Interconnection Topologies for Multi-Agent Coordination under Leader-Follower Framework *

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Abstract

In this paper, the formation control of networks of multiple agents is studied via controllability, where the network is under leader-follower structure with some agents taking the leader role and others being followers interconnected via neighbor-based rule. It is shown that the controllability of a multi-agent system is uniquely determined by the topology structure of interconnection graph, and the investigation of which comes down to that for a multi-agent system with the interconnection graph being connected. Based on these observations, two kinds of interconnection graph topologies are characterized, under which the network of multiple agents is uncontrollable, revealing to some extent how the controllability, and accordingly the formation control, are affected by the interconnection topology between agents. Finally, a necessary and sufficient condition in terms of eigenvector is presented. The results also touch upon the selection of leaders and are illustrated by several examples.

Keywords: Multi-agent systems, controllability, local interactions, leader-follower structure.

1 Introduction

Recently, the study of networked systems has caused great attention in the literature. This is because in the real world, collective behavior in swarms of entities is ubiquitous, for example in biological swarms, ants and birds often work and live together; and in the cooperative control and coordination of multiple robots or unmanned aerial vehicles, decision-making must be performed by multiple collaborating agents. As a special cooperative behavior of numbers of

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interacting dynamic agents, such behavior has advantages in increasing the chance of finding food and avoiding predators and other risks, etc. Understanding the cooperative and operational principles of such systems may provide useful ideas for developing formation control of unmanned air vehicles, underwater vehicles, satellite clusters and so on. Accordingly, researchers have started focusing their attention on modeling and understanding the cooperative principles of such collective behavior, as well as their potential engineering applications (e.g. [1, 5, 6, 14, 17]).

In the last decade, a number of researchers have investigated the formation control problem from various perspectives, e.g. [3, 10, 15, 8, 9, 16, 18]. In [16], the controllability was put forward for the first time for formation control of multi-agent systems, in which the controllability of a multi-agent system means that the system can be steered from one state to another any one through certain regulations. The spirit is to transform the formation control into a classical controllability problem for fixed topology, and a switched controllability problem for switching topology. To date, few results have been available along this line in the literature. In [7], the controllability was characterized by graph theory. In [12], the controllability problem was studied under both fixed and switching topologies for continuous-time case, and then for discrete-time case [11]. Different from the classical control system, the dynamical behavior of networked systems heavily relies on how the network is connected, i.e. the topology structure of the network. In particular, with respect to the controllability problem, how the controllability is affected by the interconnection topology structure among agents is a fundamental problem. The investigation of this problem is at the very outset. So the research on controllability of multi-agent systems calls for extensive exploration of properties for the topology structure of interconnection graph. This motivates the study in the paper.

In this paper, we consider a multi-agent system under leader-follower structure, where some agents take the leader role and others are followers interconnected via neighbor-based rule. The leaders are unaffected by the followers and do not abide by the agreement protocal whereas the followers are influenced by the leaders directly or indirectly. We first show that controllability is a property uniquely determined by the interconnection topology. Then a necessary and sufficient condition is derived for the multi-agent system to be controllable by dividing the overall system into some controllable connected components. As a consequence, the controllability problem is simplified to the investigation of that for a connected interconnection subgraph since all the connected components constitute the whole interconnection graph. Finally, two kinds of interconnection topologies are constructed to identify the uncontrollability of networks, revealing to some extent how the controllability, and accordingly the formation control, are affected by the interconnection topology among agents. A principle is also given for the selection of leaders to satisfy the necessary condition on controllability and a necessary and sufficient condition in terms of eigenvector is presented. The results are illustrated by several examples.

The paper is organized as follows: Section 2 is a brief review of graph theoretic terminologies and the controllability problem is formulated in this section. Section 3 follows with the main results. Finally, we briefly summarize the results of the paper in Section 4.

2 Preliminaries

2.1 Graph preliminaries

An undirected graph $\mathcal{G}$ consists of a node set $\mathcal{V} = \{v_1, \cdots, v_\chi\}$ and an edge set $\mathcal{E} = \{(v_i, v_j)|v_i, v_j \in \mathcal{V}\}$, where an edge is an unordered pair of distinct nodes of $\mathcal{V}$. A graph with node set $\mathcal{V}$ is said to be a graph on $\mathcal{V}$, and it can be visually depicted by drawing a dot for each node and a line for each edge. The number of nodes of a graph $\mathcal{G}$ is its order, and its total number of edges is its degree. If we use $|\cdot|$ to denote cardinality, we have that the order of $\mathcal{G}$ is $|\mathcal{V}(\mathcal{G})|$, or simply $|\mathcal{V}|$, and its degree of $\mathcal{G}$ is $|\mathcal{E}(\mathcal{G})|$, or $|\mathcal{E}|$. Two nodes v_i and v_j are neighbors if $(v_i, v_j) \in \mathcal{E}$, and the neighboring relation is indicated with $v_j \sim v_i$. In this case we say that v_j is a neighbor of v_i. The number of neighbors of each node is its valency or degree. If all the nodes of $\mathcal{G}$ are pairwise adjacent, then $\mathcal{G}$ is complete. Here it is assumed that there are no self-loops, i.e. $(v_i, v_i) \notin \mathcal{E}$, and there are no multiple edges between any pair of distinct nodes. A path $v_{i_0}v_{i_1}\cdots v_{i_s}$ is a finite sequence of nodes such that $v_{i_{k-1}} \sim v_{i_k}$, $k = 1, \cdots, s$, and a graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be two graphs. We call $\mathcal{G}'$ a subgraph of $\mathcal{G}$ (and $\mathcal{G}$ a supergraph of $\mathcal{G}'$) if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$, and we denote this by $\mathcal{G}' \subseteq \mathcal{G}$. A subgraph $\mathcal{G}'$ is said to be induced from the original graph $\mathcal{G}$ if $\mathcal{E}' = \mathcal{E} \cap \mathcal{V}' \times \mathcal{V}'$. In other words, it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. $\mathcal{G}' \subseteq \mathcal{G}$ is a spanning subgraph of $\mathcal{G}$ if $\mathcal{V}' = \mathcal{V}$. An undirected graph is said to be connected if there exists a path between any two distinct nodes of the graph. An induced subgraph of an undirected graph, which is maximal and connected, is said to be a connected component of the undirected graph. A graph is said to be simple if it is undirected, without loops and multiple edges.

The adjacency matrix $\mathcal{A}(\mathcal{G})$ of $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathcal{V}|$ matrix of whose (i, j)-entry is 1 if (v_i, v_j) is one of $\mathcal{G}$'s edges and 0 if it is not. Any undirected graph can be represented by its adjacency matrix, $\mathcal{A}(\mathcal{G})$, which is a symmetric matrix with 0-1 elements. The valency matrix $\triangle(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, j)-entry is the valency of node v_i. The incidence matrix $\mathrm{In}(\mathcal{G})$ of $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathcal{E}|$ matrix, with one row for each node and one column for each edge. Suppose edge $e = (v_i, v_j)$. Then column e of $\mathrm{In}(\mathcal{G})$ is zero except for the ith and jth entries, which are $+1$ and -1, respectively. The Laplacian matrix $\mathcal{L}(\mathcal{G})$ (simply, $\mathcal{L}$) of a graph $\mathcal{G}$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected, unweighted graph without graph loops (i, i) or multiple edges from one node to another, is an $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrix with one row and column for each node defined by

$$\mathcal{L}(\mathcal{G})_{i,j} = \begin{cases} d_i, & \text{if } i = j \text{ (number of incident edges)} \\ -1, & \text{if } i \neq j \text{ and } \exists \text{ edge } (v_i, v_j) \\ 0, & \text{otherwise.} \end{cases}$$

Given a graph $\mathcal{G}$, its associated matrices $\mathrm{In}(\mathcal{G})$ and $\mathcal{L}(\mathcal{G})$ have the following properties: (a) $\mathcal{L}(\mathcal{G})$ is always symmetric and positive semidefinite; (b) zero is always a eigenvalue of $\mathcal{L}(\mathcal{G})$ with $\mathbf{1}_n$, the vector of ones, being the associated eigenvector, and the algebraic multiplicity of the zero eigenvalue is equal to the number of connected components in the graph; (c) $\mathrm{In}(\mathcal{G})(\mathrm{In}(\mathcal{G}))^T = \mathcal{L}(\mathcal{G})$, and $\mathcal{L}(\mathcal{G}) = \triangle(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. Throughout the paper, we will abuse the language by referring to the eigenvalues and eigenvectors of $\mathcal{L}(\mathcal{G})$ as those of $\mathcal{G}$.

