A NOTE ON A CAMOUFLAGE PURSUIT PROBLEM

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ABSTRACT. Motion camouflage is a pursuit strategy whereby a predator moves towards a prey while appearing stationary to the prey except for the change in its perceived cross section as it approaches. If the effect of cross section size with distance is ignored then this means that the target is unable to discern that the aggressor is moving . The aggressor appears to be at its initial position or is camouflaged by a stationary object in the background. We shall derive a closed form solution to the problem of camouflage pursuit for a particular situation. Although general differential equations have already been derived for this strategy they have not been solved in closed form.

1. INTRODUCTION

Motion camouflage is a pursuit strategy whereby a predator moves towards a prey while appearing stationary to the prey except for the change in its perceived cross section as it approaches. Throughout the pursuit the pursuer remains between the prey and a stationary point, which the pursuer is using to camouflage itself. If the effect of size with distance is ignored then this means that the target is unable to discern that the aggressor is moving. The aggressor appears to be at its initial position or is camouflaged by a stationary object in the background. This type of pursuit strategy is observed in insects like dragonflies. Recently, an interesting article by Duncan Graham-Rowe [1] in the New Scientist, has explained why this strategy is now also seen as a very effective way of using missile defence systems. In two papers Justh and Krishnaprasad [4] and Wei, Justh, and Krishnaprasad [5] deal with the biological steering laws and the various pursuit strategies that are used by insects and defence systems in detail as well as giving a good overview of the subject and relevant references to the literature.They derive some approximate mathematical results which are based on these steering laws. In a series of very interesting articles Glendinning [2], [3] discussed the hidden or camouflage pursuit problem from a mathematical point of view. He was able to derive the general differential equation that described the path of the pursuer in cartesian coordinates; and then numerically solve the differential equation to confirm the effectiveness of the camouflage pursuit in comparison to other methods of pursuit like direct-line-of-sight-pursuit that is used by heat seeking missiles. He showed that there is no known closed form solution of the equations as a function of time; here we show that by passing to intrinsic coordinates, closed form solutions to the phase curves can be obtained. We shall consider a particular pursuit situation and derive a closed form solution in quadratures which can be put in terms of an elliptic integral of the second kind. This will be achieved by using intrinsic coordinates to formulate

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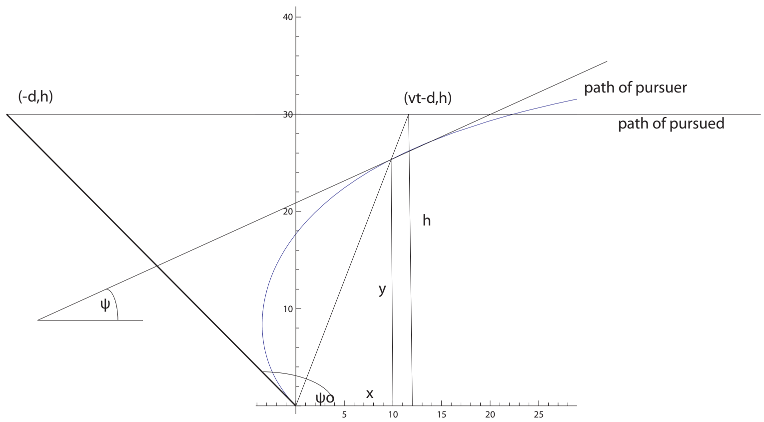

FIGURE 1. Geometry of the path of the target and the pursuer.

and solve the differential equation problem. The solution for the pursuit path and the time and distance can be quickly calculated and displayed by using a simple Mathematica programme.

