Application of Local and Non-Local Approaches to Multiple Fatigue Crack Initiation and Propagation

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ABSTRACT. *Fatigue strength conditions presented in terms of normalized equivalent stress functionals defined on loading processes are used to unite the stages of material damage with fatigue crack initiation and multiple crack propagation under arbitrary loading history. Examples of employing the local form of the functionals associated with the Palmgren-Miner linear damage accumulation rule and the power-type S–N diagram to a periodic crack system are given and shortcomings of the local approach are pointed out. A non-local approach free from the shortcomings is described. Equations for curvilinear crack growth rate vectors taking into account the whole damage history ahead of the crack are presented for multiple cracks under mixed-mode loading.*

INTRODUCTION

A common practice of a cyclic fatigue life local analysis includes usually two steps. First, a crack initiation cycle number $n*$ is determined from a fatigue strength condition expressed in terms of a damage measure based on a cycle stress range. A crack of a length a_0 is supposed to appear at a point $y*$ in a body where and when the fatigue strength condition is violated. Then the Paris type equation is used for prediction of the crack propagation from the initial value a_0 to separation of the body into pieces or to unstable crack growth. However the value a_0 being a key issue for the fatigue crack propagation prediction is often not clearly fixed and the material parameters of the strength condition of the first step seem to be completely unrelated to the Paris law parameters. On the other hand, by using the Paris type equation, one can describe neither the scale effect for short cracks nor the influence of the fatigue damage during the previous cycles on the crack propagation rate.

Trying to overcome the shortcomings, a local united approach based on an extension of the fatigue strength conditions under homogeneous stress to the crack propagation stage was employed in [1], its limitations were outlined on an example of a single fatigue crack initiation and propagation. To avoid those drawbacks, a non-local modification merging a special form of the static non-local approach [2] with the functional description of brittle cyclic strength similar to [3-4] was then employed in [1]. This allowed to analyze strength and durability under oscillating in time

homogeneous and highly inhomogeneous stress fields and to predict both the crack initiation and its propagation through the damaged material as a united process.

In this paper, we apply the local and non-local approaches to initiation and propagation a periodic system of multiple fatigue cracks. Note that some other particular non-local approaches were used for predicting fatigue life in [5-9].

1. LOCAL BRITTLE STRENGTH AND DURABILITY CONDITIONS

Let us consider a cyclic process in a body Ω represented as a cycle sequence $\{\sigma_{ij}^c(m,x)\}_{m=1,2...}$ of connected closed but generally non-coinciding loops (cycles) $\sigma_{ij}^c(m,x) = [\sigma_{ij}(x,\tau); \tau_{m-1} \le \tau \le \tau_m]$ in the stress space, $\sigma_{ij}(x,\tau_m) = \sigma_{ij}(x,\tau_{m-1})$, where $m=1,2,...$ is the cycle number, $x \in \Omega$. Then the cyclic fatigue can be described in terms of the cycle number n (instant n) as a discrete or continuous time-like parameter.

To describe fatigue crack initiation and propagation, we will analyse the brittle strength, that is strength at a particular point y along a particular infinitesimal plane with a normal vector $\vec{\zeta}$ at that point. The local fatigue brittle strength condition *for a plane* $\vec{\zeta}$ *at a point* $y \in \Omega$ can be taken in the form [4], $\underline{\Delta}(\{\sigma^c(\cdot,y)\}; n, y, \vec{\zeta}) < 1$. Here $\underline{\Delta}(\{\sigma^c(\cdot,y)\}; n, y, \vec{\zeta})$ is a local brittle *Cyclic Normalized Equivalent Stress Functional* (CNESF) defined on sequences $\{\sigma_{ij}^c(m,x)\}_{m=1,2...}$ and is positively homogeneous in $\{\sigma_{ij}^c\}$ and non-decreasing in n. It is considered as a material characteristic.

