# Note on a result of Morse and Bolt

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Abstract

A result given without derivation by Morse and Bolt in [Review of Modern Physics 16 (1944) pp 70-750] pertaining to the reflection of a spherical sound wave from an absorbent surface is investigated. It is shown that the result as given is not quite accurate.

In their comprehensive review of room acoustics Morse and Bolt [1] considered the effect of absorbent walls on sound produced in a room. They investigated theoretically, the simple model of an infinite plane absorbent wall with a spherical sound source. For the reflection of a point sound source, with harmonic variation $e^{-i\omega t}$* by a locally reacting plane they derived the correct solution (Morse and Bolt [1], section 53).

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2ik\beta \int_h^\infty e^{ik\beta(t-h)} \frac{e^{ikR(t)}}{R(t)} dt, \quad \mathrm{Im}(\beta)>0. \tag{1}$$

where

$R(t) = \sqrt{r^2 + (t+z)^2}$ ; and the pressure $p$ is given by

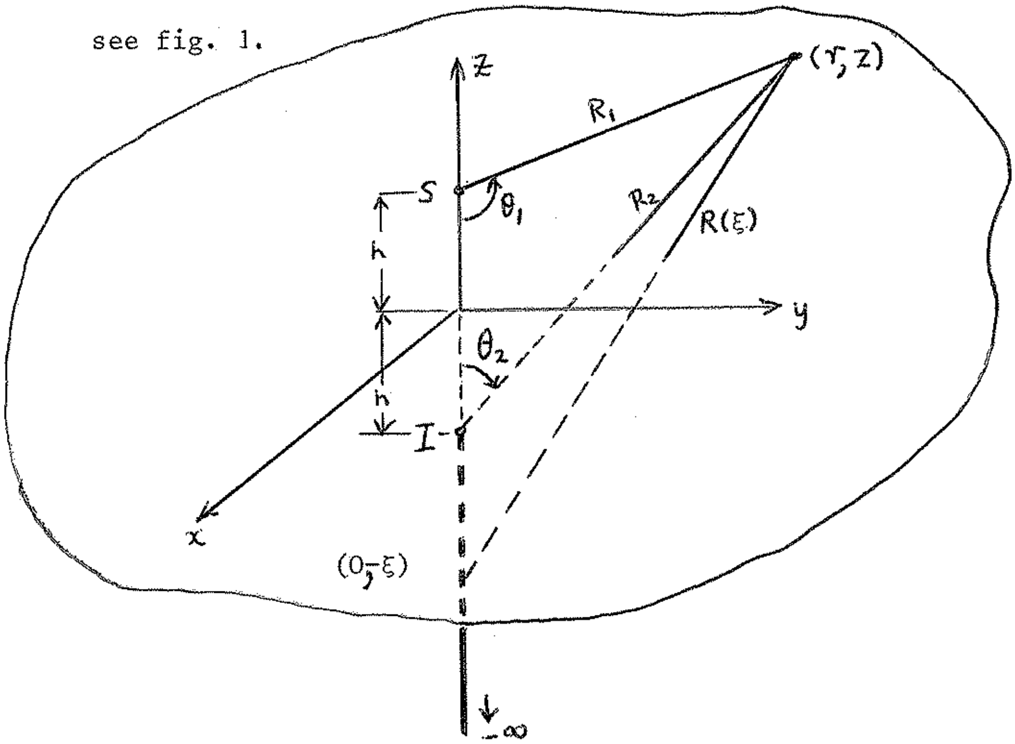
$p = \rho\frac{\partial\phi}{\partial t} = -i\omega\rho\phi$ , and the acoustic velocity $\underset{\sim}{u}$ by $\underset{\sim}{u} = -\mathrm{grad}\ \phi$.

(*Footnote: We shall supress the time variation $e^{-i\omega t}$ in the rest of the paper)

The impedance boundary condition on z = 0 requires the satisfaction of

$$\frac{\partial\phi}{\partial z} + ik\beta\phi = 0 \qquad \text{on } z = 0 \text{ , } \operatorname{Re}\beta > 0 \text{ .} \tag{2}$$

see fig. 1.




$$R(\xi) = \sqrt{r^2+(z+\xi)^2} = \sqrt{R_2^2+(\xi-h)^2-2R_2(\xi-h)\cos(\theta_2-\pi)}\,,\quad R_2 = \sqrt{r^2+(z+h)^2}\,,\quad R_1 = \sqrt{r^2+(z-h)^2}$$

$$r^2 = x^2+y^2 \text{ , } r = R_2 \sin\theta_2 \text{ , } z+h = R_2\cos\theta_2 \text{ , } 0 \le \theta_2 \le \pi/2 \text{ .}$$

fig. 1.

