The revelation principle and regularity conditions

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Abstract

The revelation principle asserts that every outcome brought by a mechanism is realized by a truthful direct mechanism. The present paper investigates the regularity conditions of these two mechanisms in the continuous space of the agent's type. It questions what regularity condition a general mechanism confers upon a direct mechanism through the revelation principle. By so doing, we elucidate the limit of the revelation principle.

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1 Introduction

The revelation principle states that every equilibrium outcome with a mechanism is realized by a truthful direct mechanism. On the strength of this principle, mechanism design literature focuses upon a direct mechanism in its analysis. Until now, no attention seems to have been paid to the regularity condition (e.g, continuity and so forth) transmitted to the direct mechanism from the indirect one via the revelation principle.

When the agent's type space is discreet, the matter is not problematic since we need not assume any mathematical structure—functional space, topology–on the direct mechanism, in search of the optimal direct mechanism. By contrast, with the continuous type space, the thing is different. Here, while solving for the optimal direct mechanism, one usually draws upon calculus of variation or optimal control, which requires, at least, absolute continuity on the state variable.

To apply these optimization techniques, we transform the implementability condition on the direct mechanism, in which process we are bound to assume absolute continuity on the mechanism. On the other hand, the direct mechanism derived from an indirect mechanism through the revelation principle is not assured to be continuous still less absolutely continuous. This implies that equivalence between an indirect mechanism and a direct mechanism(due to the revelation principle) should be taken with reservations.

Probably, the only indirect mechanism to have been actually analyzed is the non-linear price scheme. In search of the optimal scheme, one usually performs pointwise optimization, which may give rise to a discontinuous marginal price and thus a non-differentiable price scheme. As mentioned above, the direct mechanism deduced from this price scheme is not certain to satisfy the regularity condition(absolute continuity). Furthermore, a continuous price scheme does not necessarily generate a continuous direct mechanism. Hence a serious challenge to the standard direct mechanism approach in the theory of mechanism design.

This article investigates into these issues on the regularity conditions which have not drawn until now due theoretical attention.

The next section presents the setting and shows that continuity does not necessarily pass over from the general mechanism to the direct mechanism. Section 3 gives the examples of mechanism design theory. Section 4 studies a weaker regularity condition than continuity. The final section concludes the article.

2 The model

There is an agent and a principal. Θ is the agent's *type* space. The type is the agent's private information and unobservable to the principal. Y is an *allocation* space. If the principal is a monopolistic seller of a commodity and the agent is the buyer, for instance, the type is the buyer's appreciation for the commodity and the allocation space consists of a quality(quantity) and price space. If the principal is a firm and the agent is the employee, the type is the buyer's cost parameter in producing an ouput and the allocation space consists of the space of the output quantity and the salary.

It is assumed that both the spaces Θ and Y are a Hausdorff space and that Θ is a compact space. The agent's utility function is assumed to be continuous:

$$U(y, \theta) : Y \times \Theta \to \mathbb{R}.$$

M is a metric space which we call *message* space. A *mechanism* is a function $y(m)$ from M to Y. A *direct mechanism* is a mechanism with the message space M coinciding with Θ. Otherwise, the mechanism is an indirect mechanism.

Definition 1. *A direct mechanism $y(\theta) : \Theta \to Y$ is implementable if and only if for any θ and $\theta' \in \Theta$,*

$$U(y(\theta), \theta) \geq U(y(\theta'), \theta)$$

We pose

$$V(\theta) := \sup_{m \in M} U(y(m), \theta).$$

For instance, if $y(M)$ is compact, then the supremum is attained as the maximum.

