Robust H_∞ Filtering for Discrete Nonlinear Stochastic Systems with Time-Varying Delay

Yurong Liu, Zidong Wang* and Xiaohui Liu

Abstract

In this paper, we are concerned with the robust H_∞ filtering problem for a class of nonlinear discrete time-delay stochastic systems. The system under study involves parameter uncertainties, stochastic disturbances, time-varying delays and sector-like nonlinearities. The problem addressed is the design of a full-order filter such that, for all admissible uncertainties, nonlinearities and time-delays, the dynamics of the filtering error is constrained to be robustly asymptotically stable in the mean square, and a prescribed H_∞ disturbance rejection attenuation level is also guaranteed. By using the Lyapunov stability theory and some new techniques, sufficient conditions are first established to ensure the existence of the desired filtering parameters. These conditions are dependent on the lower and upper bounds of the time-varying delays. Then, the explicit expression of the desired filter gains is described in terms of the solution to a linear matrix inequality (LMI). Finally, a numerical example is exploited to show the usefulness of the results derived.

Keywords

Stochastic system; H_∞ filtering; Robust filtering; Time-varying delays; Lyapunov-Krasovskii functional; Linear matrix inequality.

I. Introduction

The optimal filtering theory has been well studied for more than three decades, and has been successfully applied in various branches of science and engineering such as the areas of control design and signal processing. Much focus has been directed to dynamical systems subject to stationary Gaussian input and measurement noise processes [1], where the celebrated Kalman filtering can be applied. When there are uncertainties in either the exogenous input signals or the system model, the robust filtering problem comes into the scene and several techniques have been proposed with respect to various filtering performance criteria, such as the H_∞ specification, the minimum variance requirement and the so-called admissible variance constraint, see [6,7,11,18,25–28,30] and the references therein. On the other hand, since time delay is commonly encountered in various engineering systems and is frequently a source of instability and poor performance, in the past few years, there has been rapidly growing interest in robust and/or H_∞ filtering for linear systems with certain types of time-delays, see [2] for a survey. In the stochastic framework, for example, the Kalman filter design problem has been tackled in [19, 20, 31] for *linear* continuous- and discrete-time time-delay systems.

In another research front of nonlinear system theory, nonlinear filtering has been an attractive topic of subject for many years. For some recent works in the deterministic case, we refer the reader to [4, 16, 17]. For the stochastic case, the nonlinear filtering problem has received considerable attention, and a number of traditional approaches have been proposed in the literature, such as Gram-charlier expansion, Edgeworth

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Y. Liu is with the Department of Mathematics, Yangzhou University, Yangzhou 225002, P. R. China.

Z. Wang and X. Liu are with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom.

*All correspondences concerning this paper should be addressed to Z. Wang. (Email: Zidong.Wang@brunel.ac.uk)

expansion, extended Kalman filters, weighted sum of gaussian densities, generalized least-squares approximation and statistically linearized filters, see [5] for a survey. Among others, some later developments include the bound-optimal filters, exponentially bounded filters, exact finite dimensional filters, approximations by Markov chains, minimum variance filters, approximation of the Kushner equation, wavelet transform, etc. It is remarkable that, Tarn and Rasis [21] have tackled the nonlinear filtering problem through the concepts of observer for stochastic nonlinear systems, and have proposed an important stochastic stability approach to designing the observers with guaranteed convergence. In [3], the radial basis function neural networks have been exploited to approximate and estimate the nonlinear stochastic dynamics, and systematic procedures have been provided. In [29], the asymptotic stability problem for a general class of nonlinear stochastic time-delay systems has been thoroughly investigated. In [8, 22–24], the filtering problems have been studied for some continuous-time nonlinear stochastic *time-delay* systems.

It is well known that discrete-time systems play a very important role in digital signal analysis and processing. However, despite its importance, up to now, the robust H_∞ filtering problem for general nonlinear *discrete* time-delay systems has not been fully investigated and the relevant results have been very few. In [9], the output-feedback stabilization problem has been neatly solved for discrete-time systems with time-varying delay in the state, and a stability condition has been proposed that is dependent on the minimum and maximum delay bounds. Furthermore, in [10], the problem of robust H_∞ filtering has been thoroughly studied for discrete stochastic time-delay systems with parameter uncertainties and nonlinear disturbances, where the parameter uncertainty is assumed to be of the polytopic-type and the nonlinearity satisfies global Lipschitz conditions. Sufficient conditions for the existence of such filters have been formulated in [10] in terms of a set of linear matrix inequalities, upon which admissible filters can be obtained from the solution of a convex optimization problem. Nevertheless, the robust H_∞ filtering problem for time-delay stochastic systems with sector-like nonlinearities and norm-bounded uncertainties has not yet received much research attention and remains open.

In this paper, we are concerned with the robust H_∞ filtering problem for a class of nonlinear discrete time-delay stochastic systems. The system under study involves parameter uncertainties, stochastic disturbances, time-varying delays and inherent *sector*-like nonlinearities. Note that, among different descriptions of the nonlinearities, the so-called *sector nonlinearity* [13] has gained much attention for *deterministic* systems, and both the control analysis and model reduction problems have been investigated, see [12, 14, 15]. We aim at designing a full-order filter such that, for all admissible uncertainties, nonlinearities and time-delays, the dynamics of the estimation error is constrained to be robustly asymptotically stable in the mean square, and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We first investigate the sufficient conditions for the filtering error system to be stable in the mean square, and then derive the explicit expression of the desired controller gains. A numerical example is provided to demonstrate the proposed design method.

Notations: Throughout this paper, $\mathbb{N}^+$ stands for the set of nonnegative integers; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. The asterisk $\star$ in a matrix is used to denote term that is induced by symmetry. Matrices, if not explicitly specified, are assumed to have compatible dimensions. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation

Consider, on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, the following uncertain nonlinear stochastic system with time delays of the form:

$$\begin{aligned}
(\Sigma): x(k+1) &= A(k)x(k) + A_d(k)x(k-d(k)) + E(k)f(x(k)) + E_d(k)f_d(x(k-d(k))) + D_1(k)v(k) \\
&\quad + [G(k)x(k) + G_d(k)x(k-d(k)) + H(k)f(x(k)) + H_d(k)f_d(x(k-d(k))) \\
&\quad + D_2(k)v(k)]w(k), \qquad (1)\\
y(k) &= C(k)x(k) + \phi(Kx(k)) + C_d(k)x(k-d(k)) + g(Kx(k-d(k))) + D(k)v(k), \qquad (2)\\
z(k) &= Lx(k), \qquad (3)\\
x(j) &= \psi(j),\ j = -d_M, -d_M+1, ..., -1, 0, \qquad (4)
\end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $y(k) \in \mathbb{R}^m$ is the output or measurement; $z(k) \in \mathbb{R}^q$ is the signal to be estimated; $w(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$\mathbb{E}[w(k)] = 0, \quad \mathbb{E}[w^2(k)] = 1, \quad \mathbb{E}\{w(i)w(j)\} = 0\ (i \neq j) \qquad (5)$$

For the exogenous disturbance signal $v(k) \in \mathbb{R}^p$, it is assumed that $v(\cdot) \in l_{e2}([0,\infty); \mathbb{R}^p)$, where $l_{e2}([0,\infty); \mathbb{R}^p)$ is the space of non-anticipatory square-summable stochastic process $f(\cdot) = (f(k))_{k\in\mathbb{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathbb{N}}$ with the following norm:

$$\|f\|_{e_2} = \left\{\mathbb{E}\sum_{k=0}^{\infty}|f(k)|^2\right\}^{1/2} = \left\{\sum_{k=0}^{\infty}\mathbb{E}|f(k)|^2\right\}^{1/2}.$$

For system (Σ), the positive integer $d(k)$ denotes the time-varying delay satisfying

$$d_m \leq d(k) \leq d_M,\ k \in \mathbb{N}^+, \qquad (6)$$

where the lower bound d_m and the upper bound d_M are known positive integers. $\psi(j),\ j = -d_M, -d_M + 1, ..., -1, 0$, are the initial conditions, which are assumed to be independent of the process $\{w(\cdot)\}$.