For example, the CNESF associated with the power S-N diagram $n^* = (\Delta\sigma / \sigma_{R1}^{*\Delta})^b$ and the *Palmgren-Miner* linear accumulation rule has form [1,4],

$$\underline{\Delta}(\{\sigma^c(\cdot,y)\}; n, y, \vec{\zeta}) = \frac{1}{\sigma_{R1}^{*\Delta}} \left\{ \int_0^n [\Delta\sigma_{\zeta\zeta}(m,y,\vec{\zeta})]^b \, dm \right\}^{1/b} \tag{1}$$

where $\Delta\sigma_{\zeta\zeta}(m,y,\vec{\zeta})$ is the normal stress range on the plane $\vec{\zeta}$ at the point y during the cycle m; b is a non-negative material constant and $\sigma_{R1}^{*\Delta}$ is a material parameter depending only on the asymmetry ratio $R(y,\vec{\zeta}) = \sigma_{\zeta\zeta\min}(m,y,\vec{\zeta}) / \sigma_{\zeta\zeta\max}(m,y,\vec{\zeta})$.

Let a an open domain $\Omega(n)$ occupied by the body at an instant n has a boundary $\Gamma(n)$ consisting of an initial body boundary $\Gamma(0)$ and of a new crack surface $Y^*(n)$ occurring and growing during the cyclic process. Then we have the following conditions for the fatigue crack initiation and propagation,

$$\underline{\Delta}(\{\sigma^c(\cdot;\Gamma(\cdot),y)\}; n, y, \vec{\zeta}) < 1, \qquad y \in \Omega(n) \quad \forall \vec{\zeta}, \tag{2}$$

$$\underline{\Delta}(\{\sigma^c(\cdot;\Gamma(\cdot),y)\}; n, y, \vec{\zeta}^*(y)) = 1, \qquad y \in Y^*(n) \tag{3}$$

$$\sigma_{ij}(n;\Gamma(n),y)\zeta_j^*(y) = 0, \qquad y \in Y^*(n) \tag{4}$$

where $\vec{\zeta}^*(y)$ is the vector normal to the crack surface $Y^*(n)$.

Assuming a smooth dependence of $\underline{\Lambda}(\{\sigma^c(\cdot;\Gamma(\cdot),y)\};n,y,\vec{\zeta})$ on $\vec{\zeta}$ and using Eqs (2)-(3), the fracture plane unit normal $\vec{\zeta}^*(y)$ can be determined from the equations

$$\left.\frac{\partial\underline{\Lambda}(\{\sigma^c(\cdot;\Gamma(\cdot),y)\};n,y,\vec{\zeta})}{\partial\zeta_j}\right|_{\vec{\zeta}=\vec{\zeta}^*(y)}=0, \qquad \left|\vec{\zeta}^*(y)\right|=1, \qquad y\in Y^*(n) \qquad (5)$$

If the direction of crack growth is a priori known, e.g. from symmetry, then there is no need to determine $\vec{\zeta}^*$.

If there is an analytical or numerical method of the stress field calculation for any crack set Y^*, relations (2)-(5) allow to describe crack propagation and particularly crack path for single as well as multiple cracks under mixed loading.

2. LOCAL DURABILITY ANALYSIS FOR A PERIODIC SYSTEM OF CRACKS

2.1 Symmetric Plane Problem for Periodic Fatigue Crack Initiation and Propagation.
Analysis of a 2D body with one edge crack of a length $a(m)$ or one central crack of a length $2a(m)$ or two symmetric edge cracks of a length $a(m)$ was reduced in [1] to solution of one linear integral Volterra equation. In this paper, we extend the analysis to a $2l$-periodic 2D-problem for a system of collinear straight cracks of a length $2a(m)$ each, already existing or appearing during the fatigue process, symmetrically posed in a symmetrically loaded infinite strip (or plane) with periodic (particularly straight) boundary, Fig.1. Here $2l$ is the distance between the crack centers. Thus the geometry is also described by only one parameter $a(m)$, i.e. $\Gamma(m)=\Gamma(a(m))$, and the fatigue crack propagation path is straight with a normal vector $\vec{\zeta}^*=\{0,1\}$.
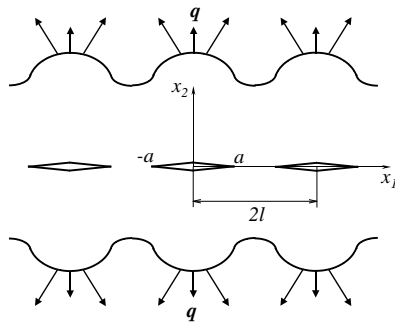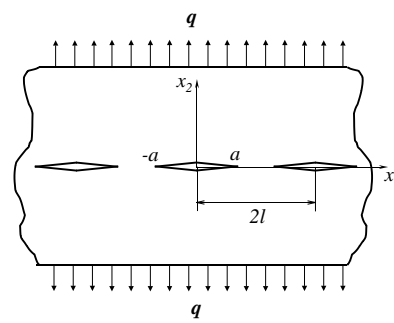

Figure 1.   Figure 2.

