Orthogonality relations for fluid-structural waves in a 3-D, rectangular duct with flexible walls

J. B. Lawrie

Department of Mathematical Sciences
Brunel University
Uxbridge, UB8 3PH, UK
May 2009

Abstract
An exact expression for the fluid-coupled structural waves that propagate in a three-dimensional, rectangular waveguide with elastic walls is presented in terms of the non-separable eigenfunctions, $\psi_n(y, z)$. It is proved that these eigenfunctions are linearly dependent and that an eigenfunction expansion representation of a suitably smooth function, $f(y, z)$, converges point-wise to that function. Orthogonality results for the derivatives $\psi_{ny}(a, z)$ are derived which, together with a partial orthogonality relation for $\psi_n(y, z)$, enable the solution of a wide range of acoustic scattering problems. Two prototype problems, of the type typically encountered in two-part scattering problems, are solved and numerical results showing the displacement of the elastic walls are presented.

Keywords: Three-dimensional waveguide, flexible wall, elastic plate, orthogonality relation, acoustic scattering, mode-matching.

1 Introduction

The propagation of acoustic waves along ducts or pipes has long been of interest to scientists and engineers. This is partly because ducting systems, such as those used for heating, ventilation and air-conditioning (HVAC), provide ideal channels for the transmission noise through structures such as buildings and aircraft. Acoustic scattering is a feature of ducting systems that becomes relevant whenever there is an abrupt change in geometry or material property. At such a discontinuity the incident sound field undergoes both reflection and transmission, usually initiating all propagating higher-order modes in the process. Capitalising on this phenomena, much research has been carried out into the effective design of reactive silencers for ducting systems. In recent years there has been increasing interest in the effect that wave-bearing boundaries have on the propagating noise and, in this context, much of the analytic work concerns two-dimensional ducts or those circular cylindrical geometries. A wide range of analytic techniques have been employed to study this class of problem including the Wiener-Hopf technique (Rawlins, 2007), Fourier methods (Huang, 2002) and mode-matching (Lawrie & Guled, 2006).

Whilst the Wiener-Hopf technique has long been a convenient tool by which to tackle a wide range of two part-problems, analytic mode-matching has only recently been established as a viable tool for two-dimensional structures in which the boundary conditions contain high-order derivatives. The reason for this is that the eigenfunctions for such systems do not satisfy standard orthogonality conditions, such as that for the set of functions $\cos(n\pi y/a), n = 0, 1, 2, \ldots$. Orthogonality relations (OR) do, however, exist albeit of non-standard form and Lawrie & Abrahams (1999) were amongst the first authors to demonstrate this. Since then this class of OR has arisen in a number of different physical situations, see for example, Evans & Porter (2003), Manam *et. al.* (2006), Kaplunov *et al* (2004).

Problems involving the propagation of guided waves in three-dimensional media are often more difficult to analyse. Recent studies include those of Zakharov (2008) who derived ORs for 3-D waves in visco-elastic laminates and Zernov & Kaplunov (2008) who studied 3-D edge waves on plates. Three-dimensional guided waves are of particular relevance to the HVAC industry since many HVAC ducts have a rectangular cross-section. For this reason, there is a vast body of literature concerning the propagation of acoustic waves in such ducts. Most of the literature, however, deals with ducts that have rigid walls and the primary reason for this is the inherent difficulties involved in analysing those with flexible walls. Articles that do address the effects of flexible duct walls include those by Cabelli (1984), Lawrie & Abrahams (2002), Huang & Choy (2005) and Martin *et al* (2004), the latter of which is primarily an experimental investigation of the wall displacement.

This article is concerned with sound propagation in a three dimensional duct, of rectangular cross-section, formed by three rigid walls and closed by a thin elastic plate (see figure 1). The analysis herein builds upon the work of

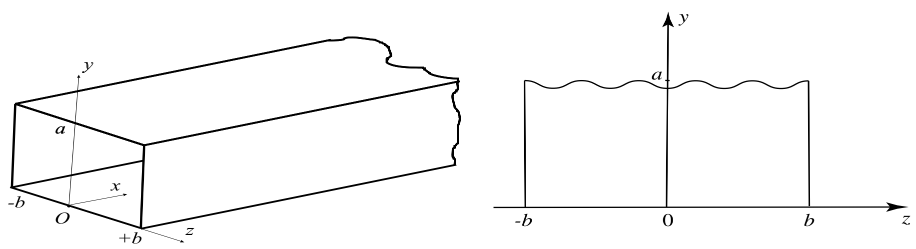

Figure 1: Duct geometry and yz cross-section.

Lawrie & Abrahams (2002) who first presented a non-separable ansatz for the duct modes together with a partial OR. That article offered a pioneering insight into the three-dimensional case, it was clear, however, that the partial OR is insufficient to enable the efficient solution of some typical scattering problems. The partial OR is based on that for the underlying two-dimensional system, (see Lawrie & Abrahams, 1999) and thus, whilst incorporating the appropriate orthogonality properties in the y direction, it neglects any such properties for the z direction. Recent developments, however, in the theory under-pinning such two-dimensional systems, Lawrie (2007), has paved the way for a better understanding of their three-dimensional counterparts. Thus, the aim of this article is two-fold. Firstly, it is intended to establish the analytic properties of the eigenfunctions for the three dimensional waveguide shown in figure 1. Secondly, it is intended to investigate the potential use of these eigenfunctions in mode-matching problems for three dimensional geometries.

In §2, following Lawrie & Abrahams (2002), the eigenfunctions and dispersion relation are presented. Some significant new results are derived in §3: it is proved that the eigenfunctions are linearly dependent and that an eigenfunction expansion of a suitable function, $f(y, z)$, converges point-wise to that function. In §4 the derivatives $\psi_{ny}(a, z)$, $n = 0, 1, 2, \ldots$ are considered. It is demonstrated that these functions are also linearly dependent and some new orthogonality results are presented. Two prototype problems are studied in §5. Both of these involve a semi-infinite duct of the type shown in figure 1. In the first an acoustic field is generated by a prescribed pressure at the end of the duct and the edge of the plate is assumed to be simply supported (pin-jointed). The second is forced by a prescribed velocity field and spring-like edge conditions are applied. Some concluding remarks are presented in §6.

2 Travelling wave solutions

In this section the non-separable ansatz for the duct modes, posed by Lawrie & Abrahams (2002), is presented. The duct occupies the region $-\infty < x < \infty$, $0 \le y \le a$, $-b \le z \le b$ where (x, y, z) are Cartesian coordinates that have been non-dimensionalised with respect to k^{-1}, k being the fluid wavenumber. An elastic plate of infinite length and width $2b$ bounds the duct at $y = a$, $-b \le z \le b$, whilst the remaining three sides are rigid (figure 1). A compressible fluid of density ρ and sound speed c occupies the interior region of the duct, but the exterior region is *in vacuo*. The travelling waves are assumed to have harmonic time dependence, $\exp(-it)$, where t has been non-dimensionalised with respect to ω^{-1}, with $\omega = ck$. Without loss of generality, it can be assumed that the duct modes propagate in the positive x direction, so that the non-dimensional, time-independent velocity potential, $\phi(x, y, z)$, assumes the form

$$\phi(x, y, z) = \sum_{n=0}^{\infty} B_n \psi(s_n, y, z) e^{is_n x} \tag{1}$$

where B_n is the amplitude of the nth travelling wave, s_n is the axial wavenumber (as yet unknown, but assumed to be either positive real or have positive imaginary part) and the nonseparable reduced potential ψ is to be determined. It is convenient, in the first instance, to treat the wavenumber as a continuous variable s rather than a discrete set of values, s_n. Thus, when written in terms of $\psi(s,y,z)$, Helmholtz' equation reduces to

$$\left\{\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+1-s^2\right\}\psi(s,y,z)=0. \tag{2}$$

The normal component of fluid velocity vanishes at the three rigid walls which, on assuming even eigenmodes, $\phi(x,y,-z)=\phi(x,y,z)$, implies:

$$\frac{\partial\psi}{\partial y}=0, \qquad y=0, \qquad 0\le z\le b, \tag{3}$$

$$\frac{\partial\psi}{\partial z}=0, \qquad z=0,\ b, \quad 0\le y\le a. \tag{4}$$

The boundary condition that describes the deflections of the thin elastic plate bounding the top of the duct is

$$\left\{\left(\frac{\partial^2}{\partial z^2}-s^2\right)^2-\mu^4\right\}\psi_y-\alpha\psi=0, \qquad y=a,\ 0\le z\le b \tag{5}$$

where μ is the *in vacuo* plate wavenumber and α a fluid loading parameter. Thus, $\mu^4=c^2h\rho_p/(Bk^2)$ and $\alpha=c^2\rho/(Bk^3)$ where $B=Eh^3/\{12(1-\nu^2)\}$ is the plate bending stiffness in which E is Young's modulus, ρ_p is the density of the plate, and ν is Poisson's ratio. Details pertaining to the derivation of (5) are given by Grant & Lawrie (2000) and the references therein.