In system (Σ), $L \in \mathbb{R}^{q\times n}$ and $K \in \mathbb{R}^{m\times n}$ are constant matrices, and the matrices $A(k), A_d(k), E(k), E_d(k), D_1(k), G(k), G_d(k), H(k), H_d(k), D_2(k), C(k), C_d(k)$ and $D(k)$ are time-varying matrices, which are assumed to be of the form:

$$\begin{aligned}
&A(k) = A + \Delta A(k),\ A_d(k) = A_d + \Delta A_d(k),\ E(k) = E + \Delta E(k),\ E_d(k) = E_d + \Delta E_d(k),\\
&G(k) = G + \Delta G(k),\ G_d(k) = G_d + \Delta G_d(k),\ H(k) = H + \Delta H(k),\ H_d(k) = H_d + \Delta H_d(k),\\
&D_1(k) = D_1 + \Delta D_1(k),\ D_2(k) = D_2 + \Delta D_2(k),\ C(k) = C + \Delta C(k),\ C_d(k) = C_d + \Delta C_d(k),\\
&D(k) = D + \Delta D(k).
\end{aligned}$$

Here, $A, A_d, E, E_d, D_1, G, G_d, H, H_d, D_2, C, C_d$ and D are known real constant matrices; $\Delta A(k), \Delta A_d(k), \Delta H(k), \Delta H_d(k), \Delta D_1(k), \Delta G(k), \Delta G_d(k), \Delta D_2(k), \Delta C(k), \Delta C_d(k)$ and $\Delta D(k)$ are unknown matrices representing time-varying parameter uncertainties, which are assumed to satisfy the following conditions:

$$\begin{bmatrix} \Delta A(k) & \Delta A_d(k) & \Delta E(k) & \Delta E_d(k) & \Delta D_1(k) \\ \Delta G(k) & \Delta G_d(k) & \Delta H(k) & \Delta H_d(k) & \Delta D_2(k) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F_1(t) \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix}, \qquad (7)$$

$$\begin{bmatrix} \Delta C(k) & \Delta C_d(k) & \Delta D(k) \end{bmatrix} = M_3 F_2(k) \begin{bmatrix} N_6 & N_7 & N_8 \end{bmatrix}, \qquad (8)$$

where $M_i (i = 1, 2, 3)$ and $N_i (i = 1, 2, ..., 8)$ are known real constant matrices and $F_i(k) (i = 1, 2)$ is the unknown time-varying matrix-valued function subject to the following condition:

$$F_i^T(k)F_i(k) \leq I,\ \forall k \in \mathbb{N}^+, i = 1, 2. \qquad (9)$$

Remark 1: The conditions (7)-(9) are referred to as the *admissible conditions*. These conditions have been frequently used to describe parameter uncertainties in many papers dealing with filtering and control problems for uncertain systems, see e.g. [6, 11, 19, 20, 22, 25–28].

The vector-valued nonlinear functions f, f_d, ϕ, g, are assumed to satisfy the following *sector-bounded* conditions:

$$[f(x) - R_1x]^T[f(x) - R_2x] \leq 0, \ \forall x \in \mathbb{R}^n, \tag{10}$$

$$[f_d(x) - S_1x]^T[f_d(x) - S_2x] \leq 0, \ \forall x \in \mathbb{R}^n, \tag{11}$$

$$[\phi(y) - U_1y]^T[\phi(y) - U_2y] \leq 0, \ \forall y \in \mathbb{R}^m, \tag{12}$$

$$[g(y) - W_1y]^T[g(y) - W_2y] \leq 0, \ \forall y \in \mathbb{R}^m, \tag{13}$$

where $R_1, R_2, S_1, S_2 \in \mathbb{R}^{n\times n}$, and $U_1, U_2, W_1, W_2 \in \mathbb{R}^{m\times m}$ are known real constant matrices, and $R = R_1 - R_2$, $S = S_1 - S_2$, $U = U_1 - U_2$ and $W = W_1 - W_2$ are symmetric positive definite matrices.

Remark 2: It is customary that the nonlinear functions f, f_d, ϕ, g, are said to belong to sectors $[R_1,\ R_2]$, $[S_1,\ S_2]$, $[U_1,\ U_2]$ and $[W_1,\ W_2]$, respectively [13]. The nonlinear descriptions in (10)-(12) are quite general that include the usual Lipschitz conditions as a special case. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied, see e.g. [12, 14, 15].

In this paper, we are concerned with the estimate $\hat{z}(k)$ of the signal $z(k)$ from the measured output $y(k)$. The full-order filter to be considered is given as follows:

$$(\Sigma_f): \qquad \hat{x}(k+1) = A_f\hat{x}(k) + B_fy(k), \tag{14}$$

$$\hat{z}(k) = L\hat{x}(k), \tag{15}$$

where $\hat{x}(k) \in \mathbb{R}^n$ and $\hat{z} \in \mathbb{R}^q$, and the constant matrices A_f and B_f are filter parameters to be determined.

Let $\tilde{x}(k) = x(k) - \hat{x}(k)$ and $\tilde{z}(k) = z(k) - \hat{z}(k)$. Then, from the systems (Σ) and (Σ_f), the filter error dynamics can be described by

$$\begin{aligned}
(\Sigma_e): x(k+1) &= A(k)x(k) + A_d(k)x(k-d(k)) + E(k)f(x(k)) + E_d(k)f_d(x(k-d(k))) + D_1(k)v(k) \\
&\quad + [G(k)x(k) + G_d(k)x(k-d(k)) + H(k)f(x(k)) + H_d(k)f_d(x(k-d(k))) \\
&\quad + D_2(k)v(k)]w(k), \\
\tilde{x}(k+1) &= \tilde{C}(k)x(t) + A_f\tilde{x}(k) + \tilde{C}_d(k)x(k-d(k)) + E(k)f(x(k)) + E_d(k)f_d(x(k-d(k))) \\
&\quad - B_f\phi(Kx(k)) - B_fg(K(k-d(k))) + \tilde{D}(k)v(t) + [G(k)x(k) + G_d(k)x(k-d(k)) \\
&\quad + H(k)f(x(k)) + H_d(k)f_d(x(k-d(k))) + D_2(k)v(k)]w(k), \\
\tilde{z}(k) &= L\tilde{x}(k), \\
x(j) &= \psi(j),
\end{aligned}$$

where $\tilde{C}(k) = A(k) - A_f - B_fC(k)$, $\tilde{C}_d(k) = A_d(k) - B_fC_d(k)$, and $\tilde{D}(k) = D_1(k) - B_fD(k)$.

The aim of this paper is to develop techniques to deal with the robust H_∞ filtering problem for uncertain discrete nonlinear stochastic systems (Σ) with time-varying delays. More specifically, given a disturbance attenuation level $\gamma > 0$, we like to design the parameters A_f and B_f of the filter (Σ_f) such that, in the presence of admissible uncertainties, time delays and nonlinearities, the following two requirements are satisfied:

(1) The filter error system (Σ_e) with $v(k) = 0$ is robustly asymptotically stable in the mean square.

(2) The filter error satisfies $\|\tilde{z}\|_{e_2} \leq \gamma\|v\|_{e_2}$ for any nonzero $v(\cdot) \in l_{e_2}([0, +\infty); \mathbb{R}^{n\times m})$ and all uncertainties.

III. Main Results

The following lemmas are essential in establishing our main results.

Lemma 1: Let $\mathcal{D}, \mathcal{S}$ and F be real matrices of appropriate dimensions with F satisfying $F^TF \le I$. Then, for any scalar $\varepsilon > 0$,

$$\mathcal{D}F\mathcal{S} + (\mathcal{D}F\mathcal{S})^T \le \varepsilon^{-1}\mathcal{D}\mathcal{D}^T + \varepsilon\mathcal{S}^T\mathcal{S}.$$

Lemma 2: (Schur Complement) Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^T$ and $\Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$$

if only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

First of all, let us deal with the stability analysis issue of the filtering error system (Σ_e), and derive a sufficient condition in the form of LMI so as to guarantee the robust mean-square asymptotic stability for the system (Σ_e) with $v(k) = 0$.

