Chapter 1

Complex Networks

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Abstract

This Chapter contains a brief introduction to complex networks, and in particular to small world and scale free networks. We show how to apply the replica method developed to analyse random matrices in statistical physics to calculate the spectral densities of the adjacency and Laplacian matrices of a scale free network. We use the effective medium approximation to treat networks with finite mean degree and discuss the local properties of random matrices associated with complex networks.

1.1 Introduction

Graphs are incorporated into many theories in physics, and as our society becomes ever more globalised and inter-connected, the study of graphs forms an integral part of the systematic study of a vast array of human, social, economic, technological, biological and physical systems. The graph theory developed by Erdös and Rényi, [Erd60] considers vertices and edges linked with a fixed probability p. This yields networks with a Poisson degree distribution. Since then Watts observed that many real networks exhibit properties in which there is a small finite number of steps between any two vertices in the network, but some local order, so that if vertex A is connected to vertex B and B is connected to vertex C then there is a good chance that A is connected to C. Networks with these two properties are referred to as small world networks. Albert and Barabási [Alb02] observed that many real networks had a degree distribution that had a fat tail, which is that the degree distribution decays at a rate slower than exponential. Furthermore, Barabási and Albert [Bar99] introduced a model that built a network with a power-law degree distribution, and networks of this type have become known as scale free networks. These two classes of networks are the two best understood complex networks. Any network that is neither regular nor of Erdös and Rényi type is normally described as complex.

1.1.1 Small world networks

A small world is a loosely defined concept normally associated with systems in which most vertices are not neighbours of one another but where it is only a small finite number of steps between any two vertices, and where there is a degree of local order. The small world phenomenon, in which strangers are often linked by a mutual acquaintance, is captured in the phrase six handshakes from the president in which nearly everyone in the world is around six acquaintances from the president. Networks exhibiting the small world phenomenon include social networks, the world wide web (the network of webpages and html links) and gene expression networks. The local ordering property of small world networks is usually associated with regular networks such as a 2-d square lattice. The finite number of steps between any two vertices is associated with completely connected graphs. In this sense small world networks can be thought of as being intermediate between a completely connected graphs and regular lattices. Most scale free networks exhibit the small world phenomenon but few small world networks are scale free. In 1998 Watts and Strogatz [Wat98] characterised all graphs by the clustering coefficient C, defined by

$$C = 3\frac{\text{Number of Triangles}}{\text{Number of Connected Triples}}\ , \tag{1.1.1}$$

and by the mean vertex-to-vertex distance. C is the average probability that two of ones friends are friends themselves ($C = 1$ on a fully connected graph, everyone knows everyone else). Watts and Strogatz observed that while Erdös and Rényi graphs had a small clustering coefficient and small mean vertex-to-vertex distance, many real graphs had a clustering coefficient larger than those seen in random graphs, but had a similar mean vertex-to-vertex distance. They then introduced a model that exhibited this property. Later, a related model was introduced by Newman and Watts [New99]. This is built as follows;

- Let N vertices be connected in a circle;

- Each of several neighbours is connected by a unit length edge;
- Then each of these edges is rewired with probability φ to a randomly chosen vertex.

This network has the following properties

- $\varphi = 0$ is a regular lattice;
- $\varphi = 1$ is an Erdös and Rényi graph;
- Exhibits small world property for $0 < \varphi < 1$.
- Average shortest distance behaves as $\sim N$ for $\varphi = 0$ and $\sim \log N$ for $\varphi > 0$.

Obviously this model mechanism can be generalised to any regular 2-d or 3-d graph. Models of this type are difficult to formulate analytically, and only a few basic properties have been obtained analytically, in contrast to both Erdös and Rényi graph and scale free networks.

1.1.2 Scale free networks

A scale free network has a degree distribution that is asymptotically power-law. That is, the number $N(k)$ of vertices with degree k behaves like $k^{-\lambda}$ for large k with $2 < \lambda$. Many real world networks are scale free including the world wide web, the internet, the citation graph, the science collaboration graph, the actor collaboration graph and the phone call graph. The latter is of course big business for telecommunications firms. Scale free networks are hugely heterogeneous structures, with, depending on the position of the cut-off in the network model, nodes with degrees ranging from 1 to $N^{\frac{1}{\lambda-1}}$. A list of cut-offs for different scale networks is given in [Dor08]. There are several different ways to build a model scale free graph. These include the Barabási-Albert model [Bar99], the static model introduced by Goh, Kahng and Kim [Goh01b] and the grown scale free graph with statistically defined modularity introduced by Tadic [Tad01]. In this Chapter we consider uncorrelated networks, there is discussion of causality and homogeneity in complex networks in [Bia05, Bia05].