The physical interpretation of (1) is as follows: The first term represents the point source at S. The second term is the same source located at the image point I. The integral represents a line of weighted point sources which stretch from the image point to $-\infty$. The integral represents an image line source along the z-axis from $-h$ to $-\infty$.

We remark that the condition $\operatorname{Im}\beta > 0$, arbitrary $\operatorname{Re}\beta$ , in the expression (1) is only required to ensure the convergence of the integral. The boundary condition (2) for the absorbent plane only requires that $\operatorname{Re}\beta > 0$. We can relax the condition $\operatorname{Im}\beta > 0$ to allow for arbitrary sign of $\operatorname{Im}\beta$, by requiring $\operatorname{Re}\beta > 0$, See Appendix.

In that case one gets instead of (1):

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2ik\beta \int_h^{i\infty} e^{ik\beta(t-h)} \frac{e^{ikR(t)}}{R(t)}\,dt \,, \qquad (3)$$

$$-\frac{\pi}{2} \le \operatorname{Arg} R(t) \le \frac{\pi}{2}, \quad \operatorname{Re}\beta > 0.$$

Thus the image line source has imaginary location. The expression (1) does not give rise to any surface waves, whereas if $\operatorname{Im}\beta < 0$ then the expression (3) may well give rise to surface waves, when carrying out steepest descent asymptotic evaluation of the integral. See for a full discussion Thomasson [2].

Expression suitable for large $|\beta|$ and $|kR_2| \gg 1$.

We shall here try to derive the expression given, without derivation, by Morse and Bolt for large $|\beta|$.

In the expression (3) we shall use the results (Watson [3] p.366 formulae (9) and (10))

$$\frac{e^{ikR(t)}}{R(t)} = ik \sum_{n=0}^{\infty} (-1)^n (2n+1) j_n(k(t-h)) h_n^{(1)}(kR_2) P_n(\cos\theta_2), \quad R_2 > |t-h| \,,$$

$$= ik \sum_{n=0}^{\infty} (-1)^n (2n+1) j_n(kR_2) h_n^{(1)}(k(t-h)) P_n(\cos\theta_2), \quad |t-h| > R_2 \,, \qquad (4a,b)$$

where $R(t) = \sqrt{R_2^2 + (t-h)^2 - 2R_2(t-h)\cos(\pi-\theta_2)}$ ,

$$j_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+\frac{1}{2}}(z) \quad , \quad h_n^{(1)} = \sqrt{\frac{\pi}{2z}}\, H_{n+\frac{1}{2}}^{(1)}(z) \,,$$

where $J_{n+\frac{1}{2}}(z)$ , $H_{n+\frac{1}{2}}^{(1)}(z)$ , and $P_n(z)\,(=(-1)^n P_n(-z))$ are the Bessel functions of the first kind, the third kind and the Legendre function of the first kind respectively. Substituting (4) into (3) and making the change of variable $t = u+h$ gives

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2k\beta \sum_{n=0}^{\infty} (-1)^{n+1}(2n+1)P_n(\cos\theta_2)\times$$

$$\times \left\{ h_n^{(1)}(kR_2)\sqrt{\frac{\pi k}{2}} \int_0^{iR_2} e^{ik\beta u} \frac{J_{n+\frac{1}{2}}(ku)}{u^{\frac{1}{2}}}\, du \right.$$

$$\left. + j_n(kR_2)\sqrt{\frac{\pi k}{2}} \int_{iR_2}^{i\infty} e^{ik\beta u} \frac{H_{n+\frac{1}{2}}(ku)}{u^{\frac{1}{2}}}\, du \right\} .$$

Replacing the variable of integration u by the substitution $-iku = t$ and using the fact that (Watson [3] p.77 and p.78 )

$$J_{n+\frac{1}{2}}(iz) = i^{n+\frac{1}{2}} I_{n+\frac{1}{2}}(z) \ , \ -\pi < \arg z \le \pi/2$$

$$H_{n+\frac{1}{2}}(iz) = \frac{2}{\pi i}\, i^{-(n+\frac{1}{2})} K_{n+\frac{1}{2}}(z), \ -\pi < \arg z \le \pi/2$$

gives

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2k\beta \sum_{n=0}^{\infty} (-i)^{n+1}(2n+1)P_n(\cos\theta_2)\times$$