Theorem 1: Let the filter parameters A_f and B_f be given and the admissible conditions hold. Then, the filtering error system (Σ_e) with $v(t) = 0$ is robustly asymptotically stable in the mean square if there exist three positive definite matrices P_1, P_2, Q and six positive constant scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \varepsilon_1, \varepsilon_2$ such that the following LMI holds:

$$\Psi < 0, \tag{16}$$

where

$$\Psi = \begin{bmatrix}
\Omega & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & -P_2 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_1 & 0 & \Theta & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_2 & 0 & \Xi_3 & \Xi_4 & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_5 & 0 & \Xi_6 & \Xi_7 & \Xi_8 & \star & \star & \star & \star & \star & \star & \star \\
-\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star & \star \\
0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star & \star \\
P_1A & 0 & P_1A_d & P_1E & P_1E_d & 0 & 0 & -P_1 & \star & \star & \star & \star \\
\Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2E & P_2E_d & -Y & -Y & 0 & -P_2 & \star & \star & \star \\
\hat{P}G & 0 & \hat{P}G_d & \hat{P}H & \hat{P}H_d & 0 & 0 & 0 & 0 & -\hat{P} & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^TP_1 & M_1^TP_2 & M_2^T\hat{P} & -\varepsilon_1 I & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^TY^T & 0 & 0 & -\varepsilon_2 I
\end{bmatrix},$$

with

$$\breve{R}_1 = (R_1^TR_2 + R_2^TR_1)/2;\ \breve{R}_2 = -(R_1^T + R_2^T)/2; \tag{17}$$

$$\breve{S}_1 = (S_1^TS_2 + S_2^TS_1)/2;\ \breve{S}_2 = -(S_1^T + S_2^T)/2; \tag{18}$$

$$\breve{U}_1 = (K^TU_1^TU_2K + K^TU_2^TU_1K)/2;\ \breve{U}_2 = -(K^TU_1^T + K^TU_2^T)/2; \tag{19}$$

$$\breve{W}_1 = (K^TW_1^TW_2K + K^TW_2^TW_1K)/2; \breve{W}_2 = -(K^TW_1^T + K^TW_2^T)/2; \tag{20}$$

$$X = P_2A_f; Y = P_2B_f; \hat{P} = P_1 + P_2; \tag{21}$$

$$\Sigma_{\tilde{C}} = P_2A - X - YC;\ \Sigma_{\tilde{C}_d} = P_2A_d - YC_d; \tag{22}$$

$$\Omega = -P_1 + (d_M - d_m + 1)Q - \lambda_1\breve{R}_1 - \lambda_3\breve{U}_1 + \varepsilon_1N_1^TN_1 + \varepsilon_2N_6^TN_6, \tag{23}$$

$$\Theta = -Q - \lambda_2\breve{S}_1 - \lambda_4\breve{W}_1 + \varepsilon_1N_2^TN_2 + \varepsilon_2N_7^TN_7, \tag{24}$$

$$\Xi_1 = \varepsilon_1N_2^TN_1 + \varepsilon_2N_7^TN_6,\ \Xi_2 = -\lambda_1\breve{R}_2^T + \varepsilon_1N_3^TN_1, \tag{25}$$

$$\Xi_3 = \varepsilon_1N_3^TN_2,\ \Xi_4 = -\lambda_1 I + \varepsilon_1N_3^TN_3, \tag{26}$$

$$\Xi_5 = \varepsilon_1N_4^TN_1,\ \Xi_6 = -\lambda_2\breve{S}_2^T + \varepsilon_1N_4^TN_2, \tag{27}$$

$$\Xi_7 = \varepsilon_1N_4^TN_3,\ \Xi_8 = -\lambda_2 I + \varepsilon_1N_4^TN_4. \tag{28}$$

Proof: For the stability analysis of the system (Σ_e), we construct the following Lyapunov-Krasovskii functional:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{29}$$

where

$$\begin{aligned} V_1(k) &= x^T(k)P_1x(k), & (30)\\ V_2(k) &= \tilde{x}^T(k)P_2\tilde{x}(k), & (31)\\ V_3(k) &= \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i), & (32)\\ V_4(k) &= \sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j}^{k-1} x^T(i)Qx(i). & (33)\end{aligned}$$

Calculating the difference of $V(k)$ along the system (Σ_e) with $v(k) = 0$ and taking the mathematical expectation, we have

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\} + \mathbb{E}\{\Delta V_4(k)\}, \tag{34}$$

where

$$\begin{aligned} \mathbb{E}\{\Delta V_1(k)\} &= \mathbb{E}\{\Delta V_1(k+1) - \Delta V_1(k)\}\\ &= \mathbb{E}\left\{\mathcal{F}_0^T(k)P_1\mathcal{F}_0(k) + \mathcal{G}_0^T(k)P_1\mathcal{G}_0(k) - x^T(k)P_1x(k)\right\}, & (35)\\ \mathbb{E}\{\Delta V_2(k)\} &= \mathbb{E}\{\Delta V_2(k+1) - \Delta V_2(k)\}\\ &= \mathbb{E}\left\{\tilde{\mathcal{F}}_0^T(k)P_2\tilde{\mathcal{F}}_0(k) + \mathcal{G}_0^T(k)P_2\mathcal{G}_0(k) - \tilde{x}^T(k)P_2\tilde{x}(k)\right\}, & (36)\end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_0(k) &= A(k)x(k) + A_d(k)x(k-d(k)) + E(k)f(x(k)) + E_d(k)f_d(x(k-d(k))), & (37)\\ \tilde{\mathcal{F}}_0(k) &= \tilde{C}(k)x(t) + A_f\tilde{x}(k) + \tilde{C}_d(k)x(k-d(k)) + E(k)f(x(k)) + E_d(k)f_d(x(k-d(k)))\\ &\quad - B_f\phi(Kx(k)) - B_fg(Kx(k-d(k))), & (38)\\ \mathcal{G}_0(k) &= G(k)x(k) + G_d(k)x(k-d(k)) + H(k)f(x(k)) + H_d(k)f_d(x(k-d(k))), & (39)\end{aligned}$$

and, furthermore, $\mathbb{E}\{\Delta V_3(k)\}$ and $\mathbb{E}\{\Delta V_4(k)\}$ are computed as follows:

$$\begin{aligned} &\mathbb{E}\{\Delta V_3(k)\} = \mathbb{E}\{V_3(k+1) - V_3(k)\} = \mathbb{E}\left\{\sum_{i=k-d(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)\right\}\\ &= \mathbb{E}\left\{x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{i=k-d(k+1)+1}^{k-1} x^T(i)Qx(i) - \sum_{i=k-d(k)+1}^{k-1} x^T(i)Qx(i)\right\}\\ &= \mathbb{E}\left\{x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{i=k-d_m+1}^{k-1} x^T(i)Qx(i)\right.\\ &\quad \left. + \sum_{i=k-d(k)+1}^{d_m} x^T(i)Qx(i) - \sum_{i=k-d(k)+1}^{k-1} x^T(i)Qx(i)\right\}\\ &\le \mathbb{E}\left\{x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{i=k-d_M+1}^{k-d_m} x^T(i)Qx(i)\right\}, \qquad (40)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\{\Delta V_4(k)\} &= \mathbb{E}\{V_4(k+1)-V_4(k)\}\\ &= \mathbb{E}\left\{\sum_{j=k-d_M+2}^{k-d_m+1}\sum_{i=j}^{k}x^T(i)Qx(i)-\sum_{j=k-d_M+1}^{k-d_m}\sum_{i=j}^{k-1}x^T(i)Qx(i)\right\}\\ &= \mathbb{E}\left\{\sum_{j=k-d_M+1}^{k-d_m}\sum_{i=j+1}^{k}x^T(i)Qx(i)-\sum_{j=k-d_M+1}^{k-d_m}\sum_{i=j}^{k-1}x^T(i)Qx(i)\right\}\\ &= \mathbb{E}\left\{\sum_{j=k-d_M+1}^{k-d_m}\big(x^T(k)Qx(k)-x^T(j)Qx(j)\big)\right\}\\ &= \mathbb{E}\left\{(d_M-d_m)x^T(k)Qx(k)-\sum_{i=k-d_M+1}^{k-d_m}x^T(i)Qx(i)\right\}. \end{aligned} \tag{41}$$