In this chapter we briefly introduce two models of scale free graphs, then use the replica method to examine the adjacency matrix and the Laplacian associated with a complex network built using the static model. We then show how the effective medium approximation can be used to treat a complex network with a finite mean degree. Finally we discuss extensions of this work to other local properties of the random matrices and to other types of complex network.

1.2 Replica analysis of scale free networks

1.2.1 Degree distribution and spectral density

Let us consider a complex network with N nodes and examine the asymptotic behaviour in the limit $N \to \infty$. The connection pattern of the network is described by the adjacency matrix A, which is an $N \times N$ symmetric matrix with

$$A_{jl} = \begin{cases} 1, & \text{if } j\text{−th and } l\text{−th nodes are directly connected,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.1}$$

In order to characterise the connection pattern, various statistical quantities are studied. One of the most common quantities is the degree distribution function. The number of edges attached to the j-th node

$$k_j = \sum_{l=1}^{N} A_{jl} \tag{1.2.2}$$

is called the degree. The degree distribution function $P(k)$ is defined as

$$P(k) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(k - k_j) \right\rangle, \tag{1.2.3}$$

where the brackets stand for the average over the probability distribution function of the adjacency matrices.

If the degree distribution function $P(k)$ obeys a power law, namely, if it is proportional to $k^{-\lambda}$ with a positive exponent λ in the limit $k \to \infty$, then the network is said to be scale free. The scale free property is one of the prominent universal features of social and biological networks [Bar99].

In spectral theory of networks, the Laplacian matrix is of another interest. The Laplacian matrix L is an $N \times N$ symmetric matrix defined as

$$L_{jl} = \begin{cases} k_j, & j = l, \\ -A_{jl}, & j \neq l. \end{cases} \tag{1.2.4}$$

The Laplacian matrix has non-negative eigenvalues and the smallest eigenvalue is always zero.

The spectral density of the adjacency (or Laplacian) matrix is an important quantity for characterising the properties of the network. It is defined as

$$\rho(\mu) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(\mu - \mu_j) \right\rangle, \tag{1.2.5}$$

where $\mu_1, \mu_2, \cdots, \mu_N$ are the eigenvalues of the adjacency matrix A (or the Laplacian matrix L).

For scale free complex networks, it is expected that the spectral density of adjacency matrices obeys a power law $\rho(\mu) \propto \mu^{-\gamma}$ in the limit $\mu \to \infty$. Such a behaviour was first observed in numerical work on complex networks [Far01, Goh01a]. Then the relation between γ and λ was analytically specified as $\gamma = 2\lambda - 1$ in a work on locally tree-like networks [Dor03, Dor04]. This relation was confirmed in a study on the static network model [Rod05].

In the following, after a brief description of a growth model, the static model of complex networks is introduced. Then the replica method in statistical physics is applied to the static model and the spectral densities of the adjacency and Laplacian matrices are evaluated.

1.2.2 Models of scale free networks

Barabási and Albert invented a growth model (BA model) of complex networks [Bar99], in which the n-th node with m edges is newly introduced at a time t_n, where $n = 1, 2, 3, \cdots$. Let us discuss the degree distribution of the BA model, neglecting the fluctuation of k_j (so that $\langle k_j \rangle = k_j$). Each of m edges is attached to the old nodes in the network, according to the rule of preferential attachment. That is, the attachment probability Π_j of the j-th old node is proportional to the degree:

$$\Pi_j = \frac{k_j}{\sum_{l=1}^{n-1} k_l}, \quad j = 1, 2, \cdots, n-1, \tag{1.2.6}$$

so that a difference equation

$$k_j(t_n) - k_j(t_{n-1}) = m\Pi_j = m\frac{k_j}{\sum_{l=1}^{n-1} k_l} \tag{1.2.7}$$

follows. As m edges are attached at each time when a node is introduced, an asymptotic estimate $\sum_{l=1}^{n-1} k_l \sim 2mn$ holds for large n. Let us suppose that the time difference $\Delta t = t_n - t_{n-1}$ is a constant. Then, in the continuous limit $\Delta t \to 0$ with a fixed $t = n\Delta t$, we obtain a differential equation

$$\frac{\partial k_j(t)}{\partial t} = \frac{k_j}{2t}. \tag{1.2.8}$$

One can solve this equation with the initial condition $k_n(t_n) = m$ as

$$k_j(t_n) = m\sqrt{\frac{n}{j}}. \tag{1.2.9}$$

That is, the degree of the j-th node is proportional to $j^{-1/2}$ at a fixed time.

