Robust H_∞ Control for A Class of Nonlinear Discrete Time-Delay Stochastic Systems with Missing Measurements ⋆

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Abstract

This paper is concerned with the problem of robust H_∞ output feedback control for a class of uncertain discrete-time delayed nonlinear stochastic systems with missing measurements. The parameter uncertainties enter into all the system matrices, the time-varying delay is unknown with given low and upper bounds, the nonlinearities satisfy the sector conditions, and the missing measurements are described by a binary switching sequence that obeys a conditional probability distribution. The problem addressed is the design of an output feedback controller such that, for all admissible uncertainties, the resulting closed-loop system is exponentially stable in the mean square for the zero disturbance input and also achieves a prescribed H_∞ performance level. By using the Lyapunov method and stochastic analysis techniques, sufficient condition are first derived to guarantee the existence of the desired controllers, and then the controller parameters are characterized in terms of linear matrix inequalities (LMIs). A numerical example is exploited to show the usefulness of the results obtained.

Key words: Discrete nonlinear stochastic system; H_∞ output feedback control; Time-varying delays; Missing measurements; Lyapunov-Krasovskii functional; Linear matrix inequality.

1 Introduction

Time delay, stochasticity and nonlinearity are arguably three of the main sources that contribute to the complexity of a system. First, various engineering systems (e.g. electrical networks, turbojet engines, microwave oscillators, nuclear reactors and hydraulic systems) have the characteristics of time delay in signal transmissions. So far, the stability analysis and robust control for dynamic time-delay systems have attracted a number of researchers over the past years. Second, as stochastic modeling has had extensive applications in control and communication problems, the stability analysis problem for linear stochastic *time-delay* systems has been studied by many authors, see e.g. [2, 7, 8, 13, 19]. Third, it is well known that nonlinearities exist universally in practical systems, and therefore nonlinear control has been an ever hot topic in the past few decades. It is worth mentioning that, among different descriptions of the nonlinearities, the so-called *sector nonlinearity* [14] has gained much attention for *deterministic* systems, and both the control analysis and model reduction problems have been investigated, see [11, 15].

Recently, the control problem of nonlinear stochastic systems has stirred renewed research interests, and a variety of nonlinear stochastic systems have been investigated by different approaches [3, 4, 16, 28]. Most recently, in [1, 22, 30], an H_∞-type theory has been developed for a large class of continuous- and discrete-time nonlinear stochastic systems. As for nonlinear stochastic time-delay systems, the relevant literature has not been very much. Most of the existing results have been only concerned with the stability analysis issue, see e.g. [5, 29]. The performance indices of the controlled systems, such as the robustness and disturbance rejection attenuation

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level, have not received enough attention despite their importance in practical applications.

In almost all literature mentioned above, the assumption of consecutive measurements has been made implicitly. Unfortunately, in many practical applications, such an assumption does not hold. For example, due to sensor temporal failure or network transmission delay/loss, at certain time points, the system measurement may contain *noise only*, indicating that the real signal is missing. Such a phenomenon is also called dropout or intermittence in networked control systems, see e.g. [23, 26]. One of the most popular ways to describe the missing measurement is to view it as a Bernoulli distributed (binary switching) white sequence specified by a conditional probability distribution in the output equation. The Bernoulli distribution description was first proposed in [18] to deal with the optimal recursive filtering problem, and has then been used in [12, 23–26] for various control and filtering problems of *linear* systems with probabilistic measurement losses. However, so far, the robust H_∞ output feedback control problem for *nonlinear stochastic time-delayed* systems with missing measurements has not been fully investigated, especially for discrete-time cases, which motivates us to shorten such a gap in the present investigation.

In this paper, we consider the robust H_∞ control problem for a class of uncertain discrete stochastic time-delay systems involving sector nonlinearities and missing measurements. The parameter uncertainties are assumed to be norm-bounded, the delays are time-varying, and the sector nonlinearities appear in the system states and delayed states. Similar to [12, 23–26], the missing measurements are characterized as a binary switching sequence satisfying a conditional probability distribution. An effective linear matrix inequality (LMI) approach is proposed to design the output feedback controllers such that, for all admissible nonlinearities, time-delays and probabilistic measurement missing, the overall uncertain closed-loop system is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We first establish the sufficient conditions for the uncertain nonlinear stochastic time-delay systems to be stable in the mean square, and then derive the explicit expression of the desired controller gains. A numerical example is provided to show the usefulness and effectiveness of the proposed design method.

Notations: Throughout this paper, $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^+$ stands for the set of nonnegative integers; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. $l_2[0,\ \infty)$ denotes the space of square summable infinite vector sequences. The notation $\|\cdot\|$ stands for the usual $l_2[0,\ \infty)$ norm while $|\cdot|$ refers to the Euclidean vector norm. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2 Problem Formulation

Consider the following uncertain discrete nonlinear system with time delays of the form:

$$
\begin{aligned}
(\Sigma') : x(k+1) =& A(k)x(k) + A_d(k)x(k-d(k)) \\
&+ E(k)f(x(k)) + E_d(k)f_d(x(k-d(k))) \\
&+ B_1u(k) + D_1w(k), \\
\tilde{y}(k) =& Cx(k) + C_dx(k-d(k)) + \phi(Sx(k)), \\
z(k) =& Lx(k) + B_2u(k), \\
x(j) =& \psi(j),\ j = -d_M, -d_M+1, ..., -1, 0,
\end{aligned}
$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, $z(k) \in \mathbb{R}^p$ is the controlled output, and $w(k) \in \mathbb{R}^q$ is the exogenous disturbance signal which is assumed to belong to $l_2[0, \infty)$.

For the system (Σ'), the positive integer $d(k)$ denotes the time-varying delay satisfying

$$
d_m \leq d(k) \leq d_M,\ k \in \mathbb{N}^+ \tag{1}
$$

where d_m and d_M are known positive integers. $\psi(j)$ $(j = -d_M, -d_M+1, ..., -1, 0)$ are the initial conditions. $B_1, B_2, C, C_d, D_1, D_2, S$ and L are known real constant matrices, and the matrices $A(k), A_d(k), E(k)$ and $E_d(k)$ are time-varying matrices of the form $A(k) = A+\Delta A(k)$, $A_d(k) = A_d + \Delta A_d(k)$, $E(k) = E + \Delta E(k)$, $E_d(k) = E_d + \Delta E_d(k)$. Here, the constant matrices A, A_d, E and E_d are known while $\Delta A(k), \Delta A_d(k), \Delta A(k), \Delta A_d(k)$ are unknown matrices representing time-varying parameter uncertainties that are assumed to satisfy the following *admissible condition*:

$$
\begin{aligned}
&\begin{bmatrix} \Delta A(k) & \Delta A_d(k) & \Delta E(k) & \Delta E_d(k) \end{bmatrix} \\
&\qquad = MF(k) \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix},
\end{aligned} \tag{2}
$$

where M and N_i $(i = 1, 2, 3, 4)$ are known real constant matrices, and $F(k)$ is the unknown time-varying matrix-valued function subject to $F^T(k)F(k) \leq I$, $\forall k \in \mathbb{N}^+$.