2. Formulation of particular pursuit problem

The specific two dimensional pursuit problem we shall consider takes place in a vertical plane with cartesian coordinates (x, y). A target(prey) is moving in a straight line $y = h[m]$ with a constant speed $v[ms^{-1}]$. This target is being pursued by a predator moving with a constant speed $u[ms^{-1}]$. Initially, at time $t = 0$, the predator is located at the origin $(0, 0)$ and the prey is located at the point $(-d, h), d > 0, h > 0$. The initial angle of elevation between the predator at $(0, 0)$ and the prey at $(-d, h)$ is $\pi - \psi_0$. For $t > 0$ the pursuer pursues the target employing the camouflage strategy. Thus throughout the pursuit the pursuer remains between the prey and the initial stationary point $(0, 0)$ (against which the pursuer is camouflaging itself). The geometry of the situation is shown in Figure 1.

Let the position of the pursuer at time t be (x, y) then from the Figure 1 we have by similar triangles that

$$\frac{h}{vt - d} = \frac{y}{x}. \tag{1}$$

Now differentiating with respect to time t, and using Newton's notation $\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$, gives

$$\frac{v}{h} = \frac{\dot{x}y - x\dot{y}}{y^2},$$

or

$$\frac{vy^2}{h} = \dot{x}y - x\dot{y}.$$

Now differentiate again with respect to time gives

$$2\frac{v}{h}y\dot{y} = \ddot{x}y - \ddot{y}x = \ddot{x} - \ddot{y}\frac{vt-d}{h}. \tag{2}$$

3. Solution of the differential equation for path of pursuit

We shall now convert this equation (2) into intrinsic coordinates ψ and s, where ψ is the angle that the tangent to the pursuing curve makes with the x-axis and s is the distance traveled by the pursuer at time t, see Figure 1. Note that $0 < \psi < \psi_0 < \pi$. The relationship between the intrinsic coordinates and cartesian coordinates is as follows

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \cos\psi\dot{s} = u\cos\psi.$$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \sin\psi\dot{s} = u\sin\psi.$$

Thus

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d(u\cos\psi)}{dt} = -u\sin\psi\dot{\psi}.$$

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d(u\sin\psi)}{dt} = -u\cos\psi\dot{\psi}.$$

By substituting these results into the equation (2) we get

$$2\frac{v}{h}u\sin\psi = -u\sin\psi\dot{\psi} - \frac{vt-d}{h}u\cos\psi\dot{\psi},$$

or by rearranging we get the basic differential equation

$$\frac{dt}{d\psi} + \frac{1}{2}t\cot\psi = -\frac{h}{2v} + \frac{1}{2}\frac{d}{v}\cot\psi. \tag{3}$$

This first order differential equation (3) can be integrated by multiplying across by the integrating factor $\sqrt{\sin\psi}$. Thus we get

$$\frac{d(t\sqrt{\sin\psi})}{d\psi} = -\frac{h\sqrt{\sin\psi}}{2v} + \frac{1}{2}\frac{d}{v}\frac{\cos\psi}{\sqrt{\sin\psi}}.$$

Integrating from an initial angle $\psi_0 (0 < \psi_0 < \pi)$ when $s = 0$ at $t = 0$ we have

$$t\sqrt{\sin\psi} = -\frac{h}{2v}\int_{\psi_0}^{\psi}\sqrt{\sin\nu}d\nu + \frac{1}{2}\frac{d}{v}\int_{\psi_0}^{\psi}\frac{d(\sin\nu)}{\sqrt{\sin\nu}}.$$

The time passed by the pursuer is thus given by

$$t = -\frac{h}{2v\sqrt{\sin\psi}}\int_{\psi_0}^{\psi}\sqrt{\sin\nu}d\nu + \frac{d}{v}[1 - \frac{\sqrt{\sin\psi_0}}{\sqrt{\sin\psi}}],$$

and by using the fact that $s = ut$ the distance traveled by the pursuer in this time t is given by

$$s(\psi) = \frac{uh}{2v\sqrt{\sin\psi}}\int_{\psi}^{\psi_0}\sqrt{\sin\nu}d\nu + \frac{ud}{v}[1 - \frac{\sqrt{\sin\psi_0}}{\sqrt{\sin\psi}}],$$

and since, $d = h\cot(\pi - \psi_0)$ then

$$s(\psi) = \frac{uh}{2v\sqrt{\sin\psi}}\int_{\frac{\pi}{2}-\psi_0}^{\frac{\pi}{2}-\psi}\sqrt{\cos\nu}d\nu - \frac{uh\cot\psi_0}{v}[1 - \frac{\sqrt{\sin\psi_0}}{\sqrt{\sin\psi}}].$$