In order to have a degree k_j smaller than k, we need to have $j > n(m/k)^2$. Therefore the number $\nu(k)$ of the nodes with the degrees smaller than k is

$\nu(k) = \sum_{j>n(m/k)^2}^{n} 1 = n(1-(m/k)^2)$. Then the degree distribution function $P(k)$ is estimated as

$$P(k) = \frac{\partial}{\partial k}\frac{\nu(k)}{n} = 2\frac{m^2}{k^3}, \tag{1.2.10}$$

so that the BA model has the exponent $\lambda = 3$.

Goh, Kahng and Kim introduced a static model (GKK model) simulating the growth model with a fixed number of nodes [Goh01b]. In the GKK model, the upper (lower) triangular elements of the adjacency matrix are assumed to be independently distributed. Suppose that there are N nodes and that the j-th node is assigned a probability

$$P_j = \frac{j^{-\alpha}}{\sum_{j=1}^{N} j^{-\alpha}} \sim (1-\alpha)N^{\alpha-1}j^{-\alpha}, \quad 0 < \alpha < 1. \tag{1.2.11}$$

In each step two nodes are chosen with the assigned probabilities and connected unless they are already connected. In order to have a mean degree p, as we shall see below, such a step is repeated $pN/2$ times. Then the j-th and l-th nodes are connected with a probability

$$f_{jl} = 1 - (1 - 2P_jP_l)^{pN/2} \sim 1 - \mathrm{e}^{pNP_jP_l}, \tag{1.2.12}$$

so that the adjacency matrix A of the network is distributed according to a probability distribution function

$$\mathcal{P}_{jl}(A_{jl}) = (1 - f_{jl})\delta(A_{jl}) + f_{jl}\delta(A_{jl} - 1), \quad j < l. \tag{1.2.13}$$

A useful asymptotic estimate [Kim05] in the limit $N \to \infty$ for averaging over $\mathcal{P}_{jl}(A_{jl})$ is

$$\ln\left\langle \exp\left(-i\sum_{j<l}^{N} A_{jl}t_{jl}\right)\right\rangle \sim pN\sum_{j<l}^{N} P_jP_l(e^{-it_{jl}} - 1) \tag{1.2.14}$$

where t_{jl} is a parameter independent of N. The remainder term is $O(1)$ for $0 < \alpha < 1/2$, $O((\ln N)^2)$ for $\alpha = 1/2$ and $O(N^{2-(1/\alpha)}\ln N)$ for $1/2 < \alpha < 1$. As a special case, we find an estimate

$$F_j(t) \equiv \ln\left\langle e^{-i\sum_{l=1}^{N} A_{jl}t}\right\rangle \sim pNP_j(e^{-it} - 1). \tag{1.2.15}$$

Now let us calculate the degree distribution. Using the definition (1.2.2) and asymptotic estimate (1.2.15), one obtains [Lee04]

$$\langle k_j \rangle = \sum_{l=1}^{N}\langle A_{jl}\rangle = i\frac{\partial}{\partial t}F_j(t)\bigg|_{t=0} \sim pNP_j, \tag{1.2.16}$$

so that the mean degree of the j-th node is proportional to $j^{-\alpha}$. Hence one can expect that the case $\alpha = 1/2$ approximates the BA model. As announced before, the mean degree is $(1/N)\sum_{j=1}^{N}\langle k_j\rangle = p$. Moreover it follows from (1.2.3), (1.2.11) and (1.2.15) that the degree distribution function is

$$\begin{aligned} P(k) &= \frac{1}{2\pi N}\sum_{j=1}^{N}\int dt\; e^{ikt+F_j(t)} \\ &\sim \frac{1}{2\pi}\int dt\int_0^1 dx\;\exp\left\{ikt + p(1-\alpha)x^{-\alpha}(e^{-it}-1)\right\}. \end{aligned} \quad (1.2.17)$$

Then, in the limit $k \to \infty$, we find

$$P(k) \sim \int_0^1 dx\;\delta\left\{k - p(1-\alpha)x^{-\alpha}\right\} = \frac{\{p(1-\alpha)\}^{1/\alpha}}{\alpha}\frac{1}{k^{1+(1/\alpha)}}. \quad (1.2.18)$$

Therefore the exponent λ of the GKK model is equal to $1 + (1/\alpha)$. When we put $\alpha = 1/2$, λ is identified with the corresponding exponent of the BA model, as expected.