Let the origin of the Cartesian coordinate system $\{x_1,x_2\}$ coincides with the center of a crack. Let an external multi-axial self-similar cyclic loading be represented in the form $\Delta q(m,x)=\Delta q_0(m)|\hat{q}(x_1)|$, where $\Delta q_0(m)$ is a scalar function and $\hat{q}(x_1)$ is a $2l$-periodic and symmetric vector. Assuming the crack growth per cycle is small, we can neglect the distortion of the stress cycle shape during one cycle and write

$\Delta\sigma_{ij}(m,y) = \Delta q_0(m)\,|\,\hat{\sigma}_{ij}(a(m),y)\,|$. The stress field $\hat{\sigma}_{ij}(a(m),y)$ induced by the loading $\hat{q}(x_1)$ is supposed to be available analytically or numerically.

Let us take CNESF in the form (1), then the equation for the crack initiation moment n_0^* according to Eq. (3) is

$$\left|\hat{\sigma}_{22}(a_0,y^*)\right|^b \int_0^{n_0^*}\left|\Delta q_0(m)\right|^b dm = \left(\sigma_{R1}^{*\Delta}\right)^b, \qquad (6)$$

where $a_0 = 0$ if there is no crack initially in the body, y^* is the tip of an already existing crack or the stress concentration point where the crack will initiate. If there exists an initial crack system with $a(0) = a_0 \neq 0$, then Eq. (6) implies $n_0^* = 0$ due to the stress singularity at the crack tip, $\left|\hat{\sigma}_{22}(a_0,a_0)\right| = \infty$, i.e. the cracks start to propagate *without any delay* after the load application.

Let the origin of the coordinate system be in the middle of a crack. Then the coordinate of the crack tip is $y_1^* = a(n)$ and the dependence $a(n)$ for the developing crack length is to be obtained from (3), which is reduced to the following *non-linear* Volterra integral equation of the first kind for $a(n) \le l$,

$$\int_{n_0^*}^{n}\left|\hat{\sigma}_{22}(a(m),a(n))\right|^b \left|\Delta q_0(m)\right|^b dm = \left(\sigma_{R1}^{*\Delta}\right)^b - \left|\hat{\sigma}_{22}(a_0,a(n))\right|^b \int_0^{n_0^*}\left|\Delta q_0(m)\right|^b dm . \qquad (7)$$

We can change variables in (7) similar to Zobnin and Rabotnov (see [10] where a solution of a corresponding creep crack problem is presented for *b*=1) and arrive at the following non-convolution *linear* Volterra equation of the first kind to be solved for $g(a) = \left[\Delta q_0(m(a)) / \sigma_{R1}^{*\Delta}\right]^b dm(a)/da,$

$$\int_{a_0}^{a(n)}\left|\hat{\sigma}_{22}(a,a(n))\right|^b g(a)da = 1 - \frac{\left|\hat{\sigma}_{22}(a_0,a(n))\right|^b}{\left|\hat{\sigma}_{22}(a_0,a_0)\right|^b}, \quad a_0 \le a(n) \le l . \qquad (8)$$

2.2 Periodic Collinear Cracks in an Infinite Plane Under Uniform Loading
Consider now a more particular example for *2l* - periodic collinear straight cracks with a length $2a(m)$ in an infinite plate. Let a uniform cyclic traction with a range $\Delta q(m,x) = \Delta q_0(m)$ be applied at infinity normal to the crack line, Fig.2. For an elastic body, the normal stress range $\Delta\sigma_{22}(m,x_1)$ ahead of the crack has the form [11],

$$\Delta\sigma_{22}(m;a(m),y_1) = \frac{\Delta K_1(m;a(m))\sin(\frac{\pi y_1}{2l})}{\sqrt{2l\,tg(\frac{\pi a(m)}{2l})\left(\sin^2(\frac{\pi y_1}{2l}) - \sin^2(\frac{\pi a(m)}{2l})\right)}}, \qquad (9)$$

$$\Delta K_1(m;a(m)) = \Delta q_0(m)\sqrt{2l\,tg\frac{\pi a(m)}{2l}}, \qquad a(m) \le l .$$

where ΔK_1 is the mode 1 stress intensity factor range. At $l \to \infty$, Eq. (9) gives the stress distribution in the corresponding problem with a single crack.

