A Delay-Dependent Approach to H_∞ Filtering for Stochastic Delayed Jumping Systems with Sensor Nonlinearities

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Abstract

In this paper, a delay-dependent approach is developed to deal with the stochastic H_∞ filtering problem for a class of Itô type stochastic time-delay jumping systems subject to both the sensor nonlinearities and the exogenous nonlinear disturbances. The time delays enter into the system states, the sensor nonlinearities and the external nonlinear disturbances. The purpose of the addressed filtering problem is to seek an H_∞ filter such that, in the simultaneous presence of nonlinear disturbances, sensor nonlinearity as well as Markovian jumping parameters, the filtering error dynamics for the stochastic time-delay system is stochastically stable with a guaranteed disturbance rejection attenuation level γ. By using Itô's differential formula and the Lyapunov stability theory, we develop a linear matrix inequality approach to derive sufficient conditions under which the desired filters exist. These conditions are dependent on the length of the time delay. We then characterize the expression of the filter parameters, and use a simulation example to demonstrate the effectiveness of the proposed results.

Keywords

H_∞ filter; Markovian switching; sensor nonlinearity; delay-dependent technique; linear matrix inequality; nonlinear disturbance.

I. Introduction

In practice, when measuring the system output through sensors, the sensor nonlinearities usually result from the harsh environments such as uncontrollable elements (e.g., variations in flow rates, temperature, etc.) and aggressive conditions (e.g., corrosion, erosion, and fouling, etc.) [19]. In real-world applications, nonlinearity is an inevitable feature for some sensors, for example, accelerometers, temperature sensors, image sensors and strain gauges. Since the sensor nonlinearity cannot be simply ignored and often lead to poor performance of the controlled system, many researchers have been investigating the analysis and synthesis problems for various systems with sensor nonlinearities [3, 5, 13, 16].

It is well known that the time delay exists commonly in dynamic systems and is frequently a source of instability. Therefore, in recent years, much work has been done about time-delay systems [1, 6, 7, 17, 18, 20, 23, 24, 26]. In particular, in the case that time delays are known and small, the linear matrix inequality (LMI) technique has been extensively used to derive delay-dependent stability criteria, see [4–7, 10, 28] for some recent publications. On the other hand, during the past few decades, stochastic modeling has come to play an important role in many branches of science such as biology, economics and engineering applications.

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Consequently, the time delay systems with stochastic perturbations have drawn a lot of attentions from researchers working in related areas, see [1, 21, 22] and references therein.

Markovian jump systems are the hybrid systems with two components in the state [11]. The first one refers to the mode which is described by a continuous-time finite-state Markovian process, and the second one refers to the state which is represented by a system of differential equations. The jump systems have the advantage of modeling the dynamic systems subject to abrupt variation in their structures, such as component failures or repairs, sudden environmental disturbance, changing subsystem interconnections, operating in different point of a nonlinear plant. Recently, filtering and control for Markovian jump systems with or without nonlinear disturbances have drawn some research attentions, see [17, 23, 24, 27, 28] for some related results. Note that exogenous nonlinear disturbances may result from the linearization process of an originally highly nonlinear plant or may be an external nonlinear input, and therefore exist in many real-world systems.

The filter design problem has long been one of the key problems in the areas of control and signal processing. The purpose of the filtering problem is to estimate the unavailable state variables (or a linear combination of the states) of a given system through noisy measurements. During the past four decades, the filtering problem has been extensively investigated for a variety of complex systems, such as deterministic delay systems [1,8,9], Markovian jumping delay systems [17, 20, 23] and stochastic delay systems [22, 24], to name just a few. When both the Markovian jump parameters and time delays appear in the stochastic systems, the H_∞ filtering problem has been studied in [24], where some useful stochastic stability conditions have been proposed by an LMI technique. In [23], the robust H_∞ filter design problem has been investigated for stochastic time-delay systems with missing measurements. However, up to now, the H_∞ filtering problem for stochastic time-delay systems with both Markovian switching and sensor nonlinearities have not been adequately addressed yet, which still remains as an interesting research topic.

In this paper, we aim to solve the H_∞ filter design problem for a class of stochastic time-delay systems with nonlinear disturbances, sensor nonlinearities and Markovian jumping parameters. Both the filter analysis and synthesis problems are tackled. A delay-dependent approach is developed to design the H_∞ filter for the stochastic delay jumping systems such that, for the addressed nonlinear disturbances and sensor nonlinearities, the filtering error system is stochastically stable with a prescribed disturbance rejection attenuation level γ. By using Itô's differential formula and the Lyapunov stability theory, sufficient conditions for the solvability of the filter design problem are derived in term of linear matrix inequalities (LMIs). These conditions are dependent on the information of the time delay, which can be easily checked by resorting to available software packages. A numerical example and the corresponding simulation results are exploited to demonstrate the effectiveness of the proposed filter design method.

