,k}\right)e_{i,k|k-1} \\ & + L_{i,k}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\delta\left(\xi_{i,k}, j\right) + \delta\left(\xi_{j,k}, i\right)\right)e_{j,k|k-1} \\ \xi_{i,k|k}^{(2)} \triangleq {} & K_{i,k}\left(\sum_{j=1}^{p}\beta_{i,j,k}\bar{C}_{i,j,k}x_k + v_{i,k}\right). \end{aligned}$$

It follows from Assumptions 1–3 that $\xi_{i,k|k}^{(1)}$ and $\xi_{i,k|k}^{(2)}$ are uncorrelated. According to (16), we arrive at

$$\begin{aligned} & \mathbb{E}\left\{\xi_{i,k|k}^{(2)}\left(\xi_{i,k|k}^{(2)}\right)^T\right\} \\ & = K_{i,k}R_{i,k}K_{i,k}^T + \sum_{j=1}^{N}K_{i,k}\bar{C}_{i,j,k}X_k\bar{C}_{i,j,k}^T K_{i,k}^T. \end{aligned} \tag{19}$$

In light of (13)–(15), applying Assumption 2, Lemma 2, and the properties of the Kronecker delta function yields

$$\begin{aligned} & \mathbb{E}\left\{\xi_{i,k|k}^{(1)}\left(\xi_{i,k|k}^{(1)}\right)^T\right\} \\ & \preceq \eta_i\left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\pi_{j,i}L_{i,k}\right)P_{i,k|k-1} \\ & \quad \times\left(I - K_{i,k}C_{i,k} - L_{i,k} - \sum_{j=1}^{N}\pi_{j,i}L_{i,k}\right)^T \\ & \quad + \eta_i\sum_{j=1}^{N}\left(\pi_{j,i} - \pi_{j,i}^2\right)L_{i,k}P_{i,k|k-1}L_{i,k}^T \\ & \quad + \sum_{\substack{j=1\\ j\neq i}}^{N}\eta_j\left(\pi_{i,j} + \pi_{j,i} + 2\pi_{i,j}\pi_{j,i}\right)L_{i,k}P_{j,k|k-1}L_{i,k}^T. \end{aligned} \tag{20}$$

Utilizing (18)–(19), (20), and $P_{i,k|k-1} \preceq \Sigma_{i,k|k-1}\ \ (\forall i \in \mathcal{V})$, we have

$$P_{i,k|k} = \mathbb{E}\left\{\xi_{i,k|k}^{(1)}\left(\xi_{i,k|k}^{(1)}\right)^T\right\} + \mathbb{E}\left\{\xi_{i,k|k}^{(2)}\left(\xi_{i,k|k}^{(2)}\right)^T\right\} \preceq \Sigma_{i,k|k}, \tag{21}$$

which, along with (17), completes the proof. ■

### B. Design of Estimator Gains

As stated in (21), the upper bound of the error covariance, i.e., $\Sigma_{i,k|k}$, is dependent on both $K_{i,k}$ and $L_{i,k}$. To improve the overall estimation precision, we present the following theorem for designing the estimator gains.

*Theorem 3:* Given positive scalars $a_1, a_2, \cdots, a_{N-1}$, let $\eta_1, \eta_2, \cdots, \eta_N$ be computed using (13)–(15). Denote

$$K_{i,k}^* = \left(I - \left(1 + \sum_{j=1}^{N}\pi_{j,i}\right)L_{i,k}^*\right)\Sigma_{i,k|k-1}C_{i,k}^T N_{i,k}^{-1} \tag{22}$$

$$L_{i,k}^* = \eta_i\left(1 + \sum_{j=1}^{N}\pi_{j,i}\right)M_{i,k}S_{i,k}^{-1} \tag{23}$$

where

$$N_{i,k} \triangleq C_{i,k}\Sigma_{i,k|k-1}C_{i,k}^T + \frac{1}{\eta_i}R_{i,k} + \frac{1}{\eta_i}\sum_{j=1}^{p}\bar{C}_{i,j,k}X_k\bar{C}_{i,j,k}^T \tag{24}$$

$$M_{i,k} \triangleq \Sigma_{i,k|k-1} - \Sigma_{i,k|k-1}C_{i,k}^T N_{i,k}^{-1}C_{i,k}\Sigma_{i,k|k-1} \tag{25}$$

$$T_{i,k} \triangleq \eta_i \sum_{j=1}^{N} \left(\pi_{j,i} - \pi_{j,i}^2\right) \Sigma_{i,k|k-1} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \eta_j \left(\pi_{i,j} + \pi_{j,i} + 2\pi_{i,j}\pi_{j,i}\right) \Sigma_{j,k|k-1} \tag{26}$$

$$S_{i,k} \triangleq \eta_i \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right)^2 M_{i,k} + T_{i,k}. \tag{27}$$

Then, assigning $K_{i,k} = K_{i,k}^*$ and $L_{i,k} = L_{i,k}^*$ minimizes $\operatorname{tr}\left\{\Sigma_{i,k|k}\right\}$, which is aimed at reducing the level of estimation error.