2.2 Problem formulation

The multi-agent system is given by

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, N \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \ldots, n_l \end{cases} \tag{1}$$

which consists of $N + n_l$ agents with simple, first order dynamics; where x_i is the state of the ith agent, $i = 1, \cdots, N + n_l$. The dimension of x_i could be arbitrary, as long as it is the same for all agents. We will analyze only the one-dimensional case for the sake of simplification of presentation. The analysis is valid for any dimension n, with the difference being that expressions should be rewritten in terms of Kronecker products. The following definition of interconnection graph is employed to describe the interconnection network once the linkages between agents are given.

Definition 1. *[16] The interconnection graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is being defined as an undirected graph consisting of a set of nodes, $\mathcal{V} = \{v_1, \ldots, v_N, v_{N+1}, \ldots, v_{N+n_l}\}$, indexed by the agents in the group, and a set of edges, $\mathcal{E} = \{(v_i, v_j) \in \mathcal{V} \times \mathcal{V} | \, v_i \sim v_j\}$, containing unordered pairs of nodes that correspond to interconnected agents.*

Let $\mathcal{N}_i$ be the neighboring set of v_i, i.e. $\mathcal{N}_i = \{j \mid v_i \sim v_j; j \neq i\}$. Then, under the following protocol,

$$u_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j), \quad i = 1, \cdots, N + n_l, \tag{2}$$

the multi-agent system (1) reads

$$\dot{x} = -\mathcal{L}x, \tag{3}$$

where $\mathcal{L}$ is the Laplacian matrix of interconnection graph, $x = (x_1, \cdots, x_{N+n_l})^T$ is the stack vector of all the agent states.

Definition 2. *The topology of an interconnection graph $\mathcal{G}$ is said to be fixed if each node of $\mathcal{G}$ has a fixed neighbor set.*

Take $x_{N+1}, \cdots, x_{N+n_l}$ to play the leaders role, and assume that interconnections with the leaders are unidirectional, that is, the leaders' neighbors still obey (2), but the leaders are indifferent, and are free to pick u_{N+j} arbitrarily, $j = 1, \ldots, n_l$. Now, let us rename the agents as follows:

$$\begin{cases} y_i \stackrel{\Delta}{=} x_i, & i = 1, \ldots, N \\ z_j \stackrel{\Delta}{=} x_{N+j}, & j = 1, \ldots, n_l \end{cases}$$

With y being the stack vector of all y_i, z the stack vector of all z_j, and u the stack vector of all u_{N+j}, $j = 1, \ldots, n_l$, the system can be written in the form:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} \mathcal{F} & \mathcal{R} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

where $\mathcal{F}$ is the matrix obtained from $\mathcal{L}$ after deleting the last n_l rows and n_l columns, and $\mathcal{R}$ is the $N \times n_l$ submatrix consisting of the first N elements of the deleted columns. Then the dynamics of the followers that correspond to the y component of the equation can be extracted as

$$\dot{y} = -\mathcal{F}y - \mathcal{R}z. \tag{4}$$

Definition 3. *The multi-agent system (1) is said to be controllable under leaders $x_{N+j}, j = 1, \ldots, n_l$, and fixed topology if system (4) is controllable.*

An interconnection graph $\mathcal{G}$ is said to be controllable if its corresponding multi-agent system is controllable. In subsequent arguments, we also indicate the multi-agent system (1) under fixed topology with matrix pair $(\mathcal{F}, \mathcal{R})$, and $\mathcal{F}, \mathcal{R}$ are said to be the corresponding system matrix and the control input matrix of the multi-agent system, respectively. Throughout the paper, we do not discriminate the eigenvalues (eigenvectors) of $\mathcal{L}$ from those of the associated interconnection graph $\mathcal{G}$. Once linkages between agents are known, an interconnection graph, and accordingly the fixed topology can be then determined in association with a multi-agent system. In contrast, given an interconnection graph, one can write out a corresponding multi-agent system, with interconnections between agents characterized by the graph. In this sense, we say that a multi-agent system has a one-to-one correspondence to an interconnection graph.

For the interconnection graph $\mathcal{G}$, denote by $\mathcal{G}_f$ and $\mathcal{G}_l$ the follower and leader subgraphs of $\mathcal{G}$, which are induced, respectively, by the follower and leader node sets. Let $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$ stand for the γ connected components in the follower subgraph $\mathcal{G}_f$, the following definition is introduced in [8] and [9], which is shown therein to be prerequisite to the investigation of controllability.

Definition 4. (leader-follower connected topology) *An interconnection graph $\mathcal{G}$ is said to be leader-follower connected if for each connected component $\mathcal{G}_{c_i}$ in the follower subgraph $\mathcal{G}_f$, there exists a leader in the leader subgraph $\mathcal{G}_l$, so that there is an edge between this leader and a node in $\mathcal{G}_{c_i}$, $i = 1, \cdots, \gamma$.*

3 Main results

In this section, we will first present a basic fact on controllability in Proposition 1. Then a necessary and sufficient condition is derived in the following Theorem 1. Based on these preliminary observations, the investigation of controllability is reduced to that for a connected interconnection graph. Finally, two kinds of interconnection topologies are constructed to identify the uncontrollability of networks.

Proposition 1. *The controllability of multi-agent system (1) is invariant under any labeling of the nodes in interconnection graph $\mathcal{G}$ if the interconnection topology of $\mathcal{G}$ and the leader positions in $\mathcal{G}$ are fixed.*

Proof. The nodes in the interconnection graph $\mathcal{G}$ are first labeled by $v_1, \cdots, v_N, v_{N+1}, \cdots, v_{N+n_l}$ with $v_{N+1}, \cdots, v_{N+n_l}$ representing the n_l leaders. Let $i_1, \cdots, i_N, i_{N+1}, \cdots, i_{N+n_l}$ be an arbitrary permutation of $1, \cdots, N+n_l$. We then relabel the nodes in $\mathcal{G}$ as $v_{i_1}, \cdots, v_{i_N}, v_{i_{N+1}}, \cdots, v_{i_{N+n_l}}$ and the corresponding interconnection graph is denoted by $\mathcal{G}'$, where $v_{i_{N+1}}, \cdots, v_{i_{N+n_l}}$ denote the leaders. In other words, the node v_j in $\mathcal{G}$ is relabeled as v_{i_j} in $\mathcal{G}'$, $i_j \in \{1, \cdots, N+n_l\}, i_j \neq$

i_k if $j \neq k$. Note that $\mathcal{G}$ and $\mathcal{G}'$ have the same topology structure since the interconnection topology of $\mathcal{G}$ is fixed.