Substituting (35)-(41) into (34) results in

$$\begin{aligned}\mathbb{E}\{\Delta V(k)\} &\leq \mathbb{E}\Big\{\mathcal{F}_0^T(k)P_1\mathcal{F}_0(k)+\mathcal{G}_0^T(k)P_1\mathcal{G}_0(k)+x^T(k)\Big[-P_1+(d_M-d_m+1)Q\Big]x(k)\\ &\quad -x^T(k-d(k))Qx(k-d(k))+\tilde{\mathcal{F}}_0^T(k)P_2\tilde{\mathcal{F}}_0(k)+\mathcal{G}_0^T(k)P_2\mathcal{G}_0(k)-\tilde{x}^T(k)P_2\tilde{x}(k)\Big\}\\ &= \mathbb{E}\big\{\xi_0^T(k)\Psi_1(k)\xi_0(k)+\xi_0^T(k)\bar{F}_0^T(k)P_1\bar{F}_0(k)\xi_0(k)+\xi_0^T(k)\tilde{F}_0^T(k)P_2\tilde{F}_0(k)\xi_0(k)\\ &\quad +\xi_0^T(k)\bar{G}_0^T(k)\hat{P}\bar{G}_0(k)\xi_0(k)\big\}, \end{aligned} \tag{42}$$

where $\hat{P}$ is defined in (21) and

$$\begin{aligned}\xi_0(k) &= [x^T(k)\ \ \tilde{x}^T(k)\ \ x^T(k-d(k))\ \ f^T(x(k))\ \ f_d^T(x(k-d(k)))\ \ \phi^T(Kx(k))\ \ g^T(Kx(k-d(k)))]^T,\\ \bar{F}_0(k) &= [A(k)\quad 0\quad A_d(k)\quad E(k)\quad E_d(k)\quad 0\quad 0],\\ \tilde{F}_0(k) &= [\tilde{C}(k)\quad A_f\quad \tilde{C}_d(k)\quad E(k)\quad E_d(k)\quad -B_f\quad -B_f],\\ \bar{G}_0(k) &= [G(k)\quad 0\quad G_d(k)\quad H(k)\quad H_d(k)\quad 0\quad 0],\\ \Psi_1(k) &= \begin{bmatrix}\Omega_1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -P_2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -Q & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},\end{aligned}$$

with $\Omega_1=-P_1+(d_M-d_m+1)Q$.

Notice (10) implies

$$\begin{bmatrix}x(k)\\ f(x(k))\end{bmatrix}^T\begin{bmatrix}\breve{R}_1 & \breve{R}_2\\ \breve{R}_2^T & I\end{bmatrix}\begin{bmatrix}x(k)\\ f(x(k))\end{bmatrix}\leq 0, \tag{43}$$

where $\breve{R}_1,\breve{R}_2$ are defined in (17)

Similarly, it follows from (11)-(13) that

$$\begin{bmatrix} x(k-d(k)) \\ f_d(x(k-\tau(k))) \end{bmatrix}^T \begin{bmatrix} \breve{S}_1 & \breve{S}_2 \\ \breve{S}_2^T & I \end{bmatrix} \begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix} \leq 0, \tag{44}$$

$$\begin{bmatrix} x(k) \\ \phi(Kx(k)) \end{bmatrix}^T \begin{bmatrix} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{bmatrix} \begin{bmatrix} x(k) \\ \phi(Kx(k)) \end{bmatrix} \leq 0, \tag{45}$$

$$\begin{bmatrix} x(k-\tau(k)) \\ g_d(Kx(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{bmatrix} \begin{bmatrix} x(k-d(k)) \\ g_d(Kx(k-d(k))) \end{bmatrix} \leq 0, \tag{46}$$

where $\breve{S}_1, \breve{S}_2, \breve{U}_1, \breve{U}_2, \breve{W}_1$ and $\breve{W}_2$ are defined in (18)-(20).

From (42)-(46), it follows that

$$\begin{aligned} \mathbb{E}\{\Delta V(k)\} \leq\ & \mathbb{E}\{\Delta V(k)\} - \mathbb{E}\Bigg\{\lambda_1 \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} \breve{R}_1 & \breve{R}_2 \\ \breve{R}_2^T & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\ & + \lambda_2 \begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \breve{S}_1 & \breve{S}_2 \\ \breve{S}_2^T & I \end{bmatrix} \begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix} \\ & + \lambda_3 \begin{bmatrix} x(k) \\ \phi(Kx(k)) \end{bmatrix}^T \begin{bmatrix} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{bmatrix} \begin{bmatrix} x(k) \\ \phi(Kx(k)) \end{bmatrix} \\ & + \lambda_4 \begin{bmatrix} x(k-d(k)) \\ g(Kx(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{bmatrix} \begin{bmatrix} x(k-d(k)) \\ g(Kx(k-d(k))) \end{bmatrix}\Bigg\} \\ =\ & \mathbb{E}\left\{\xi_0^T(k)\left[\Psi_2(k) + \bar{F}_0^T(k)P_1\bar{F}_0(k) + \tilde{F}_0^T(k)P_2\tilde{F}_0(k) + \bar{G}_0^T(k)\hat{P}\bar{G}_0(k)\right]\xi_0(k)\right\}, \end{aligned} \tag{47}$$

where

$$\Psi_2(k) = \begin{bmatrix} \Omega_2 & 0 & 0 & -\lambda_1\breve{R}_2 & 0 & -\lambda_3\breve{U}_2 & 0 \\ 0 & -P_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Theta_1 & 0 & -\lambda_2\breve{S}_2 & 0 & -\lambda_4\breve{W}_2 \\ -\lambda_1\breve{R}_2^T & 0 & 0 & -\lambda_1 I & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2\breve{S}_2^T & 0 & -\lambda_2 I & 0 & 0 \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & 0 \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I \end{bmatrix},$$

where

$$\begin{aligned} \Omega_2 &= -P_1 + (d_M - d_m + 1)Q - \lambda_1\breve{R}_1 - \lambda_3\breve{U}_1, \\ \Theta_1 &= -Q - \lambda_2\breve{S}_1 - \lambda_4\breve{W}_1. \end{aligned}$$

We know from Lyapunov stability theory that, in order to ensure the asymptotic stability of the system (Σ_e) with $v(k) = 0$, we need to show $\Psi_2(k) + \bar{F}_0^T(k)P_1\bar{F}_0(k) + \tilde{F}_0^T(k)P_2\tilde{F}_0(k) + \bar{G}_0^T(k)\hat{P}\bar{G}_0(k)\} < 0$ which, by

Lemma 2 (Schur Complement), is equivalent to

$$\begin{aligned}
\Psi_3(k) &= \begin{bmatrix} \Psi_2(k) & \bar{F}_0^T(k)P_1 & \tilde{F}_0^T(k)P_2 & \bar{G}_0^T(k)P_3 \\ P_1\tilde{F}_0(k) & -P_1 & 0 & 0 \\ P_2\tilde{F}_0(k) & 0 & -P_2 & 0 \\ P_3G_0(k) & 0 & 0 & -P_3 \end{bmatrix} \\
&= \begin{bmatrix} \Omega_2 & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & -P_2 & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \Theta_1 & \star & \star & \star & \star & \star & \star & \star \\ -\lambda_1\breve{R}_2^T & 0 & 0 & -\lambda_1 I & \star & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_2\breve{S}_2^T & 0 & -\lambda_2 I & \star & \star & \star & \star & \star \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star \\ P_1A(k) & 0 & P_1A_d(k) & P_1E(k) & P_1E_d(k) & 0 & 0 & -P_1 & \star & \star \\ P_2\tilde{C}(k) & X & P_2\tilde{C}_d(k) & P_2E(k) & P_2E_d(k) & -Y & -Y & 0 & -P_2 & \star \\ \hat{P}G(k) & 0 & \hat{P}G_d(k) & \hat{P}H(k) & \hat{P}H_d(k) & 0 & 0 & 0 & 0 & -\hat{P} \end{bmatrix} \\
&\leq 0.
\end{aligned}$$

