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MATRIX WIENER-HOPF-HILBERT FACTORISATION*

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Abstract. A method is described for effecting the explicit Wiener-Hopf factorisation of a class of (2×2)-matrices. The class is determined such that the factorisation problem can be reduced to a matrix Hilbert problem which involves an upper or lower triangular matrix. Then the matrix Hilbert problem can be further reduced to three scalar Hilbert problems on a half-line, which are solvable in the standard manner.

Key words. Wiener-Hopf-Hilbert method, diffraction, matrix factorisation

AMS(MOS) subject classifications. 30A88, 78, 15A21

1. Introduction. In a recent paper by Rawlins and Williams [1] (see also Rawlins [2]), it was shown how a class of (2×2)-matrices could be explicitly factorised. In this paper a different class of matrices is constructively factorised. By using the idea of Rawlins [3] and evaluating the matrix to be factorised on both sides of an assumed branch cut that commonly arises in diffraction problems, the problem of factorisation reduces to a matrix Hilbert problem along the branch cut. In the work of Rawlins and Williams [1], and Rawlins [2], the form of the original matrix was chosen so that the matrix Hilbert problem was reducible to two uncoupled scalar Hilbert problems. These could be solved without difficulty by the well-known methods given in Muskhelishvili's book on singular integral equations [4]. The reduction to these two scalar Hilbert problems required that the *two* diagonal element of the matrix involved in the Hilbert problem were zero. However, it is known, see Gohberg and Krein [5], that upper and lower triangular matrix Hilbert problems can also be solved explicitly. Thus we need only require *one* off-diagonal element of the matrix Hilbert problem to vanish, in order to effect a Wiener-Hopf factorisation of the original matrix. It is conjectured that the technique of matrix factorisation of the present class of (2×2)-matrices may go some way towards an eventual solution of some hitherto intractable diffraction problems. The truth of this conjecture hinges on the growth at infinity of the factor matrices. The consideration of such behaviour is more appropriate in the actual application of the method to diffraction problem and is therefore, omitted here.

We mention that the type of matrix factorised in this paper does not fall into the class considered by Daniele [6], Rawlins [7]. Jones [8] has devised a method for the Wiener-Hopf factorisation of a special type of (2×2)-matrix, that ensures that the Wiener-Hopf factors commute. In addition, the factors of various matrices whose Wiener-Hopf factors do not commute were also determined by Jones [8]. It is possible that by appropriate pre- and post-multiplying a matrix (which is susceptible to Jones' method) by appropriate analytic matrices the Wiener-Hopf factorisation can be carried out for the matrices considered here by his approach. However, the result obtained here seems to be different from that of Jones [8], and it is not clear to me how one could prove the equivalence of the two results. The difference is apparent in the scalar factorisation problem. In Jones [8] the classical approach by Cauchy's theorem leads to a solution for the factors expressed in terms of Cauchy integrals along a line parallel to the real axis in the strip of analyticity. On the other hand, the approach used here through the Hilbert problem leads to a solution involving Cauchy

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integrals along a branch cut, i.e. along a half line. The strip of analyticity is not strictly necessary in the present approach. This would indicate that the present method would be suitable for problems without dissipation. Pioneering and important work on Wiener–Hopf–Hilbert factorisation of matrices has been carried out by Hurd [9] and his coworkers. Jones [10] has extended the class of (2×2)-matrices whose factors commute to a class of $(n\times n)$-matrices whose factors commute.

In § 2 of the paper a general matrix will be considered, and its general form is appropriately specified in order that the Wiener–Hopf factorisation problem reduces to a triangular matrix Hilbert problem. In § 3 this class of matrices will be constructively factorised by solving appropriate Hilbert problems. In § 4 some remarks are made vis-a-vis direct Wiener–Hopf factorisation of the present class of matrices.