Denote by $\mathcal{L}$ and $\mathcal{L}'$ the corresponding Laplacian matrix of $\mathcal{G}$ and $\mathcal{G}'$, respectively. It follows from the same topology structure of $\mathcal{G}$ and $\mathcal{G}'$ that there exists a permutation matrix P such that

$$\mathcal{L}' = P\mathcal{L}P^T, \tag{5}$$

where $P = \left[e_{i_1}, \cdots, e_{i_N}, \cdots, e_{i_{N+n_l}}\right]^T$, e_{i_j} is the i_jth identity vector with dimension $N+n_l$. By definition, the multi-agent system matrix $\mathcal{F}'$ and the control input matrix $\mathcal{R}'$ associated with $\mathcal{G}'$ are

$$\mathcal{F}' = E'\mathcal{L}'E'^T, \;\; \mathcal{R}' = E'\mathcal{L}'T',$$

where $E' = [e_{i_1}, \cdots, e_{i_N}]^T$, $T' = \left[e_{i_{N+1}}, \cdots, e_{i_{N+n_l}}\right]$, that is, E'^T is obtained by eliminating the columns of the identity matrix I_{N+n_l} that correspond to the leader nodes in $\mathcal{G}'$, and T' is constructed by grouping these eliminated columns in a new matrix. Similarly, with respect to $\mathcal{G}$, one has

$$\mathcal{F} = E\mathcal{L}E^T, \;\; \mathcal{R} = E\mathcal{L}T, \tag{6}$$

where $E = [e_1, \cdots, e_N]^T$, $T = [e_{N+1}, \cdots, e_{N+n_l}]$. By (5),

$$\mathcal{F}' = E'P\mathcal{L}P^TE'^T. \tag{7}$$

Let $E'P = [e_{j_1}, \cdots, e_{j_N}]^T$. Since the interconnection topology of $\mathcal{G}$ and the positions of leaders in $\mathcal{G}$ are both fixed, $\mathcal{G}$ and $\mathcal{G}'$ have the same topology structure and the same leader positions. Accordingly, $\{j_1, \cdots, j_n\}$ constitutes a permutation of $\{1, \cdots, N\}$, where the latter is the index set associated with the N row vectors of E. In other words, $E'P$ and E have the same row vectors with the difference on their ordering in the corresponding matrix. Accordingly, there exists an N-by-N permutation matrix W such that $E = WE'P$. Combining this with (6) and (7) gives rise to

$$\mathcal{F} = WE'P\mathcal{L}P^TE'^TW^T = W\mathcal{F}'W^T. \tag{8}$$

Also, it follows from the same positions of leaders and the construction of control input matrices that $W\mathcal{R}'$ and $\mathcal{R}$ have the same column vectors with the difference on their ordering in the associated matrix. Hence, there exists an n_l-by-n_l permutation matrix V such that

$$W\mathcal{R}' = \mathcal{R}V. \tag{9}$$

By (8) and (9), the relationship between the two controllable matrices is derived as follows:

$$\begin{aligned}\mathcal{C} &= \left[\mathcal{R}, \mathcal{F}\mathcal{R}, \cdots, \mathcal{F}^{N-1}\mathcal{R}\right] \\ &= W\left[W^T\mathcal{R}, \mathcal{F}'W^T\mathcal{R}, \cdots, \mathcal{F}'^{N-1}W^T\mathcal{R}\right] \\ &= W\left[W^T\mathcal{R}V, \mathcal{F}'W^T\mathcal{R}V, \cdots, \mathcal{F}'^{N-1}W^T\mathcal{R}V\right] diag\left\{V^T, \cdots, V^T\right\} \\ &= W\left[\mathcal{R}', \mathcal{F}'\mathcal{R}', \cdots, \mathcal{F}'^{N-1}\mathcal{R}'\right] diag\left\{V^T, \cdots, V^T\right\} \\ &= W\mathcal{C}' diag\left\{V^T, \cdots, V^T\right\},\end{aligned}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are controllable matrices of the multi-agent system associated, respectively, with $\mathcal{G}$ and $\mathcal{G}'$. Since both W and V are permutation matrices, $rank\,\mathcal{C} = rank\,\mathcal{C}'$. This completes the proof. □

Remark 1. *The above assertion indicates that controllability is a property uniquely determined by the interconnection topology. Although intuitively reasonable, it is, as far as we know, not clarified in the literature. The proposition then provides a formal confirmation on the fact.*

To illustrate Proposition 1, we give the following example.

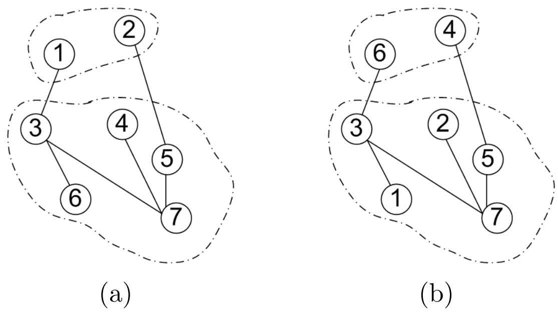


Figure 1: Two interconnection graphs with the same interconnection topology and the same leader positions.

Example 1. *Denote by $\mathcal{G}$ and $\mathcal{G}'$ the two interconnection graphs depicted, respectively, in (a) and (b) of Fig.1. The graphs show that $\mathcal{G}$ and $\mathcal{G}'$ have the same interconnection topology structure and the same leader positions. It can be seen that $N = 5, n_l = 2$. Let $P = [e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}, e_{i_5}, e_{i_6}, e_{i_7}]^T$ with $i_1 = 6, i_2 = 4, i_3 = 3, i_4 = 2, i_5 = 5, i_6 = 1, i_7 = 7$, where e_{i_j} is the i_jth identity vector with dimension seven. It can be verified that $\mathcal{L}' = P\mathcal{L}P^T$, where $\mathcal{L}'$ and $\mathcal{L}$ are Laplacian matrices of $\mathcal{G}'$ and $\mathcal{G}$, respectively. The system matrix and the control input matrix of $\mathcal{G}'$ and $\mathcal{G}$ are, respectively, $\mathcal{F}' = E'\mathcal{L}'E'^T, \mathcal{R}' = E'\mathcal{L}'T'$ and $\mathcal{F} = E\mathcal{L}E^T, \mathcal{R} = E\mathcal{L}T$, where $E' = [e_1, e_2, e_3, e_5, e_7]^T$, $T' = [e_4, e_6]$; $E = [e_3, e_4, e_5, e_6, e_7]^T$, $T = [e_1, e_2]$. Computations show that $\mathcal{F}' = E'P\mathcal{L}P^TE'^T$, with $E'P = [e_6, e_4, e_3, e_5, e_7]^T$. Obviously, $E'P$ and E have the same row vectors $e_3^T, e_4^T, e_5^T, e_6^T, e_7^T$ with the difference on their ordering in the corresponding matrix. In fact, $E = WE'P$, where $W = [e_3, e_2, e_4, e_1, e_5]^T$ is a permutation matrix. Then $\mathcal{F} = W\mathcal{F}'W^T$, $W\mathcal{R}' = \mathcal{R}[w_2, w_1]$, where w_i is the ith identity vector with dimension two, $i = 1, 2$. As a consequence, the ranks of the two controllable matrices associated with the two interconnection graphs are identical.*

Suppose $\mathcal{G}^{(1)}, \ldots, \mathcal{G}^{(\delta)}$ stand for the δ connected components of the interconnection graph $\mathcal{G}$. Leaders $x_{N+1}, \cdots, x_{N+n_l}$ are chosen according to the following principle.

Principle 1. *[8] For each connected component $\mathcal{G}^{(i)}$, the node set of the leader subgraph $\mathcal{G}_l$ contains at least one node of $\mathcal{G}^{(i)}$, $i = 1, \cdots, \delta$.*

Remark 2. *To analyze controllability, leaders should be selected only in accordance with the principle. Otherwise, the necessary condition on the feasibility of controllability, i.e. the connectedness between the leader and follower subgraphs, cannot be fulfilled (readers are referred to the Theorem 1 of [8] for details).*

When leaders are chosen according to Principle 1, each connected component $\mathcal{G}^{(i)}$ of $\mathcal{G}$ can be partitioned into two subgraphs: one is $\mathcal{G}_l^{(i)}$, the other is $\mathcal{G}_f^{(i)}$, with $\mathcal{G}_l^{(i)}$, $\mathcal{G}_f^{(i)}$ being, respectively, the induced leader and follower subgraph of $\mathcal{G}^{(i)}$.