The vertical displacement of the plate is proportional to $\phi_y(x,a,z)$, where the subscript y here and henceforth denotes differentiation. Further, the elastic plate is assumed to be clamped to the adjacent rigid sides of the duct, hence, the displacement and gradient along these edges are zero and it follows that:

$$\frac{\partial\psi}{\partial y}=0, \qquad y=a,\ z=b, \tag{6}$$

$$\frac{\partial^2\psi}{\partial y\partial z}=0, \qquad y=a,\ z=b. \tag{7}$$

These conditions will henceforth be referred to as the *corner conditions.* Note that alternative conditions, such as free plate edges, could equally well have been chosen.

Lawrie & Abrahams (2002) propose the following ansatz for $\psi(s,y,z)$

$$\psi(s,y,z)=\sum_{m=0}^{\infty}E_m(s)Y_m(y)\cosh(\tau_m(s)z). \tag{8}$$

This respects the fact that ψ is chosen to be even in z, see (4). The functions $Y_m(y)$, $m=0,1,2,\ldots$, from (2) and (3), satisfy the following simple ordinary differential equation and boundary condition,

$$Y_m''(y)-\gamma_m^2Y_m(y)=0 \quad \text{and} \quad Y_m'(0)=0, \tag{9}$$

which yields $Y_m(y) = \cosh(\gamma_m y)$, where γ_m is related to $\tau_m(s)$ through $\tau_m^2(s) + \gamma_m^2 + 1 - s^2 = 0$. A further condition for $Y_m(y)$ can be obtained by ensuring that ψ satisfies (5). On substituting (8) into (5) it is found that the quantities γ_m, $m = 0, 1, 2, \ldots$ are defined by

$$K(\gamma_m) = \{(\gamma_m^2 + 1)^2 - \mu^4\} Y_m'(a) - \alpha Y_m(a) = 0, \tag{10}$$

where the $'$ denotes differentiation with respect to the argument, in this case y. Note that, this notation is used only to denote the derivative for functions of one variable, otherwise the full partial derivative or the subscript notation is used.

Although *non-Sturm-Liouville*, the eigen-system specified by (9) and (10) is well studied. It is a straightforward procedure to show that the appropriate OR for this system is

$$\alpha \int_0^a Y_m(y) Y_j(y)\, dy = C_j \delta_{jm} - (\gamma_m^2 + \gamma_j^2 + 2) Y_j'(a) Y_m'(a) \tag{11}$$

where δ_{jm} is the usual Kronecker delta and

$$C_j = 2(\gamma_j^2 + 1)[Y_j'(a)]^2 + \tfrac{1}{2}\alpha \gamma_j^{-2} Y_j'(a) Y_j(a) + \tfrac{1}{2}\alpha a. \tag{12}$$

Furthermore, the eigenfunctions $Y_j(y)$, $j = 0, 1, 2, \ldots$ have the following properties:

$$\sum_{n=0}^{\infty} \frac{Y_n'(a) Y_n(y)}{C_n} = \sum_{n=0}^{\infty} \frac{\gamma_n^2 Y_n'(a) Y_n(y)}{C_n} = 0; \tag{13}$$

$$\sum_{n=0}^{\infty} \frac{\gamma_n^2 \left[Y_n'(a)\right]^2}{C_n} = 1; \tag{14}$$

$$\alpha \sum_{n=0}^{\infty} \frac{Y_n(v) Y_n(y)}{C_n} = \delta(y - v) + \delta(y + v) + \delta(y + v - 2a), \tag{15}$$

where $\delta(.)$ is the usual Dirac delta function. These results are established by Lawrie (2007)[1] and, in view of their significance to the analysis that follows, it is useful to relate the notation of that article to that used herein. The above eigensystem is retrieved from the general system considered by Lawrie on setting $P_a(s) = s^4 - \mu^4$, $Q_a(s) = -\alpha$, $P_0(s) = 1$, $Q_0(s) = 0$. It should also be noted that the definition of C_m above differs from that in Lawrie (2007) by a multiplicative α.

Expression (11) can now be employed to impose the condition of no normal velocity on the rigid face at $z = b$, see (4). On multiplying by $E_j(s)\tau_j(s)\sinh(\tau_j(s)b)$

[1]The reader is advised that there are errors of sign in equations (3.24) and (3.26) of Lawrie (2007). These do not, however, impinge upon the results presented in that article.

and summing over j, it is found that

$$\alpha \int_0^a Y_m(y) \sum_{j=0}^{\infty} E_j(s)\tau_j(s)\sinh(\tau_j(s)b)Y_j(y)\, dy \quad = \quad -Y_m'(a)P(s) \tag{16}$$

$$-Y_m'(a)(\gamma_m^2+2)\sum_{j=0}^{\infty} E_j(s)\tau_j(s)\sinh(\tau_j(s)b)Y_j'(a) \quad + \quad E_m(s)\tau_m(s)\sinh(\tau_m(s)b)C_m$$

where

$$P(s) = \sum_{j=0}^{\infty} E_j(s)\tau_j(s)\sinh(\tau_j(s)b)\gamma_j^2 Y_j'(a). \tag{17}$$

The first and third terms of (16) are zero (due to boundary condition (4) and corner constraint (7)) and, it follows that

$$E_m(s) = \frac{Y_m'(a)P(s)}{C_m\tau_m(s)\sinh(\tau_m(s)b)}. \tag{18}$$

The ansatz for $\psi(s, y, z)$, (8), can be multiplied by any arbitrary constant (or function of s) and so, without loss of generality, it is chosen that $P(s) = 1$. It follows that the duct modes have the form

$$\psi(s, y, z) = \sum_{m=0}^{\infty} \frac{Y_m'(a)Y_m(y)\cosh(\tau_m(s)z)}{C_m\tau_m(s)\sinh(\tau_m(s)b)}. \tag{19}$$

Note that (17) and (18) confirm (14). Equation (19) contains the quantity s which corresponds to the axial wavenumber of the mode and is, as yet, unspecified. Condition (6), however, has not been imposed and it is this that gives rise to the dispersion relation, that is

$$L(s) = \sum_{m=0}^{\infty} \frac{[Y_m'(a)]^2\cosh\{(s^2-\gamma_m^2-1)^{1/2}b\}}{C_m(s^2-\gamma_m^2-1)^{1/2}\sinh\{(s^2-\gamma_m^2-1)^{1/2}b\}} = 0. \tag{20}$$

Despite its complicated form, it is a straightforward, if tedious, procedure to numerically solve (20) and the roots s_n, $n = 0, 1, 2, \ldots$ are the admissible axial wavenumbers for the duct modes. Further, it is straightforward to show that:

(i) for every root s_n there is another root $-s_n$;

(ii) there is a finite number of real roots;

(iii) there is an infinite number of roots located on $\Re(s) = 0, \Im(s) > 0$;

(iv) there is an infinite number of roots with non-zero real and imaginary parts.