In this paper, without loss of generality, we always assume that $f(0) = 0$, $f_d(0) = 0$ and $\phi(0) = 0$. For vector-valued functions f, f_d, ϕ, we assume:

$$
\begin{aligned}
&[f(x) - f(y) - U_1(x-y)]^T \\
&\quad \times [f(x) - f(y) - U_2(x-y)] \leq 0,\ \forall x, y \in \mathbb{R}^n,
\end{aligned} \tag{3}
$$

$$[f_d(x) - f_d(y) - V_1(x-y)]^T \times [f_d(x) - f_d(y) - V_2(x-y)] \le 0,\ \forall x, y \in \mathbb{R}^n, \quad (4)$$

$$[\phi(x) - \phi(y) - W_1(x-y)]^T \times [\phi(x) - \phi(y) - W_2(x-y)] \le 0,\ \forall x, y \in \mathbb{R}^m, \quad (5)$$

where $U_1, U_2, V_1, V_2 \in \mathbb{R}^{n\times n}$, $W_1, W_2 \in \mathbb{R}^{m\times m}$ are known real constant matrices, and $U_1 - U_2, V_1 - V_2$, $W_1 - W_2$ are positive definite matrices.

Remark 1 *It is customary that the nonlinear functions f, f_d, ϕ are said to belong to sectors $[U_1,\ U_2]$, $[V_1,\ V_2]$ and $[W_1,\ W_2]$, respectively (see [14]). The nonlinear descriptions in (3)-(5) are quite general that include the usual Lipschitz conditions as a special case. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied, see e.g. [11, 15].*

In the system (Σ'), $\tilde{y}(k) \in \mathbb{R}^m$ is the *ideal* system output that always contains the real signal. However, as discussed in the introduction, in practical engineering systems (e.g., networked control systems) the system output usually contains probabilistic missing data. In such a case, the *actual* system output can be described preferably by

$$\begin{aligned} y(k) &= \gamma_k \tilde{y}(k) + D_2 w(k) \\ &= \gamma_k \left[C x(k) + C_d x(k-d(k)) + \phi(Sx(k)) \right] \\ &\quad + D_2 w(k), \end{aligned} \quad (6)$$

where the stochastic variable $\gamma_k \in \mathbb{R}$ is a Bernoulli distributed white sequence specified by the following distribution law:

$$\text{Prob}\{\gamma_k = 1\} = \mathbb{E}\{\gamma_k\} = \beta, \quad (7)$$

$$\text{Prob}\{\gamma_k = 0\} = 1 - \mathbb{E}\{\gamma_k\} = 1 - \beta \quad (8)$$

with $\beta > 0$ a known constant. Obviously, for the stochastic variable γ_k, we have the mean value $\mathbb{E}\{\gamma_k\} = \beta$ and variance $\sigma^2 = \beta(1-\beta)$.

Remark 2 *The system measurement mode (6)-(8), which can be used to represent missing measurements or uncertain observations, was first introduced in [18], and has been subsequently studied in many papers, see e.g. [12, 23–26]. Note that when the real signal is missing (i.e., $\gamma_k = 0$), the system measurement contains noise only. Such a case does happen in practice. For example, in target tracking control, due to high maneuverability of the tracked target, there may be a nonzero probability that any observation consists of noise alone if the target is absent, i.e., the measurements are not consecutive but contain missing observations. On the other hand, there are still other ways to model the missing phenomenon, such as those using randomly delayed sensor output and probabilistic jumps.*

Replacing $\tilde{y}(k)$ in (Σ') by $y(k)$, we have the following system to be investigated in this paper:

$$\begin{aligned} (\Sigma) : x(k+1) =& A(k)x(k) + A_d(k)x(k-d(k)) \\ &+ E(k)f(x(k)) + E_d(k)f_d(x(k-d(k))) \\ &+ B_1 u(k) + D_1 w(k), \\ y(k) =& \gamma_k \left[Cx(k) + C_d x(k-d(k)) + \phi(Sx(k)) \right] \\ &+ D_2 w(k), \\ z(k) =& Lx(k) + B_2 u(k), \\ x(j) =& \psi(j),\ j = -d_M, -d_M+1, ..., 0 \end{aligned}$$

In this paper, for the system (Σ), we consider the following full-order dynamic output feedback controller:

$$\begin{aligned} (\Sigma_o) : \hat{x}(k+1) &= A_K \hat{x}(k) + B_K y(k), \\ u(k) &= C_K \hat{x}(k), \end{aligned}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the controller state, A_K, B_K and C_k are the controller parameters to be determined. Due to the existence of the stochastic variable $\gamma_k \in \mathbb{R}$, we will need the following definition for the exponential stability in the mean square.

Definition 1 *The controlled system (Σ) is said to be exponentially stable in the mean square if, in case of $w(k) = 0$ and for all admissible uncertainties, there exist constants $\mu > 0$ and $0 < \alpha < 1$ such that*

$$\mathbb{E}|x(k)|^2 \le \mu \alpha^k \sup_{-d_M \le i \le 0} \mathbb{E}|\psi(i)|^2.$$

The combination of the controller (Σ_o) and the system (Σ) yields the following closed-loop system:

$$\begin{aligned} (\Sigma_c) : \eta(k+1) =& \tilde{A}(k)\eta(k) + \tilde{A}_d(k) H \eta(k-d(k)) \\ &+ \bar{E}(k) f(x(k)) + \bar{E}_d(k) f_d(x(k-d(k))) \\ &+ \gamma_k \bar{B}_K \phi(Sx(k)) + Dw(k), \\ z(k) =& \bar{L}\eta(k), \end{aligned}$$

where

$$\eta_k = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}, H = \begin{bmatrix} I_{n\times n} \\ 0_{n\times n} \end{bmatrix}^T, \tilde{A}(k) = \begin{bmatrix} A(k) & B_1 C_K \\ \gamma_k B_K C & A_K \end{bmatrix},$$

$$\tilde{A}_d(k) = \begin{bmatrix} A_d(k) \\ \gamma_k B_K C_d \end{bmatrix}, \bar{E}(k) = \begin{bmatrix} E(k) \\ 0 \end{bmatrix}, \bar{E}_d(k) = \begin{bmatrix} E_d(k) \\ 0 \end{bmatrix},$$

$$\bar{B}_K = \begin{bmatrix} 0 \\ B_K \end{bmatrix},\ D = \begin{bmatrix} D_1 \\ B_K D_2 \end{bmatrix},\ \bar{L} = \begin{bmatrix} L^T \\ C_K^T B_2^T \end{bmatrix}^T.$$