Denote by $\mathcal{L}_{i_1,\cdots,i_m}$ the principal submatrix obtained by selecting the i_1th, $\ldots$, i_mth rows and columns of the Laplacian matrix $\mathcal{L}$, and assume that $\mathcal{G}_f^{(i)}$ is on the node set $\{v_{n_{i-1}+1},\ldots,v_{n_i}\}$, with $n_0=0, n_\delta=N$, $i=1,\ldots,\delta$. Then for an interconnection graph $\mathcal{G}$ with leaders chosen according to Principle 1, the following lemma reveals a property with respect to the Laplacian matrix of $\mathcal{G}$.

Lemma 1. *$\mathcal{L}_{1,\cdots,N}$ is positive definite and*

$$\mathcal{L}_{1,\cdots,N}=diag\left\{\mathcal{L}_{1,\cdots,n_1},\mathcal{L}_{n_1+1,\cdots,n_2},\cdots,\mathcal{L}_{n_{\delta-1}+1,\cdots,N}\right\},$$

where $\mathcal{L}_{1,\cdots,n_1},\mathcal{L}_{n_1+1,\cdots,n_2},\cdots,\mathcal{L}_{n_{\delta-1}+1,\cdots,N}$ are all positive definite submatrices too.

Proof. The assertion is a combination statement of Lemmas 1,2 in [8]. □

Theorem 1. *The multi-agent system (1) is controllable under fixed topology and leaders x_{N+1}, $\cdots,x_{N+n_l}$ if and only if each connected component $\mathcal{G}^{(i)}$ is controllable, $i=1,\cdots,\delta$.*

Proof. For the simplification of presentation, it is assumed without loss of generality that $\delta=3$, that is, the interconnection graph $\mathcal{G}$ consists of three connected components. The general situation can be proved in the same manner.

Suppose $\mathcal{G}^{(1)}$ is on the node set $\{v_1,\ldots,v_{n_1},v_{n_3+1},\ldots,v_{n_4}\}$, with $\{v_1,\ldots,v_{n_1}\}$ and $\{v_{n_3+1},\ldots,$ $v_{n_4}\}$ being, respectively, the follower and leader set of $\mathcal{G}^{(1)}$. Similarly, assume $\mathcal{G}^{(i)}$ is on the node set $\{v_{n_{i-1}+1},\ldots,v_{n_i},v_{n_{i+2}+1},\ldots,v_{n_{i+3}}\}$, with $\{v_{n_{i-1}+1},\ldots,v_{n_i}\}$ and $\{v_{n_{i+2}+1},\ldots,v_{n_{i+3}}\}$ being its follower and leader set, respectively, where $i=2,3$, and $n_3=N$, $n_6=N+n_l$. Then, since $\mathcal{G}^{(i)}, i=1,2,3$, are the three connected components of $\mathcal{G}$, $\{v_1,\ldots,v_{n_1},\ldots,v_{n_3}\}$ and $\{v_{n_3+1},\ldots,v_{n_4},\ldots,v_{n_6}\}$ constitute the follower and leader node set of $\mathcal{G}$, respectively.

It follows from Lemma 1 that

$$\mathcal{F}=diag\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\},$$

where $\mathcal{F}_i=\mathcal{L}_{n_{i-1}+1,\cdots,n_i}, i=1,2,3$; $n_0=0, n_3=N$. Set $\widetilde{n}_i\overset{\Delta}{=}n_i-n_{i-1}$, $i=1,\cdots,2\delta$; the control input matrix can be correspondingly partitioned as

$$\mathcal{R}=\left[\mathcal{R}_1^T,\mathcal{R}_2^T,\mathcal{R}_3^T\right]^T$$

with

$$\begin{aligned}\mathcal{R}_1&=[\mathcal{R}_{11},0_{\widetilde{n}_1\times\widetilde{n}_5},0_{\widetilde{n}_1\times\widetilde{n}_6}],\ \mathcal{R}_{11}:\widetilde{n}_1\times\widetilde{n}_4,\\ \mathcal{R}_2&=[0_{\widetilde{n}_2\times\widetilde{n}_4},\mathcal{R}_{22},0_{\widetilde{n}_2\times\widetilde{n}_6}],\ \mathcal{R}_{22}:\widetilde{n}_2\times\widetilde{n}_5,\\ \mathcal{R}_3&=[0_{\widetilde{n}_3\times\widetilde{n}_4},0_{\widetilde{n}_3\times\widetilde{n}_5},\mathcal{R}_{33}],\ \mathcal{R}_{33}:\widetilde{n}_3\times\widetilde{n}_6.\end{aligned}$$

The controllable matrix $\mathcal{C}$ can then be written as

$$\mathcal{C} = \begin{bmatrix} \mathcal{R}_1 \ \mathcal{F}_1\mathcal{R}_1 \cdots \mathcal{F}_1^{N-1}\mathcal{R}_1 \\ \mathcal{R}_2 \ \mathcal{F}_2\mathcal{R}_2 \cdots \mathcal{F}_2^{N-1}\mathcal{R}_2 \\ \mathcal{R}_3 \ \mathcal{F}_3\mathcal{R}_3 \cdots \mathcal{F}_3^{N-1}\mathcal{R}_3 \end{bmatrix}$$

$$= \begin{bmatrix} [\mathcal{R}_{11}, 0, 0] \ [\mathcal{F}_1\mathcal{R}_{11}, 0, 0] \cdots [\mathcal{F}_1^{N-1}\mathcal{R}_{11}, 0, 0] \\ [0, \mathcal{R}_{22}, 0] \ [0, \mathcal{F}_2\mathcal{R}_{22}, 0] \cdots [0, \mathcal{F}_2^{N-1}\mathcal{R}_{22}, 0] \\ [0, 0, \mathcal{R}_{33}] \ [0, 0, \mathcal{F}_3\mathcal{R}_{33}] \cdots [0, 0, \mathcal{F}_3^{N-1}\mathcal{R}_{33}] \end{bmatrix}. \tag{10}$$

The specific structure of the controllable matrix indicated in (10) yields the following observation

$$\begin{aligned} &rank\,\mathcal{C} \\ =&rank[\mathcal{R}_{11}, \mathcal{F}_1\mathcal{R}_{11}, \cdots, \mathcal{F}_1^{N-1}\mathcal{R}_{11}] + rank[\mathcal{R}_{22}, \mathcal{F}_2\mathcal{R}_{22}, \cdots, \mathcal{F}_2^{N-1}\mathcal{R}_{22}] \\ &+ rank[\mathcal{R}_{33}, \mathcal{F}_3\mathcal{R}_{33}, \cdots, \mathcal{F}_3^{N-1}\mathcal{R}_{33}]. \end{aligned} \tag{11}$$

Denote $\mathcal{C}_i \overset{\Delta}{=} [\mathcal{R}_{ii}, \mathcal{F}_i\mathcal{R}_{ii}, \cdots, \mathcal{F}_i^{\widetilde{n}_i-1}\mathcal{R}_{ii}],\ i = 1, \cdots, \delta$; it follows from the Cayley-Hamilton theorem that

$$rank\,\mathcal{C}_i = [\mathcal{R}_{ii}, \mathcal{F}_i\mathcal{R}_{ii}, \cdots, \mathcal{F}_i^{N-1}\mathcal{R}_{ii}],\ i = 1, \cdots, \delta.$$

By (11),

$$rank\,\mathcal{C} = rank\,\mathcal{C}_1 + rank\,\mathcal{C}_2 + rank\,\mathcal{C}_3. \tag{12}$$