Notation In this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$; $|\cdot|$ refers to the Euclidean norm in $\mathbb{R}^n$, and $\|\cdot\|_2$ stands for the usual $L_2[0,\infty)$ norm. Let $\tau>0$, $C([-\tau,0];\mathbb{R}^n)$ denote the family of continuous functions ϕ from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\phi\|=\sup_{-\tau\le\theta\le 0}|\phi(\theta)|$, and I denote the identity matrix of compatible dimension . The notation $X\ge Y$ (respectively, $X>Y$) where X and Y are symmetric matrices, means that $X-Y$ is positive semi-definite (respectively, positive definite). For a matrix M, M^T represents its transpose, $\lambda_{\max}(M)$ (respectively, $\lambda_{\min}(M)$) stands for its maximum (respectively, minimum) eigenvalue and its operator norm is denoted by $\|M\|=\sup\{|Mx|:|x|=1\}=\sqrt{\lambda_{\max}(M^TM)}$. $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathcal{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\tau,0];\mathbb{R}^n)$-valued random variables $\xi=\{\xi(\theta):-\tau\le\theta\le0\}$ such that $\sup_{-\tau\le\theta\le0}\mathbb{E}|\xi(\theta)|^p<\infty$, where $\mathbb{E}\{x\}$ stands for the expectation of stochastic variable x. The shorthand diag$(M_1,\ M_2,\ ,...,\ M_n)$

denotes a block diagonal matrix with diagonal blocks being the matrices M_1, M_2, , ..., M_n. In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions .

II. Problem Formulation

Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with the following transition probabilities:

$$P\{r(t+\Delta t) = j : r(t) = i\} = \begin{cases} \gamma_{ij}\Delta t + O(\Delta t) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta t + O(\Delta t) & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \to 0} O(\Delta t)/\Delta t = 0$. Here, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j=1, j\neq i}^{N} \gamma_{ij}$.

Consider the following stochastic time-delay system with both the sensor nonlinearity and Markovian switching:

$$\begin{aligned} (\Sigma): dx(t) &= [A(r(t))x(t) + A_d(r(t))x(t-\tau) + B_1(r(t))v(t) + f(x(t), x(t-\tau), r(t))]\, dt \\ &\quad + E(r(t))x(t)d\omega(t) && (1) \\ y(t) &= \psi(u) + B_2(r(t))v(t) && (2) \\ z(t) &= L(r(t))x(t) && (3) \\ x(t) &= \phi(t),\ r(t) = r(0), \quad \forall\, t \in [-\tau, 0], && (4) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^r$ is the measured output, $z(t) \in \mathbb{R}^q$ is the controlled output, $v(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $L_2[0, \infty)$. $f(\cdot, \cdot, \cdot)$ is an unknown nonlinear exogenous disturbance input, $\psi(\cdot)$ represents the sensor nonlinearity, and $u = C(r(t))x(t) + C_d(r(t))x(t-\tau)$. $\omega(t)$ is a one-dimensional Brownian motion satisfying $\mathbb{E}\{d\omega(t)\} = 0$ and $\mathbb{E}\{d\omega^2(t)\} = dt$. The constant τ is a real time delay satisfying $0 \leq \tau < \infty$, and $\phi(t) \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ is an initial function. For a fixed mode $r(t) \in S$, $A(r(t))$, $A_d(r(t))$, $B_1(r(t))$, $B_2(r(t))$, $E(r(t))$, $C(r(t))$, $C_d(r(t))$, $L(r(t))$ are constant matrices with appropriate dimensions.

Assumption 1: For a fixed system mode, there exist known real constant mode-dependent matrices $M_1(r(t)) \in \mathbb{R}^{n\times n}$ and $M_2(r(t)) \in \mathbb{R}^{n\times n}$ such that the unknown nonlinear vector function $f(\cdot, \cdot, \cdot)$ satisfies the following boundedness condition:

$$|f(x(t), x(t-\tau), r(t))| \leq |M_1(r(t))x(t)| + |M_2(r(t))x(t-\tau)| \tag{5}$$

Remark 1: Exogenous nonlinear time-varying disturbances, which may exist in many real-world systems, have been dealt with in many papers such as [23, 24]. In Assumption 1, the nonlinear disturbance term $f(x(t), x(t-\tau), r(t))$ in (5) contains the delayed term, which is more general than that studied in [23, 24]. Note that the H_∞ filtering problem for stochastic delayed jumping systems with such kind of nonlinear exogenous disturbances has not been thoroughly investigated in the literature.