2. Determination of the class of matrices whose Wiener–Hopf factorisation reduces to a triangular matrix Hilbert problem on a half-line. Consider the general (2×2)-matrix.

$$\mathbf{A}(\alpha)=\begin{pmatrix} a_{11}(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha)\end{pmatrix},$$

where the elements $a_{ij}(\alpha)$, $i, j=1,2$ are functions of the complex variable α. These functions will be assumed to have only branch point singularities; specifically we shall assume that the branch points arise through the function $\gamma=\sqrt{\alpha^2-k^2}$, where k has positive real and imaginary parts, and the branch cuts C and C' lie along the half-lines C: $\alpha=-k-\delta$, C': $\alpha=k+\delta$, $\delta\geqq 0$. The elements $a_{ij}(\alpha)$ are also assumed to be analytic functions in the cut α-plane; and det $\mathbf{A}(\alpha)\neq 0$ within the strip $-k_i<\mathrm{Im}\,(\alpha)<k_i$ where k_i denotes the imaginary part of k. The occurrence of a complex k with $\mathrm{Im}\,(k)>0$ is traditional in Wiener–Hopf-type problems and is needed to have a common strip of analyticity; in the final solution the complex k is removed by taking the limit as $\mathrm{Im}\,k\downarrow 0$.

The Wiener–Hopf factorisation problem requires the determination of (2×2)-matrices $\mathbf{U}(\alpha)$ and $\mathbf{L}(\alpha)$, whose elements are analytic for $\mathrm{Im}\,(\alpha)>-k_i$ and $\mathrm{Im}\,(\alpha)<k_i$ respectively, such that

$$\mathbf{A}(\alpha)=\mathbf{U}(\alpha)\mathbf{L}^{-1}(\alpha) \tag{1}$$

$\mathbf{U}(\alpha)$ and $\mathbf{L}(\alpha)$ are also required to be nonsingular in their respective regions of analyticity. Obviously any matrix $\mathbf{A}'(\alpha)$ which can be expressed in the form

$$\mathbf{A}'(\alpha)=\mathbf{C}_U(\alpha)\mathbf{A}(\alpha)\mathbf{B}_L(\alpha), \tag{2}$$

where $\mathbf{B}_L(\alpha)$ and $\mathbf{C}_U(\alpha)$ are matrices whose elements are analytic functions of α for $\mathrm{Im}\,(\alpha)<k_i$ and $\mathrm{Im}\,(\alpha)>-k_i$ respectively, can also be factorised if $\mathbf{A}(\alpha)$ can be factorised. In order to effect the factorisation, it will be assumed that $\mathbf{U}(\alpha)$ is analytic except along the branch cut C through $\alpha=-k$, whilst $\mathbf{L}(\alpha)$ is analytic except along the branch cut C' through $\alpha=k$. Evaluation of equation (1) on both sides of the cut $C(C$: $\alpha=-k-\delta$, $\delta\geqq 0)$ through $\alpha=-k$ gives, on using the suffices $\pm$ to denote values evaluated on the upper and lower sides of C,

$$\mathbf{A}_+(\alpha)=\mathbf{U}_+(\alpha)\mathbf{L}^{-1}(\alpha), \tag{3}$$

$$\mathbf{A}_-(\alpha)=\mathbf{U}_-(\alpha)\mathbf{L}^{-1}(\alpha), \tag{4}$$

($\mathbf{L}(\alpha)$ is analytic except along the branch cut C' through $\alpha=k$ and therefore takes the same values on both sides of C). Eliminating $\mathbf{L}(\alpha)$ between (3) and (4) gives the matrix Hilbert problem:

$$\mathbf{U}_+(\alpha)=\mathbf{G}(\alpha)\mathbf{U}_-(\alpha), \qquad \alpha\in C, \tag{5}$$

where

$$\mathbf{G}(\alpha)=\mathbf{A}_+(\alpha)\mathbf{A}_-^{-1}(\alpha).$$

More explicitly

$$\begin{pmatrix} g_{11}(\alpha) & g_{12}(\alpha) \\ g_{21}(\alpha) & g_{22}(\alpha) \end{pmatrix} = \frac{1}{\det \mathbf{A}_-(\alpha)} \begin{pmatrix} a_{11}^+a_{22}^- - a_{12}^+a_{21}^- & a_{12}^+a_{11}^- - a_{11}^+a_{12}^- \\ a_{21}^+a_{22}^- - a_{22}^+a_{21}^- & a_{22}^+a_{11}^- - a_{21}^+a_{12}^- \end{pmatrix}.$$