*Proof:* For simplicity, denote

$$F_{i,k} \triangleq I - \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right) L_{i,k}.$$

Using the method of completing the square, (21) can be rearranged as

$$\begin{aligned}
&\Sigma_{i,k|k} \\
&= \eta_i K_{i,k} N_{i,k} K_{i,k}^T + \eta_i F_{i,k} \Sigma_{i,k|k-1} F_{i,k}^T \\
&\quad - \eta_i K_{i,k} C_{i,k} \Sigma_{i,k|k-1} F_{i,k}^T - \eta_i F_{i,k} \Sigma_{i,k|k-1} C_{i,k}^T K_{i,k}^T \\
&\quad + \eta_i \sum_{j=1}^{N} \left(\pi_{j,i} - \pi_{j,i}^2\right) L_{i,k} \Sigma_{i,k|k-1} L_{i,k}^T \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} \eta_j \left(\pi_{i,j} + \pi_{j,i} + 2\pi_{i,j}\pi_{j,i}\right) L_{i,k} \Sigma_{j,k|k-1} L_{i,k}^T \\
&= \eta_i \left(K_{i,k} - F_{i,k}\Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1}\right) N_{i,k} \\
&\qquad \times \left(K_{i,k} - F_{i,k}\Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1}\right)^T + \eta_i F_{i,k} M_{i,k} F_{i,k}^T \\
&\quad + \eta_i \sum_{j=1}^{N} \left(\pi_{j,i} - \pi_{j,i}^2\right) L_{i,k} \Sigma_{i,k|k-1} L_{i,k}^T \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} \eta_j \left(\pi_{i,j} + \pi_{j,i} + 2\pi_{i,j}\pi_{j,i}\right) L_{i,k} \Sigma_{j,k|k-1} L_{i,k}^T \\
&= \eta_i \left(K_{i,k} - F_{i,k}\Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1}\right) N_{i,k} \\
&\qquad \times \left(K_{i,k} - F_{i,k}\Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1}\right)^T \\
&\quad + \eta_i M_{i,k} + L_{i,k} S_{i,k} L_{i,k}^T \\
&\quad - \eta_i \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right) \left(L_{i,k} M_{i,k} + M_{i,k} L_{i,k}^T\right) \\
&= \eta_i \left(K_{i,k} - F_{i,k}\Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1}\right) N_{i,k} \\
&\qquad \times \left(K_{i,k} - F_{i,k}\Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1}\right)^T + \eta_i M_{i,k} \\
&\quad + \left(L_{i,k} - \eta_i \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right) M_{i,k} S_{i,k}^{-1}\right) S_{i,k} \\
&\qquad \times \left(L_{i,k} - \eta_i \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right) M_{i,k} S_{i,k}^{-1}\right)^T \\
&\quad + \eta_i M_{i,k} - \eta_i^2 \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right)^2 M_{i,k} S_{i,k}^{-1} M_{i,k}.
\end{aligned} \tag{28}$$

It can be seen from (28) that simultaneously assigning

$$L_{i,k} = \eta_i \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right) M_{i,k} S_{i,k}^{-1} = L_{i,k}^* \tag{29}$$

$$K_{i,k} = F_{i,k} \Sigma_{i,k|k-1} C_{i,k}^T N_{i,k}^{-1} = K_{i,k}^* \tag{30}$$

minimizes $\operatorname{tr}\left\{\Sigma_{i,k|k}\right\}$, which completes the proof. ■

*Corollary 1:* After applying the designed estimator gains given in (22)–(23), $\Sigma_{i,k|k-1}$ and $\Sigma_{i,k|k}$ satisfy the following recursion:

$$\Sigma_{i,k|k-1} = A_{k-1} \Sigma_{i,k-1|k-1} A_{k-1}^T + \sum_{j=1}^{p} \bar{A}_{j,k-1} X_{k-1} \bar{A}_{j,k-1}^T + B_{k-1} Q_{k-1} B_{k-1}^T \tag{31}$$

$$\Sigma_{i,k|k} = \eta_i M_{i,k} - \eta_i^2 \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right)^2 M_{i,k} S_{i,k}^{-1} M_{i,k} \tag{32}$$

with initial condition $\Sigma_{i,0|0} = P_0$ and $M_{i,k}$, $S_{i,k}$ defined in (25), (27), respectively.