Assumption 2: The nonlinear function $\psi(\cdot)$ in stochastic systems (1)-(4) represents the sector nonlinearities satisfying the following sector condition:

$$(\psi(u) - K_1(r(t))u)^T (\psi(u) - K_2(r(t))u) \leq 0,\ \forall\, u \in \mathbb{R}^n, \tag{6}$$

where the matrices $K_1(r(t)) \geq 0$ and $K_2(r(t)) \geq 0$ $(K_2(r(t)) > K_1(r(t)))$ are given mode-dependent constant diagonal matrices.

Remark 2: As in [13], it is customary to say that the nonlinear function belongs to a sector $[K_1(r(t)), K_2(r(t))]$. The nonlinear description in (6) is quite general that include the usual Lipschitz condition as a special case.

Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied, see e.g. [5,12,15].

For technical convenience, the nonlinear function $\psi(u)$ can be decomposed into a linear and a nonlinear part as

$$\psi(u) = \psi_s(u) + K_1(r(t))u, \tag{7}$$

where the nonlinear part $\psi_s(u)$ belongs to the set Ψ_s given by

$$\Psi_s = \{\psi_s(u) : \psi_s^T(u)(\psi_s(u) - K(r(t))u) \le 0\}, \tag{8}$$

with $K(r(t)) = K_2(r(t)) - K_1(r(t)) > 0$.

In this paper, in order to estimate $z(t)$, we are interested in designing a filter of the following structure:

$$\begin{aligned} (\Sigma_f) : d\hat{x}(t) &= F(r(t))\hat{x}(t)dt + G(r(t))y(t)dt & (9) \\ \hat{z}(t) &= H(r(t))\hat{x}(t), & (10) \end{aligned}$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\hat{z}(t) \in \mathbb{R}^q$, and $F(r(t))$, $G(r(t))$ and $H(r(t))$ are filter parameters to be determined.

Note that the set S consists of different operation modes of the system (1)-(4) for each possible values of $r(t) = i,\ i \in S$. In the sequel, we denote the matrix associated with the ith mode by

$$W_i \triangleq W(r(t) = i),$$

where the matrix W could be A, A_d, B_1, B_2, E, C, C_d, L, M_1, M_2, K_1, K_2, K, F, G or H.

Let the filter estimation error be $e(t) = z(t) - \hat{z}(t)$. By augmenting the state variables

$$\xi(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \quad \xi_\tau = \begin{bmatrix} x(t-\tau) \\ \hat{x}(t-\tau) \end{bmatrix},$$

and combining (Σ) and (Σ_f), we obtain the filtering error dynamics as follows:

$$\begin{aligned} (\Sigma_e) : d\xi(t) &= \left[\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_iv(t) + N^T f(x(t), x(t-\tau), i) + \bar{G}_i\psi_s(u)\right] dt \\ &\quad + \bar{E}_iN\xi(t)d\omega(t) & (11) \\ e(t) &= \bar{L}_i\xi(t) & (12) \end{aligned}$$

where

$$\begin{aligned} \bar{A}_i &= \begin{bmatrix} A_i & 0 \\ G_iK_{1i}C_i & F_i \end{bmatrix}, \ \bar{A}_{di} = \begin{bmatrix} A_{di} \\ G_iK_{1i}C_{di} \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} B_{1i} \\ G_iB_{2i} \end{bmatrix}, \\ \bar{G}_i &= \begin{bmatrix} 0 \\ G_i \end{bmatrix}, \ \bar{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \ \bar{L}_i = \begin{bmatrix} L_i & -H_i \end{bmatrix}, \ N = \begin{bmatrix} I & 0 \end{bmatrix}. \end{aligned}$$

For the purpose of presentation simplification, we define a new state variable

$$\eta(t) = \bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_iv(t) + N^T f(x(t), x(t-\tau)) + \bar{G}_i\psi_s(u), \tag{13}$$

and then the systems (11) can be rewritten as

$$d\xi(t) = \eta(t)dt + \bar{E}_iN\xi(t)d\omega(t). \tag{14}$$

Observe the system (11)-(12) and let $\xi(t;\zeta)$ denote the state trajectory from the initial data $\xi(\theta)=\zeta(\theta)$ on $-\tau\leq\theta\leq 0$ in $L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$. Obviously, $\xi(t,0)\equiv 0$ is the trivial solution of system (11)-(12) corresponding to the initial data $\zeta=0$.