For convenience, we denote

$$\bar{A}(k) = \begin{bmatrix} A(k) & B_1 C_K \\ \beta B_K C & A_K \end{bmatrix},\ \bar{A}_d(k) = \begin{bmatrix} A_d(k) \\ \beta B_K C_d \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0 & 0 \\ B_K C & 0 \end{bmatrix},\ \bar{C}_d = \begin{bmatrix} 0 \\ B_K C_d \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A & B_1 C_K \\ \beta B_K C & A_K \end{bmatrix},\ \bar{A}_d = \begin{bmatrix} A_d \\ \beta B_K C_d \end{bmatrix}, \tag{9}$$

$$\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix},\ \bar{E}_d = \begin{bmatrix} E_d \\ 0 \end{bmatrix},$$

$$\Delta\bar{A}(k) = \begin{bmatrix} \Delta A(k) & 0 \\ 0 & 0 \end{bmatrix},\ \bar{A}_d(k) = \begin{bmatrix} \Delta A_d(k) \\ 0 \end{bmatrix},$$

$$\Delta\bar{E}(k) = \begin{bmatrix} \Delta E(k) \\ 0 \end{bmatrix},\ \Delta\bar{E}_d(k) = \begin{bmatrix} \Delta E_d(k) \\ 0 \end{bmatrix},$$

$$\bar{M} = [M^T \quad 0]^T,\ \bar{N}_1 = [N_1 \quad 0].$$

It is clear that

$$\bar{A}(k) = \bar{A} + \Delta\bar{A}(k),\quad \bar{A}_d(k) = \bar{A}_d + \Delta\bar{A}_d(k),$$
$$\bar{E}(k) = \bar{E} + \Delta\bar{E}(k),\quad \bar{E}_d(k) = \bar{E}_d + \Delta\bar{E}_d(k).$$

In this paper, we shall focus on the robust stabilization problem whose purpose is to design a full-order exponentially stable H_∞ controller for the system (Σ) via output feedback. More specifically, an output feedback controller Σ_o is to be designed such that (i) the closed-loop system (Σ_c) is exponentially stable in the mean square; and (ii) Under zero initial condition, the controlled output z satisfies $\|z\|_2 \le \gamma\|w\|_2$ for any nonzero $w \in l_2$, where $\gamma > 0$ is a prescribed constant.

The following lemmas will be used in establishing our main results.

Lemma 1 *Let $\mathcal{D}, \mathcal{S}$ and F be real matrices of appropriate dimensions with $F^T F \le I$. Then, for any scalar $\varepsilon > 0$, we have $\mathcal{D}F\mathcal{S} + (\mathcal{D}F\mathcal{S})^T \le \varepsilon^{-1}\mathcal{D}\mathcal{D}^T + \varepsilon\mathcal{S}^T\mathcal{S}$.*

3 Main Results

First of all, let us deal with the robust stability analysis problem and derive a sufficient condition under which the closed-loop system (Σ_c) with given controller and $w(k) = 0$ is robustly exponentially stable in the mean-square.

Theorem 1 *Let the controller parameters A_K, B_K and C_K be given and the admissible condition hold. Then, the closed-loop system (Σ_c) with $w(t) = 0$ is robustly exponentially stable in the mean square if there exist two matrices $P > 0$, $Q > 0$ and two positive constant scalars ε, λ such that the following matrix inequality holds:*

$$\Psi = \begin{bmatrix} \Pi & * & * & * & * & * & * & * \\ 0 & \Pi_a & * & * & * & * & * & * \\ -\breve{U}_2^T H & 0 & -I & * & * & * & * & * \\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & * & * & * & * \\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I & * & * & * \\ \bar{A} & \bar{A}_d & \bar{E} & \bar{E}_d & 0 & \Pi_b & * & * \\ \sigma\bar{C} & \sigma\bar{C}_d & 0 & 0 & \sigma\bar{B}_K & 0 & -P^{-1} & * \\ \bar{N}_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0, \tag{10}$$

with $\Pi_a = -Q - \lambda\breve{V}_1$ and $\Pi_b = -P^{-1} + \varepsilon\bar{M}\bar{M}^T$, where

$$\breve{U}_1 = (U_1^T U_2 + U_2^T U_1)/2;\ \breve{U}_2 = -(U_1^T + U_2^T)/2; \tag{11}$$
$$\breve{V}_1 = (V_1^T V_2 + V_2^T V_1)/2; \breve{V}_2 = -(V_1^T + V_2^T)/2; \tag{12}$$
$$\mathcal{W}_1 = (S^T W_1^T W_2 S + S^T W_2^T W_1 S)/2; \tag{13}$$
$$\mathcal{W}_2 = -(S^T W_1^T + S^T W_2^T)/2; \tag{14}$$
$$\begin{aligned}\Pi = & -P + (d_M - d_m + 1)H^T Q H - H^T \breve{U}_1 H \\ & -H^T \mathcal{W}_1 H,\end{aligned} \tag{15}$$

and $\sigma = \sqrt{\beta(1-\beta)}$ with $\beta = \mathbb{E}\{\gamma_k\}$.

Proof: For the stability analysis of the system (Σ_c), we construct the following Lyapunov-Krasovskii functional:

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{16}$$

where $V_1(k) = \eta^T(k)P\eta(k)$, $V_2(k) = \sum_{i=k-d(k)}^{k-1} \eta^T(i)H^T Q H\eta(i)$ and $V_3(k) = \sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j}^{k-1} \eta^T(i)H^T Q H\eta(i)$.

Calculating the difference of $V(k)$ along the system (Σ_c) with $w(k) = 0$ and taking the mathematical expectation, we have

$$\mathbb{E}\{\Delta V(k)\} = \sum_{i=1}^{3} \mathbb{E}\{\Delta V_i(k)\}, \tag{17}$$

where

$$\begin{aligned}\mathbb{E}\{\Delta V_1(k)\} &= \mathbb{E}\{V_1(k+1) - V_1(k)\} \\ &= \mathbb{E}\Big\{\hat{\mathcal{A}}_0^T(k)P\hat{\mathcal{A}}_0(k) + \sigma^2\hat{\mathcal{B}}_0^T(k)P\hat{\mathcal{B}}_0(k) \\ &\quad -\eta^T(k)P\eta(k)\Big\}\end{aligned} \tag{18}$$

with

$$\begin{aligned}\hat{\mathcal{A}}_0(k) = & \bar{A}(k)\eta(k) + \bar{A}_d(k)H\eta(k-d(k)) + \bar{E}(k)f(x(k)) \\ & +\bar{E}_d(k)f_d(x(k-d(k))) + \beta\bar{B}_K\phi(Sx(k)),\end{aligned} \tag{19}$$
$$\hat{\mathcal{B}}_0(k) = \bar{C}\eta(k) + \bar{C}_d H\eta(k-d(k)) + \bar{B}_K\phi(Sx(k)). \tag{20}$$