$$\times \left\{ h_n^{(1)}(kR_2)\sqrt{\frac{\pi}{2}} \int_0^{kR_2} e^{-\beta t} \frac{I_{n+\frac{1}{2}}(t)}{t^{\frac{1}{2}}}\, dt \right.$$

$$\left. + (-1)^{n+1} j_n(kR_2)\sqrt{\frac{2}{\pi}} \int_{kR_2}^{\infty} e^{-\beta t} \frac{K_{n+\frac{1}{2}}(t)}{t^{\frac{1}{2}}}\, dt \right\} ,$$

$$= \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2k\beta \sum_{n=0}^{\infty} (-i)^{n+1}(2n+1)P_n(\cos\theta_2)\times$$

$$\times \left\{ h_n^{(1)}(kR_2)\sqrt{\frac{\pi}{2}} \int_0^{\infty} e^{-\beta t} \frac{I_{n+\frac{1}{2}}(t)}{t^{\frac{1}{2}}}\, dt \right.$$

$$\left. + \frac{1}{\sqrt{kR_2}} \int_{kR_2}^{\infty} e^{-\beta t} \left[ (-1)^{n+1} J_{n+\frac{1}{2}}(kR_2) K_{n+\frac{1}{2}}(t) - (\pi/2) H_{n+\frac{1}{2}}(kR_2) I_{n+\frac{1}{2}}(t) \right] \frac{dt}{t^{\frac{1}{2}}} \right\} .$$

By using the result (Watson [3] p.387)

$$\sqrt{\frac{\pi}{2}} \int_0^{\infty} e^{-\beta t} \frac{I_{n+\frac{1}{2}}(t)}{t^{\frac{1}{2}}}\, dt = Q_n(\beta) \ , \ \mathrm{Re}\,\beta > 1 \ , \ n > -1 \ ,$$

where $Q_n$ is the Legendre function of the second kind; and making a change of integration variable $t = kR_2u$ the above expression becomes

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2k\beta \sum_{n=0}^{\infty} (-i)^{n+1}(2n+1)P_n(\cos\theta_2)h_n^{(1)}(kR_2)Q_n(\beta)$$

$$+ 2k\beta \sum_{n=0}^{\infty} (-i)^{n+1}(2n+1)P_n(\cos\theta_2)\times$$

$$\times \left\{\int_1^{\infty} e^{-kR_2\beta u}\left[(-1)^{n+1}J_{n+\frac{1}{2}}(kR_2)K_{n+\frac{1}{2}}(kR_2u)-(\pi/2)H_{n+\frac{1}{2}}(kR_2)I_{n+\frac{1}{2}}(kR_2u)\right]\frac{du}{u^{\frac{1}{2}}}\right\},$$

$$\operatorname{Re}\beta > 1.$$

Noting that

$$P_o(\cos\theta_2) = 1\ ,\ h_o^{(1)}(kR_2) = \frac{e^{ikR_2}}{ikR_2}\ ,\ Q_o(\beta) = \tfrac{1}{2}\ln\left(\frac{\beta+1}{\beta-1}\right),$$

the previous expression can be written

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2}\left\{1 - \beta\ln\left(\frac{\beta+1}{\beta-1}\right)\right\}$$

$$+ 2k\beta \sum_{n=1}^{\infty} (-i)^{n+1}(2n+1)h_n^{(1)}(kR_2)P_n(\cos\theta_2)Q_n(\beta)$$

$$+ 2k\beta \sum_{n=0}^{\infty} (-i)^{n+1}(2n+1)P_n(\cos\theta_2)\times$$

$$\times \int_1^{\infty} e^{-kR_2\beta u}\left[(-1)^{n+1}J_{n+\frac{1}{2}}(kR_2)K_{n+\frac{1}{2}}(kR_2u)-(\pi/2)H^{(1)}_{n+\frac{1}{2}}(kR_2)I_{n+\frac{1}{2}}(kR_2u)\right]\frac{du}{u^{\frac{1}{2}}}.$$

$$\operatorname{Re}\beta > 1. \qquad (5)$$

The second and third terms of the above expression correspond precisely to the result stated by Morse and Bolt [1] p.143 formula (8.11), as the reflected field. However they do not have the extra integral term derived above. It is suspected that they failed to break up the range of integration into the regions $|t-h| \gtrless R_2$. If one simply substituted (4a) and assumed $kR_2 \to \infty$ one would also get their result. However such an approach is asymptotic and all terms with $kR_2$ appearing should be asymptotically expanded. The other possibility is that they dropped the condition $kR_2 \to \infty$.