Let us suppose periodic cyclic traction, $\Delta q(m) = \Delta q_0 = const$. Then Eq. (6) implies the fracture cycle number for an infinite plane without crack is $n_\infty^* = (\sigma_1^{*\Delta} / \Delta q_0)^b$ under the considered loading. As was mentioned above, $n_0^* = 0$ if there exists an initial crack. Let $\tilde{n} = n / n_\infty^*$ be the normalized cycle number. After substituting stress range (9) into Eq. (7) and a change of variables, the latter equation can be solved by the Laplace transform *under the assumption b<2*, giving

$$\frac{da(\tilde{n})}{d\tilde{n}} = \frac{\Delta K_1^2(a(\tilde{n}))}{\Delta q_0^2 \sin(b\pi/2)} \left[\frac{\Delta K_1^4(a(\tilde{n}))}{\Delta K_1^4(a_0)} \frac{\cos^2 \dfrac{\pi a(\tilde{n})}{2l}}{\cos^2 \dfrac{\pi a_0}{2l}} - 1 \right]^{1-b/2}, \quad a_0 \le a(n) \le l. \qquad (10)$$

The results are presented on Figs 3 and 2 for $b = 1.5$ and different values of l. As one can see from Eq. (10) and the graphs, the solution degenerates into the solution [1] of the corresponding problem with a single crack when $l \to \infty$.
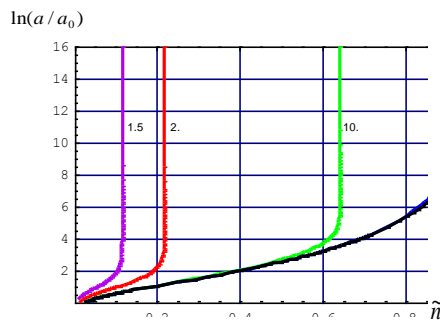

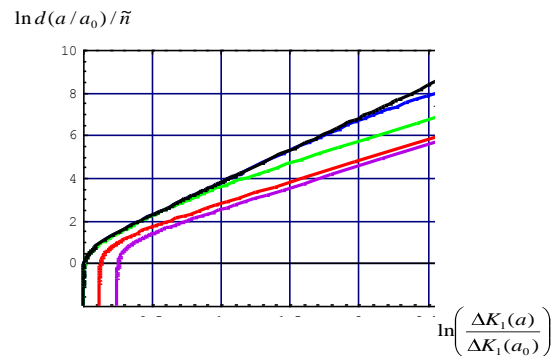Figure 3. Length of fatigue 2*l*-periodic crack vs. cycle number for different *l* (local approach)

Figure 4. Fatigue 2*l*-periodic crack growth rate vs. stress intensity factor range for different *l* (local approach).

The crack growth rate given by Eq. (10) looks like the Paris type law, whose parameters, however, are not the material constants but depend on *l*, a_0 and Δq_0 (see also [1]).

The obtained solution is valid only for $b < 2$ and blows up (predicting instant unstable crack propagation) when $b \to 2$, that is, it is not able to describe the fatigue crack propagation for common structural materials with experimentally determined values of S-N diagram constant *b* (usually $b \ge 4$). The local approach does not also predict the fatigue crack start delay observed experimentally. A way to overcome those shortcomings is an application of a non-local approach.