Similarly, it is not difficult to check that

$$\begin{aligned}\mathbb{E}\{\Delta V_2(k)\} \le \mathbb{E}\Big\{ & \eta^T(k)H^T Q H\eta(k) \\ & -\eta^T(k-d(k))H^T Q H\eta(k-d(k)) \\ & + \sum_{i=k-d_M+1}^{k-d_m} \eta^T(i)H^T Q H\eta(i)\Big\},\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_3(k)\} = \mathbb{E}\Big\{ & (d_M - d_m)\eta^T(k)H^T Q H\eta(k) \\ & - \sum_{i=k-d_M+1}^{k-d_m} \eta^T(i)H^T Q\eta(i)\Big\}.\end{aligned} \tag{22}$$

Substituting (18)-(22) into (17) leads to

$$\mathbb{E}\{\Delta V(k)\} \leq \mathbb{E}\Big\{\xi_0^T(k)\Psi_1(k)\xi_0(k) + \xi_0^T(k)\mathcal{A}_0^T(k)P \times\mathcal{A}_0(k)\xi_0(k) + \sigma^2\xi_0^T(k)\mathcal{B}_0^T P\mathcal{B}_0\xi_0(k)\Big\}, \quad (23)$$

where

$$\begin{aligned}\xi_0(k) &= [\eta^T(k) \quad x^T(k-d(k)) \quad f^T(x(k)) \quad f_d^T(x(k-d(k))) \quad \phi^T(Sx(k))]^T,\\ \mathcal{A}_0(k) &= [\bar{A}(k) \quad \bar{A}_d(k) \quad \bar{E}(k) \quad \bar{E}_d(k) \quad 0],\\ \mathcal{B}_0 &= [\bar{C} \quad \bar{C}_d \quad 0 \quad 0 \quad \bar{B}_K],\\ \Psi_1(k) &= \mathrm{diag}\,\{\Pi_1, -Q, 0, 0, 0\}\end{aligned}$$

with $\Pi_1 = -P + (d_M - d_m + 1)H^T QH$.

It follows readily from (3)-(5) that

$$\begin{bmatrix}\eta(k)\\ f(x(k))\end{bmatrix}^T \begin{bmatrix}H^T\breve{U}_1H & H^T\breve{U}_2\\ \breve{U}_2^T H & I\end{bmatrix}\begin{bmatrix}\eta(k)\\ f(x(k))\end{bmatrix} \leq 0, \quad (24)$$

$$\begin{bmatrix}x(k-d(k))\\ f_d(x(k-d(k)))\end{bmatrix}^T \begin{bmatrix}\breve{V}_1 & \breve{V}_2\\ \breve{V}_2^T & I\end{bmatrix}\begin{bmatrix}x(k-d(k))\\ f_d(x(k-d(k)))\end{bmatrix} \leq 0, \quad (25)$$

$$\begin{bmatrix}\eta(k)\\ \phi(Sx(k))\end{bmatrix}^T \begin{bmatrix}H^T\mathcal{W}_1H & H^T\mathcal{W}_2\\ \mathcal{W}_2^T H & I\end{bmatrix}\begin{bmatrix}\eta(k)\\ \phi(Sx(k))\end{bmatrix} \leq 0, \quad (26)$$

where $\breve{U}_1, \breve{U}_2, \breve{V}_1, \breve{V}_2, \mathcal{W}_1$ and $\mathcal{W}_2$ are defined in (11) and (13). Then, it implies from (23), (24) and (26) that

$$\begin{aligned}&\mathbb{E}\{\Delta V(k)\}\\ &\leq \mathbb{E}\Big\{\xi_0^T(k)\Psi_1(k)\xi_0(k) + \xi_0^T(k)\mathcal{A}_0^T(k)P\mathcal{A}_0(k)\xi_0(k)\\ &\quad +\sigma^2\xi_0^T(k)\mathcal{B}_0^T P\mathcal{B}_0\xi_0(k)\Big\}\\ &\quad - \mathbb{E}\left\{\begin{bmatrix}\eta(k)\\ f(x(k))\end{bmatrix}^T \begin{bmatrix}H^T\breve{U}_1H & H^T\breve{U}_2\\ \breve{U}_2^T H & I\end{bmatrix}\begin{bmatrix}\eta(k)\\ f(x(k))\end{bmatrix}\right.\\ &\quad + \lambda\begin{bmatrix}x(k-d(k))\\ f_d(x(k-d(k)))\end{bmatrix}^T \begin{bmatrix}\breve{V}_1 & \breve{V}_2\\ \breve{V}_2^T & I\end{bmatrix}\begin{bmatrix}x(k-d(k))\\ f_d(x(k-d(k)))\end{bmatrix}\\ &\quad \left.+ \begin{bmatrix}\eta(k)\\ \phi(Sx(k))\end{bmatrix}^T \begin{bmatrix}H^T\mathcal{W}_1H & H^T\mathcal{W}_2\\ \mathcal{W}_2^T H & I\end{bmatrix}\begin{bmatrix}\eta(k)\\ \phi(Sx(k))\end{bmatrix}\right\}\\ &= \mathbb{E}\Big\{\xi_0^T(k)\Big[\Psi_2(k) + \mathcal{A}_0^T(k)P\mathcal{A}_0(k) + \sigma^2\mathcal{B}_0^T P\mathcal{B}_0\Big]\xi_0(k)\Big\}, \quad (27)\end{aligned}$$

where

$$\Psi_2(k) = \begin{bmatrix}\Pi & 0 & -H^T\breve{U}_2 & 0 & -H^T\mathcal{W}_2\\ 0 & -Q-\lambda\breve{V}_1 & 0 & -\lambda\breve{V}_2 & 0\\ -\breve{U}_2^T H & 0 & -I & 0 & 0\\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & 0\\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I\end{bmatrix}.$$

It is clear from $\Psi < 0$ that there exists a sufficiently small scalar $\varepsilon_0 > 0$ such that

$$\Psi + \varepsilon_0 \mathrm{diag}\{I_{2n\times 2n}, 0\} < 0. \quad (28)$$

In order to deal with the exponential stability, we shall first prove that (28) indicates

$$\Psi_2(k) + \varepsilon_0\mathrm{diag}\{I_{2n\times 2n}, 0\} + \mathcal{A}_0^T(k)P\mathcal{A}_0(k) + \sigma^2\mathcal{B}_0^T P\mathcal{B}_0 < 0. \quad (29)$$