The problem (5) reduces to an upper or lower traingular matrix Hilbert problem along C, if the condition $g_{12}(\alpha)\equiv 0$ or $g_{21}(\alpha)\equiv 0$ is satisfied. That is

$$a_{ij}^+(\alpha)a_{ii}^-(\alpha)=a_{ii}^+(\alpha)a_{ij}^-(\alpha), \qquad i=1, j=2 \text{ or } i=2, j=1, \alpha\in C;$$

or ignoring the trivial case of $a_{ij}^{\pm}(\alpha)\equiv 0$, and assuming $a_{ij}(\alpha)\neq 0$ on C,

$$\left(\frac{a_{ii}(\alpha)}{a_{ij}(\alpha)}\right)^+ - \left(\frac{a_{ii}(\alpha)}{a_{ij}(\alpha)}\right)^- = 0, \qquad \alpha\in C. \tag{6}$$

We shall assume that $a_{ij}(\alpha)$ can have isolated zeros in the cut α-plane, (but not on C), and consequently $a_{ii}(\alpha)/a_{ij}(\alpha)$ can have isolated poles in the cut α-plane. Then provided $a_{ii}(\alpha)/a_{ij}(\alpha)=0(|k+\alpha|^{-\mu})$, $0\leqq\mu<1$ as $\alpha\to -k$, a solution of (6) follows immediately (see Muskhelishvili [4, § 15]) as

$$a_{ii}(\alpha)=a_{ij}(\alpha)F_i(\alpha). \tag{7}$$

Here the function $F_i(\alpha)$ is analytic except along the branch cut C' through $\alpha=k$,[1] and except for poles at the zeros of $a_{ij}(\alpha)$; the multiplicity of these poles is not greater than the multiplicity of the corresponding zeros. Thus since we can express $F_i(\alpha)=r_i(\alpha)/s_i(\alpha)$ where $r_i(\alpha)$ and $s_i(\alpha)$ are analytic except possibly along the branch cut C' then $\mathbf{G}(\alpha)$ will be of upper or lower triangular form if:

$$\text{(i)}\quad \mathbf{A}(\alpha)=\begin{pmatrix} a_{11}(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha)s_2(\alpha) & a_{21}(\alpha)r_2(\alpha) \end{pmatrix}, \tag{8}$$

or

$$\text{(ii)}\quad \mathbf{A}(\alpha)=\begin{pmatrix} a_{12}(\alpha)r_1(\alpha) & a_{12}(\alpha)s_1(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) \end{pmatrix}; \tag{9}$$

and $\det \mathbf{A}(\alpha)\neq 0$ in the cut α-plane.

We shall now carry out the explicit Wiener–Hopt factorisation of the matrix given in case (i) above. The procedure for factorising the matrix given in case (ii) will be completely analogous.

3. Wiener–Hopf factorisation of the matrix defined by (8). We assume the matrix $\mathbf{A}(\alpha)$ has the form (8) and the same general properties as outlined in the first paragraph of § 2. If we carry out the same evaluation on the branch cut C, as described in § 2, the equation (5) reduces to the upper triangular matrix Hilbert problem

$$\mathbf{U}_+(\alpha)=\mathbf{G}(\alpha)\mathbf{U}_-(\alpha), \qquad \alpha\in C, \tag{10}$$

[1] If $\mathbf{A}(\alpha)=\mathbf{A}(-\alpha)$ for α in the cut plane, then it is not difficult to show (see Rawlins [3]) that $F_i(\alpha)$ must also be analytic along the branch cut C'.

where

$$g_{11}(\alpha)=\frac{(a_{11}(\alpha)r_2(\alpha)-a_{12}(\alpha)s_2(\alpha))^+}{(a_{11}(\alpha)r_2(\alpha)-a_{12}(\alpha)s_2(\alpha))^-},$$

(11)

$$g_{12}(\alpha)=\frac{a_{12}^+(\alpha)a_{11}^-(\alpha)-a_{11}^+(\alpha)a_{12}^-(\alpha)}{a_{21}^-(\alpha)(a_{11}(\alpha)r_2(\alpha)-a_{12}(\alpha)s_2(\alpha))^-},$$

$$g_{21}(\alpha)=0, \qquad g_{22}(\alpha)=\frac{a_{21}^+(\alpha)}{a_{21}^-(\alpha)},$$

(12)

$$\mathbf{U}(\alpha)=\begin{pmatrix} u_{11}(\alpha) & u_{12}(\alpha) \\ u_{21}(\alpha) & u_{22}(\alpha) \end{pmatrix}.$$