We pose the maximizer of the utility function as

$$N(\theta) := \{m \in M | V(\theta) = U(y(m), \theta)\}.$$

The *selection* of a set valued map $N(\theta)$ is a single valued map $s(\theta)$ such that

$$s(\theta) \in N(\theta).$$

Proposition 1 (Revelation Principle). *Let $N(\theta)$ be non-empty for any $\theta \in \Theta$. Whatever selection $s(\theta)$ is chosen, it holds good that*

$$V(\theta) = U(y(s(\theta)), \theta).$$

$y(s(\theta))$ is an implementable direct mechanism as a function from Θ to Y. This is the justification, whereby literature on mechanism design usually concentrates upon the direct mechanism. On the strength of it, it searches for the optimal direct mechanism while claiming that all the outcomes by an indirect mechanism are replicated by a direct mechanism. However, in search of the optimal direct mechanism, we are obliged to transform the implementability condition into a manageable form(see Proposition 7) in order to set up the maximization problem.[1] For this transformation, we assume, at least, absolute continuity on the direct mechanism. The problem is that $y(s(\theta))$ is not certain to satisfy continuity even if $y(\cdot)$ is continuous. In other words, it is not assured that $N(\theta)$ possesses a continuous selection even in the most favourable situation.

[1] Calculus of variation or optimal control in most cases.

We have several fundamental propositions, for which proof the reader is referred to Aubin and Cellina (1984).

Proposition 2 (Maximum principle). *If $y(M)$ is compact, $V(\theta)$ is continuous.*

We have a stronger result.

Proposition 3. *Let Θ and Y be a metric space. If U is Lipschitzean on $Y \times \Theta$, so is V on Θ.*

Proof. Take θ_1 and $\theta_2 \in \Theta$. For any $\epsilon > 0$, there is y_1 such that $V(\theta_1) \leq U(y_1, \theta_1) + \epsilon$. Then, we have

$$\begin{aligned} V(\theta_1) - V(\theta_2) &\leq U(y_1, \theta_1) - U(y_1, \theta_2) + \epsilon, \\ &\leq ld(\theta_1, \theta_2) + \epsilon. \end{aligned}$$

where l is a Lipschitz constant of U. ϵ can be arbitrary and the proposition follows.

□

It follows, in this case, that V is absolutely continuous. In economic theory, in search of the optimal implementable mechanism, we resort to calculus of variations(or optimal control) and V is treated as a state variable, which is required to be absolutely continuous. Therefore, if the agent's utility function U is Lipschitzean, V fulfils the regularity condition for the use of the techniques.

Proposition 4. *If $y(M)$ is compact, $N(\theta)$ is upper semicontinuous.*[2]

Note first that no regularity is assumed upon y except that $y(M)$ is compact. The compactness must not be so strong an assumption in our mechanism context, where evey type of agent sends the best message m so as to be allocated the best $y(m)$. That condition ensures the existence of the best message.

[2]The definitions of semicontinuity are relegated to the appendix.

According to the proposition, if M is compact and the mechanism $y(m)$ is continuous, $N(\theta)$ is upper semicontinuous. The problem is that an upper semicontinuous set-valued map is not certain to have a continuous selection even under a fairly strong condition.[3] In other words, the implementable direct mechanism deduced from $y(m)$ is not necessarily continuous. However, in the case where $N(\theta)$ is single-valued, we have the following:

Corollary 1. *If $y(M)$ is compact and if $N(\theta)$ is a single valued map, $N(\theta)$ is continuous.*

For instance, if with all θ, $U(y(m), \theta)$ is strictly concave in m, then $N(\theta)$ is single-valued and indeed continuous.

In view of the above facts, in the next section, we shall take a look at the examples of models in mechanism design theory.

3 Examples

3.1 The case of $U(y, \theta) = q\theta - p$

Rochet (1985) gives implementability conditions with a more general form of U without any regularity condition on the mechanism $y(\theta)$. For our purpose, though, the following specification is enough.

Let us suppose that $y = (p, q)$ and $Y = Q \times P$ where $Q \subset \mathbb{R}^L$ for $L \geq 1$ is a closed and bounded quality(quantity) space and that $P = \mathbb{R}$ is a price space. Suppose also that $\Theta \subset \mathbb{R}^L$ is a convex set.