Before formulating the problem to be investigated, we first introduce the following stability concepts for the augmented system (11)-(12).

Definition 1: For the system (11)-(12) and every $\zeta\in L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, the trivial solution is said to be *mean-square asymptotically stable* if

$$\lim_{t\to\infty}\mathbb{E}|\xi(t)|^2=0; \tag{15}$$

and is said to be *mean-square exponentially stable* if there exist scalars $\alpha>0$ and $\beta>0$ such that

$$\mathbb{E}|x(t,\zeta)|^2\leq\alpha e^{\beta t}\sup_{-2\tau\leq\theta\leq 0}\mathbb{E}|\zeta(\theta)|^2. \tag{16}$$

Definition 2: Given a scalar $\gamma>0$, the filter error system (11)-(12) with sensor nonlinearity is said to be *stochastically stable with disturbance attenuation level* γ if it is mean-square exponentially stable and, under zero initial conditions, $\|e(t)\|_{\mathbb{E}_2}<\gamma\|v(t)\|_2$ holds for all nonzero $v(t)\in L_2[0,\infty)$, where

$$\|e(t)\|_{\mathbb{E}_2}:=\left(\mathbb{E}\left\{\int_0^\infty|e(t)|^2dt\right\}\right)^{1/2}.$$

The purpose of this paper is to design an H_∞ filter of the form (9)-(10) for the system (1)-(4) such that, for all admissible time delays, exogenous nonlinear disturbances, sensor nonlinearities and Markovian jumping parameters, the filtering error system (11)-(12) is stochastically stable with disturbance attenuation level γ, where the criteria are dependent on the length of time delay.

III. Main Results

A. Filter analysis

Firstly, let us give the following lemmas which will be used in the proofs of our main results in this paper.

Lemma 1: (Schur Complement) [2] Given constant matrices $\Sigma_1,\Sigma_2,\Sigma_3$ where $\Sigma_1=\Sigma_1^T$ and $0<\Sigma_2=\Sigma_2^T$. Then $\Sigma_1+\Sigma_3^T\Sigma_2^{-1}\Sigma_3<0$ if and only if

$$\begin{bmatrix}\Sigma_1 & \Sigma_3^T\\ \Sigma_3 & -\Sigma_2\end{bmatrix}<0,$$

or, equivalently

$$\begin{bmatrix}-\Sigma_2 & \Sigma_3\\ \Sigma_3^T & \Sigma_1\end{bmatrix}<0.$$

Lemma 2: [25] Let $x\in\mathbb{R}^n, y\in\mathbb{R}^n$ and $\varepsilon>0$. Then, we have

$$x^Ty+y^Tx\leq\varepsilon x^Tx+\varepsilon^{-1}y^Ty.$$

Lemma 3: [25] Let Φ_1, Φ_2, Φ_3 and $\Xi>0$ be given constant matrices with appropriate dimensions, Then for any scalar $\varepsilon>0$ satisfying $\varepsilon I-\Phi_2^T\Xi\Phi_2>0$, we have

$$[\Phi_1+\Phi_2\Phi_3]^T\Xi[\Phi_1+\Phi_2\Phi_3]\leq\Phi_1^T[\Xi^{-1}-\varepsilon^{-1}\Phi_2\Phi_2^T]^{-1}\Phi_1+\varepsilon\Phi_3^T\Phi_3.$$

In the following theorem, the delay-dependent technique and an LMI method are used to deal with the stability analysis problem for the H_∞ filter design of the stochastic system (1)-(4), and a sufficient condition is derived that ensures the solvability of the H_∞ filtering problem.