Note that $\Psi_3(k)$ can be rewritten as follows:

$$\Psi_3(k) = \Psi_3 + \Delta\Psi_3(k), \tag{48}$$

where

$$\Psi_3 = \begin{bmatrix} \Omega_2 & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & -P_2 & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \Theta_1 & \star & \star & \star & \star & \star & \star & \star \\ -\lambda_1\breve{R}_2^T & 0 & 0 & -\lambda_1 I & \star & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_2\breve{S}_2^T & 0 & -\lambda_2 I & \star & \star & \star & \star & \star \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star \\ P_1A & 0 & P_1A_d & P_1E & P_1E_d & 0 & 0 & -P_1 & \star & \star \\ \Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2E & P_2E_d & -Y & -Y & 0 & -P_2 & \star \\ \hat{P}G & 0 & \hat{P}G_d & \hat{P}H & \hat{P}H_d & 0 & 0 & 0 & 0 & -\hat{P} \end{bmatrix},$$

with $\Sigma_{\tilde{C}}$ and $\Sigma_{\tilde{C}_d}$ being defined in (22), and

$$\Delta\Psi_3 = \begin{bmatrix} 0 & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star \\ P_1\Delta A(k) & 0 & P_1\Delta A_d(k) & P_1\Delta E(k) & P_1\Delta E_d(k) & 0 & 0 & 0 & \star & \star \\ \Delta\Sigma_{\tilde{C}}(k) & 0 & \Delta\Sigma_{\tilde{C}_d}(k) & P_2\Delta E(k) & P_2\Delta E_d(k) & 0 & 0 & 0 & 0 & \star \\ \hat{P}\Delta G(k) & 0 & \hat{P}\Delta G_d(k) & \hat{P}\Delta H(k) & \hat{P}\Delta H_d(k) & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with $\Delta\Sigma_{\tilde{C}}(k) = P_2\Delta A(k) - Y\Delta C(k)$ and $\Delta\Sigma_{\tilde{C}_d}(k) = P_2\Delta A_d(k) - Y\Delta C_d(k)$.

Let

$$\bar{M}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^T P_1 & M_1^T P_2 & M_2^T \hat{P} \end{bmatrix}^T, \quad (49)$$

$$\bar{M}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^T Y^T & 0 \end{bmatrix}^T, \quad (50)$$

$$\bar{N}_1 = \begin{bmatrix} N_1 & 0 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (51)$$

$$\bar{N}_2 = \begin{bmatrix} N_6 & 0 & N_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (52)$$

Using Eq. (7) and Lemma 2, one can have

$$\begin{aligned} \Delta\Psi_3 &= \bar{M}_1 F_1(k) \bar{N}_1 + \bar{N}_1^T F_1^T(k) \bar{M}_1^T - \bar{M}_2 F_2(k) \bar{N}_2 - \bar{N}_2^T F_2^T(k) \bar{M}_2^T \\ &\leq \varepsilon_1^{-1} \bar{M}_1 \bar{M}_1^T + \varepsilon_2^{-1} \bar{M}_2 \bar{M}_2^T + \varepsilon_1 \bar{N}_1^T \bar{N}_1 + \varepsilon_2 \bar{N}_2^T \bar{N}_2. \end{aligned} \quad (53)$$

It is implied from (48) and (53) that

$$\Psi_3(k) \leq \Psi_4 + \varepsilon_1^{-1} \bar{M}_1 \bar{M}_1^T + \varepsilon_2^{-1} \bar{M}_2 \bar{M}_2^T, \quad (54)$$

where

$$\Psi_4 = \begin{bmatrix} \Omega & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & -P_2 & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \Xi_1 & 0 & \Theta & \star & \star & \star & \star & \star & \star & \star & \star \\ \Xi_2 & 0 & \Xi_3 & \Xi_4 & \star & \star & \star & \star & \star & \star & \star \\ \Xi_5 & 0 & \Xi_6 & \Xi_7 & \Xi_8 & \star & \star & \star & \star & \star & \star \\ -\lambda_3 \breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_4 \breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & \star & \star & \star \\ P_1 A & 0 & P_1 A_d & P_1 E & P_1 E_d & 0 & 0 & P_1 D_1 & -P_1 & \star & \star \\ \Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2 E & P_2 E_d & -Y & -Y & \Sigma_{\tilde{D}} & 0 & -P_2 & \star \\ \hat{P} G & 0 & \hat{P} G_d & \hat{P} H & \hat{P} H_d & 0 & 0 & \hat{P} D_2 & 0 & 0 & -\hat{P} \end{bmatrix}$$

and $\Omega, \Theta, \Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6, \Xi_7, \Xi_8$ are defined in (23)-(28).

Now, it follows from Lemma 2 (schur complement) that (16) (i.e. $\Psi < 0$) is equivalent to the fact that the right-hand side of (54) is negative definite. Therefore, we arrive at the conclusion that $\Psi_3(k) < 0$, which indicates that the filtering error system (Σ_e) with $v(k) = 0$ is robustly stable in the mean square. ∎

Next, we consider the H_∞ performance of the filtering error system (Σ_e).

Theorem 2: Let the filter parameters A_f and B_f be given and $\gamma > 0$ be a positive constant. Then, the filtering error system (Σ_e) is robustly asymptotically stable in the mean square for $v(k) = 0$ and satisfies $\|\tilde{z}\|_{e_2} \leq \gamma \|v\|_{e_2}$ for any nonzero $v(\cdot) \in l_{e_2}([0, +\infty); \mathbb{R}^{n \times m})$ if there exist three positive definite matrices P_1, P_2, Q and eight positive constant scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ such that the following LMI holds:

$$\Phi_0 < 0, \quad (55)$$

where

$$\Phi_0 = \left[\begin{array}{ccccccccccccccc}
\Omega & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & \Upsilon & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_1 & 0 & \Theta & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_2 & 0 & \Xi_3 & \Xi_4 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_5 & 0 & \Xi_6 & \Xi_7 & \Xi_8 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
-\lambda_3 \breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & 0 & -\lambda_4 \breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & \star & \star & \star & \star & \star & \star & \star \\
P_1 A & 0 & P_1 A_d & P_1 E & P_1 E_d & 0 & 0 & P_1 D_1 & -P_1 & \star & \star & \star & \star & \star & \star \\
\Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2 E & P_2 E_d & -Y & -Y & \Sigma_{\tilde{D}} & 0 & -P_2 & \star & \star & \star & \star & \star \\
\hat{P} G & 0 & \hat{P} G_d & \hat{P} H & \hat{P} H_d & 0 & 0 & \hat{P} D_2 & 0 & 0 & -\hat{P} & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^T P_1 & M_1^T P_2 & M_2^T \hat{P} & -\varepsilon_1 I & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^T Y^T & 0 & 0 & -\varepsilon_2 I & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^T P_1 & M_1^T P_2 & M_2^T \hat{P} & 0 & 0 & -\varepsilon_3 I & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^T Y^T & 0 & 0 & 0 & 0 & -\varepsilon_4 I
\end{array}\right],$$

with

$$\Upsilon = -P_2 + L^T L, \tag{56}$$

$$\Gamma = -\gamma^2 I + \varepsilon_3 N_5^T N_5 + \varepsilon_4 N_8^T N_8, \tag{57}$$

$$\Sigma_{\tilde{D}} = P_2 D_1 - Y D, \tag{58}$$

and $\breve{R}_1, \breve{R}_2, \breve{S}_1, \breve{S}_2, \breve{U}_1, \breve{U}_2, \breve{W}_1, \breve{W}_2, X, Y, \Omega, \Sigma_{\tilde{C}}, \Sigma_{\tilde{C}_d}, \Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6, \Xi_7, \Xi_8$ are defined as in Theorem 1.

Proof: First, it is easy to see that $\Phi_0 < 0$ implies that $\Psi < 0$ and, therefore, according to Theorem 1, the filtering error system (Σ_e) with $v(k) = 0$ is robustly asymptotically stable in the mean square.

Next, let us deal with the H_∞ performance of the system (Σ_e). Introduce the same Lyapunov-Krasovskii functional as in Theorem 1:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{59}$$

where $V_1(k), V_2(k), V_3(k), V_4(k)$ are defined in (30)-(33).