In fact, by the Schur complement, the above inequality is equivalent to

$$\Psi_3(k) = \begin{bmatrix}\Pi+\varepsilon_0 I & * & * & * & * & * & *\\ 0 & -Q-\lambda\breve{V}_1 & * & * & * & * & *\\ -\breve{U}_2^T H & 0 & -I & * & * & * & *\\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & * & * & *\\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I & * & *\\ \bar{A}(k) & \bar{A}_d(k) & \bar{E}(k) & \bar{E}_d(k) & 0 & -P^{-1} & *\\ \sigma\bar{C} & \sigma\bar{C}_d & 0 & 0 & \sigma\bar{B}_K & 0 & -P^{-1}\end{bmatrix} < 0,$$

which can be decomposed as follows:

$$\Psi_3(k) = \Psi_3 + \Delta\Psi_3(k), \quad (30)$$

where

$$\Psi_3 := \begin{bmatrix}\Pi+\varepsilon_0 I & * & * & * & * & * & *\\ 0 & -Q-\lambda\breve{V}_1 & * & * & * & * & *\\ -\breve{U}_2^T H & 0 & -I & * & * & * & *\\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & * & * & *\\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I & * & *\\ \bar{A} & \bar{A}_d & \bar{E} & \bar{E}_d & 0 & -P^{-1} & *\\ \sigma\bar{C} & \sigma\bar{C}_d & 0 & 0 & \sigma\bar{B}_K & 0 & -P^{-1}\end{bmatrix} < 0, \quad (31)$$

$$\Delta\Psi_3(k) := \begin{bmatrix}0 & * & * & * & * & * & *\\ 0 & 0 & * & * & * & * & *\\ 0 & 0 & 0 & * & * & * & *\\ 0 & 0 & 0 & 0 & * & * & *\\ 0 & 0 & 0 & 0 & 0 & * & *\\ \Delta\bar{A}(k) & \Delta\bar{A}_d(k) & \Delta\bar{E}(k) & \Delta\bar{E}_d(k) & 0 & 0 & *\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}. \quad (32)$$

From (2) and Lemma 1, it follows that

$$\begin{aligned}\Delta\Psi_3(k) &= \tilde{M}F(k)\tilde{N} + (\tilde{M}F(k)\tilde{N})^T\\ &\leq \varepsilon\tilde{M}\tilde{M}^T + \varepsilon^{-1}\tilde{N}^T\tilde{N},\end{aligned} \quad (33)$$

where $M = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \bar{M}^T \quad 0]^T$ and $\tilde{N} = [\bar{N}_1 \quad N_2 \quad N_3 \quad N_4 \quad 0 \quad 0 \quad 0]$. With (31) and (33), one has

$$\Psi_3(k) \leq \Psi_4 + \varepsilon^{-1}\tilde{N}^T\tilde{N}, \tag{34}$$

where

$$\Psi_4 := \begin{bmatrix} \Pi + \varepsilon_0 I & * & * & * & * & * & * \\ 0 & \Pi_a & * & * & * & * & * \\ -\breve{U}_2^T H & 0 & -I & * & * & * & * \\ 0 & -\lambda \breve{V}_2^T & 0 & -\lambda I & * & * & * \\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I & * & * \\ \bar{A} & \bar{A}_d & \bar{E} & \bar{E}_d & 0 & \Pi_b & * \\ \sigma\bar{C} & \sigma\bar{C}_d & 0 & 0 & \sigma\bar{B}_K & 0 & -P^{-1} \end{bmatrix} \tag{35}$$

with Π_a and Π_b defined in Theorem 1.

By Schur complement, the inequality (28) is true if and only if the right-hand side of (34) is negative definite, which implies that $\Psi_3(k) < 0$. Therefore, (29) holds, and it follows from (27) and (29) that

$$\mathbb{E}\{\Delta V(k)\} \leq -\varepsilon_0 \mathbb{E}\{|\eta_k|^2\}. \tag{36}$$

We are now in a position to proceed with the exponential stability analysis of the system (Σ_c). According to the definition of $V(k)$, we have

$$\mathbb{E}\{V(k)\} \leq \rho_1 \mathbb{E}\{|\eta_k|^2\} + \rho_2 \sum_{i=k-d_M}^{k-1} \mathbb{E}\{|\eta_i|^2\}, \tag{37}$$

where $\rho_1 = \lambda_{\max}(P)$ and $\rho_2 = (d_M - d_m + 1)\lambda_{\max}(Q)$. For any scalar $\mu > 1$, the above inequality, together with (36), implies that

$$\begin{aligned} &\mu^{k+1}\mathbb{E}\{V(k+1)\} - \mu^k \mathbb{E}\{V(k)\} \\ &= \mu^{k+1}\mathbb{E}\{\Delta V(k)\} + \mu^k(\mu - 1)\mathbb{E}\{V(k)\} \\ &\leq \omega_1(\mu)\mu^k \mathbb{E}\{|\eta_k|^2\} + \omega_2(\mu) \sum_{i=k-d_M}^{k-1} \mu^k \mathbb{E}\left\{|\eta_i|^2\right\}, \end{aligned} \tag{38}$$

where $\omega_1(\mu) = -\mu\varepsilon_0 + (\mu - 1)\rho_1$ and $\omega_2(\mu) = (\mu - 1)\rho_2$.

Furthermore, for $N \geq d_M + 1$, summing up both sides of (38) from 0 to $N - 1$ with respect to k, we have

$$\begin{aligned} \mu^N \mathbb{E}\{V(N)\} - \mathbb{E}\{V(0)\} &\leq \omega_1(\mu) \sum_{k=0}^{N-1} \mu^k \mathbb{E}\{|\eta_k|^2\} \\ &+ \omega_2(\mu) \sum_{k=0}^{N-1} \sum_{i=k-d_M}^{k-1} \mu^k \mathbb{E}\left\{|\eta_i|^2\right\}. \end{aligned} \tag{39}$$

Note that for $d_M \geq 1$,

$$\begin{aligned} &\sum_{k=0}^{N-1} \sum_{i=k-d_M}^{k-1} \mu^k \mathbb{E}\left\{|\eta_i|^2\right\} \\ &\leq \Big(\sum_{i=-d_M}^{-1} \sum_{k=0}^{i+d_M} + \sum_{i=0}^{N-1-d_M} \sum_{k=i+1}^{i+d_M} + \sum_{i=N-1-d_M}^{N-1} \sum_{k=i+1}^{N-1} \Big) \mu^k \mathbb{E}\Big\{|\eta_i|^2\Big\} \\ &\leq d_M \sum_{i=-d_M}^{-1} \mu^{i+d_M} \mathbb{E}\left\{|\eta_i|^2\right\} + d_M \sum_{i=0}^{N-1-d_M} \mu^{i+d_M} \mathbb{E}\left\{|\eta_i|^2\right\} \\ &\quad + d_M \sum_{i=N-1-d_M}^{N-1} \mu^{i+d_M} \mathbb{E}\left\{|\eta_i|^2\right\} \\ &\leq d_M \mu^{d_M} \max_{-d_M \leq i \leq 0} \mathbb{E}\{|\psi(i)|^2\} + d_M \mu^{d_M} \sum_{i=0}^{N-1} \mu^i \mathbb{E}\left\{|\eta_i|^2\right\} \end{aligned} \tag{40}$$