This last expression is also the intrinsic equation of the path of pursuit. It is also worth noting that since

$$\int_0^{\varphi}\sqrt{\cos w}dw = \int_0^{\varphi}\sqrt{1-2(\sin w/2)^2}dw = 2\int_0^{\varphi/2}\sqrt{1-2(\sin w)^2}dw = 2E[\varphi/2, 2],$$

where $E[\phi, m]$ is the incomplete elliptic integral of the second kind, then we can also represent these expressions for time and path length in the form

$$(4) \qquad t(\psi) = -\frac{h}{v\sqrt{\sin\psi}}(E[\frac{\pi}{4} - \frac{\psi}{2}, 2] - E[\frac{\pi}{4} - \frac{\psi_0}{2}, 2]) - \frac{h\cot\psi_0}{v}[1 - \frac{\sqrt{\sin\psi_0}}{\sqrt{\sin\psi}}].$$

$$(5) \qquad s(\psi) = \frac{uh}{v\sqrt{\sin\psi}}(E[\frac{\pi}{4} - \frac{\psi}{2}, 2] - E[\frac{\pi}{4} - \frac{\psi_0}{2}, 2]) - \frac{uh\cot\psi_0}{v}[1 - \frac{\sqrt{\sin\psi_0}}{\sqrt{\sin\psi}}].$$

4. Some properties of the solution

The cartesian equation for of the path of pursuit is given by using the parametric representation

$$x = \int_0^s \cos\psi ds = \int_{\psi_0}^{\psi}\cos w\frac{ds(w)}{dw}dw.$$

$$y = \int_0^s \sin\psi ds = \int_{\psi_0}^{\psi}\sin w\frac{ds(w)}{dw}dw.$$

In particular since

$$(6) \qquad \begin{aligned} \frac{ds(w)}{dw} &= \frac{uh}{2v}\frac{d}{dw}(\frac{1}{\sqrt{\sin w}}\int_w^{\psi_0}\sqrt{\sin\mu}d\mu - 2\cot\psi_0[1 - \frac{\sqrt{\sin\psi_0}}{\sqrt{\sin\psi}}]), \\ &= -\frac{uh}{2v}(1 + \frac{\cos w}{2(\sin w)^{\frac{3}{2}}}\int_w^{\psi_0}\sqrt{\sin\mu}d\mu + \frac{\cos\psi_0\cos w}{\sqrt{\sin\psi_0}(\sin w)^{\frac{3}{2}}}); \end{aligned}$$

then we get

$$y = -\frac{uh}{2v}\int_{\psi_0}^{\psi}(\sin w + \frac{\cos w}{2\sqrt{\sin w}}\int_w^{\psi_0}\sqrt{\sin\mu}d\mu + \frac{\cos\psi_0\cos w}{\sqrt{\sin\psi_0}\sqrt{\sin w}})dw,$$

which on carrying out a few integrations gives

$$y = \frac{uh}{2v}(\cos\psi - \cos\psi_0) - \frac{uh}{4v}\int_{\psi_0}^{\psi}\frac{\cos w}{\sqrt{\sin w}}\int_w^{\psi_0}\sqrt{\sin\mu}d\mu dw + \frac{uh\cos\psi_0}{v}[1 - \frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}].$$

By making the change of integration variable $\mu = \frac{\pi}{2} - \sigma$ we get

(7)

$$y = \frac{uh}{2v}(\cos\psi - \cos\psi_0) + \frac{uh}{4v}\int_{\psi}^{\psi_0}\frac{\cos w}{\sqrt{\sin w}}\int_{\frac{\pi}{2}-\psi_0}^{\frac{\pi}{2}-w}\sqrt{\cos\sigma}d\sigma dw + \frac{uh\cos\psi_0}{v}[1 - \frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}].$$