1.2.3 Partition function

In order to calculate the spectral densities of the adjacency and Laplacian matrices of the GKK model, let us apply the replica method [Rod88, Bra88] in statistical physics (see Chapter 8). To begin with, we rewrite (1.2.5) in the form

$$\rho(\mu) = \frac{2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\mu}\langle\ln Z(\mu + i\epsilon)\rangle, \quad \epsilon \downarrow 0 \quad (1.2.19)$$

in terms of the partition function

$$Z(\mu) = \int\prod_{j=1}^{N}d\phi_j\;\exp\left(\frac{i}{2}\mu\sum_{j=1}^{N}\phi_j^2 - \frac{i}{2}\sum_{jl}^{N}J_{jl}\phi_j\phi_l\right). \quad (1.2.20)$$

Here J is the adjacency matrix A or the Laplacian matrix L. Since there is a relation

$$\langle\ln Z\rangle = \lim_{n\to 0}\frac{\ln\langle Z^n\rangle}{n}, \quad (1.2.21)$$

we wish to evaluate

$$\langle Z^n\rangle = \int\prod_{j=1}^{N}d\vec{\phi}_j\;\exp\left(\frac{i}{2}\mu\sum_{j=1}^{N}\vec{\phi}_j^2\right)\left\langle\exp\left(-\frac{i}{2}\sum_{jl}^{N}J_{jl}\vec{\phi}_j\cdot\vec{\phi}_l\right)\right\rangle. \quad (1.2.22)$$

In term of the replica variables $\phi_j^{(k)}$, the vector $\vec{\phi}_j$ and the measure $d\vec{\phi}_j$ are defined as

$$\vec{\phi}_j = (\phi_j^{(1)}, \phi_j^{(2)}, \cdots, \phi_j^{(n)}), \quad d\vec{\phi}_j = d\phi_j^{(1)}d\phi_j^{(2)}\cdots d\phi_j^{(n)}. \quad (1.2.23)$$

Let us introduce

$$\tilde{c}_j(\vec{\phi}) = \delta(\vec{\phi} - \vec{\phi}_j) \tag{1.2.24}$$

and an auxiliary function $c_j(\vec{\phi})$ with a normalisation

$$\int d\vec{\phi}\ c_j(\vec{\phi}) = 1. \tag{1.2.25}$$

Then the relation (1.2.14) yields an asymptotic estimate

$$\langle Z^n \rangle \sim \int \prod_{j=1}^{N} d\vec{\phi}_j \int \prod_{j=1}^{N} \mathcal{D}c_j(\vec{\phi}) \prod_{j=1}^{N} \prod_{\vec{\phi}} \delta(c_j(\vec{\phi}) - \tilde{c}_j(\vec{\phi})) e^{S_1 + S_2} \tag{1.2.26}$$

in the limit $N \to \infty$. Here S_1 and S_2 are defined as

$$S_1 = \frac{i}{2}\mu \sum_{j=1}^{N} \int d\vec{\phi}\ c_j(\vec{\phi}) \vec{\phi}^2 \tag{1.2.27}$$

and

$$S_2 = \frac{pN}{2} \sum_{jl}^{N} P_j P_l \int d\vec{\phi} \int d\vec{\psi}\ c_j(\vec{\phi}) c_l(\vec{\psi}) (f(\vec{\phi}, \vec{\psi}) - 1) \tag{1.2.28}$$

with

$$f(\vec{\phi}, \vec{\psi}) = \begin{cases} e^{-i\vec{\phi}\cdot\vec{\psi}}, & \text{if } J \text{ is the adjacency matrix } A, \\ e^{-(i/2)(\vec{\phi}-\vec{\psi})^2}, & \text{if } J \text{ is the Laplacian matrix } L. \end{cases} \tag{1.2.29}$$