Then, from (39) and (40), one has

$$\begin{aligned} \mu^N \mathbb{E}\{V(N)\} \leq \mathbb{E}\{V(0)\} &+ \left[\omega_1(\mu) + d_M \mu^{d_M} \omega_2(\mu)\right] \\ &\times \sum_{k=0}^{N-1} \mu^k \mathbb{E}\{|\eta_k|^2\} + d_M \mu^{d_M} \omega_2(\mu) \\ &\times \max_{-d_M \leq i \leq 0} \mathbb{E}\{|\psi(i)|^2\}. \end{aligned} \tag{41}$$

Let $\rho_0 = \lambda_{\min}(P)\}$ and $\rho = \max\{\rho_1, \ \rho_2\}$. It is obvious that

$$\mathbb{E}\{V(N)\} \geq \rho_0 \mathbb{E}\{|\eta_N|^2\}. \tag{42}$$

It also follows from (37) that

$$\mathbb{E}\{V(0)\} \leq \rho \max_{-d_M \leq i \leq 0} \mathbb{E}\{|\psi(i)|^2\}. \tag{43}$$

In addition, it can be verified that there exists a scalar $\mu_0 > 1$ such that

$$\omega_1(\mu_0) + d_M \mu_0^{d_M} \omega_2(\mu_0) = 0. \tag{44}$$

Substituting (42)-(44) into (41), we obtain

$$\mathbb{E}\{|\eta_N|^2\} \leq c_0 \left(\frac{1}{\mu_0}\right)^N \max_{-d_M \leq i \leq 0} \mathbb{E}\{|\psi(i)|^2\},$$

where $c_0 = \frac{1}{\rho_0}\left(\rho + d_M \mu_0^{d_M} \omega_2(\mu_0)\right)$. This indicates that the closed-loop system (Σ_c) with $w(k) = 0$ is robustly exponentially stable in the mean square. The proof is complete.

Next, we will analyze the H_∞ performance of the closed-loop system (Σ_c).

Theorem 2 *Let the controller parameters A_K, B_K and C_K be given and γ be a prespecified positive constant. Then, the closed-loop system (Σ_c) is robustly exponentially stable in the mean square for $w(k)=0$ and satisfies $\|z\|_2 \le \gamma\|w\|_2$ under the zero initial condition for any nonzero $w \in l_2[0,+\infty)$, if there exist two positive definite matrices P, Q and two positive scalars ε, λ such that the following matrix inequality holds:*

$$\Phi = \begin{bmatrix} \Upsilon & * & * & * & * & * & * & * & * \\ 0 & \Pi_a & * & * & * & * & * & * & * \\ -\breve{U}_2^T H & 0 & -I & * & * & * & * & * & * \\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & * & * & * & * & * \\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & * & * & * \\ \bar{A} & \bar{A}_d & \bar{E} & \bar{E}_d & 0 & D & \Pi_b & * & * \\ \sigma\bar{C} & \sigma\bar{C}_d & 0 & 0 & \sigma\bar{B}_K & 0 & 0 & -P^{-1} & * \\ \bar{N}_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{45}$$

where $\Pi_a, \Pi_b, \breve{L}_1, \breve{L}_2, \breve{U}_1, \breve{U}_2, \mathcal{W}_1$ and $\mathcal{W}_2$ are defined as in Theorem 1, and

$$\Upsilon = -P + (d_M - d_m + 1)H^T Q H - H^T \breve{U}_1 H - H^T \mathcal{W}_1 H + \bar{L}^T \bar{L}.$$

Proof. It is not difficult to verify that $\Phi < 0$ implies $\Psi < 0$. According to Theorem 1, the closed-loop system (Σ_c) is robustly exponentially stable in the mean square.

We now deal with the H_∞ performance of the closed-loop system. Construct the same Lyapunov-Krasovskii functional candidate $V(k)$ as in Theorem 1. A similar calculation as in Theorem 1 leads to

$$\begin{aligned}\mathbb{E}\{\Delta V(k)\} \le\, & \mathbb{E}\{\xi^T(k)\Phi_1(k)\xi(k) + \xi^T(k)\mathcal{A}^T(k)P\mathcal{A}(k)\xi(k) \\ & + \sigma^2 \xi^T(k)\mathcal{B}^T P \mathcal{B}\xi(k)\},\end{aligned} \tag{46}$$

where

$$\begin{aligned}\xi(k) &= [\eta^T(k) \quad x^T(k-d(k)) \quad f^T(x(k)) \\ &\qquad f_d^T(x(k-d(k))) \quad \phi^T(Sx(k)) \quad w^T(k)]^T, \\ \mathcal{A}(k) &= [\bar{A}(k) \quad \bar{A}_d(k) \quad \bar{E}(k) \quad \bar{E}_d(k) \quad 0 \quad D], \\ \mathcal{B} &= [\bar{C} \quad \bar{C}_d \quad 0 \quad 0 \quad \bar{B}_K \quad 0], \\ \Phi_1(k) &= \operatorname{diag}\{-P + (d_M - d_m + 1)H^T Q H, -Q, 0, 0, 0, 0\}.\end{aligned}$$

In order to deal with the H_∞ performance of the system (Σ_c), we introduce

$$J(n) = \mathbb{E}\sum_{k=0}^{n}\left[z^T(k)z(k) - \gamma^2 w^T(k)w(k)\right], \tag{47}$$

where n is non-negative integer.

Under the zero initial condition, from (24)-(26), (46) and (47), one has

$$\begin{aligned} J(n) &= \mathbb{E}\sum_{k=0}^{n}\left[z^T(k)z(k) - \gamma^2 w^T(k)w(k) + \Delta V(k)\right] \\ &\quad - \mathbb{E}V(n+1) \\ &\le \mathbb{E}\sum_{k=0}^{n}\left[\eta^T(k)\bar{L}^T\bar{L}\eta(k) - \gamma^2 w^T(k)w(k) + \Delta V(k)\right] \\ &\le \mathbb{E}\sum_{k=0}^{n}[\xi^T(k)\Phi_2(k)\xi(k) + \xi^T(k)\mathcal{A}^T(k)P\mathcal{A}(k)\xi(k) \\ &\quad + \sigma^2\xi^T(k)\mathcal{B}^T P\mathcal{B}\xi(k)], \end{aligned} \tag{48}$$

where

$$\Phi_2(k) = \begin{bmatrix} \Upsilon & 0 & -H^T\breve{U}_2 & 0 & -H^T\mathcal{W}_2 & 0 \\ 0 & -Q-\lambda\breve{V}_1 & 0 & -\lambda\breve{V}_2 & 0 & 0 \\ -\breve{U}_2^T H & 0 & -I & 0 & 0 & 0 \\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & 0 & 0 \\ -\mathcal{W}_2^T H & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}.$$

By Schur complement and along the same line as in the proof of Theorem 1, we can show that $J(n) < 0$. The details are omitted here for space saving. Letting $n \to \infty$, we obtain $\|z\|_2 \le \gamma\|w\|_2$. This completes the proof of the theorem.