In order that (1) represents only waves that travel in the positive x direction and/or decay exponentially as $x \to \infty$, the convention is adopted that the $+s_n$ roots have either $\Re(s_n) > 0, \Im(s_n) = 0$ or $\Im(s_n) > 0$. They are ordered sequentially, real roots first, starting with the largest real root and then by increasing imaginary part. Thus, s_0 is always the largest real root. It is important to note that for any complex root, say s_c, lying in the first quadrant of the complex s-plane, then $-s_c^*$, where * indicates the complex conjugate, also lies in the upper half plane. Such pairs are positioned in the sequence of roots according to the magnitude of their imaginary part and, furthermore, in the order s_c followed by $-s_c^*$. It is worthwhile commenting that that these roots lie on curves reminiscent of the parabolic arcs observed in studies of 2-D elastic waveguides, see for example Besserer & Malischewsky (2004). Finally, it is assumed that $s_n \neq 0$ and that no root is repeated. Where appropriate, the notation $\psi_n(y, z)$ will henceforth be used to indicate that s has been replaced by s_n in the function $\psi(s, y, z)$. Likewise, $\tau_m(s_n)$ becomes τ_{mn} etc.

In the next two sections several properties of the eigenfunctions $\psi_n(y, z)$, $n = 0, 1, 2, \ldots$ and their derivatives $\psi_{ny}(a, z)$, $n = 0, 1, 2, \ldots$ are presented. In §5, these results are used to solve two prototype scattering problems.

3 Analytic properties of the eigenfunctions

Lemma: The set of eigenfunctions $\psi_n(y, z)$, $n = 0, 1, 2, \ldots$ satisfy the following orthogonality relation

$$\begin{aligned} \alpha \int_0^b \int_0^a \psi_n(y, z)\psi_\ell(y, z)\, dy\, dz \quad &= \quad D_\ell \delta_{\ell n} - \int_0^b \{\psi_{nyyy}(a, z)\psi_{\ell y}(a, z) \\ + \psi_{ny}(a, z)\psi_{\ell yyy}(a, z) \quad &+ \quad 2\psi_{ny}(a, z)\psi_{\ell y}(a, z)\}\ dz \end{aligned} \tag{21}$$

where $D_n = -L'(s_n)/(2s_n)$ and $\delta_{\ell n}$ is the Kronecker delta function.

Proof: This result (first presented by Lawrie & Abrahams, 2002) is verified by substituting the series expressions for $\psi_n(y, z)$ and $\psi_m(y, z)$ (that is (19) with s replaced by s_n and s_m respectively) into the left hand side of (21). On interchanging the order of summation and integration and using (11), expression (21) follows. □

Theorem: The eigenfunctions $\psi_n(y, z)$ described above are linearly dependent for $0 \leq y \leq a$, $0 \leq z \leq b$.

Proof: In order to prove this result it is necessary only to determine one infinite sum of the eigenfunctions which is zero. Such a sum can be constructed by analysing the families of poles in the integrand of a suitably chosen integral. The method is the same as that used by Lawrie (2007). The appropriate integral is

$$I_1(y, z, w) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{s\psi_y(s, a, w)Z_1(s, y, z)}{L(s)}\, ds = 0, \tag{22}$$

where $0 \leq y \leq a$, $0 \leq w, z \leq b$ and the path of integration is indented above(below) any poles on the negative(positive) real axis. Note that $\psi_y(s, a, w)$ is obtained directly on differentiating (19) with respect to y , $L(s)$ is given by

(20) and

$$Z_1(s,y,z) = \sum_{m=0}^{\infty} \frac{Y_m'(a)Y_m(y)\sinh(\tau_m(s)z)}{C_m(s^2-\gamma_m^2-1)\sinh(\tau_m(s)b)}, \tag{23}$$

which, on differentiating with respect to z, yields $\psi(s,y,z)$.

The integrand has three families of poles.

i) Those corresponding to the admissible wavenumbers s_n, $n = 0, 1, 2\ldots$ and defined by $L(s) = 0$.

ii) Those corresponding to $s^2 - \gamma_m^2 - 1 = 0$.

iii) A doubly infinite family of poles defined by $\sinh(\tau_m(s)b) = 0$. That is, occurring when $s = (1+\gamma_m^2-(n\pi/b)^2)^{1/2} = \nu_{mn}$, $m = 0, 1, 2, \ldots$, $n = 1, 2, 3\ldots$. The path of integration can be deformed onto a semi-circular arc of radius $R >> 1$ in the upper half plane. On evaluating the residue contributions for the poles crossed and noting that there is no contribution from the arc as $R \to \infty$, it is found that

$$\begin{aligned} &- \sum_{n=0}^{\infty} \frac{\psi_{ny}(a,w)Z_1(s_n,y,z)}{D_n} + \frac{z}{b}\sum_{m=0}^{\infty}\frac{Y_m'(a)Y_m(y)}{C_m} \\ &+ \frac{2}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{Y_m'(a)Y_m(y)(-1)^n\sin(n\pi z/b)}{C_m\, n}\left\{\frac{\psi_y(s,a,w)}{L(s)}\right\}_{s=\nu_{mn}} = 0. \end{aligned} \tag{24}$$

It is straightforward to show that

$$\lim_{s\to\nu_{mn}} \frac{\psi_y(s,a,w)}{L(s)} = (-1)^n\cos(\frac{n\pi w}{b}), \tag{25}$$

and thus expression (24) becomes

$$\sum_{n=0}^{\infty}\frac{\psi_{ny}(a,w)Z_1(s_n,y,z)}{D_n} = \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{Y_m'(a)Y_m(y)}{C_m}\sum_{n=0}^{\infty}\frac{\cos(n\pi w/b)\sin(n\pi z/b)}{\epsilon_n n} \tag{26}$$

where $\epsilon_0 = 2$ and $\epsilon_n = 1$, $n > 0$. It is interesting to observe that this now takes the form of a separable function: an infinite sum of the eigenfunctions $Y_m(y)$ times an infinite sum of the functions $\sin(n\pi z/b)$. The latter can be evaluated, thus

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{\psi_{ny}(a,w)Z_1(s_n,y,z)}{D_n} &= \sum_{m=0}^{\infty}\frac{Y_m'(a)Y_m(y)}{C_m} \\ &\times \; \{1 - H(w-z) + H(w+z-2b) - H(-z-w)\} \end{aligned} \tag{27}$$

where $H(.)$ is the usual Heaviside function, defined such that $H(0) = 1/2$. On differentiating this with respect to z, it is found that

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{\psi_{ny}(a,w)\psi_n(y,z)}{D_n} &= \sum_{m=0}^{\infty}\frac{Y_m'(a)Y_m(y)}{C_m} \\ &\times \; \{\delta(w-z) + \delta(w+z-2b) + \delta(-z-w)\}\,. \end{aligned} \tag{28}$$

Thus, the sum of interest has been expressed as a product of a sum of the eigenfunctions $Y_m(y)$ and some delta functions. Due to the linear dependence of the functions $Y_m(y)$, see (13), the sum in the right hand side of (28) is in fact zero. It follows that

$$\sum_{n=0}^{\infty} \frac{\psi_{ny}(a,w)\psi_n(y,z)}{D_n} = 0, \quad 0 \le y \le a, \quad 0 \le w, z \le b. \qquad \Box \tag{29}$$

This result demonstrates that the eigenfunctions $\psi_n(y,z)$ are linearly dependent. In fact, (29) is not the only such sum. On replacing $\psi_y(s,a,w)$ with $\psi_{yyy}(s,a,w)$ in the integrand of (22) and repeating the above calculation, it is found that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\psi_{nyyy}(a,w)\psi_n(y,z)}{D_n} &= \sum_{m=0}^{\infty} \frac{\gamma_m^2 Y_m'(a) Y_m(y)}{C_m} \qquad (30) \\ &\times \;\; \{\delta(w+z-2b) + \delta(-z-w) + \delta(w-z)\} \end{aligned}$$

which on using (13) reduces to

$$\sum_{n=0}^{\infty} \frac{\psi_{nyyy}(a,w)\psi_n(y,z)}{D_n} = 0, \quad 0 \le y \le a, \quad 0 \le w, z \le b. \tag{31}$$