(8)

$$y = \frac{uh}{2v}(\cos\psi - \cos\psi_0) + \frac{uh}{2v}\int_{\psi}^{\psi_0} \frac{\cos w}{\sqrt{\sin w}}(E[\frac{\pi}{4} - \frac{w}{2}] - E[\frac{\pi}{4} - \frac{\psi_0}{2}])dw + \frac{uh\cos\psi_0}{v}[1 - \frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}].$$

We can also interchange the order of integration in the double integral in the expression (7) to give the equivalent representation

$$y = \frac{uh}{2v}(\cos\psi - \cos\psi_0) + \frac{uh}{4v}\int_{\frac{\pi}{2}-\psi_0}^{\frac{\pi}{2}-\psi} \sqrt{\cos\sigma}d\sigma \int_{\psi}^{\frac{\pi}{2}-\sigma} \frac{\cos w}{\sqrt{\sin w}}dw + \frac{uh\cos\psi_0}{v}[1 - \frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}],$$

$$y = \frac{uh}{v}(\cos\psi - \cos\psi_0) - \frac{uh}{2v}\sqrt{\sin\psi}\int_{\frac{\pi}{2}-\psi_0}^{\frac{\pi}{2}-\psi} \sqrt{\cos\sigma}d\sigma + \frac{uh\cos\psi_0}{v}[1 - \frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}].$$

$$(9) \quad y = \frac{uh}{v}(\cos\psi - \cos\psi_0\frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}) - \frac{uh}{2v}\sqrt{\sin\psi}(E[\frac{\pi}{4} - \frac{\psi}{2}, 2] - E[\frac{\pi}{4} - \frac{\psi_0}{2}, 2]).$$

It is interesting to note that by eliminating y between (8) and (9) we derive an interesting identity for the integral of an elliptic function in terms of an elliptic function.

Clearly from the geometry of the problem for an initial angle $\psi_0 = \pi/2$ then ψ decreases monotonically as the distance s increases. This is not obvious for an arbitrary initial angle of attack. This can be proved analytically as follows. From the expression (6) we need to prove that for $0 < \psi < \psi_0 < \pi$

$$\frac{\cos\psi}{2(\sin\psi)^{\frac{3}{2}}}F(\psi) > 0,$$

where

$$F(\psi) = \int_{\psi}^{\psi_0} \sqrt{\sin\mu}d\mu + 2\frac{(\sin\psi)^{\frac{3}{2}}}{\cos\psi} + 2\frac{\cos\psi_0}{\sqrt{\sin\psi_0}}.$$

Then equivalently we need to prove that $F(\psi) > 0$ for $0 < \psi < \psi_0 < \pi/2$; and that $F(\psi) < 0$ for $\pi/2 < \psi < \psi_0 < \pi$. To this end we take the derivative of $F(\psi)$ giving

$$F'(\psi) = -\sqrt{\sin\psi} + 3\sqrt{\sin\psi} + \frac{2(\sin\psi)^{\frac{5}{2}}}{(\cos\psi)^2} = \frac{2\sqrt{\sin\psi}}{(\cos\psi)^2} > 0.$$

Thus $F(\psi)$ is a monotonic increasing function of ψ in the range $0 < \psi < \pi$.
For $0 < \psi < \psi_0 < \pi/2$ then

$$\int_0^{\psi_0} \sqrt{\sin\nu}d\nu + 2\frac{\cos\psi_0}{\sqrt{\sin\psi_0}} < F(\psi) < \frac{2}{\cos\psi_0\sqrt{\sin\psi_0}};$$

hence $F(\psi) > 0$.
For $\pi/2 < \psi < \psi_0 < \pi$ then

$$-\infty < F(\psi) < \frac{2}{\cos\psi_0\sqrt{\sin\psi_0}};$$

hence $F(\psi) < 0$. We also remark that from the expression (9) we can determine the angle ψ if capture occurs. Clearly capture will occur when $y = h$. Thus from (9) we have the analytic condition for capture is that the equation:

(10)

$$G(\psi) = -h + \frac{uh}{v}(\cos\psi - \cos\psi_0) + \frac{uh}{2v}\sqrt{\sin\psi}\int_{\psi_0}^{\psi}\sqrt{\sin\nu}\,d\nu + \frac{uh\cos\psi_0}{v}[1 - \frac{\sqrt{\sin\psi}}{\sqrt{\sin\psi_0}}] = 0,$$

has a root. We shall first show that $G(\psi)$ is a monotonic decreasing function of ψ for $0 < \psi < \psi_0 < \pi$. To this end it can be shown that

$$G'(\psi) = -\frac{uh\cos\psi}{4v\sin\psi}(2\frac{(\sin\psi)^{\frac{3}{2}}}{\cos\psi} + 2\frac{\cos\psi_0}{\sqrt{\sin\psi_0}} + \int_{\psi}^{\psi_0}\sqrt{\sin\nu}\,d\nu), \tag{11}$$

$$G'(\psi) = -\frac{uh\cos\psi}{4v\sin\psi}F(\psi).$$

Thus by using the properties of $F(\psi)$ derived above, the monotonicity of $G(\psi)$ follows. We also note that

$$-h < G(\psi) < \frac{h}{v}(u - v).$$

It follows that there will only be one root for $u \geq v$ and none for $u < v$. Hence provided the pursuit can carry on without time or energy constraint the predator will capture the prey for $u \geq v$, but the prey will escape capture for $u < v$. Knowing the above properties we can dictate the capture distance in terms of ψ. For example we can plot the graph of $s(\psi)$ (for a given angle of prey sighting ψ_0), against ψ by using (5), see Figure 2, and read off the value of ψ corresponding to a required distance of capture s. Alternatively, from an analytic point of view for a given angle ψ_0 the capture angle ψ is given by solving the transcendental equation (10), $G(\psi) = 0$.

5. Numerical and graphical results

Some graphs of a few situations, where capture or interception of target by the pursuer occurs at the point of intersection of the straight line of the target and the curved path of the pursuer, are shown below by using Mathematica. In Figure 2 the plot of the distance $s(\psi)$ against ψ is given for $u = 1.1, v = 1, h = 100, \psi_0 = 3\pi/4$; the corresponding target line and curve of pursuit with the capture intersection shown is given in Figure 3. In Figures 4 to 6 the paths of pursuit for values of the parameter $u = 1, u = 1.1, u = 2$ are given for $\psi_0 = 7\pi/8, 6\pi/8, ...\pi/8$ and $v = 1$. It can be seen that the larger the initial angle ψ_0 that the pursuer initiates the pursuit the sooner the target is intercepted. In Figure 7 the graph of $v = 1, \psi_0 = 3\pi/4$ for the various values of $u = 1, 2, ...7$ are given. In figure 8 the plots of the pursuit paths for direct-line-of-sight and camouflage pursuit are shown. The curve for the direct-line-of-sight-pursuit is calculated from the paper of Eliezer and Barton [6]. It can be seen that the camouflage pursuit path captures the prey before the direct-line-of-sight-pursuit path. We notice from these graphs that the target is captured sooner the earlier the pursuer sees, and starts its pursuit,that is, the larger the angle ψ_0.