Finally, we are ready to deal with the design of H_∞ output feedback controller for the system (Σ) and give the main result of this paper in the following theorem where the technique of linearizing change of variables [9, 10] is used.

Theorem 3 *Let $\gamma > 0$ be a given positive constant. Then, for the nonlinear stochastic system (Σ) with missing measurements, an output feedback controller can be designed such that the closed-loop system (Σ_c) is robustly exponentially stable in the mean-square for $w(k)=0$ and also satisfies $\|z\|_2 \le \gamma\|w\|_2$ under the zero initial condition for any nonzero w, if there exist four real constant matrices $Y > 0$, Ω, Γ and Λ, and a scalar $\lambda > 0$ such that, for given matrices $X > 0, Q > 0$ and a scalar $\varepsilon > 0$, the following LMI holds:*

$$\Phi_0 := \begin{bmatrix} \Phi_{011} & \Phi_{21}^T \\ \Phi_{21} & \Phi_{22} \end{bmatrix} < 0 \tag{49}$$

where

$$\Phi_{011} = \begin{bmatrix} \mathit{\Pi} & * & * & * & * & * \\ 0 & -Q-\lambda\breve{V}_1 & * & * & * & * \\ \Xi_1 & 0 & -I & * & * & * \\ 0 & -\lambda\breve{V}_2^T & 0 & -\lambda I & * & * \\ \Xi_2 & 0 & 0 & 0 & -I & * \\ 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

$$
\Phi_{021} = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & 0 & \Theta_5 \\ \Theta_6 & \Theta_7 & 0 & 0 & \Theta_8 & 0 \\ \Xi_3 & N_2 & N_3 & N_4 & 0 & 0 \\ \Xi_4 & 0 & 0 & 0 & 0 & 0 \\ \Xi_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\Phi_{022} = \begin{bmatrix} \mathit{\Pi} & * & * & * & * & * \\ 0 & \mathit{\Pi} & * & * & * & * \\ 0 & 0 & -\varepsilon I & * & * & * \\ 0 & 0 & 0 & -I & * & * \\ 0 & 0 & 0 & 0 & -\hat{Q}^{-1} & * \\ \Xi_6 & 0 & 0 & 0 & 0 & -\varepsilon^{-1} I \end{bmatrix},
$$

$$
\mathit{\Pi} = \begin{bmatrix} -X & -I \\ -I & -Y \end{bmatrix}, \ \Theta_1 = \begin{bmatrix} AX + B_1\Lambda & A \\ \Omega & YA + \beta\Gamma C \end{bmatrix},
$$

$$
\Theta_2 = \begin{bmatrix} A_d \\ YA_d + \beta\Gamma C_d \end{bmatrix}, \ \Theta_3 = \begin{bmatrix} E \\ YE \end{bmatrix}, \ \Theta_4 = \begin{bmatrix} E_d \\ YE_d \end{bmatrix},
$$

$$
\Theta_5 = \begin{bmatrix} D_1 \\ YD_1 + \Gamma D_2 \end{bmatrix}, \ \Theta_6 = \sigma \begin{bmatrix} 0 & 0 \\ \Gamma CX & \Gamma C \end{bmatrix},
$$

$$
\Theta_7 = \sigma \begin{bmatrix} 0 \\ \Gamma C_d \end{bmatrix}, \ \Theta_8 = \sigma \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \ \Xi_1 = -\breve{U}_2^T [X \ I],
$$

$$
\Xi_2 = -\mathcal{W}_2^T [X \ I], \ \Xi_3 = N_1 [X \ I], \ \Xi_4 = [LX + B_2\Lambda \ L],
$$
$$
\Xi_5 = [X \ I], \ \Xi_6 = \varepsilon M^T [I \ Y],
$$
$$
\hat{Q} = (d_M - d_m + 1)Q - \breve{U}_1 - \mathcal{W}_1,
$$

and $\breve{L}_1, \breve{L}_2, \breve{U}_1, \breve{U}_2, \mathcal{W}_1$ and $\mathcal{W}_2$ are defined as in Theorems 1.

In this case, the parameters of the output feedback controller can be designed as

$$
A_K = R^{-1}(\Omega - YAX - \beta\Gamma CX + YB_1\Lambda)G^{-T}, \tag{50}
$$
$$
B_K = R^{-1}\Gamma, \ C_K = \Lambda G^{-T}, \tag{51}
$$

where G and R are any matrices satisfying

$$
RG^T = I - YX. \tag{52}
$$

Proof: First, it follows readily from (49) that

$$
\mathit{\Pi} = \begin{bmatrix} -X & -I \\ -I & -Y \end{bmatrix} < 0,
$$

which, by Schur complement, implies that $Y - X^{-1} > 0$ and $I - YX$ is nonsingular. Therefore, there always exist nonsingular matrices G and R such that Eq. (52) holds. Now, adopting the similar method as in [6], we define

$$
\mathit{\Pi}_1 = \begin{bmatrix} X & I \\ G^T & 0 \end{bmatrix}, \ \mathit{\Pi}_2 = \begin{bmatrix} I & Y \\ 0 & R^T \end{bmatrix}, \ P = \mathit{\Pi}_2 \mathit{\Pi}_1^{-1}, \tag{53}
$$

and then

$$
P = \begin{bmatrix} Y & R \\ R^T & Z \end{bmatrix}, \tag{54}
$$

where $Z = G^{-1}X(Y - X^{-1})XG^{-T} > 0$.

It is clear that

$$
Z - R^T Y^{-1} R = R^T (XY - I)^{-1} (Y - X^{-1}) (YX - I)^{-1} R > 0,
$$

which, by Schur complement, implies that $P > 0$.