Lemma: The eigenfunctions $\psi_n(y,z)$ described by (19)–(20) have the property

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\psi_n(y,z)\psi_n(v,w)}{D_n} &= \{\delta(v-y) + \delta(v+y) + \delta(v+y-2a)\} \qquad (32) \\ &\times \;\; \{\delta(w+z-2b) + \delta(z+w) + \delta(w-z)\}, \\ &\quad 0 \le v, y \le a, \quad 0 \le z, w \le b. \end{aligned}$$

Proof: As for theorem 3.2, this result can be proved by analysing the families of poles in the integrand of a suitably chosen integral. The appropriate integral is

$$I_2(y,z,w) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{s\psi_{sy}(s,a,w) Z_2(s,y,z)}{L(s)}\, ds = 0, \tag{33}$$

where $0 \le y \le a$, $0 \le w, z \le b$ and the path of integration is indented above(below) any poles on the negative(positive) real axis and

$$Z_2(s,y,z) = \sum_{m=0}^{\infty} \frac{Y_m'(a) Y_m'(y) \sinh(\tau_m(s) z)}{C_m \gamma_m^2 (s^2 - \gamma_m^2 - 1) \sinh(\tau_m(s) b)}. \tag{34}$$

Note that, on differentiating (34) with respect to both y and z and putting $s = s_n$, $\psi_n(y,z)$ is retrieved. The pole structure of this integrand is the same as that for $I_1(y,z,w)$ and, on deforming the path of integration onto a semi-circular arc of radius $R >> 1$ in the upper half plane and evaluating the residue contributions for the poles crossed, it is found that, as $R \to \infty$

$$\sum_{n=0}^{\infty} \frac{\psi_n(v,w)Z_2(s_n,y,z)}{D_n} = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{Y_m(v)Y_m'(y)}{C_m\gamma_m^2} \sum_{n=0}^{\infty} \frac{\cos(n\pi w/b)\sin(n\pi z/b)}{\epsilon_n n}. \tag{35}$$

Again the variables have been separated, thus, the sum of interest has been expressed in the form "a sum dependent on y times a sum that is dependent on z". It follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\psi_n(v,w)Z_2(s_n,y,z)}{D_n} &= \sum_{m=0}^{\infty} \frac{Y_m(v)Y_m'(y)}{C_m\gamma_m^2} \\ &\times \{1 - H(w-z) - H(-z-w) + H(w+z-2b)\}. \end{aligned} \tag{36}$$

Then, on differentiating with respect to both y and z, and using (9) and (15), it is found that

$$\begin{aligned} \alpha \sum_{n=0}^{\infty} \frac{\psi_n(v,w)\psi_n(y,z)}{D_n} &= \{\delta(y-v) + \delta(y+v) + \delta(y+v-2a)\} \\ &\times \{\delta(z-w) + \delta(z+w) + \delta(w+z-2b)\}. \quad \square \end{aligned} \tag{37}$$

Theorem: Given that the coefficients

$$\begin{aligned} B_n &= \frac{1}{D_n}\left\{\alpha \int_0^b \int_0^a f(v,w)\psi_n(v,w)\, dvdw \right. \\ &+ \left. \int_0^b [f_y(a,w)\psi_{nyyy}(a,w) + f_{yyy}(a,w)\psi_{ny}(a,w) + 2f_y(a,w)\psi_{ny}(a,w)]\ dw \right\} \end{aligned}$$

exist, where $f(y,z)$ is any function that is three times differentiable on the domain $0 \le y \le a$, $0 \le z \le b$ and the eigenfunctions $\psi_n(y,z)$ are defined by (19)–(20), then the series

$$\sum_{n=0}^{\infty} B_n\psi_n(y,z) \tag{38}$$

converges point-wise to $f(y,z)$ for $0 \le y \le a$, $0 \le z \le b$.

Proof: Assume that a suitably smooth function $f(y,z)$, $0 \le y \le a$, $0 \le z \le b$ can be expressed as an eigenfunction expansion in terms of $\psi_n(y,z)$, $n = 0, 1, 2, \ldots$. Let $F_N(y,z)$ be the sum of the first N terms of this eigenfunction expansion, thus

$$F_N(y,z) = \sum_{n=0}^{N} B_n\psi_n(y,z). \tag{39}$$

On substituting the coefficients B_n into (39), it is found that

$$\begin{aligned}\sum_{n=0}^{N} B_n\psi_n(y,z) &= \sum_{n=0}^{N}\frac{1}{D_n}\left\{\alpha\int_0^b\int_0^a f(v,w)\psi_n(v,w)\,dvdw\right. \qquad (40)\\ &+ \int_0^b \Big[f_y(a,w)\psi_{nyyy}(a,w)+f_{yyy}(a,w)\psi_{ny}(a,w)\\ &+ \left. 2f_y(a,w)\psi_{ny}(a,w)\Big]\,dw\right\}\psi_n(y,z)\end{aligned}$$

which, after interchanging the order of differentiation and integration and then letting $N\to\infty$, may be written as

$$\begin{aligned}\sum_{n=0}^{\infty} B_n\psi_n(y,z) &= \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(v,w)\sum_{n=0}^{\infty}\frac{\psi_n(y,z)\psi_n(v,w)}{D_n}\,dvdw \qquad (41)\\ &+ \int_0^b\left\{f_y(a,w)\sum_{n=0}^{\infty}\frac{\psi_{nyyy}(a,w)\psi_n(y,z)}{D_n}\right.\\ &+ \left.\Big[f_{yyy}(a,w)+2f_y(a,w)\Big]\sum_{n=0}^{\infty}\frac{\psi_{ny}(a,w)\psi_n(y,z)}{D_n}\right\}\,dw\end{aligned}$$

where

$$\mathcal{F}(v,w)=f(v,w)H(v)H(a-v)H(w)H(b-w). \qquad (42)$$

Now, on utilizing (29), (31) and lemma 3.3 it is found that

$$\begin{aligned}\sum_{n=0}^{\infty} B_n\psi_n(y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(v,w)\{\delta(v-y)+\delta(v+y)+\delta(v+y-2a)\}\\ &\quad\times\{\delta(w+z-2b)+\delta(z+w)+\delta(w-z)\}\,dvdw\\ &= f(y,z),\quad 0\le y\le a,\quad 0\le z\le b. \qquad \Box \qquad (43)\end{aligned}$$

4 The derivatives of the eigenfunctions

It has been proved that a function $f(y,z)$ can be represented as an eigenfunction expansion in terms of the eigenfunctions $\psi_n(y,z)$. The partial OR, (21), utilises, however, only the orthogonality properties of the functions $Y_n(y)$. That is, it neglects any orthogonality properties in the z dependence of $\psi_n(y,z)$. Yet, on differentiating (28) and (30) with respect to y, setting $y=a$ and using (13) and (14), the following two results are obtained:

$$\sum_{n=0}^{\infty}\frac{\psi_{ny}(a,w)\psi_{ny}(a,z)}{D_n}=0, \qquad (44)$$

and

$$\sum_{n=0}^{\infty}\frac{\psi_{nyyy}(a,w)\psi_{ny}(a,z)}{D_n}=\delta(w+z-2b)+\delta(-z-w)+\delta(w-z) \qquad (45)$$

where $0 \leq w, z \leq b$.