### C. Performance Analysis

In this part, we conduct performance analysis to evaluate factors influencing the estimation accuracy. Before proceeding further, we present the matrix inversion lemma as an essential tool.

*Lemma 3 (Matrix Inversion Lemma, [13]):* For invertible matrices $A, C$ and matrices $U, V$ with compatible dimensions, the following identity holds:

$$(A + UCV)^{-1} = A^{-1} - A^{-1} U \left(C^{-1} + V A^{-1} U\right)^{-1} V A^{-1}.$$

For node $i$, consider two scenarios where the only difference is that the covariance matrices of the additive measurement noise are given as $R_{i,k} = R_{i,k}^{(a)}$ and $R_{i,k} = R_{i,k}^{(b)}$, respectively. Following Theorem 3 and Corollary 1, the upper bounds of estimation error covariance matrices with minimized trace are denoted as $\Sigma_{i,k|k}^{(a)}$ and $\Sigma_{i,k|k}^{(b)}$. Using the above notations, the following theorem is established.

*Theorem 4:* If $R_{i,k}^{(a)} < R_{i,k}^{(b)}$, then $\Sigma_{i,k|k}^{(a)} < \Sigma_{i,k|k}^{(b)}$, i.e., $\Sigma_{i,k|k}$ is monotonically increasing with respect to $R_{i,k}$.

*Proof:* By combining (24)–(25) and applying the matrix inversion lemma, we have

$$M_{i,k}^{-1} = \Sigma_{i,k|k-1}^{-1} + \eta_i C_{i,k}^T \left(R_{i,k} + \sum_{j=1}^{p} \bar{C}_{i,j,k} X_k \bar{C}_{i,j,k}^T\right)^{-1} C_{i,k}. \tag{33}$$

According to (27), (32) can be rewritten as

$$\Sigma_{i,k|k} = \eta_i M_{i,k} - \eta_i M_{i,k} \left(\eta_i M_{i,k} + \frac{T_{i,k}}{\left(1 + \sum_{j=1}^{N} \pi_{j,i}\right)^2}\right) \eta_i M_{i,k}, \tag{34}$$

which, by applying the matrix inversion lemma, leads to

$$\begin{aligned}
\Sigma_{i,k|k}^{-1} &= \frac{M_{i,k}^{-1}}{\eta_i} + \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right)^2 T_{i,k}^{-1} \\
&= \frac{\Sigma_{i,k|k-1}^{-1}}{\eta_i} + \left(1 + \sum_{j=1}^{N} \pi_{j,i}\right)^2 T_{i,k}^{-1} \\
&\quad + C_{i,k}^T \left(R_{i,k} + \sum_{j=1}^{p} \bar{C}_{i,j,k} X_k \bar{C}_{i,j,k}^T\right)^{-1} C_{i,k}.
\end{aligned} \tag{35}$$

Replacing $R_{i,k}$ in (35) with $R_{i,k}^{(a)}$ and $R_{i,k}^{(b)}$, respectively, we arrive at

$$\left(R_{i,k}^{(a)} + \sum_{j=1}^{p} \bar{C}_{i,j,k} X_k \bar{C}_{i,j,k}^T\right)^{-1} > \left(R_{i,k}^{(b)} + \sum_{j=1}^{p} \bar{C}_{i,j,k} X_k \bar{C}_{i,j,k}^T\right)^{-1},$$

which indicates that

$$\left(\Sigma^{(a)}_{i,k|k}\right)^{-1} > \left(\Sigma^{(b)}_{i,k|k}\right)^{-1}$$

or, equivalently, $\Sigma^{(a)}_{i,k|k} < \Sigma^{(b)}_{i,k|k}$. Then, the proof is complete. ■

For node $i$, let $j \in \{1, 2, \cdots, p\}$ be given. Similarly, consider two cases where $\bar{C}_{i,j,k} = \bar{C}^{(a)}_{i,j,k}$ and $\bar{C}_{i,j,k} = \bar{C}^{(b)}_{i,j,k}$, respectively. Then, $\Sigma_{i,k|k}$ can be computed as $\Sigma^{(a)}_{i,k|k}$ and $\Sigma^{(b)}_{i,k|k}$. Along a similar line of the proof of Theorem 4, we obtain the following result readily.