Evaluating the matrix expression (10) and equating corresponding elements of the matrices on both sides of the equality sign gives the following equations:

$$\text{(13)} \qquad \left.\begin{aligned} u_{1j}^+(\alpha)&=g_{11}(\alpha)u_{1j}^-(\alpha)+g_{12}(\alpha)u_{2j}^-(\alpha),\\ u_{2j}^+(\alpha)&=g_{22}(\alpha)u_{2j}^-(\alpha),\end{aligned}\right\} \quad \alpha\in C, \quad j=1,2.$$

The four equations (13) can clearly be solved if the coupled system

$$\left.\begin{aligned} &\text{(14)} \qquad v_1^+(\alpha)=g_{11}(\alpha)v_1^-(\alpha)+g_{12}(\alpha)v_2^-(\alpha),\\ &\text{(15)} \qquad v_2^+(\alpha)=g_{22}(\alpha)v_2^-(\alpha),\end{aligned}\right\} \quad \alpha\in C,$$

can be solved. The equation (15) is a standard Hilbert problem whose fundamental solution is given directly by the methods of Muskhelishvili [4, Chap. 10]. Similarly we can determine the fundamental solution of the standard auxiliary Hilbert problem

$$\text{(16)} \qquad v^+(\alpha)=g_{11}(\alpha)v^-(\alpha), \qquad \alpha\in C,$$

for $v(\alpha)$. Then the equation (14) can be written as

$$\text{(17)} \qquad u^+(\alpha)-u^-(\alpha)=g_{12}(\alpha)v_2^-(\alpha)/v^+(\alpha), \qquad \alpha\in C,$$

where

$$u(\alpha)=v_1(\alpha)/v(\alpha).$$

In the equation (17) the right-hand side is a known quantity and therefore we have a standard Hilbert problem whose fundamental solution is given by using the techniques described in Muskhelishvili [4, Chap. 10]. Suppose therefore we have found fundamental solutions $v_2^{(0)}(\alpha)$, $v^{(0)}(\alpha)$ and $u^{(0)}(\alpha)$ of the equations (15), (16) and (17), respectively. To determine the general solution, we set $v_2(\alpha)=v_2^{(0)}(\alpha)v_2^*(\alpha)$, $v(\alpha)=v^{(0)}(\alpha)v^*(\alpha)$, and $u(\alpha)=u^{(0)}(\alpha)+u^*(\alpha)$; then we are led to the Hilbert problems

$$[v_2^*(\alpha)]^+=[v_2^*(\alpha)]^-, \qquad [v^*(\alpha)]^+=[v^*(\alpha)]^-,$$

and

$$[u^*(\alpha)]^+-[u^*(\alpha)]^-=g_{12}(\alpha)[v_2^{(0)}(\alpha)]^-/[v^{(0)}(\alpha)]^+\{[v_2^*(\alpha)]^-/[v^*(\alpha)]^+-1\}$$

which have a solution [Muskhelishvili [4, § 15])

$$\text{(18)} \qquad v_2^*(\alpha)=P_2(\alpha), \quad v^*(\alpha)=P_2(\alpha), \quad u^*(\alpha)=P_1(\alpha),$$

where $P_1(\alpha)$, $P_2(\alpha)$ are entire functions of α. Thus a suitably general solution of (14) and (15) is given by

$$v_2(\alpha)=P_2(\alpha)v_2^{(0)}(\alpha) \text{ and } v_1(\alpha)=u(\alpha)v(\alpha)=(u^{(0)}(\alpha)+P_1(\alpha))P_2(\alpha)v^{(0)}(\alpha).$$

A suitably general solution of the equation (13) and consequently of (10) is therefore given by

$$\mathbf{U}(\alpha)=\begin{pmatrix}(u^{(0)}(\alpha)+P_{11}(\alpha))P_{21}(\alpha)v^{(0)}(\alpha) & (u^{(0)}(\alpha)+P_{12}(\alpha))P_{22}(\alpha)v^{(0)}(\alpha)\\ P_{21}(\alpha)v_2^{(0)}(\alpha) & P_{22}(\alpha)v_2^{(0)}(\alpha)\end{pmatrix}, \tag{19}$$