On the other hand, let e_i stand for the ith identity vector with dimension $N + n_l$ and set

$$P = [e_1, \cdots, e_{n_1}, e_{n_3+1}, \cdots, e_{n_4}, e_{n_1+1}, \cdots, e_{n_2}, e_{n_4+1}, \cdots, e_{n_5}, e_{n_2+1}, \cdots, e_{n_3}, e_{n_5+1}, \cdots, e_{n_6}]^T.$$

Obviously, P is a permutation matrix. It can be verified that

$$P\mathcal{L}P^T = diag\left\{\widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2, \widetilde{\mathcal{L}}_3\right\},$$

where

$$\widetilde{\mathcal{L}}_i = \begin{bmatrix} \mathcal{F}_i & \mathcal{R}_{ii} \\ \mathcal{R}_{ii}^T & * \end{bmatrix}. \tag{13}$$

Consider the ith connected component $\mathcal{G}^{(i)}$, $i = 1, \cdots, \delta$; with its follower subgraph $\mathcal{G}_f^{(i)}$ and leader subgraph $\mathcal{G}_l^{(i)}$ on the node sets $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$ and $\{v_{n_{i+2}+1}, \ldots, v_{n_{i+3}}\}$, respectively. Concerning each connected component $\mathcal{G}^{(i)}$, we rename the nodes as follows:

$$w_1 \overset{\Delta}{=} v_{n_{i-1}+1}, \cdots, w_{\widetilde{n}_i} \overset{\Delta}{=} v_{n_i}; \quad \widetilde{n}_i \overset{\Delta}{=} n_i - n_{i-1}$$

$$w_{\widetilde{n}_i+1} \overset{\Delta}{=} v_{n_{i+2}+1}, \cdots, w_{\widetilde{n}_i+\widetilde{n}_{i+3}} \overset{\Delta}{=} v_{n_{i+3}}; \quad \widetilde{n}_{i+3} \overset{\Delta}{=} n_{i+3} - n_{i+2}.$$

It follows that there is a 'smaller' multi-agent system $(\mathcal{F}_i, \mathcal{R}_{ii})$ in association with an interconnection graph, denoted by $\widetilde{\mathcal{G}}^{(i)}$, whose follower and leader subgraphs are, respectively, on the node sets $\{w_1, \cdots, w_{\widetilde{n}_i}\}$ and $\{w_{\widetilde{n}_i+1}, \cdots, w_{\widetilde{n}_i+\widetilde{n}_{i+3}}\}$; and the linkages between agents in $\widetilde{\mathcal{G}}^{(i)}$ are the same as those in $\mathcal{G}^{(i)}$. Accordingly, the matrix $\widetilde{\mathcal{L}}_i$ shown in (13) is the Laplacian matrix of the interconnection graph $\widetilde{\mathcal{G}}^{(i)}$, and $\widetilde{\mathcal{G}}^{(i)}$ is controllable if and only if the connected component $\mathcal{G}^{(i)}$ is controllable. At the same time, it follows from (13) and the definition of $\mathcal{C}_i$ that $\mathcal{C}_i$ is the controllable matrix of the aforementioned 'smaller' multi-agent system.

Furthermore, by (12), $\mathcal{C}$ is full row rank if and only if so is each $\mathcal{C}_i$, $i = 1, \cdots, \delta$. In other words, the original multi-agent system is controllable if and only if each $\widetilde{\mathcal{G}}^{(i)}$, and accordingly each $\mathcal{G}^{(i)}$, is controllable. □

To facilitate understanding the result, we give the following example.

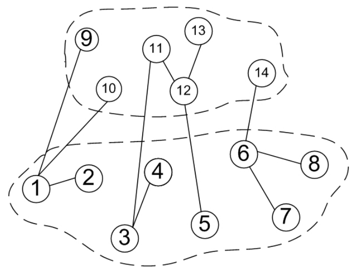


Figure 2: The interconnection graph $\mathcal{G}$

Example 2. *Consider a multi-agent system with its interconnection graph $\mathcal{G}$ depicted in Fig.2. The system matrices are*

$$\mathcal{F} = diag\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}, \ \mathcal{R} = [\mathcal{R}_1^T, \mathcal{R}_2^T, \mathcal{R}_3^T]^T,$$

with

$$\mathcal{F}_1 = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \mathcal{F}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{F}_3 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$\mathcal{R}_1 = [\mathcal{R}_{11}, 0_{2\times 3}, 0_{2\times 1}], \mathcal{R}_2 = [0_{3\times 2}, \mathcal{R}_{22}, 0_{3\times 1}], \mathcal{R}_3 = [0_{3\times 2}, 0_{3\times 3}, \mathcal{R}_{33}],$$

where

$$\mathcal{R}_{11} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, \mathcal{R}_{22} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \mathcal{R}_{33} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Set

$$P = [e_1, e_2, e_9, e_{10}, e_3, e_4, e_5, e_{11}, e_{12}, e_{13}, e_6, e_7, e_8, e_{14}]^T.$$

Computations show that

$$P\mathcal{L}P^T = diag\left\{\widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2, \widetilde{\mathcal{L}}_3\right\},$$

with $\widetilde{\mathcal{L}}_i = \begin{bmatrix} \mathcal{F}_i & \mathcal{R}_{ii} \\ \mathcal{R}_{ii}^T & \mathcal{M}_i \end{bmatrix}, i = 1, 2, 3,$ *where*

$$\mathcal{M}_1 = \mathcal{M}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{M}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

There are three connected components $\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \mathcal{G}^{(3)}$*, respectively, on the node sets* $\{v_1, v_2, v_9, v_{10}\}$, $\{v_3, v_4, v_5, v_{11}, v_{12}, v_{13}\}$, $\{v_6, v_7, v_8, v_{14}\}$*, and they correspond, respectively, to* $(\mathcal{F}_1, \mathcal{R}_{11})$, $(\mathcal{F}_2, \mathcal{R}_{22})$ *and* $(\mathcal{F}_3, \mathcal{R}_{33})$*. So* $\widetilde{n}_1 = 2, \widetilde{n}_2 = \widetilde{n}_3 = 3$*, and* $N = 8$*. It can be verified that both* $\mathcal{G}^{(1)}$ *and* $\mathcal{G}^{(2)}$ *are controllable, while* $\mathcal{G}^{(3)}$ *not, with the dimension of its controllable subspace being two. As a consequence, the overall multi-agent system is not controllable, with the dimension of its controllable subspace being seven.*

Theorem 1 can be viewed as a kind of separation principle for controllability. By Theorem 1, the controllability problem can be simplified to the investigation of that for a connected interconnection graph since each $\mathcal{G}^{(i)}$ is a connected component of the original interconnection graph. In view of this fact, we make, without loss of generality, the following assumption throughout the paper.

Assumption 1. *The interconnection graph* $\mathcal{G}$ *is connected.*

Remark 3. *The above arguments show that the controllability of multi-agent systems ought to be studied according to the following procedure:*

1) *Select leaders according to Principle 1 to satisfy first the necessary condition on controllability, i.e. the leader-follower connectedness between leader and follower subgraphs (the readers are referred to the proof of Theorem 1 in [8] for details).*

2) *With the selected leaders, the controllability problem can then be investigated, due to Theorem 1, under the assumption that the interconnection graph* $\mathcal{G}$ *is connected.*

Hence, Theorem 1 in [8] builds up a principle for the selection of leaders, and then, the Theorem 1 presented herein further simplifies the problem to the investigation of controllability only for a connected interconnection graph. This is what combining Theorem 1 established here with the one in [8] brings us.