If one allows $kR_2 \to \infty$ Re $\beta > 1$ in the expression (5) then using the asymptotic results (Watson [3] p.199)

$$J_{n+\frac{1}{2}}(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - (n+1)\ \frac{\pi}{2}) \ ,$$

$$H_{n+\frac{1}{2}}(z) \sim \sqrt{\frac{2}{\pi z}}\ i^{-(n+1)} e^{iz} \ ,$$

$$K_{n+\frac{1}{2}}(z) \sim \sqrt{\frac{\pi}{2z}}\ e^{-z} \ ,$$

$$I_{n+\frac{1}{2}}(z) \sim \frac{e^{z}}{\sqrt{2\pi z}} \ ,$$

$$\text{as } |z| \to \infty \ ;$$

$$h_n^{(1)}(z) = i^{-n-1} \frac{e^{iz}}{z} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!} (-2iz)^{-m} \ ,$$

it can be shown that the integral is $o(e^{-kR_2 \text{Re}(\beta-1)})$ and therefore

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} + 2k\beta \sum_{n=0}^{\infty} (-i)^{n+1} (2n+1) P_n(\cos\theta_2) h_n(kR_2) Q_n(\beta) + o(e^{-kR_2 \text{Re}(\beta-1)}) \qquad \text{Re}\,\beta > 1 \ , \ kR_2 \to \infty.$$

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2} \left\{1 + 2\beta \sum_{n=0}^{\infty} (-1)^{n+1} (2n+1) P_n(\cos\theta_2) Q_n(\beta) \times \right.$$
$$\left. \times \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!} (-2ikR_2)^{-m} \right\} + o(e^{-kR_2 \text{Re}(\beta-1)}) \ , \tag{6}$$

$$kR_2 \to \infty \ , \ \text{Re}\,\beta > 1 \ .$$

whose dominant term is given by

$$\phi(r,z) = \frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2}\left\{1 + 2\beta \sum_{n=0}^{\infty} (-1)^{n+1}(2n+1)P_n(\cos\theta_2)Q_n(\beta)\right\}$$

$$+ 0((kR_2)^{-2}) \, .$$

$$kR_2 \to \infty \, , \quad \mathrm{Re}\,\beta > 1 \, . \qquad (6a)$$

Therefore Morse and Bolt's result will be correct if the extra condition of validity, $kR_2$ large, be included.

The expansion (6) is valid, for all $\mathrm{Re}\,\beta > 1$. However the convergence is poor except for large $|\beta|$.

## Appendix

Here we prove the result (3). To achieve this we apply Cauchy's residue theorem to the integral

$$\int_{c_1+c_2+c_3} \frac{e^{ik\beta t + ikR(t)}}{R(t)}\, dt \quad ,$$

where initially we shall assume $\mathrm{Re}\,\beta > 0$ , $\mathrm{Im}\,\beta > 0$ , $c_1 + c_2 + c_3$ is the closed contour shown in fig 2. The multivalued function $R(t) = \sqrt{r^2 + (t+h)^2}$ is defined uniquely by $R(t) = \sqrt{(t+h+ir)}\sqrt{(t+h-ir)}$ where

$$\sqrt{(t+h+ir)} = |t+h+ir|^{\frac{1}{2}} e^{\frac{i\theta_1}{2}}, \quad \sqrt{(t+h-ir)} = |t+h-ir|^{\frac{1}{2}} e^{\frac{-i\theta_2}{2}}, \quad 0 < \theta_{1,2} < 2\pi .$$