*Corollary 2:* If $\bar{C}^{(a)}_{i,j,k} X_k \left(\bar{C}^{(a)}_{i,j,k}\right)^T < \bar{C}^{(b)}_{i,j,k} X_k \left(\bar{C}^{(b)}_{i,j,k}\right)^T$, then $\Sigma^{(a)}_{i,k|k} < \Sigma^{(b)}_{i,k|k}$.

*Proof:* The proof closely follows that of Theorem 4 and is therefore omitted here. ■

With respect to the process noise $w_k$ and the multiplicative noise $\alpha_{j,k}$, we have similar results. Specifically, consider two cases where $Q_{k-1} = Q^{(a)}_{k-1}$, $\bar{A}_{j,k-1} = \bar{A}^{(a)}_{j,k-1}$ and $Q_{k-1} = Q^{(b)}_{k-1}$, $\bar{A}_{j,k-1} = \bar{A}^{(b)}_{j,k-1}$, respectively, where $j \in \{1, 2, \cdots, p\}$. Let $\Sigma^{(a)}_{i,k|k}$ and $\Sigma^{(b)}_{i,k|k}$ be the corresponding values of $\Sigma_{i,k|k}$. Then, we have the following corollary.

*Corollary 3:* The relation $\Sigma^{(a)}_{i,k|k} < \Sigma^{(b)}_{i,k|k}$ holds if

$$\begin{aligned} &B_{k-1} Q^{(a)}_{k-1} B^T_{k-1} + \sum_{j=1}^{p} \bar{A}^{(a)}_{j,k-1} X_{k-1} \left(\bar{A}^{(a)}_{j,k-1}\right)^T \\ <&B_{k-1} Q^{(b)}_{k-1} B^T_{k-1} + \sum_{j=1}^{p} \bar{A}^{(b)}_{j,k-1} X_{k-1} \left(\bar{A}^{(b)}_{j,k-1}\right)^T . \end{aligned}$$

*Proof:* It follows directly from (31) that $\Sigma^{(a)}_{i,k|k-1} < \Sigma^{(b)}_{i,k|k-1}$ for any $i \in \mathcal{V}$, where $\Sigma^{(a)}_{i,k|k-1}$ and $\Sigma^{(b)}_{i,k|k-1}$ are the corresponding values of $\Sigma_{i,k|k-1}$. Following a similar approach to the proof of Theorem 4, it can be easily verified that $\Sigma^{(a)}_{i,k|k-1} < \Sigma^{(b)}_{i,k|k-1}$, thereby completing the proof. ■

*Remark 3:* In Theorem 4 and Corollaries 2–3, it has been rigorously established that increasing the levels of the noises (including process noise, measurement noise, and multiplicative noise) consistently degrades estimation accuracy. This result confirms the existence of a monotonic relationship between estimation performance and noise intensity.

## IV. Numerical Simulations

### A. System Setting

Since sensor networks are widely applied in power grid systems, in this section, we consider a typical circuit system in the simulation experiment [23]. Parameters of the discrete-time system model are given as follows:

$$A_k = \begin{bmatrix} 0.4231 & -0.5769 & 1.1538 \\ -0.5769 & 0.4231 & -0.3846 \\ -1.5 & 0.5 & -2.0 \end{bmatrix}, \quad F_k = \begin{bmatrix} 0.1923 \\ 0.1923 \\ 1.5 \end{bmatrix},$$

$$B_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A}_{1,k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.015 & -0.005 & 0.03 \end{bmatrix}$$

with $n_x = 3$ and $p = 1$. As to the measurement, we consider a sensor network with $N = 6$ nodes with the topology shown in Fig. 1.

The measurement matrices $C_{i,k}$ ($i \in \{1, 2, \cdots, 6\}$) are given as

$$\begin{aligned} &C_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ &C_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} -0.75 & -0.75 & -1.5 \end{bmatrix}, \\ &C_5 = \begin{bmatrix} 1.5 & 1.5 & 3 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where we assume that node 6 is unable to directly measure the system state and must rely on data exchange with its neighbors.