In the following particular setting, Rochet (1985) and Rochet (1987) derives the regularity of $V(\theta)$ from the implementability constraint without assuming any regularity on the direct mechanism:

$$U(y, \theta) = U(q, p, \theta) := q\theta - p.$$

[3]By contrast, a lower semicontinuous set-valued function possesses a continuous selection if it is of closed convex values.

Therefore, in this case, the regularity condition does not jeopardize the claim for generality of a direct mechanism.

Proposition 5. *The mechanism is almost everywhere implementable*[4] *if and only if*

$$V \text{ is convex,}$$
$$\nabla V(\theta) = q(\theta) \qquad a.e.$$

This proposition makes no regularity assumption upon the mechanism. The proof is based upon the linearity between θ and q, and the subdifferential of a convex function. In search of the optimal mechanism, replacing $q(\theta)$ with ∇V, Rochet and Chone (1998) search for the optimal V in the Sobolev space $\mathcal{H}^1$. Since the convex function is continuous in the interior of the domain, no additional regularity condition is assumed a priori. Thus, the revelation principle is totally valid.

In the one-dimensional type space, we obtain the following proposition.

Proposition 6. *Suppose that $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$. The mechanism (q, p) is implementable if and only if*

$$V \text{ is convex and absolutely continuous,}$$
$$\dot{V}(\theta) = q(\theta) \qquad a.e.$$

Proof. We only show that implementability leads to absolute continuity of $V(\theta)$. $U(y, \theta)$ is absolutely continuous in θ for every y. Thus

$$|V(\theta'') - V(\theta')| \leq \sup_{\theta\in\Theta} |U(y(\theta), \theta'') - U(y(\theta), \theta')| \leq \sup_{\theta\in\Theta} \int_{\theta'}^{\theta''} |U_\theta(y(\theta), \tilde{\theta})| d\tilde{\theta} \leq \int_{\theta'}^{\theta''} |\overline{q}| d\tilde{\theta},$$

where $\overline{q}$ is the maximum of Q.

□

[4] i.e, it holds good, for almost every θ, that $U(y(\theta), \theta) \geq U(y(\theta'), \theta)$ for any θ'.

Absolute continuity is a standard regularity condition imposed upon the state variable in the calculus of variations. Thus, in the case of the one-dimensional type space, we can resort to the calculus of variations by replacing q with $\dot{U}$. Likewise, the second equation in the proposition and absolute continuity make $q(\theta)$ measurable and integrable. As a result, we can resort to optimal control without any additional assumption.

The above two proposition are derived by use of the convexity of $U(\theta)$ and the subdifferential without any regularity assumption on the mechanism itself. In order to use this technique, we need linearity between the agent's type and the action variable q(see Rochet (1985) and Rochet (1987)).

3.2 The general case

In the case of the more general form of U, we cannot use the technique employed in the previous section and we are obliged to put some prior regularity condition upon a mechanism in order to derive the implementability condition(c.f. Guesnerie and Laffont (1984) and Fudenberg and Tirole (1993)).

Let us suppose that $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$, $Y = Q \times P$ where Q is a non-empty, compact, convex subset of $\mathbb{R}^L$ for $L \geq 1$ and $P = \mathbb{R}$.

$$U(y, \theta) := U(q, p, \theta).$$

Assumption 1. *U is strictly increasing in p. U is of class C^2.*

With these assumptions, we obtain the necessary condition for implementability.

Proposition 7. *Under Assumption 1, let the mechanism $y = (q, p)$ be absolutely continuous(or piecewise C^1). If the mechanism is implementable, then it obtains that*[5] $\left[\frac{\partial}{\partial\theta}\left(\frac{\partial_q U(q(\theta),p(\theta),\theta)}{\partial_p U(q(\theta),p(\theta),\theta)}\right)\right]\frac{dq}{d\theta}(\theta) \leq 0$*, for any θ where p and q are differentiable.*

[5] This is a product of vectors.