Now we consider the asymptotic estimate of

$$\int \prod_{j=1}^{N} d\vec{\phi}_j \prod_{j=1}^{N} \prod_{\vec{\phi}} \delta(c_j(\vec{\phi}) - \tilde{c}_j(\vec{\phi})) = \int \prod_{j=1}^{N} \mathcal{D}a_j(\vec{\phi}) \exp\left(\sum_{j=1}^{N} G_j \right)$$

with

$$G_j = 2\pi i \int d\vec{\phi}\ a_j(\vec{\phi}) c_j(\vec{\phi}) + \ln \int d\vec{\phi}\ e^{-2\pi i a_j(\vec{\phi})} \tag{1.2.30}$$

in the limit $N \to \infty$. The dominant contribution to the functional integral over $a_j(\vec{\phi})$ comes from the extremum satisfying $\delta G_j / \delta a_j = 0$. It follows that the asymptotic estimate of $\langle Z^n \rangle$ is rewritten as

$$\langle Z^n \rangle \sim \int \prod_{j=1}^{N} \mathcal{D}c_j(\vec{\phi}) e^{S_0 + S_1 + S_2} \tag{1.2.31}$$

with

$$S_0 = -\sum_{j=1}^{N} \int d\vec{\phi}\ c_j(\vec{\phi}) \ln c_j(\vec{\phi}). \tag{1.2.32}$$

1.2.4 Extremum condition

Let us next examine the functional integration over $c_j(\vec{\phi})$. It is dominated in the limit $N \to \infty$ by the extremum satisfying

$$\delta \left\{ S_0 + S_1 + S_2 + \sum_{j=1}^{N} \alpha_j \left(\int d\vec{\phi}\; c_j(\vec{\phi}) - 1 \right) \right\} = 0, \tag{1.2.33}$$

where α_j is the Lagrange multiplier ensuring the normalisation of $c_j(\vec{\phi})$. This extremum condition can be rewritten in the form

$$c_j(\vec{\phi}) = \mathcal{A}_j \exp \left\{ \frac{i}{2}\mu\vec{\phi}^2 + pNP_j \sum_{l=1}^{N} P_l \int d\vec{\psi}\; c_l(\vec{\psi})(f(\vec{\phi},\vec{\psi}) - 1) \right\}, \tag{1.2.34}$$

where $\mathcal{A}_j$ is a normalisation constant.

There have been several attempts to solve this extremum condition. In numerical work [Kue08] claimed that a superimposed Gaussian form

$$c_j(\vec{\phi}) = \int d\Pi_j(\omega) \frac{1}{(2\pi/\omega)^{n/2}} \exp\left(-\frac{\omega}{2}\vec{\phi}^2\right) \text{ with } \int d\Pi_j(\omega) = 1 \tag{1.2.35}$$

gives good agreement with the results of numerical diagonalizations.

In the limit of large mean degree p, the extremum condition can be simplified, so that it can be treated analytically [Rod05, Kim07]. In order to see the simplification, one puts a (single) Gaussian ansatz

$$c_j(\vec{\phi}) = \frac{1}{(2\pi i\sigma_j)^{n/2}} \exp\left(-\frac{1}{2i\sigma_j}\vec{\phi}^2\right) \tag{1.2.36}$$

into (1.2.34). It follows in the limit $n \to 0$ that

$$\exp\left(-\frac{1}{2i\sigma_j}\vec{\phi}^2\right) = \mathcal{A}_j \exp\left[\frac{i}{2}\mu\vec{\phi}^2 + pNP_j \sum_{l=1}^{N} P_l \left\{ h_l(\vec{\phi}) - 1 \right\}\right], \tag{1.2.37}$$

where

$$h_l(\phi) = \begin{cases} \exp\left(-\dfrac{i\sigma_l}{2}\vec{\phi}^2\right), & \text{if } J \text{ is the adjacency matrix } A, \\ \exp\left(-\dfrac{i}{2(1-\sigma_l)}\vec{\phi}^2\right), & \text{if } J \text{ is the Laplacian matrix } L. \end{cases} \tag{1.2.38}$$

Let us first consider the spectral density of the adjacency matrix. Introducing the scalings $\mu = O(p^{1/2})$, $\vec{\phi}^2 = O(p^{-1/2})$ and $\sigma_j = O(p^{-1/2})$, we take the limit of large p. Then we see from (1.2.37) that

$$\mu - \frac{1}{\sigma_j} - pNP_j \sum_{l=1}^{N} P_l\sigma_l = 0, \tag{1.2.39}$$

which determines σ_j. It follows from (1.2.19), (1.2.21) and (1.2.31) that

$$\rho(\mu) \sim -\frac{1}{N\pi}\text{Im}\sum_{j=1}^{N}\sigma_j \sim -\frac{1}{\pi}\text{Im}\int_0^1 \sigma(x)dx \tag{1.2.40}$$

gives the spectral density, where $\sigma(x) = \sigma_j$ with $x = j/N$.