Similar to the calculation in the proof of Theorem 1, we obtain the mathematical expectation of the difference of $V(k)$ along the system (Σ_c) as follows:

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\} + \mathbb{E}\{\Delta V_4(k)\}. \tag{60}$$

Here

$$\begin{aligned}\mathbb{E}\{\Delta V_1(k)\} &= \mathbb{E}\left\{\mathcal{F}^T(k)P_1\mathcal{F}(k)+\mathcal{G}^T(k)P_1\mathcal{G}(k)-x^T(k)P_1x(k)\right\}, && (61)\\ \mathbb{E}\{\Delta V_2(k)\} &= \mathbb{E}\left\{\tilde{\mathcal{F}}^T(k)P_2\tilde{\mathcal{F}}(k)+\mathcal{G}^T(k)P_2\mathcal{G}(k)-\tilde{x}^T(k)P_2\tilde{x}(k)\right\}, && (62)\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}(k) &= A(k)x(k)+A_d(k)x(k-d(k))+E(k)f(x(k))+E_d(k)f_d(x(k-d(k)))+D_1(k)v(k), && (63)\\ \tilde{\mathcal{F}}(k) &= \tilde{C}(k)x(t)+A_f\tilde{x}(k)+\tilde{C}_d(k)x(k-d(k))+E(k)f(x(k))+E_d(k)f_d(x(k-d(k)))\\ &\quad -B_f\phi(Kx(k))-B_fg(Kx(k-d(k)))+\tilde{D}(k)v(t), && (64)\\ \mathcal{G}(k) &= G(k)x(k)+G_d(k)x(k-d(k))+H(k)f(x(k))+H_d(k)f_d(x(k-d(k)))+D_2(k)v(k), && (65)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\{\Delta V_3(k)\} &\le \mathbb{E}\left\{x^T(k)Qx(k)-x^T(k-d(k))Qx(k-d(k))+\sum_{i=k-d_M+1}^{k-d_m}x^T(i)Qx(i)\right\}, && (66)\\ \mathbb{E}\{\Delta V_4(k)\} &= \mathbb{E}\left\{(d_M-d_m)x^T(k)Qx(k)-\sum_{i=k-d_M+1}^{k-d_m}x^T(i)Qx(i)\right\}. && (67)\end{aligned}$$

Substituting (61)-(67) into (60) leads to

$$\begin{aligned}\mathbb{E}\{\Delta V(k)\} &\le \mathbb{E}\Big\{\mathcal{F}^T(k)P_1\mathcal{F}(k)+\mathcal{G}^T(k)P_1\mathcal{G}(k)+x^T(k)\Big[-P_1+(d_M-d_m+1)Q\Big]x(k)\\ &\quad -x^T(k-d(k))Qx(k-d(k))+\tilde{\mathcal{F}}^T(k)P_2\tilde{\mathcal{F}}(k)+\mathcal{G}^T(k)P_2\mathcal{G}(k)-\tilde{x}^T(k)P_2\tilde{x}(k)\Big\}\\ &= \mathbb{E}\big\{\xi^T(k)\Phi_1(k)\xi(k)+\xi^T(k)\bar{F}^T(k)P_1\bar{F}(k)\xi(k)+\xi^T(k)\tilde{F}^T(k)P_2\tilde{F}(k)\xi(k)\\ &\quad +\xi^T(k)\bar{G}^T(k)\hat{P}\bar{G}(k)\xi(k)\big\}, && (68)\end{aligned}$$

where

$$\begin{aligned}\xi(k) &= [x^T(k)\ \ \tilde{x}^T(k)\ \ x^T(k-d(k))\ \ f^T(x(k))\ \ f_d^T(x(k-d(k)))\ \ \phi^T(Kx(k))\ \ g^T(Kx(k-d(k)))\ \ v^T(x(k))]^T,\\ \bar{F}(k) &= [A(k)\quad 0\quad A_d(k)\quad E(k)\quad E_d(k)\quad 0\quad 0\quad D_1(k)],\\ \tilde{F}(k) &= [\tilde{C}(k)\quad A_f\quad \tilde{C}_d(k)\quad E(k)\quad E_d(k)\quad -B_f\quad -B_f\quad \tilde{D}(k)],\\ \bar{G}(k) &= [G(k)\quad 0\quad G_d(k)\quad H(k)\quad H_d(k)\quad 0\quad 0\quad D_2(k)],\\ \Phi_1(k) &= \begin{bmatrix}\Omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -P_2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -Q & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},\end{aligned}$$

with Ω_1 being defined as in Theorem 1.

We are now ready to deal with the H_∞ performance of the filtering process. Introduce

$$J(n)=\mathbb{E}\sum_{k=0}^{n}\left[\tilde{z}^T(k)\tilde{z}(k)-\gamma^2v^T(k)v(k)\right] \tag{69}$$

where n is non-negative integer. Obviously, our goal is to show $J(n)<0$.

Under the zero initial condition, one has

$$\begin{aligned} J(n) &= \mathbb{E}\sum_{k=0}^{n}\left[\tilde{z}^T(k)\tilde{z}(k)-\gamma^2 v^T(k)v(k)+\Delta V(k)\right]-\mathbb{E}V(n+1) \\ &\le \mathbb{E}\sum_{k=0}^{n}\Big[\tilde{x}^T(k)L^T(k)L(k)\tilde{x}(k)-\gamma^2 v^T(k)v(k)+\xi^T(k)\Phi_1(k)\xi(k) \\ &\quad +\xi^T(k)\bar{F}^T(k)P_1\bar{F}(k)\xi(k)+\xi^T(k)\tilde{F}^T(k)P_2\tilde{F}(k)\xi(k)+\xi^T(k)\bar{G}^T(k)\hat{P}\bar{G}(k)\xi(k)\Big] \\ &= \mathbb{E}\sum_{k=0}^{n}\Big[\xi^T(k)\Phi_2\xi(k)+\xi^T(k)\bar{F}^T(k)P_1\bar{F}(k)\xi(k)+\xi^T(k)\tilde{F}^T(k)P_2\tilde{F}(k)\xi(k) \\ &\quad +\xi^T(k)\bar{G}^T(k)\hat{P}\bar{G}(k)\xi(k)\Big], \end{aligned} \tag{70}$$

where

$$\Phi_2(k)=\begin{bmatrix} \Omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Upsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \tag{71}$$

with Υ being defined in (56) and Ω_2 defined as in Theorem 1.

From (43)-(46), it is not difficult to see that

$$\begin{aligned} J(n) &\le J(n)-\mathbb{E}\Bigg\{\lambda_1\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T\begin{bmatrix} \breve{R}_1 & \breve{R}_2 \\ \breve{R}_2^T & I \end{bmatrix}\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\ &\quad +\lambda_2\begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix}^T\begin{bmatrix} \breve{S}_1 & \breve{S}_2 \\ \breve{S}_2^T & I \end{bmatrix}\begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix} \\ &\quad +\lambda_3\begin{bmatrix} x(k) \\ \phi(Kx(k)) \end{bmatrix}^T\begin{bmatrix} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{bmatrix}\begin{bmatrix} x(k) \\ \phi(Kx(k)) \end{bmatrix} \\ &\quad +\lambda_4\begin{bmatrix} x(k-d(k)) \\ g(Kx(k-d(k))) \end{bmatrix}^T\begin{bmatrix} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{bmatrix}\begin{bmatrix} x(k-d(k)) \\ g(Kx(k-d(k))) \end{bmatrix}\Bigg\} \\ &= \mathbb{E}\left\{\xi^T(k)\left[\Phi_3(k)+\bar{F}^T(k)P_1\bar{F}(k)+\tilde{F}^T(k)P_2\tilde{F}(k)+\bar{G}^T(k)\hat{P}\bar{G}(k)\right]\xi(k)\right\}, \end{aligned} \tag{72}$$

where

$$\Phi_3(k)=\begin{bmatrix} \Omega_2 & 0 & 0 & -\lambda_1\breve{R}_2 & 0 & -\lambda_3\breve{U}_2 & 0 & 0 \\ 0 & \Upsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Theta_1 & 0 & -\lambda_2\breve{S}_2 & 0 & -\lambda_4\breve{W}_2 & 0 \\ -\lambda_1\breve{R}_2^T & 0 & 0 & -\lambda_1 I & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2\breve{S}_2^T & 0 & -\lambda_2 I & 0 & 0 & 0 \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & 0 & 0 \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

with Ω_2 and Θ_1 being defined as in Theorem 1.