Proof. See Fudenberg and Tirole (1993). The proof assumes the mechanism to be piecewise C^1 but it is straightforward that the proof extends to absolute continuity. □

The proof builds upon almost everywhere differentiability of the mechanism due to absolute continuity(or piecewise C^1). As was remarked above, this is an annoying regularity assumption since the direct mechanism generated by the indirect mechanism is not assured to be continuous.

Assumption 2. *The following vector is component-wise non-negative: i.e. on $Q \times P \times \Theta$,*

$$\frac{\partial}{\partial \theta}\left(\frac{\partial_q U}{\partial_p U}\right) \geq 0.$$

Boundary behaviour: for any $(q, p, \theta) \in Q \times P \times \Theta$ there exists K_0 and $K_1 > 0$ such that for all q_l where $l = 1, \ldots, L$,

$$\left|\frac{\partial_{q_l} U(q, p, \theta)}{\partial_p U(q, p, \theta)}\right| \leq K_0 + K_1 |\theta|$$

uniformly in (q, p, θ).[6]

When the action space is one-dimensional, $L = 1$, we obtain the necessary and sufficient condition for implementability.

Proposition 8. *Under Assumptions 1 and 2 and $L = 1$, the absolutely continuous(or piecewise C^1) mechanism is implementable if and only if $\frac{dq}{d\theta} \geq 0$ a.e.*

On the strength of this proposition, the standard mechanism design literature works not with $(q(\theta), p(\theta))$ but $(q(\theta), V(\theta))$. It resorts to optimal control with $V(\theta)$ a state variable and $q(\theta)$ a control variable. As is stated after the previous proposition, an indirect mechanism does not necessarily generates an absolutely continuous direct mechanism; in particular, this implies that the optimal non-linear price scheme does not necessarily give rise to a direct mechanism with that regularity. Accordingly, the optimal direct

[6] Here, $(q_1, \ldots, q_L) = q \in \mathbb{R}^L$.

mechanism found by use of the above proposition(through optimal control) may not be optimal, compared to the optimal non-linear price scheme.

4 Weaker regularity

The above argument shows that the assumption that an indirect mechanism generates an absolutely continuous(or piecewise C^1) direct mechanism through the revelation principle is rather strong. The next question is whether an indirect mechanism generates a direct mechanism with a weaker regularity or a measurable direct mechanism. We refer to the theory of a measurable selection of a multi-valued map.

Definition 2. *Suppose that $(\Omega, \mathcal{A})$ is a measurable space and X is a polish(i.e. separable, complete and metric) space and that $F : \Omega \rightsquigarrow X$ is a multi-valued map with closed images. F is* measurable *if the inverse image of every open set is a measurable set: i.e, if, for any open set $O \subset X$,*

$$F^{-1}(O) := \{w \in O | F(w) \cap O \neq \emptyset\}$$

is measurable.

Proposition 9 (Measurable Selection). *Suppose that $(\Omega, \mathcal{A})$ is a measurable space and X a polish space and $F : \Omega \rightsquigarrow X$ a measurable multi-valued map with non-empty closed images. Then F possesses a measurable selection(i.e. a selection which is measurable as a single-valued function).*

Proof. See Th.8.1.3 in Aubin and Frankowska (1990). □

For the details of the characterization of the measurable multi-valued map, refer to Aubin and Frankowska (1990). Here it suffices to have

Proposition 10. *On the same assumption as in Proposition 9, if $F^{-1}(C) \in \mathcal{A}$ for every closed set $C \subset X$, then F is a measurable multi-valued map.*

Proof. Let us take an open set $O \subset X$ and define the closed set

$$C_n := \{x \in X | d(x, \complement O) \geq \frac{1}{n}\}^7.$$

Obviously, we have $O = \cup_{n \geq 1} C_n$ and we see that $F(\omega) \cap O \neq \emptyset$ holds if and only if $F(\omega) \cap C_n \neq \emptyset$ for some $n \geq 1$. In consequence,

$$F^{-1}(O) = \cup_{n \geq 1} F^{-1}(C_n) \in \mathcal{A}.$$

□

Proposition 11. *On the same assumption as in the previous proposition, if F is upper semicontinuous, then F is measurable.*

Proof. If F is upper semicontinuous, then $F^{-1}(C)$ is closed for every closed $C \subset X$. From the previous proposition it follows that F is measurable.