where $\det \mathbf{U}(\alpha)=P_{21}(\alpha)P_{22}(\alpha)v^{(0)}(\alpha)v_2^{(0)}(\alpha)(P_{11}(\alpha)-P_{12}(\alpha))$, and $P_{ij}(\alpha)$, $i,j=1,2$ are entire functions. The choice of the entire functions $P_{ij}(\alpha)$ is further restricted by the condition that $\mathbf{U}(\alpha)$ is nonsingular; and the requirement that the corresponding matrix $\mathbf{L}(\alpha)=\mathbf{A}^{-1}(\alpha)\mathbf{U}(\alpha)$ is nonsingular, and its elements should be analytic except along the branch cut C' through $\alpha=k$. In particular, the elements of $\mathbf{L}(\alpha)$ should not have poles at $\alpha=-k$. For the applications we have in mind it is sufficient to let $P_{21}=P_{22}=P_{11}=-P_{12}=1$, giving

$$\begin{aligned}&\mathbf{U}^{(0)}(\alpha)=\begin{pmatrix}(u^{(0)}(\alpha)+1)v^{(0)}(\alpha) & (u^{(0)}(\alpha)-1)v^{(0)}(\alpha)\\ v_2^{(0)}(\alpha) & v_2^{(0)}(\alpha)\end{pmatrix},\\ &\det \mathbf{U}^{(0)}(\alpha)=2v^{(0)}(\alpha)v_2^{(0)}(\alpha).\end{aligned} \tag{20}$$

In a completely analogous way it can be shown that a Wiener–Hopf factorisation of the matrix defined by (9) is given by

$$\mathbf{A}(\alpha)=\mathbf{U}(\alpha)\mathbf{L}^{-1}(\alpha)$$

where

$$\mathbf{U}(\alpha)=\begin{pmatrix}P_{11}(\alpha)v_1^{(0)}(\alpha) & P_{12}(\alpha)v_1^{(0)}(\alpha)\\ P_{11}(\alpha)v^{(0)}(\alpha)(u^{(0)}(\alpha)+P_{21}(\alpha)) & P_{12}(\alpha)v^{(0)}(\alpha)(u^{(0)}(\alpha)+P_{22}(\alpha))\end{pmatrix}, \tag{21}$$

and $P_{ij}(\alpha)$, $i,j=1,2$ are entire functions,

$$\det \mathbf{U}(\alpha)=P_{11}(\alpha)P_{12}(\alpha)v_1^{(0)}(\alpha)v^{(0)}(\alpha)(P_{22}(\alpha)-P_{21}(\alpha));\text{ and } v_1^{(0)}(\alpha),\ v^{(0)}(\alpha),$$

and $u^{(0)}(\alpha)$ are fundamental solutions of the standard Hilbert problems

$$\left.\begin{aligned}&v_1^+(\alpha)=g_{11}(\alpha)v_1^-(\alpha),\\ &v^+(\alpha)=g_{22}(\alpha)v^-(\alpha),\\ &u^+(\alpha)-u^-(\alpha)=g_{21}(\alpha)v_1^-(\alpha)/v^+(\alpha),\end{aligned}\right\}\quad \alpha\in C.$$

$g_{ij}(\alpha)$ are the elements of the lower triangular matrix $\mathbf{G}(\alpha)=\mathbf{A}_+(\alpha)\mathbf{A}_-^{-1}(\alpha)$. Imposition of the further restriction that $\mathbf{U}(\alpha)$ and $\mathbf{L}(\alpha)$ are nonsingular, and analytic everywhere except along the branch cuts C and C' respectively, dictates the choice $P_{11}=P_{12}=P_{22}=-P_{21}=1$, giving

$$\mathbf{U}^{(0)}(\alpha)=\begin{pmatrix}v_1^{(0)}(\alpha) & v_1^{(0)}(\alpha)\\ v^{(0)}(\alpha)(u^{(0)}(\alpha)-1) & v^{(0)}(\alpha)(u^{(0)}(\alpha)+1)\end{pmatrix}. \tag{22}$$

The elements of $\mathbf{U}(\alpha)$ have been constructed by assuming that the matrix $\mathbf{L}(\alpha)$ in equation (2) is continuous across C and therefore $\mathbf{L}(\alpha)$ defined by

$$\mathbf{L}(\alpha)=\mathbf{A}^{-1}(\alpha)\mathbf{U}(\alpha),$$

with the elements defined by one of the equations (19) to (22), should from the method of construction, be continuous across C. Equation (5) and the equation above gives

$$\mathbf{L}_+(\alpha)=\mathbf{A}_+^{-1}(\alpha)\mathbf{U}_+(\alpha)=\mathbf{A}_+^{-1}(\alpha)\mathbf{A}_+(\alpha)\mathbf{A}_-^{-1}(\alpha)\mathbf{U}_-(\alpha)=\mathbf{L}_-(\alpha),$$

thus verifying explicitly that $\mathbf{L}(\alpha)$ is indeed continuous across C.