3. NON-LOCAL BRITTLE STRENGTH CONDITIONS

3.1 General Description

We will suppose that strength at a point $y \in \Omega$ on a plane $\vec{\zeta}$ depends not only on the stress history at that point, $\left\{ \sigma_{ij}^c(m, y) \right\}_{m=1,2...}$ but also on the stress history in its neighborhood and generally, in the whole of the body, $\left\{ \sigma_{ij}^c(m, x) \right\}_{m=1,2...}, x \in \Omega$. A non-local brittle cyclic normalized equivalent stress functional $\underline{\Lambda}^\Theta(\{\sigma^c\}; n, \Gamma, y, \vec{\zeta})$, which is positively homogeneous in σ and non-decreasing in n, can be introduced [1,4]. It is considered as a material characteristics implicitly reflecting influence of material microstructure. Then the non-local strength condition *for a plane $\vec{\zeta}$ at a point $y \in \Omega$* takes the form $\underline{\Lambda}^\Theta(\{\sigma^c\}; n, \Gamma, y, \vec{\zeta}) < 1$.

The simplest examples of the non-local brittle CNESFs and strength conditions are obtained by replacing the local stress $\sigma_{ij}(\tau, x)$ by its non-local counterpart $\sigma_{ij}^\Theta(\tau; \Gamma, y, \vec{\zeta})$ in the corresponding local brittle CNESFs described in Section 1,

$$\underline{\Lambda}^\Theta(\{\sigma^c\}; n, \Gamma, y, \vec{\zeta}) = \underline{\Lambda}(\{\sigma^{\Theta c}(\cdot, \Gamma(\cdot), y, \vec{\zeta})\}; n, \Gamma, y, \vec{\zeta}). \tag{11}$$

The non-local stress $\sigma_{ij}^\Theta(\tau; \Gamma(\tau), y, \vec{\zeta})$ can be taken particularly as a weighted average of $\sigma_{ij}(\tau, x)$ (see [12] and also [1-2, 4-5],

$$\sigma_{ij}^\Theta(\tau; \Gamma(\tau), y, \vec{\zeta}) = \int_{\Omega^\Theta(y, \vec{\zeta}; \Gamma)} w_{ijkl}(y, x, \vec{\zeta}; \Gamma) \sigma_{kl}(\tau; x) dx \tag{12}$$

where the weight function w and the non-locality zone Ω^Θ (some neighborhood of y) are characteristics of material point y, plane $\vec{\zeta}$ and generally of the body shape Γ, such as $\int_{\Omega^\Theta(y, \vec{\zeta}; \Gamma)} w_{ijkl}(y, x, \vec{\zeta}; \Gamma) = \delta_{ik}\delta_{jl}$.

For example, $\Omega^\Theta(y, \vec{\zeta}; \Gamma)$ can be taken as a 2D disc of a diameter 2δ in a 3D body $\Omega(n)$ or as a 1D segment of a length 2δ for a 2D body $\Omega(n)$, in the plane $\vec{\zeta}$ with the centre at y, where δ is considered as a material parameter. Near the boundary $\Gamma(n)$, $\Omega^\Theta(y, \vec{\zeta}; \Gamma)$ should be taken as an intersection of the disc/segment with $\Omega(n)$.

Using the introduced brittle non-local CNESF $\Lambda^\Theta(\{\sigma^c\}; n, \Gamma, y, \vec{\zeta})$, the fatigue fracture process (the fatigue crack initiation and its propagation through the damaged material) can be described as in Section 1 after replacement there the stress tensor σ by its non-local counterpart σ^Θ.

However it is sometimes more convenient to employ for that purpose an equation for the crack rate vector instead of Eqs (2)-(4). Let us consider a 2D case for homogeneous isotropic body under a cyclic process as an example. Suppose there exist k cracks with K moving tips $y *^\chi$ ($1 \le \chi \le K; k \le K \le 2k$). We can take the total derivative of non-local counterpart of Eq. (3) at a crack tip $y *^\chi$ with respect to n and arrive for CNESF

(1) after some manipulations using the non-local counterpart of expression (5), at the following crack growth rate equation,

$$
\frac{d\,y_j^{*\chi}}{dn} = -[\Delta\sigma_{\zeta\zeta}^{\Theta}(n, y^{*\chi}(n), \vec{\zeta})]^b\, \xi_j^{\chi}(n)\left[\xi_i^{\chi}(n)\int_0^n \frac{\partial}{\partial y_i}\left(\sigma_{\zeta\zeta}^{\Theta}(m, y, \vec{\zeta})\right)\Bigg|_{y=y^{*\chi}(n)} dm\right]^{-1} \quad (13)
$$

Here is no sum in χ, and $\xi_i^{\chi}(n)$ is the outward unit vector tangent to the crack. The fracture plane is determined from the non-local counterpart of Eq. (5). Due to the integral term, Eq. (13) accounts all damage history at the point $y^{*\chi}$. Note that for an arbitrary CNESF, the crack growth rate equation is obtained after solving a system of K linear algebraic equation with respect to K scalar unknown values $|\dot{y}^{*\chi}(n)|$, where $dy_i^{*\chi}(n)/dn = \xi_i^{\chi}(n)\,|\dot{y}^{*\chi}(n)|$.