6. Conclusions

We have derived an exact solution to a camouflage pursuit problem which will be useful in comparing approximate solutions obtained by other means. Although the solution is ostensibly in two dimensions in real life where three dimensions is applicable it is suggested that pursuit will take place in a plane formed by the line of the target and the initial point of location of the prey. This could be achieved by predator reorientation by swivelling or realigning him/herself appropriately. This would need to be observed in nature to be verified. From my own experience on approaching ponds dragonflies on spotting me do realign themselves to offer the smallest cross section. To the authors knowledge this is the only know exact solution to a camouflage pursuit problem. We have derived some important analytic properties of the solution. By using the exact solution we confirm numerically the efficacy of the camouflage pursuit over the direct-line-of-sight-pursuit as a capture mechanism. Thus the methods of pursuit used by insects such as dragon flies to capture prey is more effective than heat seeking missiles that are used in defence systems. This is another case Darwin's survival of the fittest; where insects in nature have improvised a sophisticated pursuit mechanisms for survival of their species. Finally we remark it would be interesting to consider the more complex problem of finding the initial angle ψ_0 that is required to achieve capture for a given limitation on the distance $s(\psi)$ (and hence energy) of travel by the pursuer.

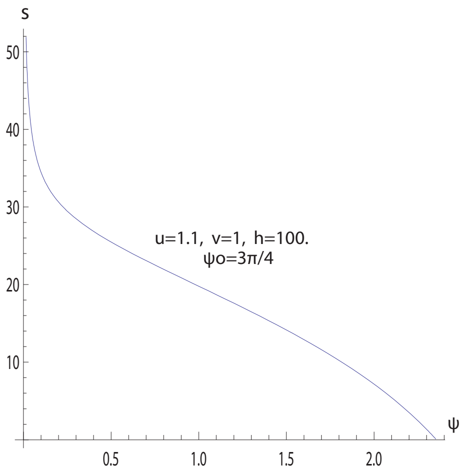

FIGURE 2. Graph of the distance $s(\psi)$ against ψ for $u = 1.1, v = 1, h = 100, \psi_0 = 3\pi/4$.

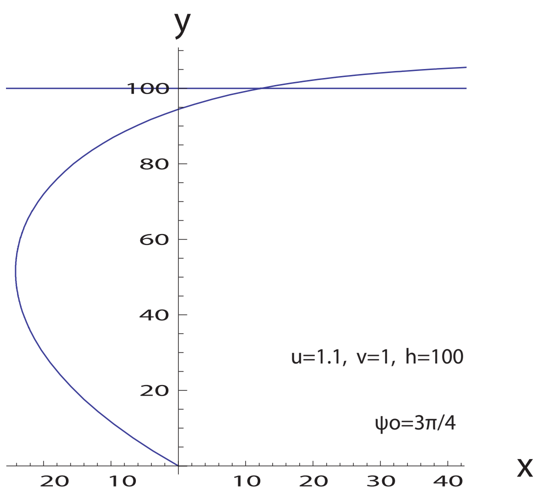

FIGURE 3. The pursuit curve and the straight line path of the target for $u = 1.1, v = 1, h = 100, \psi_0 = 3\pi/4$.

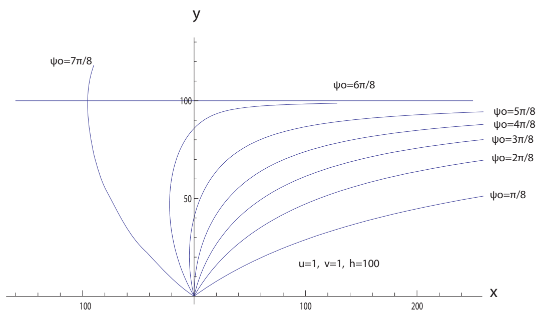

FIGURE 4. The paths of pursuit for $u = 1, v = 1, h = 100$ and various initial angles of sighting ψ_0.

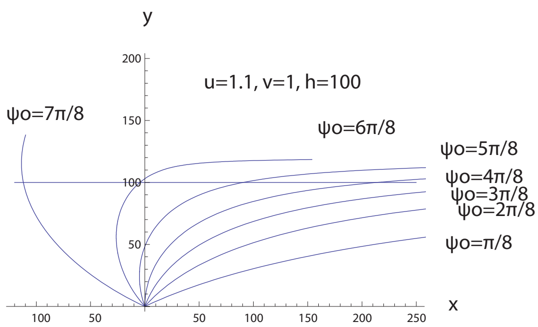

FIGURE 5. The paths of pursuit for $u = 1.1, v = 1, h = 100$ and various initial angles of sighting ψ_0.