By a tedious calculation, one can rewrite LMI (49) as

$$
\begin{bmatrix}
-\mathit{\Pi}_1^T P \mathit{\Pi}_1 & * & * & * & * & * & * & * & * & * & * & * \\
0 & -Q - \lambda \breve{V}_1 & * & * & * & * & * & * & * & * & * & * \\
-\breve{U}_2^T H \mathit{\Pi}_1 & 0 & -I & * & * & * & * & * & * & * & * & * \\
0 & -\lambda \breve{V}_2^T & 0 & -\lambda I & * & * & * & * & * & * & * & * \\
-\mathcal{W}_2^T H \mathit{\Pi}_1 & 0 & 0 & 0 & -I & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\
\mathit{\Pi}_2^T \bar{A} \mathit{\Pi}_1 & \mathit{\Pi}_2^T \bar{A}_d & \mathit{\Pi}_2^T \bar{E} & \mathit{\Pi}_2^T \bar{E}_d & 0 & \mathit{\Pi}_2^T D & -\mathit{\Pi}_2^T P^{-1} \mathit{\Pi}_2 & * & * & * & * & * \\
\sigma \mathit{\Pi}_2^T \bar{C} \mathit{\Pi}_1 & \sigma \mathit{\Pi}_2^T \bar{C}_d & 0 & 0 & \sigma \mathit{\Pi}_2^T \bar{B}_K & 0 & 0 & -\mathit{\Pi}_2^T P^{-1} \mathit{\Pi}_2 & * & * & * & * \\
\bar{N}_1 \mathit{\Pi}_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & -\varepsilon I & * & * & * \\
\bar{L} \mathit{\Pi}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & * & * \\
H \mathit{\Pi}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{Q}^{-1} & * \\
0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \bar{M}^T \mathit{\Pi}_2 & 0 & 0 & 0 & 0 & -\varepsilon^{-1} I
\end{bmatrix} < 0 \tag{55}
$$

Pre- and post-multiplying (55) by

$$\mathrm{diag}\{\Pi_1^{-T}, I, I, I, I, I, \Pi_2^{-T}, \Pi_2^{-T}, I, I, I, I\}$$

and $\mathrm{diag}\{\Pi_1^{-1}, I, I, I, I, I, \Pi_2^{-1}, \Pi_2^{-1}, I, I, I, I\}$, respectively, we can show by Schur complement that the inequality (45) holds. Therefore, from Theorem 2, the desired result follows and the proof is complete.

Remark 3 *The robust H_∞ controller design problem is solved in Theorem 3 for the addressed uncertain nonlinear stochastic time-delay systems with missing measurements. An LMI-based sufficient condition is derived for the existence of state feedback controllers, which ensures the robust exponential stability in the mean square for the resulting closed-loop system and also reduces the effect of the disturbance input on the controlled output to a prescribed level for all admissible uncertainties. The feasibility of the controller design problem can be readily checked by the solvability of an LMI, and this can be done by resorting to the Matlab LMI toolbox.*

Remark 4 *Note that Lemma 1 has been used in the proof of Theorem 1 to tackle the parameter uncertainties, hence certain conservatism might be introduced. Such conservatism can be significantly reduced by imposing more constraints on the LMI parameters ε. A selection of more general Lyapunov functional in latest literature (e.g. [10]) could reduce the possible conservatism as well. The conservatism reduction leaves an interesting topic for our future research.*

Remark 5 *It is noticed that the only stochastic term involved in the system (Σ) is from the stochastic variable γ_k in (6). If $\gamma_k = 1$ all the time, our main results will reduce to the case of robust H_∞ control for a class of deterministic nonlinear discrete time-delay systems. Such specialized results are still believed to be new for two reasons: 1) most existing LMI-based results have dealt with the nonlinearities satisfying Lipschitz conditions which are more restrictive than the sector conditions, see e.g. [9, 17]; and 2) most existing literature has been concerned with continuous-time systems, see e.g. [11, 15], and the corresponding results on discrete-time systems are relatively few. Moreover, it would be interesting to further investigate the systems with Itô-type stochastic disturbances [27], and the corresponding results will appear in the near future.*

4 Numerical Example

Consider the system (Σ) with the following parameters:

$$A = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.1 & -0.3 & 0.1 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \ A_d = \begin{bmatrix} 0.1 & -0.1 & 0 \\ 0.1 & -0.2 & 0 \\ 0 & -0.2 & -0.1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \ E_d = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \\ 0 & -0.1 \end{bmatrix}, \ D_1 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0.8 & 0.7 \\ -0.6 & 0.9 & 0.6 \end{bmatrix}, \ C_d = \begin{bmatrix} 0.9 & -0.6 & 0.8 \\ 0.5 & 0.8 & 0.7 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.6 & 0.7 & 0.2 \\ 0.5 & 0.8 & 0.3 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0.9 & -0.1 \\ 0.5 & 0.8 \end{bmatrix},$$

$$L = \begin{bmatrix} -0.1 & 0 & 0.1 \\ -0.1 & -0.1 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.2 \\ 0 & 0.3 \end{bmatrix}, \ U_1 = V_1 = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.3 & 0 \\ -0.1 & 0.1 & 0.3 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.4 \end{bmatrix}, \ U_2 = V_2 = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.1 & -0.3 & -0.1 \\ -0.1 & 0 & -0.3 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -0.3 & 0 \\ -0.1 & -0.4 \end{bmatrix}, M = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \ N_i = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}^T,$$

$(i = 1, ..., 4)$, $\beta = 0.8$, $d_m = 2$, $d_M = 3$.

In this example, the H_∞ performance level is taken as $\gamma = 0.9$. In order to design output feedback controller, we first choose $\varepsilon = 0.4$, $Q = \mathrm{diag}\{0.4, 0.5, 0.4\}$, $X = \mathrm{diag}\{0.5, 0.6, 0.5\}$.

With the above parameters and by using the Matlab LMI Toolbox, we can solve the LMI (49). Furthermore, we choose

$$G = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \ R = \begin{bmatrix} -0.5412 & 0.2249 & 0.3215 \\ 0.4105 & -0.9993 & 0.2696 \\ -0.1454 & 1.0047 & -0.5463 \end{bmatrix}.$$

Therefore, by (50)(51), the parameters of the desired output feedback controller can be designed as

$$A_K = \begin{bmatrix} 3.7847 & -2.3238 & -1.5778 \\ 2.9212 & -2.5302 & -1.0455 \\ 3.8859 & -3.0151 & -1.6630 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 2.4653 & -2.8886 \\ 1.5387 & -2.2687 \\ 1.8090 & -3.7140 \end{bmatrix},$$

$$C_K = \begin{bmatrix} -0.0197 & -0.0302 & 0.0152 \\ -0.0462 & 0.0770 & 0.0113 \end{bmatrix},$$

with which, according to Theorem 3, the addressed uncertain nonlinear stochastic time-delay systems with missing measurements is robustly exponentially stable in the mean square and the effect of the disturbance input on the controlled output is constrained to the prescribed level.

5 Conclusions

In this paper, we have studied the problem of robust H_∞ output feedback control for a class of uncertain discrete-time delayed nonlinear stochastic systems with missing measurements. An output feedback controller has been designed for all admissible uncertainties such that the resulting closed-loop system is exponentially stable in the mean square for the zero disturbance input and also achieves a prescribed H_∞ performance level. A numerical example has been exploited to show the usefulness of the results obtained. Further research directions would include the investigation on more general nonlinear systems and the extension of our main results to more complex discrete-time systems with distributed delays.

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