Let us define the scaled variables $s(x) = \sqrt{p}\ \sigma(x)$ and $E = \mu/\sqrt{p}$. Then, using (1.2.11), we can rewrite (1.2.39) as

$$s(x)x^{-\alpha} = \frac{1}{Ex^{\alpha} - c} \text{ with } c = (1-\alpha)^2 \int_0^1 s(x)x^{-\alpha}dx, \tag{1.2.41}$$

from which $s(x)$ is evaluated and the asymptotic expansions of the spectral density are derived. The results are [Rod05]

$$\rho(\mu) = \frac{1}{\pi\sqrt{p}}\left\{\frac{1}{1-\alpha^2} - \frac{1+5\alpha+18\alpha^2+20\alpha^3+16\alpha^4}{8(1-\alpha^2)^3(1+2\alpha)(1+3\alpha)}|E|^2\right\} + O(|E|^4) \tag{1.2.42}$$

for small $|E|$ and

$$\rho(\mu) \sim \frac{2}{\sqrt{p}}\frac{(1-\alpha)^{1/\alpha}}{\alpha}\frac{1}{E^{1+(2/\alpha)}} \tag{1.2.43}$$

for large E. Therefore the exponent γ is $1 + (2/\alpha)$, so that the relation with $\lambda = 1 + (1/\alpha)$ is $\gamma = 2\lambda - 1$, as anticipated from the analysis of locally tree-like networks [Dor03, Dor04].

We next analyse the spectral density of Laplacian matrices. Let us adopt the scalings $\mu = O(p)$, $\vec{\phi}^2 = O(p^{-1})$ and $\sigma_j = O(p^{-1})$. Then it follows from (1.2.37) that

$$\sigma_j = \frac{1}{\mu + i\epsilon - pNP_j}, \quad \epsilon \downarrow 0. \tag{1.2.44}$$

In terms of the scaled variable $\omega = \mu/(p(1-\alpha))$, the spectral density $\rho(\mu)$ of the Laplacian matrix is derived from (1.2.40) and (1.2.44) as [Kim07]

$$\rho(\mu) = \begin{cases} \dfrac{\omega^{-1-(1/\alpha)}}{p\ \alpha(1-\alpha)}, & \omega > 1, \\ 0, & \omega < 1. \end{cases} \tag{1.2.45}$$

The weighted versions of the adjacency and Laplacian matrices, such as the weighted Laplacian matrix

$$W_{jl} = \frac{L_{jl}}{(\langle k_j\rangle\langle k_l\rangle)^{\beta/2}} \tag{1.2.46}$$

with an exponent β, can be analysed [Kim07] in a similar way in the limit $p \to \infty$.

1.2.5 Effective medium approximation

In order to approximately treat a network with a finite mean degree p, we can employ a useful scheme called the effective medium approximation (EMA) [Sem02, Nag07, Nag08]. In the EMA, one substitutes the (single) Gaussian ansatz (1.2.36) into (1.2.27),(1.2.28) and (1.2.32) and consider the variational equation

$$\frac{\partial}{\partial \sigma_j}(S_0 + S_1 + S_2) = 0. \tag{1.2.47}$$

For the spectral density of adjacency matrices, the variational equation (1.2.47) turns into

$$\mu - \frac{1}{\sigma_j} - NpP_j \sum_{l=l}^{N} \frac{P_l \sigma_l}{1 - \sigma_j \sigma_l} = 0 \tag{1.2.48}$$

in the limit $n \to 0$. This equation can be solved by a numerical iteration method. Then we put the solution into (1.2.40) and obtain the EMA spectral density. The result is compared in Figure 1.1 with the spectral density of numerically generated adjacency matrices (averaged over 100 samples, $N = 1000$, $\alpha = 1/2$ and $p = 12$). The agreement is fairly good except around the origin.