Next, we are to present a 'partition' for the connected interconnection graph $\mathcal{G}$. Recall that $\mathcal{G}_l$ and $\mathcal{G}_f$ stand for, respectively, the leader and follower subgraph of $\mathcal{G}$. Although $\mathcal{G}$ is connected as a whole, as is not always the case for $\mathcal{G}_f$. Accordingly it can be assumed that $\mathcal{G}_f$ consists of γ connected components $\mathcal{G}_{c_1}, \cdots, \mathcal{G}_{c_\gamma}$. Let $\mathcal{G}(i)$ be an induced subgraph of $\mathcal{G}$, with its node set being the union of those associated with $\mathcal{G}_{c_i}$ and $\mathcal{G}_l$, $i = 1, \cdots, \gamma$. In other words, $\mathcal{G}(i)$ can be viewed as a 'smaller' interconnection graph with its follower subgraph being $\mathcal{G}_{c_i}$ and leader subgraph still being $\mathcal{G}_l$. Then $\mathcal{G}(1), \cdots, \mathcal{G}(\gamma)$ constitute a 'partition' of $\mathcal{G}$ in the sense that the whole interconnection graph $\mathcal{G}$ is partitioned into γ induced subgraphs $\mathcal{G}(1), \cdots, \mathcal{G}(\gamma)$, with each one having the same leader subgraph $\mathcal{G}_l$ and the union of them being $\mathcal{G}$.

To proceed, we need the following supporting lemmas.

Lemma 2. *(Theorem 1 of [8]) If multi-agent system (1) with fixed topology is controllable, then the interconnection graph is leader-follower connected, and each subgraph $\mathcal{G}(i)$ is controllable, where $i \in \{1, \ldots, \gamma\}$; γ is the number of connected components in $\mathcal{G}_f$.*

Lemma 3. *(Lemma 2.2 of [7]) Suppose the interconnection graph $\mathcal{G}$ is connected, the multi-agent system (1) is controllable if and only if $\mathcal{L}$ and $\mathcal{F}$ do not share any common eigenvalues.*

In view of this lemma, we will pursue conditions under which the Laplacian matrix $\mathcal{L}$ has multiple eigenvalues. A direct consequence of the conditions will be that $\mathcal{F}$ and $\mathcal{L}$ have common eigenvalues at least for a single leader case.

The following lemma is famous. The readers are referred to, for example, Theorem 9.1.1 of [4] for detail.

Lemma 4. (Interlacing) *Suppose M and N are real symmetric matrices of order m and n with eigenvalues $\lambda_1(M) \geq \cdots \geq \lambda_m(M)$ and $\lambda_1(N) \geq \cdots \geq \lambda_n(N)$, respectively. If M is a principal submatrix of N, then the eigenvalues of M interlace those of N, that is,*

$$\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N) \quad \text{for } i = 1, \cdots, m.$$

To characterize the desirable topology structure, we give the following definition.

Definition 5. *The κ nodes $v_{i_1}, \cdots, v_{i_\kappa}$ in the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ are said to have the same neighbor set if each of these nodes has the same set of neighbors $\{v_{i_{\kappa+1}}, \cdots, v_{i_{\kappa+\varrho}}\}$, where $v_{i_j} \in \mathcal{V}$, $i_h \neq i_j$ for $\forall h \neq j$.*

Obviously, this definition is meaningless for a single node, i.e. the case $\kappa = 1$. So the number κ is not less than two whenever the concept of the same neighbor set is mentioned.

Lemma 5. *(Lemma 2.1 of [2]) Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a graph with vertex subset $\mathcal{V}' = \{v_1, \cdots, v_\kappa\}$ having the same set of neighbors $\{v_{\kappa+1}, \cdots, v_{\kappa+\varrho}\}$, where $\mathcal{V} = \{v_1, \cdots, v_\kappa, \cdots, v_{\kappa+\varrho}, \cdots, v_n\}$. Then the Laplacian matrix of the graph $\mathcal{G}$ has at least $\kappa - 1$ equal eigenvalues and they are all equal to the cardinality ϱ of the neighbor set. Also the corresponding $\kappa - 1$ eigenvectors are*

$$[1, -1, 0, \cdots, 0]^T, [1, 0, -1, 0, \cdots, 0]^T, \cdots, [\underbrace{1, 0, \cdots, -1}_{\kappa}, 0, \cdots, 0]^T.$$

Theorem 2. *The multi-agent system (1) is uncontrollable, if there are nodes with the same neighbor set in the interconnection graph $\mathcal{G}$, and at the same time the leaders should be selected as follows:*

- *When $\kappa = 2$, i.e. there are only two nodes with the same neighbor set, the leaders are required to be selected from the remaining nodes in $\mathcal{G}$ other than the two nodes with the same neighbor set.*
- *When $\kappa \geq 3$, the number of leaders is required not greater than $\kappa - 2$ and the leaders are to be selected arbitrarily.*

Proof. Since the interconnection graph $\mathcal{G}$ is connected, any selection of leaders accords with the prerequisite of controllability, i.e., the leader-follower connectedness between the leader and follower subgraphs. Suppose each node in the subset $\{v_{i_1}, \cdots, v_{i_\kappa}\}$ has the same set of neighbors $\{v_{i_{\kappa+1}}, \cdots, v_{i_{\kappa+\varrho}}\}$, where $i_h \neq i_j$, $\forall h \neq j$. Take the permutation matrix as follows

$$P = \left[e_{i_1}, \cdots e_{i_\kappa}, e_{i_{\kappa+1}}, \cdots e_{i_{\kappa+\varrho}}, \cdots, e_{N+n_l}\right],$$

where e_s is the sth identity vector, $s = i_1, \cdots, i_{\kappa+\varrho}, \cdots, N+n_l$. Then v_{i_j} plays the same role in $\mathcal{G}$ as that the node v_j plays in another interconnection graph $\mathcal{G}'$, which corresponds to the permutation Laplacian matrix $P\mathcal{L}P'$, $j = 1, \cdots, \kappa+\varrho, \cdots, N+n_l$. In a word, it can be assumed, without loss of generality, that $\{v_1, \cdots, v_\kappa\}$ has the same set of neighbors $\{v_{\kappa+1}, \cdots, v_{\kappa+\varrho}\}$, as is due to the fact that $\mathcal{G}$ and $\mathcal{G}'$ have the same interconnection topology structure.

When $\kappa = 2$; the two nodes with the same neighbor set can be indicated with v_1, v_2, and then it follows from Lemma 5 that $[1, -1, \underbrace{0, \cdots, 0}_{N+n_l-2}]^T$ is an eigenvector of the Laplacian $\mathcal{L}$ associated with the eigenvalue ϱ. Since the n_l leaders are chosen from the remaining nodes $v_2, \cdots, v_{N+n_l}$ and the system matrix $\mathcal{F}$ is obtained from $\mathcal{L}$ by deleting the rows and columns corresponding to the leader nodes, it can be verified by straightforward computation that $[1, -1, \underbrace{0, \cdots, 0}_{N-2}]^T$ is an eigenvector of $\mathcal{F}$ associated with the same eigenvalue ϱ. So $\mathcal{L}$ and $\mathcal{F}$ share a common eigenvalue ϱ. By Lemma 3, the multi-agent system (1) is uncontrollable.

When $\kappa \geq 3$, with the selected $n_l \leq \kappa - 2$ leaders, the interconnection graph $\mathcal{G}$ can be 'partitioned', as mentioned above, into γ subgraphs $\mathcal{G}(1), \cdots, \mathcal{G}(\gamma)$. Since $v_1, \cdots, v_\kappa$ possess the same neighbor set, the induced subgraph on the node set $\{v_1, \cdots, v_\kappa, v_{\kappa+1}, \cdots, v_{\kappa+\varrho}\}$, indicated with $\widetilde{\mathcal{G}}$, is connected. As a consequence, $\widetilde{\mathcal{G}}$ must belong to a $\mathcal{G}(i)$ as long as the leaders are chosen in advance. In other words, it is a subgraph of this $\mathcal{G}(i)$, where the index i may vary with respect to differently selected leader set, $i \in \{1, \cdots, \gamma\}$.