Theorem 1: Consider the filtering error system (11)-(12) with given filter parameters. If there exist positive definite matrices $P_i > 0$, $T_i > 0$, $Q > 0$ and $R > 0$ such that the following matrix inequalities

$$\begin{bmatrix} \Omega_{1i} & 0 & P_i\bar{B}_i & P_i\bar{G}_i + N^T C_i^T K_i & \bar{\tau}\bar{A}_i^T N^T R & 0 & P_i N^T & P_i\bar{A}_{di}N \\ * & \Omega_{2i} & 0 & C_{di}^T K_i & \bar{\tau}\bar{A}_{di}^T N^T R & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & \bar{\tau}\bar{B}_i^T N^T R & 0 & 0 & 0 \\ * & * & * & -2I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{\tau}R & R & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_{1i}I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad \forall\, i \in S \qquad (17)$$

$$\begin{bmatrix} T_i & P_i\bar{A}_{di} \\ \bar{A}_{di}^T P_i & R \end{bmatrix} > 0, \quad \forall\, i \in S \qquad (18)$$

hold, where

$$\begin{aligned} \Omega_{1i} &:= P_i(\bar{A}_i + \bar{A}_{di}N) + (\bar{A}_i + \bar{A}_{di}N)^T P_i + \Sigma_{i,j=1}^N \gamma_{ij} P_j + \bar{L}_i^T \bar{L}_i + N^T Q N \\ &\quad + N^T \bar{E}_i^T P_i \bar{E}_i N + \bar{\tau} c_e N^T N + 2(\varepsilon_{1i} + \varepsilon_{2i})(N^T M_{1i}^T M_{1i} N) + \bar{\tau} T_i, \qquad (19) \\ \Omega_{2i} &:= 2(\varepsilon_{1i} + \varepsilon_{2i})(M_{2i}^T M_{2i}) - Q, \qquad (20) \end{aligned}$$

with $c_e = \max_{i\in S} \|\bar{E}_i\|^2$, then the filtering error system is stochastically stable with the disturbance attenuation level γ for $\tau \leq \bar{\tau}$ ($\bar{\tau}$ is the upper bound of the time delay).

Proof: See Appendix for detailed proof. ■

In the next subsection, our attention is focused on the design of filter parameters F_i, G_i and H_i, for $i \in S$, by using the results in Theorem 1. The explicit expression of the expected filter parameters is obtained in term of the solution to a set of LMIs.

B. Filter synthesis

The following theorem shows that the desired filter parameters can be derived by solving several LMIs.

Theorem 2: Consider the system (11)-(12). If there exist matrices $X_i > 0$, $\mathcal{Y}_i > 0$, $\bar{T}_{11i} > 0$, $\bar{T}_{22i} > 0$, $Q > 0$, $R > 0$, a matrix $\bar{T}_{12i}$ and scalars $\varepsilon_{1i} > 0$, $\varepsilon_{2i} > 0$ such that the following linear matrix inequalities

$$
\left[\begin{array}{ccccccccccccccc}
\Pi_{1i} & \Pi_{2i} & 0 & \mathcal{Y}_i B_{1i} & C_i^T K_i & \bar{\tau} A_i^T R & 0 & \mathcal{Y}_i A_{di} & \mathcal{Y}_i & E_i^T \mathcal{Y}_i & E_i^T X_i & L_i^T - H_i^T & M_{1i}^T \varepsilon_{1i} & M_{2i}^T \varepsilon_{2i} & Q \\
* & \Pi_{3i} & 0 & \Pi_{4i} & \tilde{G}_i + C_i^T K_i & \bar{\tau} A_i^T R & 0 & \Pi_{5i} & X_i & E_i^T \mathcal{Y}_i & E_i^T X_i & L_i^T & M_{1i}^T \varepsilon_{1i} & M_{2i}^T \varepsilon_{2i} & Q \\
* & * & \Omega_{2i} & 0 & C_{di}^T K_i & \bar{\tau} A_{di} R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I & 0 & \bar{\tau} B_{1i}^T R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\bar{\tau} R & \bar{\tau} R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_{2i} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_{1i} I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\mathcal{Y}_i & \mathcal{Y}_i & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -X_i & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -2\varepsilon_{1i} I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -2\varepsilon_{2i} I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -Q
\end{array}\right] < 0, \quad \forall\, i \in S \tag{21}
$$

$$
\begin{bmatrix}
\bar{T}_{11i} & \bar{T}_{12i} & \mathcal{Y}_i A_{di} \\
\bar{T}_{12i}^T & \bar{T}_{22i} & \Pi_{5i} \\
A_{di}^T \mathcal{Y}_i & \Pi_{5i}^T & R
\end{bmatrix} > 0, \quad \forall\, i \in S \tag{22}
$$