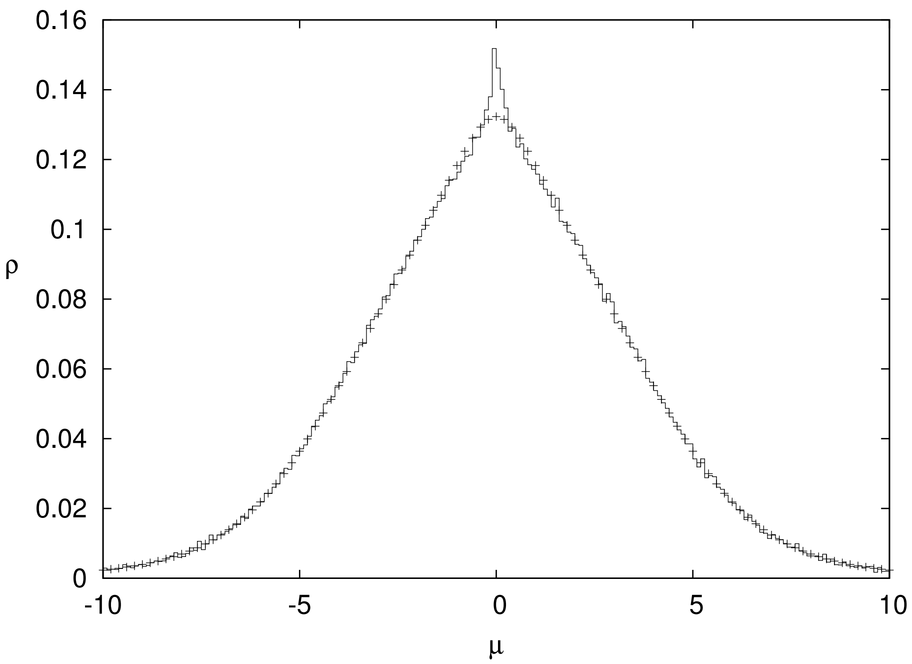

Figure 1.1: The EMA spectral density (+) and the spectral density of numerically generated adjacency matrices (histogram) with $\alpha = 1/2$ and $p = 12$.

In the limit $p \to \infty$ with a scaling $\sigma_j = O(p^{-1/2})$, the EMA equation (1.2.48) becomes the extremum condition (1.2.39), as expected. Then one can formulate a perturbative method to analytically calculate the finite p correction to the solution of (1.2.39). The result for the spectral density takes the form

$$\rho(\mu) = \frac{1}{\sqrt{p}} \left\{ \rho_0(\mu) + \frac{1}{p}\rho_1(\mu) + O\left(\frac{1}{p^2}\right) \right\}. \tag{1.2.49}$$

For large E, we have already seen in equation (1.2.43) that the unperturbed term $p^{-1/2}\rho_0(\mu)$ is $O(E^{-1-(2/\alpha)})$, whereas the first order correction $p^{-3/2}\rho_1(\mu)$ turns out [Nag08] to be $O(E^{-3-(2/\alpha)})$. Hence the spectral density is dominated by the unperturbed term in the limit $E \to \infty$.

In the limiting case $\alpha = 0$, one obtains the adjacency matrices of classical random graphs [Erd60, Rod88, Bra88]. Then the dependence of σ_j on j can be omitted so that σ_j is set to be σ. The EMA equation becomes a simple cubic equation [Sem02]

$$\mu\sigma^3 + (p-1)\sigma^2 - \mu\sigma + 1 = 0 \tag{1.2.50}$$

and gives a closed analytic solution for the spectral density. In the limit $p \to \infty$ we obtain a semi-circular density

$$\rho(\mu) = \begin{cases} \dfrac{\sqrt{4p-\mu^2}}{2\pi p}, & -2\sqrt{p} < \mu < 2\sqrt{p}, \\ 0, & \mu < -2\sqrt{p} \text{ or } \mu > 2\sqrt{p}, \end{cases} \tag{1.2.51}$$

as expected from the theory of random matrices.

In the case of Laplacian matrices, the variational equation (1.2.47) takes the form

$$\mu - \frac{1}{\sigma_j} - NpP_j \sum_{l=1}^{N} \frac{P_l}{1-\sigma_j-\sigma_l} = 0. \tag{1.2.52}$$

As before, the EMA spectral density can be evaluated from this equation by a numerical iteration method. It is compared with the spectral density of numerically generated Laplacian matrices (averaged over 100 samples, $N = 1000$, $\alpha = 1/2$ and $p = 12$) in Figure 1.2.