Expressions (44) and (45) provide useful information about the set of functions $\psi_{ny}(a, z)$. The first of these expressions indicates that these functions are linearly dependent whilst the other hints at the existence of an OR for this set of functions. Further, on differentiating (41) with respect to y, setting $y = a$ and using (29) and (44) it is found that

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n \psi_{ny}(a, z) &= \int_0^b f_y(a, w) \sum_{n=0}^{\infty} \frac{\psi_{nyyy}(a, w)\psi_{ny}(a, z)}{D_n} \, dw \\
&= \int_{-\infty}^{\infty} \mathcal{G}_y(a, w)\{\delta(w + z - 2b) + \delta(z + w) + \delta(w - z)\} \, dw \\
&= f_y(a, z), \quad 0 \leq z \leq b, \qquad (46)
\end{aligned}$$

where

$$\mathcal{G}_y(a, w) = f_y(a, w)H(w)H(b - w). \qquad (47)$$

It may be deduced, from (46), that an arbitrary, suitably differentiable function $q(z)$ defined for $0 \leq z \leq b$ can be expressed as an eigenfunction expansion in terms of the functions $\psi_{ny}(a, z)$. That is,

$$q(z) = \sum_{n=0}^{\infty} \tilde{A}_n \psi_{ny}(a, z) \qquad (48)$$

where

$$\tilde{A}_n = \frac{1}{D_n} \int_0^b q(w)\psi_{nyyy}(a, w) \, dw. \qquad (49)$$

At this point it is expedient to remark that, unlike (44) and (45), the sum

$$\sum_{n=0}^{\infty} \frac{\psi_{nyyy}(a, w)\psi_{nyyy}(a, z)}{D_n} \qquad (50)$$

is neither convergent nor can it be expressed in terms of delta functions. This an important implication for the class of function that can be represented as an eigenfunction expansion of the form (38). If (41) is differentiated three times with respect to y and y is then set to a, the divergent sum (50) arises. Thus, in order for the eigenfunction expansion (38) to be three times differentiable, it is necessary that $f_y(a, z) = 0,\ 0 \leq z \leq b$. Expression (50) also impinges upon the results that follow immediately below.

In view of (29) and (44) it is clear that the expansion (48) is not unique. The coefficient $\tilde{A}_n$, $n = 0, 1, 2, \ldots$ can be generalised as follows :

$$A_n = \frac{1}{D_n} \int_0^b \Big[p(w)\psi_n(a, w) + q(w)\psi_{nyyy}(a, w) + \lambda(w)\psi_{ny}(a, w) \Big] \, dw \qquad (51)$$

where $p(w)$ and $\lambda(w)$ are suitable but arbitrary functions. Then $q(z)$ is still given by (48) but with $\tilde{A}_n$ replaced by A_n. A natural question to ask is: *given the* *coefficients A_n, can $p(z)$ and $\lambda(z)$ be constructed as an eigenfunction expansion similar in form to (48)?* In fact, it is relatively straightforward to construct such a representation for the function $p(z)$. On multiplying (51) by $\alpha\psi_n(y,z)$ and summing over n, then integrating with respect to y, $0 \le y \le a$, and using (37), it is found that

$$p(z) = \alpha \sum_{n=0}^{\infty} A_n \chi_n(z) \tag{52}$$

where

$$\chi_n(z) = \int_0^a \psi_n(y,z)\, dy = \sum_{m=0}^{\infty} \frac{[Y_m'(a)]^2 \cosh(\tau_{mn} z)}{C_m \gamma_m^2 \tau_{mn} \sinh(\tau_{mn} b)}. \tag{53}$$

It is a little more complicated, however, to construct the function $\lambda(z)$. Since the sum (50) is not defined, it is not possible to multiply through by $\psi_{nyyy}(a,z)$ and then sum. Instead it is necessary first to construct the coefficient

$$\begin{aligned} G_n &= \frac{1}{D_n} \int_0^b \Big[p(w)\psi_n(a,w) + q(w)\psi_{nyyy}(a,w) \Big]\, dw, \\ &= \frac{1}{D_n} \sum_{m=0}^{\infty} A_m(\alpha T_{mn} + R_{nm}) \end{aligned} \tag{54}$$

where (48) and (52) have been used implying that

$$R_{nm} = \int_0^b \psi_{nyyy}(a,w)\psi_{my}(a,w)\, dw \tag{55}$$

and

$$T_{mn} = \int_0^b \chi_m(w)\psi_n(a,w)\, dw. \tag{56}$$

Then

$$A_n - G_n = \frac{1}{D_n} \int_0^b \lambda(w)\psi_{ny}(a,w)\, dw \tag{57}$$

and it follows that

$$\lambda(z) = \sum_{n=0}^{\infty} \left\{ A_n - \frac{1}{D_n} \sum_{m=0}^{\infty} A_m(\alpha T_{mn} + R_{mn}) \right\} \psi_{nyyy}(a,z). \tag{58}$$

The following example is included to demonstrate the effectiveness of the three eigenfunction expansions (48), (52) and (58). It should be noted that $q(z)$ and $\lambda(z)$ must satisfy $q'(0) = \lambda'(0) = 0$ and $q(b) = q'(b) = \lambda(b) = \lambda'(b) = 0$, they are otherwise arbitrary. The function $p(z)$, on the other hand must satisfy $p'(0) = p'(b) = 0$. These restrictions are consistent with the corner conditions satisfied by $\psi_{ny}(a,z)$, $n = 0, 1, 2, \ldots$. Thus, the functions $p(z)$, $q(z)$ and $\lambda(z)$ are chosen to be: $p(z) = 1 + (1 + \cos(\pi z/b))^2$; $q(z) = (z^2 - b^2/16)(z^2/b^2 - 1)^4$ and $\lambda(z) = (4z^2/b^2 - 1)(36z^2/b^2 - 1)(z^2/b^2 - 1)^6$. The eigenfunctions used to generate the series representations of these functions are those corresponding

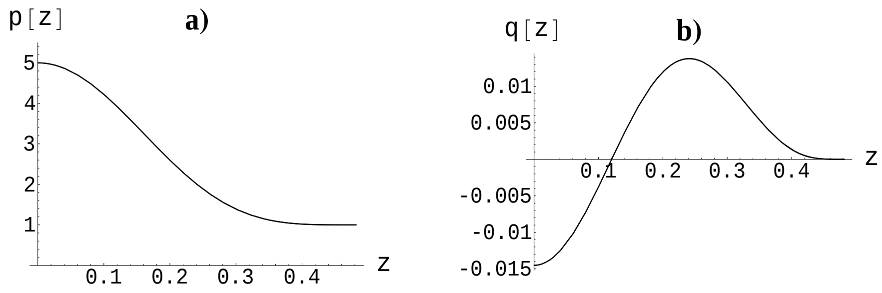

Figure 2: The functions a) $p(z)$ and b) $q(z)$ plotted with their eigenfunction expansions.

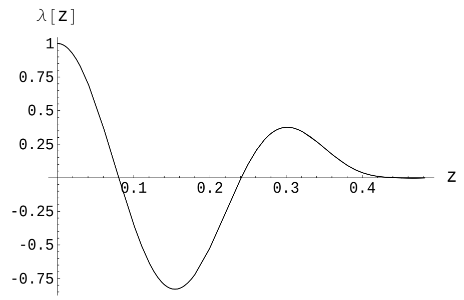

Figure 3: The function $\lambda(z)$ and its eigenfunction expansion.

to a duct with physical dimensions $ak^{-1} = 0.15m$, $bk^{-1} = 0.31m$ and bounded by an aluminium plate of thickness $0.002m$. The frequency is 85 Hz, for which the dispersion relation has two real roots. Figures 2 a) and b) show the functions $p(z)$ and $q(z)$ together with their eigenfunction representations (evaluated using 30 and 45 terms respectively). Figure 3 shows $\lambda(z)$ and its eigenfunction representation evaluated using 60 terms. In each case the eigenfunctions representations overlie the actual functions. Note that, although it is required only that $q(z)$ and $\lambda(z)$ have a zero of order two at $z = b$, the eigenfunction expansions generally converge faster the higher the order of this zero.