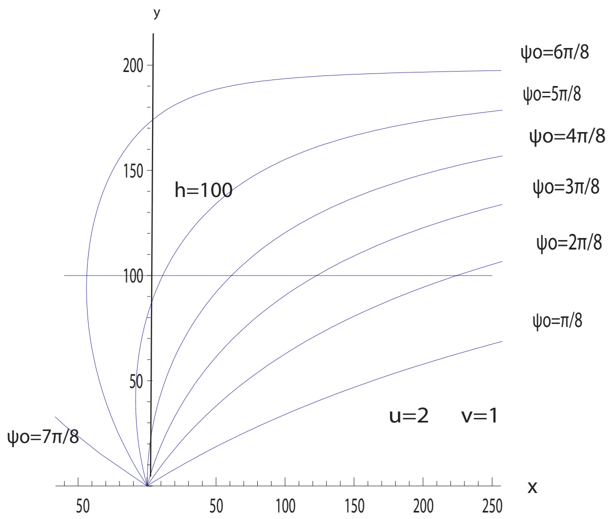

FIGURE 6. The paths of pursuit for $u = 2, v = 1, h = 200$ and various initial angles of sighting ψ_0.

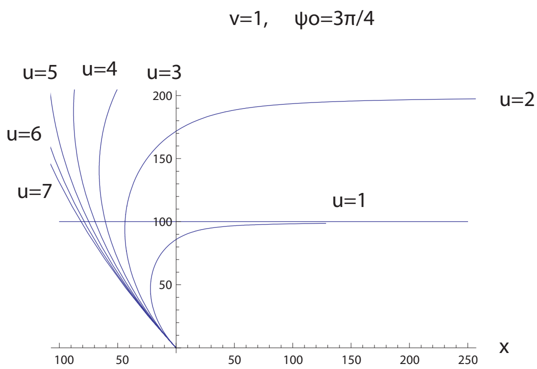

FIGURE 7. The paths of pursuit for $v = 1, h = 100$ and initial angle of sighting $\psi_0 = 3\pi/4$ for various values of pursuer speed u.

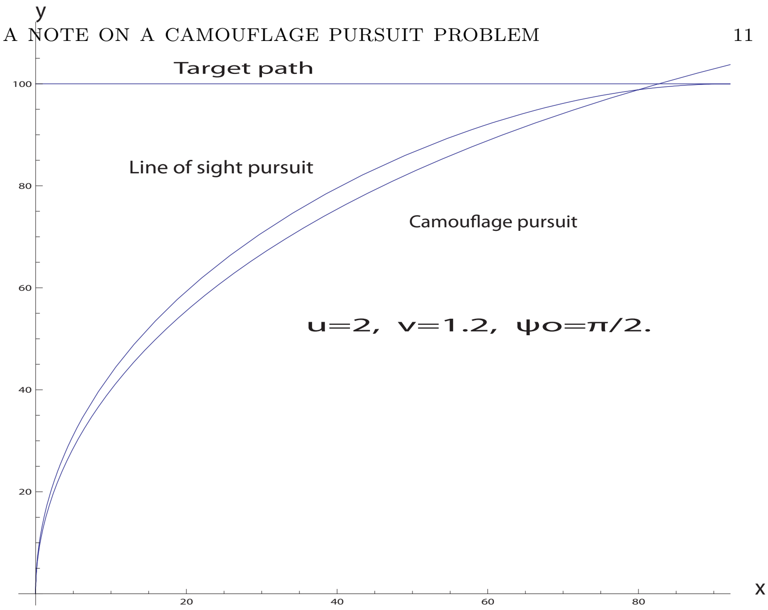

FIGURE 8. The paths of pursuit for $u = 2, v = 1.2, h = 100$ and initial angle of sighting $\psi_0 = \pi/2$, for camouflage and line-of-sight pursuit.

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