By (72), in order to guarantee $J(n) < 0$, we just need to show

$$\Phi_3(k) + \bar{F}^T(k)P_1\bar{F}(k) + \tilde{F}^T(k)P_2\tilde{F}(k) + \bar{G}^T(k)\hat{P}\bar{G}(k) < 0,$$

which, by Lemma 2 (Schur Complement), is equivalent to

$$\Phi_4(k) < 0,$$

where

$$\Phi_4(k) = \begin{bmatrix} \Omega_2 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & \Upsilon & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \Theta_1 & \star & \star & \star & \star & \star & \star & \star & \star \\ -\lambda_1\breve{R}_2^T & 0 & 0 & -\lambda_1 I & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_2\breve{S}_2^T & 0 & -\lambda_2 I & \star & \star & \star & \star & \star & \star \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & \star & \star & \star \\ P_1A(k) & 0 & P_1A_d(k) & P_1E(k) & P_1E_d(k) & 0 & 0 & P_1D_1(k) & -P_1 & \star & \star \\ P_2\tilde{C}(k) & X & P_2\tilde{C}_d(k) & P_2E(k) & P_2E_d(k) & -Y & -Y & P_2\tilde{D}(k) & 0 & -P_2 & \star \\ \hat{P}G(k) & 0 & \hat{P}G_d(k) & \hat{P}H(k) & \hat{P}H_d(k) & 0 & 0 & \hat{P}D_2(k) & 0 & 0 & -\hat{P} \end{bmatrix}.$$

Notice that $\Phi_4(k)$ can be rearranged as follows:

$$\Phi_4(k) = \Phi_4 + \Delta\Phi_4(k), \tag{73}$$

where

$$\Phi_4 = \begin{bmatrix} \Omega_2 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & \Upsilon & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \Theta_1 & \star & \star & \star & \star & \star & \star & \star & \star \\ -\lambda_1\breve{R}_2^T & 0 & 0 & -\lambda_1 I & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_2\breve{S}_2^T & 0 & -\lambda_2 I & \star & \star & \star & \star & \star & \star \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & \star & \star & \star \\ P_1A & 0 & P_1A_d & P_1E & P_1E_d & 0 & 0 & P_1D_1 & -P_1 & \star & \star \\ \Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2E & P_2E_d & -Y & -Y & \Sigma_{\tilde{D}} & 0 & -P_2 & \star \\ \hat{P}G & 0 & \hat{P}G_d & \hat{P}H & \hat{P}H_d & 0 & 0 & \hat{P}D_2 & 0 & 0 & -\hat{P} \end{bmatrix},$$

and

$$\Delta\Phi_4(k) = \begin{bmatrix} 0 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star \\ P_1\Delta A(k) & 0 & P_1\Delta A_d(k) & P_1\Delta E(k) & P_1\Delta E_d(k) & 0 & 0 & P_1\Delta D_1(k) & 0 & \star & \star \\ \Delta\Sigma_{\tilde{C}}(k) & 0 & \Delta\Sigma_{\tilde{C}_d}(k) & P_2\Delta E(k) & P_2\Delta E_d(k) & 0 & 0 & \Delta\Sigma_{\tilde{D}}(k) & 0 & 0 & \star \\ \hat{P}\Delta G(k) & 0 & \hat{P}\Delta G_d(k) & \hat{P}\Delta H(k) & \hat{P}\Delta H_d(k) & 0 & 0 & \hat{P}\Delta D_2(k) & 0 & 0 & 0 \end{bmatrix},$$

with $\Delta\Sigma_{\tilde{C}}(k) = P_2\Delta A(k) - Y\Delta C(k), \Delta\Sigma_{\tilde{C}_d}(k) = P_2\Delta A_d(k) - Y\Delta C_d(k)$ and $\Delta\Sigma_{\tilde{D}}(k) = P_2\Delta D_1(k) - Y\Delta D(k)$.

Let

$$\hat{M}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^T P_1 & M_1^T P_2 & M_2^T \hat{P} \end{bmatrix}^T, \quad (74)$$

$$\hat{M}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^T Y^T & 0 \end{bmatrix}^T, \quad (75)$$

$$\hat{N}_1 = \begin{bmatrix} N_1 & 0 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (76)$$

$$\hat{N}_2 = \begin{bmatrix} N_6 & 0 & N_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (77)$$

$$\hat{N}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_5 & 0 & 0 & 0 \end{bmatrix}, \quad (78)$$

$$\hat{N}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_8 & 0 & 0 & 0 \end{bmatrix}. \quad (79)$$

It follows easily from (7) and Lemma 2 that

$$\begin{aligned} \Delta\Phi_4(k) &= \hat{M}_1 F_1(k)\hat{N}_1 + \hat{N}_1^T F_1^T(k)\hat{M}_1^T - \hat{M}_2 F_2(k)\hat{N}_2 - \hat{N}_2^T F_2^T(k)\hat{M}_2^T \\ &\quad + \hat{M}_1 F_1(k)\hat{N}_3 + \hat{N}_3^T F_1(k)^T \hat{M}_1^T - \hat{M}_2 F_2(k)\hat{N}_4 - \hat{N}_4^T F_2^T(k)\hat{M}_2^T \\ &\le \varepsilon_1^{-1}\hat{M}_1\hat{M}_1^T + \varepsilon_2^{-1}\hat{M}_2\hat{M}_2^T + \varepsilon_3^{-1}\hat{M}_1\hat{M}_1^T + \varepsilon_4^{-1}\hat{M}_2\hat{M}_2^T \\ &\quad + \varepsilon_1\hat{N}_1^T\hat{N}_1 + \varepsilon_2\hat{N}_2^T\hat{N}_2 + \varepsilon_3\hat{N}_3^T\hat{N}_3 + \varepsilon_4\hat{N}_4^T\hat{N}_4. \end{aligned} \quad (80)$$

and then it can be obtained from (73) and (80) that

$$\Phi_4(k) \le \Phi_5 + \varepsilon_1^{-1}\hat{M}_1\hat{M}_1^T + \varepsilon_2^{-1}\hat{M}_2\hat{M}_2^T + \varepsilon_3^{-1}\hat{M}_1\hat{M}_1^T + \varepsilon_4^{-1}\hat{M}_2\hat{M}_2^T, \quad (81)$$

where

$$\Phi_5 = \begin{bmatrix} \Omega & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & \Upsilon & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \Xi_1 & 0 & \Theta & \star & \star & \star & \star & \star & \star & \star & \star \\ \Xi_2 & 0 & \Xi_3 & \Xi_4 & \star & \star & \star & \star & \star & \star & \star \\ \Xi_5 & 0 & \Xi_6 & \Xi_7 & \Xi_8 & \star & \star & \star & \star & \star & \star \\ -\lambda_3\breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star \\ 0 & 0 & -\lambda_4\breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & \star & \star & \star \\ P_1 A & 0 & P_1 A_d & P_1 E & P_1 E_d & 0 & 0 & P_1 D_1 & -P_1 & \star & \star \\ \Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2 E & P_2 E_d & -Y & -Y & \Sigma_{\tilde{D}} & 0 & -P_2 & \star \\ \hat{P}G & 0 & \hat{P}G_d & \hat{P}H & \hat{P}H_d & 0 & 0 & \hat{P}D_2 & 0 & 0 & -\hat{P} \end{bmatrix}.$$

By Lemma 2, (55) (i.e. $\Phi_0 < 0$) holds if and only if the right-hand side of (81) is negative definite, which implies $J(n) < 0$. Letting $n \to \infty$, we have

$$\|\tilde{z}\|_{e2} \leq \gamma \|v\|_{e2},$$

which completes the proof of the theorem.

■

Finally, we are in a position to solve the H_∞ filter design problem for the system (Σ). The following result can be easily accessible from Theorem 2, hence the proof is omitted.