5 Two prototype scattering problems

In this section the use of (21) and (45) for solving two typical scattering problems is demonstrated. The problems are drawn from a class often encountered when using mode-matching to solve a two part problem involving acoustic propagation in a duct - though normally for the two-dimensional case, for example Lawrie & Kirby (2006). For both prototype problems the duct is semi-infinite, lying in the region $x > 0$, but in all other respects is identical to that described at the beginning of §2. In the first case the scattered field is generated by a prescribed pressure distribution applied to the surface $x = 0$, $0 \leq y \leq a$, $0 \leq z \leq b$ and plate is simply supported along $x = 0$, $0 \leq z \leq b$ whilst for the second, the velocity is prescribed and spring-like edge conditions are applied. In both cases, numerical results are generated for a duct with physical dimensions $ak^{-1} = 0.15m$, $bk^{-1} = 0.31m$ and bounded by an aluminium plate of thickness $0.002m$.

Prescribed pressure

The fluid velocity potential can be expressed in terms of the eigenmodes $\psi_\ell(y, z)$ as

$$\phi(x, y, z) = \sum_{\ell=0}^{\infty} B_\ell \psi_\ell(y, z) e^{is_\ell x} \tag{59}$$

where the amplitude B_ℓ is determined from the applied pressure and the "pin-jointed" edge conditions:

$$\phi_y(0, a, z) = \sum_{\ell=0}^{\infty} B_\ell \psi_{\ell y}(a, z) = 0, \quad 0 \leq z \leq b; \tag{60}$$

$$\phi_{yxx}(0, a, z) = -\sum_{\ell=0}^{\infty} B_\ell s_\ell^2 \psi_{\ell y}(a, z) = 0,\ 0 \leq z \leq b. \tag{61}$$

Since the edge conditions are applied on $x = 0$, $y = a$ for $0 \leq z \leq b$, it follows that $\phi_{yz}(0, a, z) = \phi_{yzz}(0, a, z) = 0$, $0 \leq z \leq b$. Thus, due to Helmholtz's equation, $\phi_{yxx}(0, a, z) + \phi_{yyy}(0, a, z) = 0$ which enables the edge conditions to be expressed as $\phi_y(0, a, z) = \phi_{yyy}(0, a, z) = 0$, $0 \leq z \leq b$. On multiplying (21) by B_ℓ and summing over ℓ and applying the edge conditions, it is found that

$$B_n = \frac{\alpha}{D_n} \int_0^b \int_0^a f(y, z) \psi_n(y, z)\, dy\, dz \tag{62}$$

in which $f(y, z)$ is the (known) non-dimensional pressure applied to the end face $x = 0$, $0 \leq y \leq a$, $-b \leq z \leq b$.

To demonstrate the above solution, numerical results are presented for the case in which the pressure distribution applied at $x = 0$ is

$$f(y, z) = \left(\frac{4z^2}{b^2} - 1\right) \left(1 + \cos(\frac{\pi z}{b})\right)^2 \left(\frac{y^2}{2a} - \frac{a\cosh(y/a)}{\sinh(1)}\right)$$

and the frequency is 155 Hz (for which the dispersion relation has three real roots). Figures 4 a) and 4 b) show the "plate displacement" $\phi_y(x, a, z)$ obtained using (59) with the coefficients given by (62). A section of the displacement

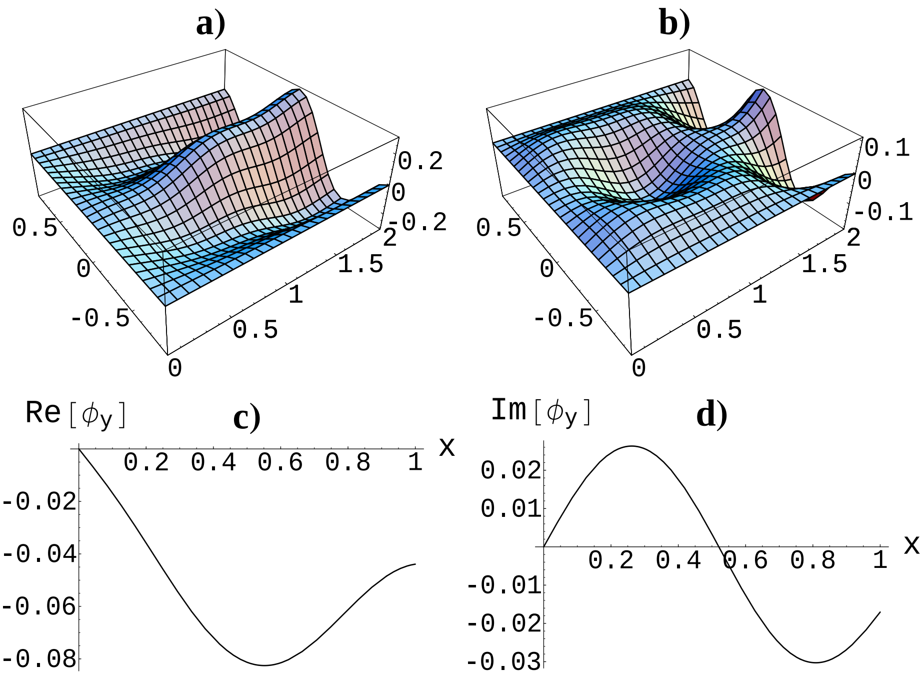

Figure 4: Pin-jointed plate: a) and b) show the real and imaginary parts of $\phi_y(x, a, z)$, $0 \leq x \leq 2$, $-b \leq z \leq b$, (b=0.879); c) and (d) show the real and imaginary parts of the section $\phi_y(x, a, b/3)$, $0 \leq x \leq 1$.

(taken in the xy-plane at $z = b/3$) is shown in figures 4 c) and d). This confirms that, as expected for a simply supported plate, the plate displacement is zero whilst the gradient is non-zero at $x = 0$. Note, for this example, all the eigenfunction expansions have been evaluated using 50 terms. Note also that the applied pressure is separable,"even" in z and has been chosen such that the z dependence satisfies the constraints discussed in §4, that is, $f(y, 0) \neq 0$ and $f(y, b) = f_z(y, b) = 0$. In addition, to these constraints it is necessary that $f_y(0, z) = f_y(a, z) = 0$. The first of the latter two is a reflection of the fact that $\psi_{ny}(0, z) = 0,\ n = 0, 1, 2\ldots$ and the second, as mentioned above, is required in order that $f_{yyy}(a, z)$ can be expressed as an eigenfunction expansion in terms of $\psi_{nyyy}(a, z)$, $n = 0, 1, 2, \ldots$.

Prescribed velocity

Similarly to the above case, the fluid velocity potential can be expressed in terms of the eigenmodes $\psi_n(y, z)$ as

$$\phi(x, y, z) = \sum_{n=0}^{\infty} \left\{ B_n - \frac{1}{s_n D_n} \int_0^b \lambda(w)\psi_{ny}(a, w)\, dw \right\} \psi_n(y, z)e^{is_n x} \tag{63}$$

where $\lambda(w)$ is unrelated to that of §4. The integral in (63) is an arbitrary eigensolution and, in view of (29), it is clear that the x derivative of this term vanishes when $x = 0$. Thus, the boundary condition at $x = 0$, $0 \leq y \leq a$, $-b \leq z \leq b$ is expressed as

$$\phi_x(0, y, z) = \sum_{n=0}^{\infty} B_n i s_n \psi_n(y, z) = f(y, z) \tag{64}$$

where $f(x, y)$ is the (known) velocity distribution applied to the end of the duct. The edge conditions to be applied are a spring-like condition, that is $\phi_{yxxx} + \beta\phi_y(0, a, z) = 0,\ 0 \leq z \leq b$ where β is proportional to the spring stiffness, together with the zero gradient condition $\phi_{yx}(0, a, z) = 0,\ 0 \leq z \leq b$. Note that, as $\beta \to \infty$ this pair of conditions tend to those for a clamped edge. Note also that the clamped edge conditions cannot be applied directly as they give rise to a Fredholm integral equation of the first kind, for which it is unclear if a unique solution exists.