3.2 Example of Non-Local Durability Analysis
Let us consider the 2D problem from Section 2.1 using the non-local durability analysis with particular non-local CNESF (11), (12) where the crack propagation plane $\vec{\zeta}^*$ is prescribed by the problem symmetry, $\Omega^{\Theta}(y, \vec{\zeta}; \Gamma)$ is the interval $(y_1 - \delta_-(y_1), y_1 + \delta_+(y_1))$ for y ahead of the crack $a(m)$, $\delta_-(y_1) = \min(\delta, |y_1 - a(m)|)$ $\delta_+(y_1) = \min(\delta, |y_1 - l|)$ and δ is a material constant. Let $w_{ijkl}(y, x, \vec{\zeta}) = w(y, x)\delta_{ij}\delta_{kl}$, where $w(y,x)$ is a bounded function, which is considered as a material characteristics to be identified. As possible approximations, one can choose e.g. $w(y,x)$ constant w.r.t. $x \in \Omega(y)$ (thus arriving at the Neuber stress averaging, cf. [12]), a piece-wise linear or a more smooth hat-shaped dependence on x.

Repeating the same reasoning as in Section 2 but now for the non-local stress $\Delta\sigma_{ij}^{\Theta}(m; a, y_1)$, we arrive at the same Eqs (6)-(8) where $\hat{\sigma}_{22}(a, y_1)$ must be replaced by $\hat{\sigma}_{22}^{\Theta}(a, y_1) = \int_{y_1 - \delta_-(y_1)}^{y_1 + \delta_+(y_1)} w(y_1, x_1)\hat{\sigma}_{22}(a, y_1)dx_1$. For a problem with initially existing crack, the crack propagation start instant n_0^* obtained from the non-local counterpart of (6) is non-zero since $\left|\sigma_{22}^{\Theta}(a_0, a_0)\right| < \infty$ at the crack tip in spite of $\left|\sigma_{22}(a_0, a_0)\right| = \infty$. For example, the start delay for a constant loading q_0 is $n_0^* = n_\infty^* \left|\hat{\sigma}_{22}^{\Theta}(a_0, a_0)\right|^{-b}$.

Differentiating the non-local counterpart of (7) w.r.t. $a(n)$ (cf. Eq. (13)), we arrive at the following linear non-convolution Volterra equation of the second kind for the unknown function $g(a)$,

$$
g(a(n)) + \int_{a_0}^{a(n)} K(a(n), a)g(a)da = Y(a(n)), \quad a_0 \le a(n) \le l, \quad (15)
$$

$$K(a(n),a) = \left| \sigma_{22}^{\Theta}(a(n),a(n)) \right|^{-b} \frac{\partial}{\partial a(n)} \left| \sigma_{22}^{\Theta}(a,a(n)) \right|^{-b},$$

$$Y(a(n)) = -\left| \sigma_{22}^{\Theta}(a_0,a_0) \right|^{-b} K(a(n),a_0).$$

For the particular problem from Section 2.2, Eq. (15) can be solved numerically similar to [1], where it was done for $l = \infty$, corresponding to the case of one crack.

CONCLUSIONS

A united description of fatigue crack initiation and propagation is principally possible using the local as well as the non-local approach, however the local approach in the considered examples can be applied only to a limited range of material fatigue parameters and cannot describe the crack start delay. The non-local approach is free of the drawbacks. When the stress fields are available analytically or numerically and the strength conditions are associated with the linear accumulation rule, the 2D problem in the both approaches can be reduced to non-linear Volterra equation(s) for the unknown crack geometry. For the crack under mixed-mode loading, equations for curvilinear crack growth rate and direction are presented taking into account the whole damage history ahead of the crack.

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