4. It was remarked in the introduction that the class of matrices (8) and (9) could possibly be factorised directly using Jones' technique [8] by expressing (8) or (9) in a suitable product form. One of the referees has indicated that this is possible by expressing them as the product of triangular matrices. Thus, for example, the matrix (8) can be written in the form

$$\mathbf{A}(\alpha)=\begin{pmatrix} a_{12}(\alpha)-a_{11}(\alpha)s(\alpha)/r(\alpha) & a_{11}(\alpha)/r(\alpha) \\ 0 & a_{21}(\alpha)\end{pmatrix}\begin{pmatrix} 0 & 1 \\ r(\alpha) & s(\alpha)\end{pmatrix}.$$

This product of matrices is now in the form of (2) where

$$\mathbf{C}_U(\alpha)=\mathbf{I},\qquad \mathbf{B}_L(\alpha)=\begin{pmatrix} 0 & 1 \\ r(\alpha) & s(\alpha)\end{pmatrix}.$$

The triangular matrix

$$\mathbf{A}'(\alpha)=\begin{pmatrix} a_{12}(\alpha)-a_{11}(\alpha)s(\alpha)/r(\alpha) & a_{11}(\alpha)/r(\alpha) \\ 0 & a_{21}(\alpha)\end{pmatrix},$$

can be explicitly factorised since

$$\begin{aligned}\begin{pmatrix} a(\alpha) & b(\alpha) \\ 0 & c(\alpha)\end{pmatrix}&=\begin{pmatrix} a_+(\alpha) & a_+(\alpha)[b(\alpha)/(c_-(\alpha)a_+(\alpha))]_R \\ 0 & c_+(\alpha)\end{pmatrix}\\ &\quad\cdot\begin{pmatrix} a_-(\alpha) & c_-(\alpha)[b(\alpha)/(c_-(\alpha)a_+(\alpha))]_L \\ 0 & c_-(\alpha)\end{pmatrix}\\ &=\begin{pmatrix} a_-(\alpha) & a_-(\alpha)[b(\alpha)/(c_+(\alpha)a_-(\alpha))]_L \\ 0 & c_-(\alpha)\end{pmatrix}\\ &\quad\cdot\begin{pmatrix} a_+(\alpha) & c_+(\alpha)[b(\alpha)/(c_+(\alpha)a_-(\alpha))]_R \\ 0 & c_+(\alpha)\end{pmatrix},\end{aligned}$$

using the notation of Jones [8]. However the above factorisation is prima facie different from that obtained in the main text of this paper. The above results involves scalar factorisation along infinite lines in a strip of regularity, whereas the method used in this paper involves scalar factorisation along semi-infinite lines. It is also worth mentioning that the Wiener–Hopf type factorisation problem, using a strip of regularity, is subsumed in the more general Hilbert factorisation problem along a line segment, see Noble [12].

Conclusions. We have presented a method for factorising matrices which arise in diffraction problems. This could offer scope for deriving closed-form solutions to hitherto unsolved diffraction problems, see Rawlins [11]. The applicability of the present method to a given matrix $\mathbf{A}'(\alpha)$ (whose elements, besides having the branch point singularities at $\alpha=\pm k$, also have poles; and whose determinant vanishes or becomes infinite in the cut α plane) can be easily determined. If $\mathbf{A}'_+(\alpha)[\mathbf{A}'_-(\alpha)]^{-1}$ is triangular, then the present method can be used to factorise the matrix $\mathbf{A}'(\alpha)$. One merely has to determine the $\mathbf{C}_U(\alpha)$ and $\mathbf{B}_L(\alpha)$ of the expression (2) which ensures that the elements of $\mathbf{A}(\alpha)$ have no poles and that $\det\mathbf{A}(\alpha)\neq 0$. This can be effected without too much difficulty by inspection.

Finally we mention that the $(n\times n)$ triangular matrix Hilbert problem can also be solved explicitly. Thus provided we can find the class of $(n\times n)$-matrices that reduce to the $(n\times n)$ triangular matrix Hilbert problem on analytic evaluation about the branch cut C, we will have effected a Wiener–Hopf factorisation of this class of $(n\times n)$-matrices.

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