Thus, on multiplying (21) by $B_\ell i s_\ell$, summing over ℓ and applying the zero gradient edge condition,

$$B_n i s_n = \frac{1}{D_n} \int_0^b \int_0^a f(y, z)\psi_n(y, z)\, dy\, dz + \frac{1}{D_n} \int_0^b f_{yyy}(a, z)\psi_{ny}(a, z)\, dz. \tag{65}$$

Analogous to the previous problem, $\phi_{yxxx}(0, a, z) + \phi_{xyyy}(0, a, z) = 0$ and hence the remaining edge condition can be expressed as $\phi_{xyyy}(0, a, z) - \beta\phi_y(0, a, z) = 0,\ 0 \leq z \leq b$ which, on using (63) can be expressed as an integral equation. It is found that

$$i\lambda(z) = f_{yyy}(a, z) - \beta \sum_{n=0}^{\infty} B_n \psi_{ny}(a, z) + \beta \int_0^b \lambda(w) \sum_{n=0}^{\infty} \frac{\psi_{ny}(a, w)\psi_{ny}(a, z)}{s_n D_n}\, dw \tag{66}$$

where (45) has been used in order to isolate the unknown function $\lambda(z)$ on the left hand side and B_n is given by (65). This is a Fredholm integral of the second kind and is straightforward to solve numerically.

To demonstrate the above solution, numerical results are presented for the case in which the velocity distribution applied at $x = 0$ is $f(y,z) = (z^2/b^2 - 1)^6(y^2/a^2 - 1)^2$. The frequency is taken to be 50 Hz (for which the dispersion relation has only one real root). Figure 5 shows the function $\lambda(z)$, $0 \le z \le b$ obtained by solving (66) numerically. It is worth noting that $\lambda(z)$ satisfies the conditions $\lambda'(0) = 0$ and $\lambda(b) = \lambda'(b) = 0$, and that, in order to ensure this, it was necessary to choose the applied velocity distribution to satisfy $f_{yyy}(a,b) = 0$. Figures 6 a) and b) show the plate displacement $\phi_y(x,a,z)$ obtained using (63)

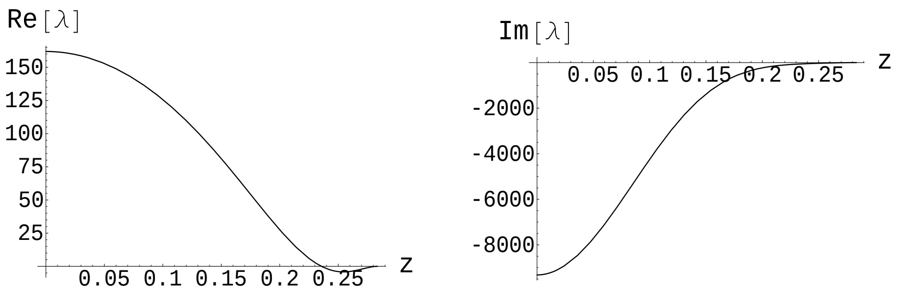

Figure 5: The real and imaginary parts of the function $\lambda(z)$ evaluated using (66).

having determined $\lambda(z)$ with $\beta = 100000$. It is clear that $\phi_y(0,a,z) \approx 0$, $-b \le z \le b$. A section of the plate displacement (taken in the xy-plane at $z = b/3$) is shown in figures 6 c) and d). This confirms that the plate displacement is close to zero and the gradient is zero at $x = 0$. Note, for this example, the eigenfunction expansions have been evaluated using a maximum of 45 terms. Finally, figure 7 shows the absolute value of the non-dimensional plate displacement at the edge $x = 0$ plotted against β for three fixed values of z, that is: $z = 0$, $b/3$, $3b/4$. When $\beta = 0$ the spring-like condition reduces to the zero force condition, whilst as β increases it is clear that, after an modest initial augmentation, the edge displacement tends to zero.

6 Discussion

The initial aim of this article was to establish the analytic properties of the eigenfunctions $\psi_n(y,z)$, $n = 0,1,2\ldots$. This has been accomplished in sections three and four where several pertinent results are derived. In §3, theorem 3.2 proved that the eigenfunctions $\psi_n(y,z)$, $n = 0,1,2\ldots$ are linearly dependent. It is interesting to note that, although non-separable themselves, certain sums

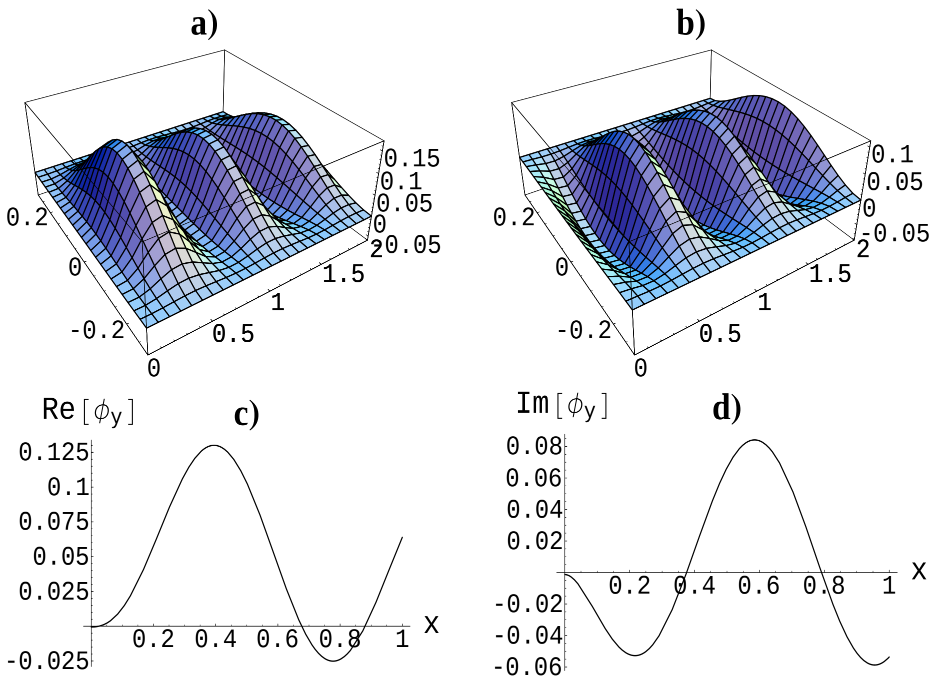

Figure 6: Spring supported plate with $\beta = 100000$: a) and b) show the real and imaginary parts of $\phi_y(x, a, z)$, $0 \leq x \leq 2$, $-b \leq z \leq b$, (b=0.284); c) and (d) show the real and imaginary parts of the section $\phi_y(x, a, b/3)$, $0 \leq x \leq 1$

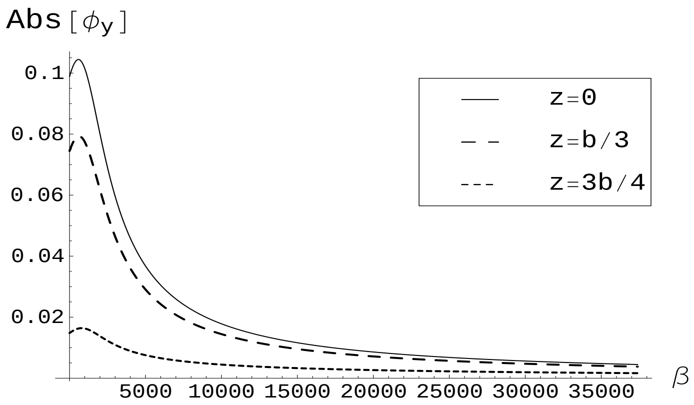

Figure 7: Edge displacement, $\phi_y(0, a, z)$, against β for $z = 0,\ b/3,\ 3b/4$.

involving products of the eigenfunctions $\psi_n(y, z)$ (and/or their derivatives) can be reduced to a separable form which enables them to be evaluated using known properties of the underlying two-dimensional system. Sums (29) and (31) are of this form and, furthermore, are valid for $0 \le w \le b$ which enables arbitrary eigensolutions, such as that used in (63), to be assembled. These eigensolutions satisfy (2)–(7) and are useful for constructing solutions that satisfy specified edge conditions, such as the spring-like condition used in §5. Theorem 3.4 proved that, with appropriate choice of the amplitude coefficients, an eigenfunction expansion of a suitable function, $f(x, y)$, converges point-wise to that function. This result is particularly significant in that it justifies the use of such expansions as a means of representing a wide class of functions.