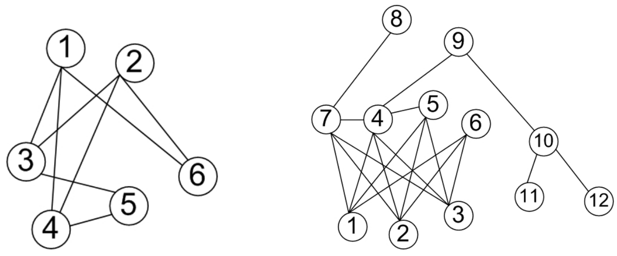
By Lemma 5, the Laplacian matrix $\mathcal{L}(i)$ associated with $\mathcal{G}(i)$ has an eigenvalue ϱ with its algebraic multiplicity at least $\kappa - 1$. For the convenience of presentation, we assume without loss of generality that $\varrho = \lambda_1 = \cdots = \lambda_{\kappa-1}$. Denote by $\mathcal{F}(i)$ the system matrix of the 'smaller' multi-agent system corresponding to $\mathcal{G}(i)$. Recall that $\mathcal{G}_{c_i}$ and $\mathcal{G}_l$ are, respectively, the follower and leader subgraph of $\mathcal{G}(i)$. If there are m_i nodes in $\mathcal{G}_{c_i}$, $\mathcal{F}(i)$ is $m_i \times m_i$ and $\mathcal{L}(i)$ is $(m_i + n_l) \times (m_i + n_l)$, where n_l is the number of leaders, i.e., the number of nodes in $\mathcal{G}_l$. It can be seen that $\mathcal{F}(i)$ is a principal submatrix of $\mathcal{L}(i)$ with order m_i. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m_i}$ be the eigenvalues of $\mathcal{F}(i)$. It follows from Lemma 4 that

$$\lambda_{n_l+1} \leq \mu_1 \leq \lambda_1$$

where $\lambda_1 \geq \cdots \geq \lambda_{m_i+n_l}$ are the eigenvalues of $\mathcal{L}(i)$. This, together with $n_l \leq \kappa - 2$, gives rise to

$$\mu_1 = \lambda_1 = \cdots = \lambda_{n_l+1} = \cdots = \lambda_{\kappa-1},$$

which means that $\mathcal{F}(i)$ and $\mathcal{L}(i)$ have at least one common eigenvalue $\mu_1 = \lambda_1 = \varrho$. In view of Lemma 3, the induced subgraph $\mathcal{G}(i)$ is uncontrollable. The multi-agent system is then uncontrollable by following Lemma 2. □



(a) (b)

Figure 3: Two interconnection graphs

Example 3. *The example is employed to illustrate Theorem 2. The interconnection graph (a) of Fig.3 corresponds to the situation* $\kappa = 2$*, where* v_1, v_2 *have the same neighbor set* $\{v_3, v_4, v_6\}$*. If* v_5, v_6 *are chosen to be leaders, computations show that* $[1, -1, 0, 0, 0, 0]^T$ *and* $[1, -1, 0, 0]^T$ *are, respectively, the eigenvector of* $\mathcal{L}$ *and* $\mathcal{F}$*, associated with the same eigenvalue* $\varrho = 3$*. The corresponding multi-agent system is uncontrollable.*

The interconnection graph (b) of Fig.3 corresponds to the situation $\kappa \geq 3$*, where each node in the subset* $\{v_1, v_2, v_3\}$ *has the same set of neighbors* $\{v_4, v_5, v_6, v_7\}$*. So* $\kappa = 3, \varrho = 4$*. Since* $\kappa = 3$*, one only need consider the case of single leader. The leader is indicated with* v_l *and falls into one of the following three cases: (a)* $v_l \in \{v_1, v_2, v_3, v_5, v_6, v_8, v_{11}, v_{12}\}$*; (b)* $v_l \in \{v_4, v_7, v_9\}$*; (c)* $v_l = v_{10}$*. In case of (a), the follower subgraph* $\mathcal{G}_f$ *is connected. Accordingly,* $\gamma = 1$*. In case of (b),* $\mathcal{G}_f$ *consists of two connected components, and then* $\gamma = 2$*. In case of (c),* $\mathcal{G}_f$ *consists of three connected components, which are denoted, respectively, by* $\mathcal{G}_{c_1}$*,* $\mathcal{G}_{c_2}$*, and* $\mathcal{G}_{c_3}$*, where* $\mathcal{G}_{c_1}$ *is on the note set* $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$*,* $\mathcal{G}_{c_2}$ *on the node* v_{11} *and* $\mathcal{G}_{c_3}$ *on the node* v_{12}*. In any case, it can be verified that the Laplacian matrix* $\mathcal{L}$ *and the system matrix* $\mathcal{F}$ *share the common eigenvalue* 4*. Accordingly, the multi-agent system with the single leader and the interconnection topology structure depicted in (b) of Fig.3 is uncontrollable regardless how the leader is selected.*

Lemma 6. *(Corollary 2.3 [13]) Let* $\mathcal{G}$ *be a graph on* n *vertices. If* $0 \neq \mu < n$ *is an eigenvalue of the Laplacian matrix associated with* $\mathcal{G}$*, then any eigenvector affords* μ *takes the value* 0 *on every vertex of degree* $n - 1$*.*

Let χ stand for the number of nodes in the interconnection graph $\mathcal{G}$. Assume that there are m nodes, say $v_{\chi-(m-1)}, \cdots, v_\chi$, in the interconnection graph $\mathcal{G}$, with each one having the degree $\chi - 1$. We have the following assertion.

Theorem 3. *The multi-agent system (1) is uncontrollable if leaders are chosen from the node set* $\{v_{\chi-(m-1)}, \cdots, v_\chi\}$*, with each one in the set having degree* $\chi - 1$*, and there is an eigenvalue of* $\mathcal{G}$ *not equal to* 0 *and* χ*. Furthermore, in this case, the dimension of the corresponding controllable subspace is one.*

Proof. As mentioned above, let $\{v_1, \cdots, v_N\}$ and $\{v_{N+1}, \cdots, v_{N+n_l}\}$ stand for the follower and leader node set, respectively, where $\chi \triangleq N + n_l$, and the n_l leaders are chosen from the set $\{v_{\chi-(m-1)}, \cdots, v_\chi\}$, $1 \leq n_l \leq m$. Let μ be an eigenvalue of interconnection graph $\mathcal{G}$ not equal to 0 and χ. By Lemma 6, any eigenvector associated with μ takes value 0 on the $(N + i)$th element, $i = 1, \cdots, n_l$. Accordingly, any eigenvector ξ associated with μ can be denoted by

$[\xi_{1,\cdots,N}^T, \underbrace{0, \cdots, 0}_{n_l}]^T$, where $\xi_{1,\cdots,N}$ is the vector consisting of the first N elements of ξ. Since $\mathcal{F}$ is a principle submatrix of the Laplacian $\mathcal{L}$, obtained by deleting the last n_l rows and n_l columns of $\mathcal{L}$, a straightforward matrix calculation shows that $\xi_{1,\cdots,N}$ is an eigenvector of $\mathcal{F}$ corresponding to the eigenvalue μ. So, $\mathcal{L}$ and $\mathcal{F}$ share a common eigenvalue μ. By Lemma 3, the multi-agent system (1) is uncontrollable.

In fact, direct computation shows that under aforementioned conditions, each row of the controllable matrix associated with the multi-agent system is in the following form

$$\left[\underbrace{1, \cdots, 1}_{n_l}, \underbrace{-n_l, \cdots, -n_l}_{n_l}, \underbrace{(-n_l)^2, \cdots, (-n_l)^2}_{n_l}, \cdots, \underbrace{(-n_l)^{N-1}, \cdots, (-n_l)^{N-1}}_{n_l}\right].$$

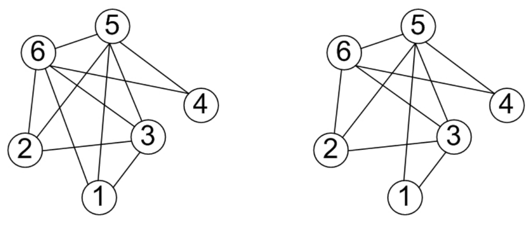
Accordingly, the dimension of the controllable subspace is one. □

A direct consequence of the result is the following corollary.