Theorem 3: Let $\gamma > 0$ be a given positive constant and the admissible conditions hold. Then, for the nonlinear stochastic system (Σ), an H_∞ filter (Σ_f) can be designed such that the filtering error system (Σ_e) is robustly mean-square asymptotically stable for $v(k) = 0$ and also satisfies $\|\tilde{z}\|_{e_2} \leq \gamma \|v\|_{e_2}$ under the zero initial condition for any nonzero $v(\cdot) \in l_{e_2}([0, +\infty); \mathbb{R}^{n \times m})$ if there exist five real constant matrices $P_1 > 0, P_2 > 0, Q > 0, X, Y$ and eight scalars $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0, \varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \varepsilon_4 > 0$ such that the following LMI holds:

$$\Phi < 0, \tag{82}$$

where

$$\Phi_0 = \left[\begin{array}{cccccccccccccccc}
\Omega & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & \Upsilon & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_1 & 0 & \Theta & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_2 & 0 & \Xi_3 & \Xi_4 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\Xi_5 & 0 & \Xi_6 & \Xi_7 & \Xi_8 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
-\lambda_3 \breve{U}_2^T & 0 & 0 & 0 & 0 & -\lambda_3 I & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & 0 & -\lambda_4 \breve{W}_2^T & 0 & 0 & 0 & -\lambda_4 I & \star & \star & \star & \star & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & \star & \star & \star & \star & \star & \star & \star \\
P_1 A & 0 & P_1 A_d & P_1 E & P_1 E_d & 0 & 0 & P_1 D_1 & -P_1 & \star & \star & \star & \star & \star & \star \\
\Sigma_{\tilde{C}} & X & \Sigma_{\tilde{C}_d} & P_2 E & P_2 E_d & -Y & -Y & \Sigma_{\tilde{D}} & 0 & -P_2 & \star & \star & \star & \star & \star \\
\hat{P} G & 0 & \hat{P} G_d & \hat{P} H & \hat{P} H_d & 0 & 0 & \hat{P} D_2 & 0 & 0 & -\hat{P} & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^T P_1 & M_1^T P_2 & M_2^T \hat{P} & -\varepsilon_1 I & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^T Y^T & 0 & 0 & -\varepsilon_2 I & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^T P_1 & M_1^T P_2 & M_2^T \hat{P} & 0 & 0 & -\varepsilon_3 I & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^T Y^T & 0 & 0 & 0 & 0 & -\varepsilon_4 I
\end{array}\right],$$

and $\breve{R}_1, \breve{R}_2, \breve{S}_1, \breve{S}_2, \breve{U}_1, \breve{U}_2, \breve{W}_1, \breve{W}_2, \Omega, \Upsilon, \Theta, \Gamma, \Sigma_{\tilde{C}}, \Sigma_{\tilde{C}_d}, \Sigma_{\tilde{D}}, \Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6, \Xi_7$ and Ξ_8 are defined as in Theorems 1 and 2. Furthermore, the filter parameters can be designed as follows

$$A_f = P_2^{-1} X, \quad B_f = P_2^{-1} Y.$$

Remark 3: The robust H_∞ filter design problem is solved in Theorem 3 for the addressed uncertain nonlinear stochastic time-delay systems. We derive an LMI-based sufficient condition for the existence of full-order filters that ensure the mean-square asymptotic stability of the resulting filtering error system and reduce the effect of the disturbance input on the estimated signal to a prescribed level for all admissible uncertainties. The feasibility of the filter design problem can be readily checked by the solvability of an LMI, which is dependent on the lower bound and upper bound of the time-varying delays. The solvability of such a delay-dependent LMI can be readily checked by resorting to the Matlab LMI toolbox. In next section, an illustrative example will be provided to show the potential of the proposed techniques.

IV. Numerical Example

In this section, a numerical example is presented to demonstrate the usefulness of the developed method on the design of robust H_∞ filter for the discrete uncertain nonlinear stochastic systems with time-varying delays.

Consider the system (Σ) with the following parameters:

$$\begin{aligned}
A &= \begin{bmatrix} 0.5 & 0 & 0.1 \\ 0.1 & -0.4 & 0.1 \\ 0.1 & 0 & -0.4 \end{bmatrix}, \; A_d = \begin{bmatrix} 0.1 & -0.1 & 0 \\ 0.1 & -0.2 & 0 \\ 0 & -0.2 & -0.1 \end{bmatrix}, \; E = H = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \\
E_d &= H_d = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix}, \; G = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & 0.2 & 0.1 \\ -0.1 & 0 & 0.1 \end{bmatrix}, \; G_d = \begin{bmatrix} -0.1 & 0 & 0.1 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.1 & 0 \end{bmatrix}, \\
L &= \begin{bmatrix} -0.1 & -0 & 0.1 \\ -0.1 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \; D_1 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \; D_2 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.2 \\ 0 & 0.3 \end{bmatrix}, \; K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0.8 & 0.7 \\ -0.6 & 0.9 & 0.6 \end{bmatrix}, \; C_d = \begin{bmatrix} 0.9 & -0.6 & 0.8 \\ 0.5 & 0.8 & 0.7 \end{bmatrix}, \; D = \begin{bmatrix} 0.9 & -0.6 \\ 0.5 & 0.8 \end{bmatrix}, \\
R_1 &= S_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix}, R_2 = S_2 = \begin{bmatrix} -0.2 & -0.1 & -0.1 \\ 0 & -0.2 & -0.1 \\ 0 & -0.1 & -0.1 \end{bmatrix}, \\
U_1 &= W_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.2 \end{bmatrix}, U_2 = W_2 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \\
M_1 &= M_2 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \; M_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \; N_1 = N_2 = N_3 = N_4 = N_6 = N_7 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}^T, \\
N_5 &= N_8 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^T, \; d_m = 2, \; d_M = 3.
\end{aligned}$$

The H_∞ performance level is taken as $\gamma = 0.9$. With the above parameters and by using the Matlab LMI

Toolbox, we solve the LMI (82), and obtain

$$
\begin{aligned}
P_1 &= \begin{bmatrix} 0.7456 & -0.3686 & 0.0016 \\ -0.3686 & 1.6489 & 0.2817 \\ 0.0016 & 0.2817 & 0.2794 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0.5242 & -0.0728 & -0.2299 \\ -0.0728 & 0.0477 & 0.0325 \\ -0.2299 & 0.0325 & 0.1676 \end{bmatrix}, \\
Q &= \begin{bmatrix} 0.1631 & -0.1086 & -0.0153 \\ -0.1086 & 0.4112 & 0.0842 \\ -0.0153 & 0.0842 & 0.0628 \end{bmatrix}, \; X = \begin{bmatrix} 0.0918 & -0.0240 & 0.0258 \\ -0.0238 & -0.0055 & 0.0118 \\ -0.0277 & 0.0022 & -0.0182 \end{bmatrix}, \\
Y &= \begin{bmatrix} 0.0733 & -0.0064 \\ 0.0016 & -0.0168 \\ -0.0285 & -0.0143 \end{bmatrix}, \; \lambda_1 = 1.1539, \; \lambda_2 = 0.6240, \; \lambda_3 = 0.2683, \; \lambda_4 = 0.1453, \\
\varepsilon_1 &= 0.5564, \; \varepsilon_2 = 0.0484, \; \varepsilon_3 = 13.5772, \; \varepsilon_4 = 1.6933.
\end{aligned}
$$

Therefore, the filtering parameters can be designed as

$$
A_f = P_2^{-1}X = \begin{bmatrix} 0.2173 & -0.1319 & 0.0590 \\ -0.2965 & -0.2322 & 0.4101 \\ 0.1903 & -0.1228 & -0.1073 \end{bmatrix}, \quad B_f = P_2^{-1}Y = \begin{bmatrix} 0.2057 & -0.1876 \\ 0.3118 & -0.4662 \\ 0.0516 & -0.2525 \end{bmatrix}.
$$

V. Conclusions

In this paper, we have studied the robust H_∞ filtering problem for a class of nonlinear discrete time-delay stochastic systems. The system under study involves parameter uncertainties, stochastic disturbances, time-varying delays and inherent sector nonlinearities. An effective linear matrix inequality (LMI) approach has been proposed to design the filters such that, for all admissible nonlinearities and time-delays, the overall uncertain filtering error dynamics is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We have first investigated the sufficient conditions for the filtering error dynamics to be stable in the mean square, and then derived the explicit expression of the desired controller gains. A numerical example has been provided to show the usefulness and effectiveness of the proposed design method.

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