In §4 the derivatives of the eigenfunctions were investigated and new orthogonality properties were presented. These results are not expressed in the form of an orthogonality relation (indeed it is not clear whether an OR, as such, exists). Instead it was demonstrated that three suitably smooth functions $p(z)$, $q(z)$ and $\lambda(z)$ can be expressed as an eigenfunction expansion in terms of the functions $\chi_n(z)$, $\psi_{ny}(a, z)$ and $\psi_{nyyy}(a, z)$ respectively, but with the same (or closely related) amplitude coefficient in each case.

The second aim of this article was to investigate the potential use of the eigenfunctions $\psi_n(y, z)$ in mode-matching problems. This issue is addressed in §5 where two prototype problems have been solved. These prototypes have been selected to be representative of the class of problem that arises in a typical mode-matching problem - in which the pressure and velocity must usually be enforced at an interface between two adjacent duct sections. Here the interface is taken to lie at $x = 0$. The first prototype deals with a prescribed pressure and pin-jointed edge conditions and this is solved exactly using the partial OR, (21). The second prototype deals with a prescribed velocity and spring-like edge conditions which tend to the usual clamped conditions as the spring constant increases. The solution is constructed as an eigenfunction expansion containing an additive eigensolution, see (63), which is expressed in terms of an unknown function $\lambda(z)$. On applying the edge conditions a Fredholm integral equation is obtained which is then solved numerically to determine $\lambda(z)$ - and hence the solution to the problem.

The purpose of this paper is not to present a parametric investigation of the prototype problems, but instead to assess whether the eigenfunction expansions considered herein provide a viable solution method to three dimensional mode-matching problems. Nevertheless, numerical results showing the plate displacement are presented for both prototype problems thereby demonstrating that this approach is indeed a potentially powerful tool for tackling three-dimensional mode-matching problems. There are, however, some issues that need to be addressed. For example, sufficient roots of the dispersion relation must be found in order for the method to be accurate. In fact, it is generally not difficult to find the complex roots but obtaining the imaginary roots is a tedious procedure. The reason for this is the structure of $L(s)$. When s is imaginary $L(s)$ has an infinite number of asymptotes and the zeros generally lie very close these and are consequently difficult to locate numerically due to the rapidly changing gradient. Furthermore, due to the poles of $L(s)$ it is difficult to apply the argument principle as a means of testing whether all roots in a given region of the complex plane have been found. On account of (44) and (45), however, the quantity R_{nm}, (55), has some interesting properties, one of which is

$$R_{nm} = \sum_{\ell=0}^{\infty} \frac{R_{n\ell} R_{\ell m}}{D_\ell}. \tag{67}$$

The sum on the right is highly sensitive to missing roots and, thus, this expression provides a useful check for whether sufficient roots of the dispersion relation have been found. With these points in mind, however, it would be useful to have some means of predicting the number and location of roots. A Weyl series, see for example Howls & Trasler (1998), would be helpful in this respect. Alternatively, following Lawrie & Kirby (2006), it may be possible to construct a “root-free” approach for certain duct geometries.

In summary, this article presents an analytic investigation into a class of eigenfunctions that is potentially of use in mode-matching problems involving the propagation of sound in 3-D rectangular ducts. The thrust of the article is in establishing the underlying properties of these eigenfunctions. It remains to be seen whether the proposed methodology is numerically efficient when applied to problems of real engineering significance - particularly when compared with modern numerical approaches such as the semi-analytic finite element (SAFE) method, see Castaings & Lowe (2008) and Moreau *et al.*(2006). Nevertheless, based on such comparisons for the 2-D case, see Kirby & Lawrie (2005), it is anticipated that this 3-D mode-matching method should prove a useful qualitative tool whilst simultaneously providing a benchmark against which to test a variety of numerical strategies.

References

[1] Besserer, H. & Malischewsky, P. 2004 *"Mode series expansions at vertical boundaries in elastic waveguides"* Wave Motion, **39**, 41–59.

[2] Cabelli, A. 1985 *"The propagation of sound in a square duct with a non-rigid wall" J. Sound Vib.* **103**, 379–394.

[3] Castaings, M & Lowe, M. 2008 *"Finite element model for wave guided along solid systems of arbitrary section coupled to infinite solid media" J. Acoust. Soc. Am.* **123**, 696–708.

[4] Evans, D. V. & Porter, R. 2003 *"Wave scattering by narrow cracks in ice sheets floating on water of finite depth"*, J.Fluid Mech. **484**, 143–165.

[5] Grant, A. D. & Lawrie J. B. (2000) *"Propagation of fluid-loaded structural waves along a duct with smoothly varying bending characteristics"* Q. Jl Mech appl. Math. **53**, 299–321.

[6] Howls, C. J. & Trasler, S. A. (1998) "Weyl's wedges" *J. Phys A*, **31(8)**, 1911-1928.

[7] Huang, L. 2002 "Modal analysis of a drumlike silencer" J. Acoust. Soc. Am. **112**, 2014–2025.

[8] Huang L. & Choy, Y. S. 2005 "Vibroacoustics of three-dimensional drum silencer " J. Acoust. Soc. Am. **118**, 2313–2320.

[9] Kaplunov, Y. D., Kirillova, I. V. & Postnova, Y. A. 2004 *"Dispersion of waves in a plane acoustic layer with flexible elastic walls."* Acoustical Physics, **50**, 694–698.

[10] Kirby, R.& Lawrie, J. B. 2005 *"A point collocation approach to modelling large dissipative silencers." J. Sound Vib.*, **286**, 313–339.

[11] Lawrie, J. B. & Abrahams, I. D. 1999 *"An orthogonality condition for a class of problem with high order boundary conditions; applications in sound/structure interaction"* Q. Jl Mech appl. Math. **52**, 161–181.

[12] Lawrie, J. B. & Abrahams, I. D. 2002 *"On the propagation and scattering of fluid-structural waves in a three-dimensional duct bounded by thin elastic walls."* Proceedings of IUTAM 2000/10, Kluwer. Eds. I.D. Abrahams et. al. ISBN 1-4020-0590-3.

[13] Lawrie, J. B. & Guled, I. M. M. 2006 *"On tuning a reactive silencer by varying the position of an internal membrane." J. Acoust. Soc. Am.*, **120**, 780–790.

[14] Lawrie, J. B. & Kirby, R. 2006 *"Mode-matching without root-finding: application to a dissipative silencer." J. Acoust. Soc. Am.*, **119**, 2050–2061.

[15] Lawrie, J. B. 2007 *"On eigenfunction expansions associated with wave propagation along ducts with wave-bearing boundaries." IMA Jl Appl Math*, **72**, 376–394.

[16] Manam, Bhattacharjee & Sahoo 2006 *"Expansion formulae in wave structure interaction problems" Proc. R. Soc. Lond. A.* **462**, 263–287.

[17] Martin, V., Cummings, A. & Gronier, C. 2004 *"Discrimination of coupled structural/acoustical duct modes by active control: principles and experimental results" J. Sound Vib.* **274**, 583–603.

[18] Moreau, L., Castaings, M., Hosten, B. & Predoi, M.V. 2006 *"An orthogonality relation-based technique for post-processing finite element predictions of waves scattering in solid waveguides"J. Acoust. Soc. Am.* **120**, 611–620.

[19] Rawlins, A. D. 2007 *"Wave propagation in a bifurcated impedance-lined cylindrical waveguide"* Journal of Engineering Mathematics **59**, 419–435.

[20] Zakharov, D. D. 2008 *"Orthogonality of 3D guided waves in viscoelastic laminates and far field evalutaion to a local acoustic source"* Internationl Journal of Solids and Structures **45**, 1788–1803.

[21] Zernov, V. & Kaplunov, J. 2008 *"Three dimensional edge-waves in plates." Proc. R. Soc. Lond.*, **A464**, 301–318.