Corollary 1. *A complete graph is uncontrollable.*

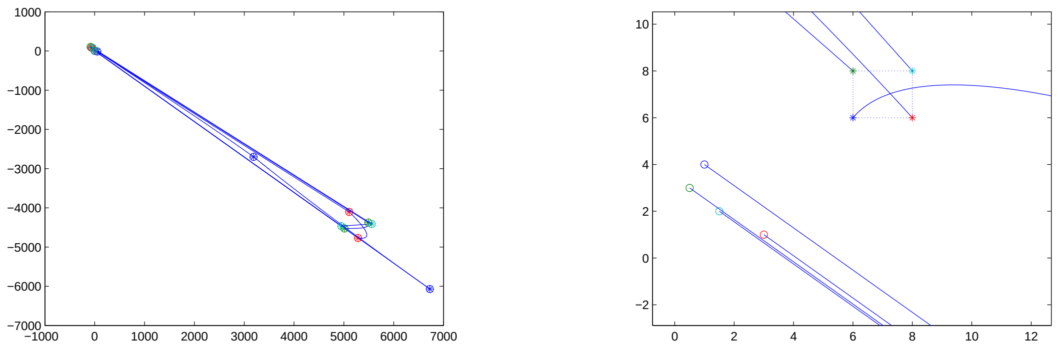
The corollary is true because each node in a complete graph has a degree of $\chi - 1$, where χ is the number of nodes in a graph.

Remark 4. *Corollary 1 is the Proposition V.1 in [16], which is employed therein to show that increased connectivity has an adverse effect on the controllability of networks. Here, Theorem 3 tells us that rather than completeness of the overall graph, the existence of one node with the degree (of connectivity) $\chi - 1$ is enough to destroy the controllability only if the very node is chosen as a leader. In this sense, Theorem 3 provides a further observation on the relationship between controllability and connectivity for networks of multiple agents. We remark that Theorem 3 can also be verified by computing the controllable matrix directly and it can be shown that under conditions of Theorem 3, the rank of the controllable matrix is one.*



(a) (b)

Figure 4: Two interconnection graphs: (a) is uncontrollable, while (b) is controllable.



(a) (b)

Figure 5: The trajectories of the four followers in the plane, with the associated interconnection graph of the multi-agent system depicted in (b) of Fig.4. Here, (a) is the whole trajectory of the system and (b) depicts the initial state and the final desired configuration of the multi-agent system, which is the magnification of the corresponding part of (a).

Example 4. *The example is employed to verify the above result. The number* χ *of nodes in the interconnection graph depicted in (a) of Fig.4 is 6, where nodes* v_5 *and* v_6 *have the same degree* $\chi - 1$*. The eigenvalues of the corresponding Laplacian matrix* $\mathcal{L}$ *are* $0, 2, 3, 5, 6, 6$*. If* v_5 *and* v_6 *are chosen to take the leader role, calculations show that the eigenvalues of* $\mathcal{F}$ *are* $2, 2, 3, 5$*. In this case,* $2, 3, 5$ *are clearly the common eigenvalues of* $\mathcal{F}$ *and* $\mathcal{L}$*. If the leader is single, say* v_5 *is the leader, it can be verified that* $\mathcal{F}$ *and* $\mathcal{L}$ *also share the common eigenvalues* $2, 3, 5$*. In both cases, the rank of the controllable matrix is one. It is interesting to point out that although the dimension of the controllable subspace corresponding to the system associated with (a) of Fig.4 is just one, the multi-agent system can be turned to be controllable by a slight modification of the original interconnection graph. For example, if the connection between node 1 and node 6 is removed, that is, the edge between 1 and 6 are deleted, see (b) of Fig.4, the original system turns to be controllable. Fig. 5 depicts the trajectories of the four controllable followers in the plane, where (a) of Fig.5 is magnified in apart in (b) of Fig.5 to observe clearly the initial state and the final desired configuration of the system.*

Next, we present a necessary and sufficient condition for the controllability of multi-agent systems.

Theorem 4. *The multi-agent system is controllable if and only if there is no eigenvector of* $\mathcal{G}$ *taking 0 on the elements corresponding to the leaders.*

Proof. The theorem can be reformulated as stating that the system is uncontrollable if and only there exists an eigenvector of $\mathcal{G}$ takes 0 on the elements corresponding to the leaders.

(*Sufficiency*) Suppose $\{v_{i_1}, \cdots, v_{i_N}\}$ and $\{v_{i_{N+1}}, \cdots, v_{i_{N+n_l}}\}$ are, respectively, the follower and leader node set. Set $E \triangleq [e_{i_1}, \cdots, e_{i_N}]^T, T \triangleq \left[e_{i_{N+1}}, \cdots, e_{i_{N+n_l}}\right]$, where e_{i_j} is the i_jth identity vector with dimension $N + n_l$. Then $\mathcal{F} = E\mathcal{L}E^T$, $\mathcal{R} = E\mathcal{L}T$. Let y be an eigenvector of $\mathcal{L}$ associated with the eigenvalue λ, with the i_jth component of y, i.e. y_{i_j}, being zero, $j =$

$N+1, \cdots, N+n_l$. It can be directly verified that $E^T E y = y$. Then, from $\mathcal{L}y = \lambda y$, one has

$$E\mathcal{L}E^T E y = \lambda E y, \;\; T^T \mathcal{L} E^T E y = 0.$$

That is, $\mathcal{F}y_1 = \lambda y_1, \mathcal{R}^T y_1 = 0$, where $y_1 \stackrel{\Delta}{=} Ey = [y_{i_1}, \cdots, y_{i_N}]^T$. By the controllability PBH criteria, the multi-agent system $(\mathcal{F}, \mathcal{R})$ is uncontrollable.

(*Necessity*) Since $\mathcal{F}$ is symmetric, its left eigenvectors are equal to the right ones. Suppose the system is uncontrollable. Then, by the PBH criteria of controllability, there exists a vector $x \in \mathbb{R}^N$ such that $\mathcal{F}x = \lambda x$ for some $\lambda \in \mathbb{R}$, with $\mathcal{R}^T x = 0$. Let

$$P \stackrel{\Delta}{=} [e_{i_1}, \cdots, e_{i_N}, e_{i_{N+1}}, \cdots, e_{i_{N+n_l}}]^T = \begin{bmatrix} E \\ T^T \end{bmatrix},$$

where E, T are matrices defined as above. It follows that P is a permutation matrix and

$$P\mathcal{L}P^T \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{F} & \mathcal{R} \\ \mathcal{R}^T & T^T \mathcal{L} T \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Accordingly, $y \stackrel{\Delta}{=} P^T \begin{bmatrix} x \\ 0 \end{bmatrix}$ is an eigenvector of $\mathcal{L}$, with the components corresponding to the leaders being zero. This completes the proof. □

Remark 5. *This Theorem characterizes the controllability from the viewpoint of eigenvector of Laplacian matrix, while Lemma 3 from the viewpoint of eigenvalue.*

4 Conclusions

In this paper, we study connections between the controllability of multi-agent systems and the topology structure of interconnection graph. It is shown that the controllability of a multi-agent system is uniquely determined by the topology structure of interconnection graph as long as the leaders are designated. Two kinds of topology structures are revealed under which the system is uncontrollable and necessary and sufficient conditions are proposed for the controllability of networks of multiple agents. One advantage of the results is that controllability, and then the feasibility of formation control for multi-agent systems can be determined straightforward from the graph topology itself. To facilitate understanding the results and notations, several examples are included in the paper. The results add to the understanding of formation control of multi-agent systems by means of the classical concept of controllability.

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