□

Proposition 12. *On the same assumption as in the previous proposition, if F is upper semicontinuous, then it has a measurable selection.*

Proof. From Propositions 10 and 12, follows the proposition.

□

Now, we can make a statement upon the mechanism.

Proposition 13. *Let there be given Y, Θ, M and $U(y, \theta)$ as in Section 2 and suppose in addition that Y is Polish. Then, if $y(M)$ is compact, $N(\theta)$ possesses a measurable selection.*

Proof. It follows from Propositions 4 and 12. □

[7] d is a distance and $\complement$ is a complement of a set.

In particular, if the mechanism $y(m)$ is continuous, we can obtain, at least, a measurable direct mechanism, $y(m(\theta))$.

Now, to the direct mechanism we can give a regularity condition.

Proposition 14. *Let there be given* Y*,* Θ *and* $U(y, \theta)$ *as in Section 2 and suppose in addition that* Y *is Polish. Then, if the direct mechanism* $y(\theta)$ *is implementable and* $y(\Theta)$ *is compact, then* $N(\theta)$ *possesses a measurable selection.*

The proposition states that even though implementable, the direct mechanism is not necessarily measurable. There exists, however, a measurable direct mechanism with which every type of agent is as well off as with $y(\theta)$. This case includes the one in Section 3.2.

5 Conclusion

We have examined the regularity condition which the direct mechanism inherits from the general mechanism through the revelation principle. Generally, the maximizer of the agent's utility function is not a single valued function in the type. This causes the direct mechanism induced not necessarily to take over the regularity condition that the general mechanism initially possesses.

6 Appendix

In the case of a single-valued map, we can equivalently define continuity either by the inverse image of an open set or the limit of a converging sequence. In the case of a multi-valued map, these two definitions are not equivalent and consequently we have the two corresponding definitions of continuity.

Definition 3. *Suppose that* X *and* Z *are topological spaces. A multi-valued map* F *from* X *to the subsets of* Z *is said to be* upper semicontinuous *at* x *if for any neighbourhood*

V of $F(x)$, there exists a neighbourhood U of x such that $F(U) \subset V$.

F is upper semicontinuous if it is upper semicontinuous at every point.

Lemma 1. *Suppose that X and Z are topological spaces and that F is a multi-valued map from X to the subsets of Z. F is upper semicontinuous if and only if the inverse image of any closed set A, i.e. $\{x|F(x) \cap A \neq \emptyset\}$ is closed.*

Proof. First, it is straightforward that F is upper semicontinuous if and only if the core of any open set B, i.e. $\{x|F(x) \subset B\}$ is open. If we denote the core and the inverse image of a set S by $F^{+1}(S)$ and $F^{-1}(S)$, then we see that $F^{+1}(\complement S) = \complement F^{-1}(S)$. The rest follows from this.

□

Definition 4. *Suppose that X and Z are topological spaces. A multi-valued map F from X to the subsets of Z is said to be* lower semicontinuous *at x if for any $y \in F(x)$ and for any neighbourhood $V(y)$ of y, there exists a neighbourhood $U(x)$ of x such that for any $x' \in U(x)$, $F(x') \cap V(y) \neq \emptyset$.*

F is lower semicontinuous if it is lower semicontinuous at every point.

Definition 5. *Let the specifications of the previous definitions be given. F is* continuous *at x if it is both lower and upper semicontinuous at x. F is continuous if it is continuous at every point.*

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