The branch cuts for $R(t)$ are shown in fig. 2 by squiggly lines

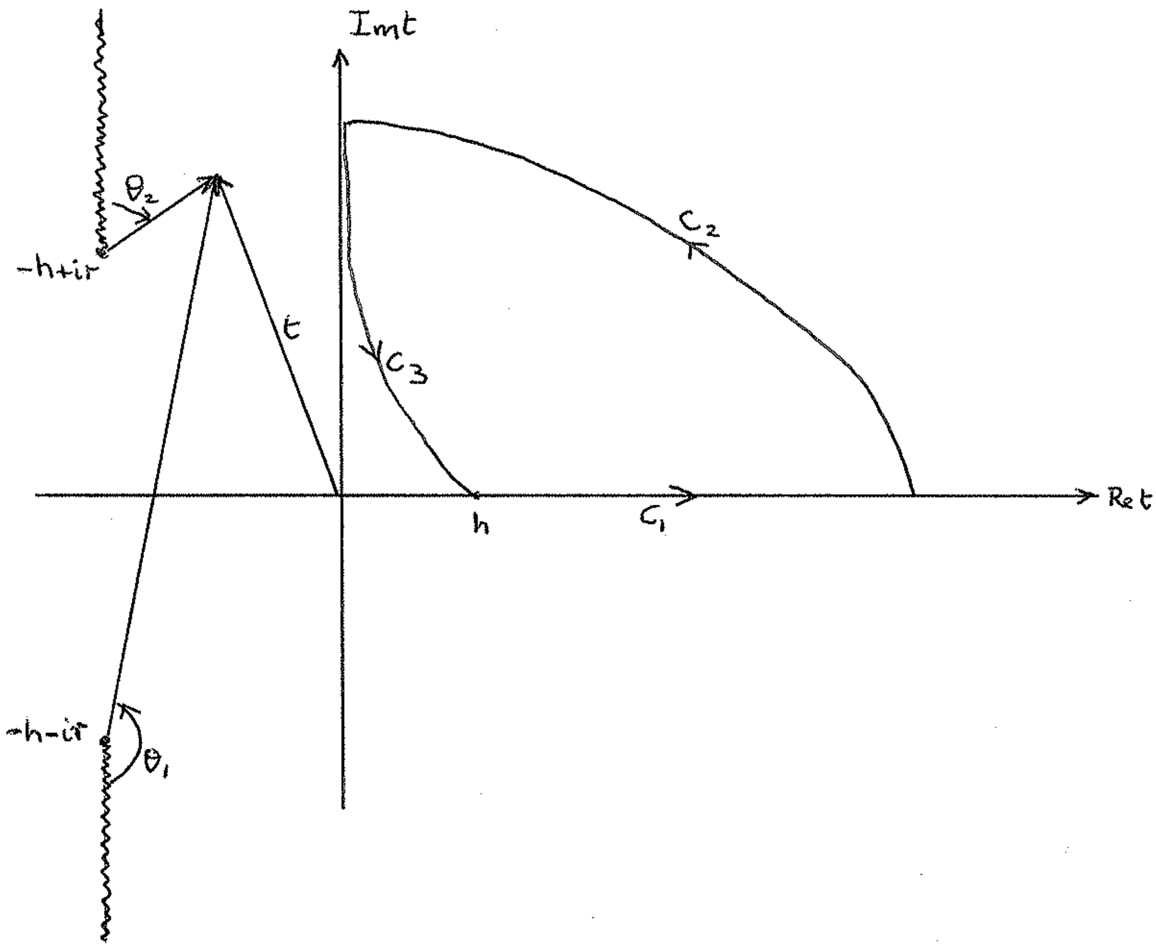



fig. 2

Since no singularities occur inside the contour $c_1 + c_2 + c_3$ Cauchy's residue theorem gives

$$\frac{1}{2\pi i}\left\{\int_{c_1} + \int_{c_2} + \int_{c_3}\right\} \frac{e^{ik\beta t + ikR(t)}}{R(t)}\, dt = 0.$$

On $c_2$ $e^{ik\beta t + ikR(t)} \sim e^{-k(\mathrm{Im}\,t(\mathrm{Re}\beta+1) + \mathrm{Re}\,t\,\mathrm{Im}\beta)} \to 0,$

as $|t| \to \infty$, $0 < \arg t < \frac{\pi}{2}$.

Thus $\displaystyle\int_h^{i\infty} \frac{e^{ik\beta t + ikR(t)}}{R(t)}\, dt = \int_h^{\infty} \frac{e^{ik\beta t + ikR(t)}}{R(t)}\, dt$ $\mathrm{Re}\,\beta > 0$, $\mathrm{Im}\,\beta > 0$.

Now the integral $\displaystyle\int_h^{i\infty} \frac{e^{ik\beta + tikR(t)}}{R(t)}\, dt$ exists for arbitrary $\mathrm{Im}\beta$, $\mathrm{Re}\,\beta > 0$.
Hence by the principle of analytic continuation the result (3) follows.

References

1. P.M. Morse and R.H. Bolt 1944 Reviews of Modern Physics 16, 70 - 150. Sound waves in rooms.

2. S.-I. Thomasson 1976 Journal Acoustical Society of America 59 (4) 780-785.
Reflection of waves from a point source by an impedance boundary.

4. G.N. Watson 1952 A treatise on Bessel functions. London. Cambridge University Press.