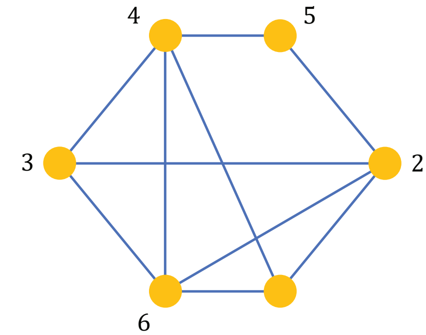


Fig. 1: Topology of the sensor network

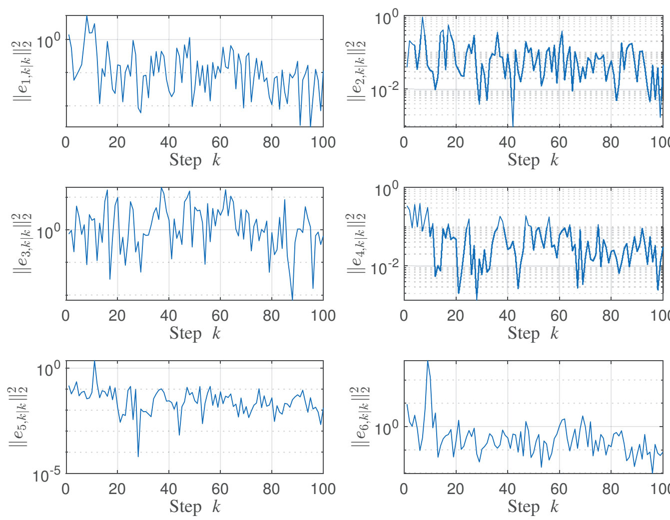


Fig. 2: Local estimation error at each node

The process noise and measurement noise, namely $w_k$ and $v_{i,k}$ with $i \in \mathcal{V}$, are set as zero-mean Gaussian white noise with covariances $Q_k = 10^{-2} I$ and $R_{i,k} = 2.5 \times 10^{-3}$. Moreover, the multiplicative noises $\alpha_{j,k}$ and $\beta_{i,j,k}$ with $i \in \mathcal{V}$ and $j \in \{1, 2, \cdots, p\}$ are subject to standard normal distribution.

### B. Assessment of Estimation Performance

The norm of estimation error at each node, denoted as $\left\|e_{i,k|k}\right\|_2^2$, is shown in Fig. 2. It is illustrated that, even under the influence of noise, the estimation error at each node is well controlled.

To guide the data exchange between nodes, the matrix $\Pi \triangleq [\pi_{i,j}]_{N \times N}$ is given by

$$\Pi = \begin{bmatrix} 0 & 0.5 & 0 & 0.3 & 0 & 0.2 \\ 0.3 & 0 & 0.2 & 0 & 0.2 & 0.3 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0.1 & 0 & 0.2 & 0 & 0.4 & 0.3 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0.2 & 0.2 & 0.4 & 0.2 & 0 & 0 \end{bmatrix}.$$

Then, the communication traffic at node 6 during the first 50 time instants under the schedule of the gossip protocol is illustrated in Fig. 3. The blue squares represent nodes that node 6 pulls data from, while orange circles stand for nodes that push data to node 6.

We also compare the performance of the proposed method with a traditional distributed Kalman filter (DKF), which is formulated as follows:

$$\hat{x}_{i,k|k-1} = A_{k-1} \hat{x}_{i,k-1|k-1} + F_{k-1} u_{k-1}$$

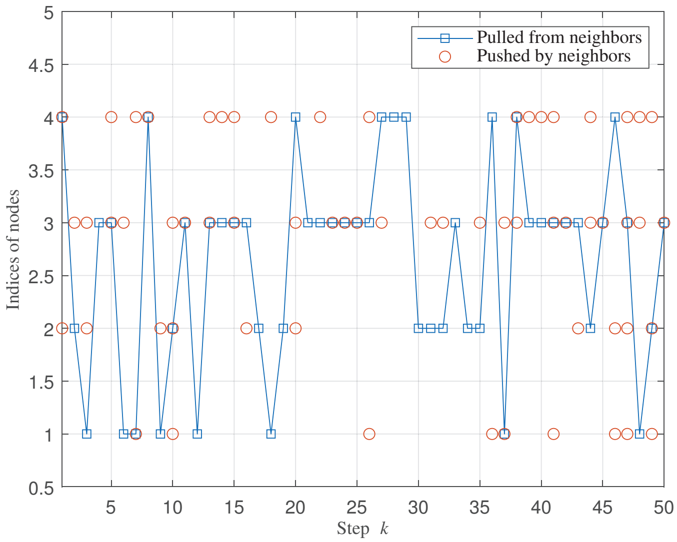


Fig. 3: Data exchange performed by node 6

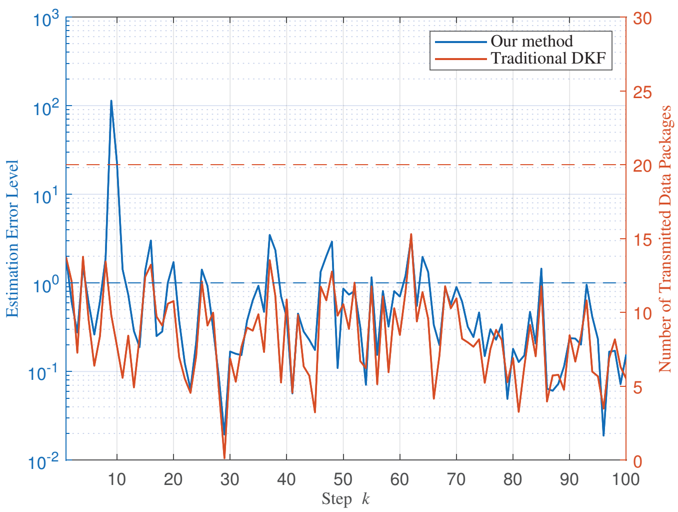


Fig. 4: Comparison of the proposed method and traditional DKF

$$\begin{aligned}\hat{x}_{i,k|k} &= \hat{x}_{i,k|k-1} + K_{i,k}\left(y_{i,k} - C_{i,k}\hat{x}_{i,k|k-1}\right) \\ &\quad + L_{i,k}\sum_{j\in\mathcal{N}_i}\left(\hat{x}_{j,k|k-1} - \hat{x}_{i,k|k-1}\right).\end{aligned}$$

The results are shown in Fig. 4, where the solid lines indicate the level of the estimation error, which is denoted as

$$\mathrm{ErrorLevel}_k \triangleq \frac{1}{N}\sum_{i=1}^{N}\left\|x_k - \hat{x}_{i,k|k}\right\|_2^2.$$

Furthermore, the dashed lines represent the consumption of communication resources, where we assume that one package is delivered when a vector $\hat{x}_{i,k|k-1}$ $(i \in \mathcal{V})$ is transmitted from one node to another. It is illustrated in Fig. 4 that the proposed method significantly reduces the consumption of communication resources at the cost of a slight performance degradation.

### C. Effects of the Gossip Protocol

As shown in the system setting, the sensor installed at node 6 is not capable of directly measuring the system, so this node must rely on data exchange between neighbors to effectively obtain state estimates. To illustrate this, we compare the estimator gain matrices of node 1 and node 6 at the last time instant. The gain matrices are as follows:

$$K_{1,100} = \begin{bmatrix} 1.1374 \\ -0.1421 \\ -1.8287 \end{bmatrix}, \quad K_{6,100} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$L_{1,100} = \begin{bmatrix} 0.1169 & -0.2765 & 0.0921 \\ -0.1175 & 0.2774 & -0.0928 \\ -0.0397 & 0.1494 & -0.0241 \end{bmatrix},$$

$$L_{6,100} = \begin{bmatrix} 0.4674 & -0.2366 & 0.0816 \\ 0.0686 & 0.2248 & 0.0567 \\ -0.0603 & 0.1828 & 0.3217 \end{bmatrix}.$$

As illustrated in the above extreme case, node 6 assigns zero gain to the sensor and completely relies on data exchange with its neighbors. In contrast, node 1 tends to “trust” its own measurements and assigns relatively small gains for information from neighbors.

## V. Conclusions

In this paper, the problem of distributed recursive state estimation has been addressed for a class of time-varying systems with multiplicative noise over sensor networks, where the push-pull-based gossip protocol is implemented. A mathematical model of the gossip protocol, specifically designed for distributed state estimation, has been developed, in which random variables are introduced to represent the process by which each node randomly selects its neighbors for bidirectional data exchange. The probabilities governing each node’s selection of neighbors have been treated as customizable protocol parameters. Through a comprehensive analysis of the estimation error, the relationship between protocol parameters and estimation error covariances has been examined. Estimator gains have been designed to minimize the trace of the upper bounds of the estimation error covariance matrices, providing a computationally efficient alternative. Performance analysis has been conducted to establish the monotonicity of estimation performance with respect to noise intensity. Finally, simulation experiments have been performed to verify the effectiveness of the proposed methodology.

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