In the limiting case of classical random graphs ($\alpha = 0$), we find a quadratic equation

$$2\mu\sigma^2 + (p-\mu-2)\sigma + 1 = 0, \tag{1.2.53}$$

which yields the EMA spectral density

$$\rho(\mu) = \begin{cases} \dfrac{\sqrt{8p-(\mu-p-2)^2}}{4\pi\mu}, & p_- < \mu < p_+, \\ 0, & \mu < p_- \text{ or } \mu > p_+ \end{cases} \tag{1.2.54}$$

with $p_\pm = p \pm 2\sqrt{2p} + 2$.

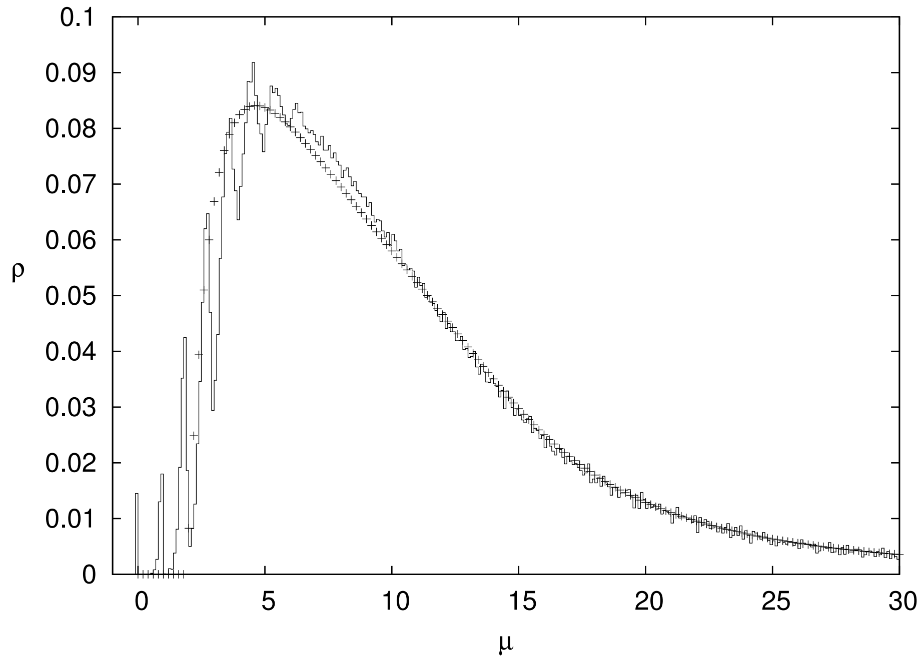

Figure 1.2: The EMA spectral density (+) and the spectral density of numerically generated Laplacian matrices (histogram) with $\alpha = 1/2$ and $p = 12$.

1.3 Local properties

In the application of random matrices to quantum physics, it is known that the local spectral distributions are universal even if the global spectral density depends on the details of each system. It is interesting to see if such strong universality also holds for the matrices associated with complex networks.

Chung, Lu and Vu rigorously analysed the largest eigenvalues of the adjacency matrices of scale free networks [Chu03a, Chu03b]. A variant of the static model was used in their work. If the exponent of the degree distribution is $\lambda > 5/2$, their result claims under certain conditions that the largest eigenvalues have power law distributions with the exponent $2\lambda - 1$.

Bandyopadhyay and Jalan, on the other hand, numerically analysed the eigenvalue correlation of the adjacency matrices of scale free as well as small world networks [Ban07, Jal07, Jal08]. The growth process of the BA model was used to generate scale free networks. Their results show that the eigenvalue spacing distributions and the spectral rigidity follow the predictions of Gaussian Orthogonal Ensemble (GOE) of random matrices (see Chapter 13). In the case of small world networks, they used the Watts-Strogatz model, in which a regular ring lattice is randomised [Wat98]. A transition toward the GOE behaviour is observed as a function of the fraction of randomised edges. Very recently [Car09], the deformation in the eigenvalue spacing distribution described in Chapter 13 has been found empirically in a small world network.

Finally, in [Mit08] Mitrovic and Tadic studied numerically the spectral properties of the adjacency and Laplacian matrices in a wide class of scale free networks with mesoscopic subgraphs. They identify signals of cyclic mesoscopic structures in the spectra. For instance, the centre of the spectra is effected by minimally connected nodes and the number of distinct modules leads to additional peaks in the Laplacian spectra.

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