hold, where

$$
\begin{aligned}
\Pi_{1i} &:= \mathcal{Y}_i A_i + A_i^T \mathcal{Y}_i + \mathcal{Y}_i A_{di} + A_{di}^T \mathcal{Y}_i + \Sigma_{j=1}^N \gamma_{ij} \mathcal{Y}_j + \bar{\tau} c_e I + \bar{\tau} \bar{T}_{11i}, \\
\Pi_{2i} &:= \mathcal{Y}_i A_i + A_i^T X_i + \mathcal{Y}_i A_{di} + A_{di}^T X_i + \Sigma_{j=1}^N \gamma_{ij} \mathcal{Y}_j + \bar{\tau} c_e I + \bar{\tau} \bar{T}_{12i}, \\
&\quad + C_i^T K_{1i} \tilde{G}_i^T + \tilde{F}_i^T + C_{di}^T K_{1i} \tilde{G}_i^T, \\
\Pi_{3i} &:= X_i A_i + A_i^T X_i + \tilde{G}_i K_{1i} C_i + C_i^T K_{1i} \tilde{G}_i^T + X_i A_{di} + \tilde{G}_i K_{1i} C_{di}, \\
&\quad + C_{di}^T K_{1i} \tilde{G}_i^T + A_{di}^T X_i + \Sigma_{j=1}^N \gamma_{ij} X_j + \bar{\tau} c_e I + \bar{\tau} \bar{T}_{22i}, \\
\Pi_{4i} &:= X_i B_{1i} + \tilde{G}_i B_{2i}, \\
\Pi_{5i} &:= X_i A_{di} + \tilde{G}_i K_{1i} C_{di},
\end{aligned} \tag{23}
$$

then the system (11)-(12) is stochastically stable with disturbance attenuation γ for $\tau \leq \bar{\tau}$. In this case, the parameters of the desired H_∞ filter (Σ_f) are given as follows:

$$F_i := (\mathcal{Y}_i - X_i)^{-1}\tilde{F}_i,\ G_i := (\mathcal{Y}_i - X_i)^{-1}\tilde{G}_i,\ H_i := \tilde{H}_i. \tag{24}$$

Proof: Define

$$P_i = \left[\begin{array}{cc} X_i & \mathcal{Y}_i - X_i \\ \mathcal{Y}_i - X_i & X_i - \mathcal{Y}_i \end{array}\right] > 0, \quad \Upsilon = \left[\begin{array}{cc} Y_i & I \\ Y_i & 0 \end{array}\right], \tag{25}$$

where $Y_i = \mathcal{Y}_i^{-1} > 0$.

From (22), we have

$$T := \Upsilon^{-T}\mathrm{diag}(Y_i,\ I)\left(\begin{array}{cc} \bar{T}_{11i} & \bar{T}_{12i} \\ \bar{T}_{12i}^T & \bar{T}_{22i} \end{array}\right)\mathrm{diag}(Y_i,\ I)\Upsilon > 0. \tag{26}$$

Pre- and post-multiplying the LMIs in (21) by $\mathrm{diag}(Y_i, \underbrace{I, ...I}_{8}, Y_i, \underbrace{I, ..., I}_{5})$, and (22) by $\mathrm{diag}(Y_i, I, I)$, we have

$$\left[\begin{array}{ccccccccccccccc}
\bar{\Pi}_{1i} & \bar{\Pi}_{2i} & 0 & B_{1i} & Y_iC_i^TK_i & \bar{\tau}Y_iA_i^TR & 0 & A_{di} & I & Y_iE_{1i}^T & Y_iE_i^TX_i & Y_iL_i^T - Y_iH_i^T & Y_iG_{1i}^T\varepsilon_{1i} & Y_iG_{2i}^T\varepsilon_{2i} & Y_iQ \\
* & \Pi_{3i} & 0 & \Pi_{4i} & \tilde{G}_i & \bar{\tau}A_i^TR & 0 & \Pi_{5i} & X_i & E_{1i}^T & E_i^TX_i & L_i^T & G_{1i}^T\varepsilon_{1i} & G_{2i}^T\varepsilon_{2i} & Q \\
* & * & \Omega_{2i} & 0 & C_{di}^TK_i & \bar{\tau}A_{di}R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma^2I & 0 & \bar{\tau}B_{1i}^TR & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\bar{\tau}R & \bar{\tau}R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_{2i}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_{1i}I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -Y_i & I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -X_i & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -2\varepsilon_{1i}I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -2\varepsilon_{2i}I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -Q
\end{array}\right] < 0, \quad \forall\, i \in S \tag{27}$$

$$\begin{bmatrix} Y_i\bar{T}_{11i} & Y_i\bar{T}_{12i} & A_{di} \\ * & \bar{T}_{22i} & X_iA_{di}+\tilde{G}_iK_{1i}C_{di} \\ * & * & R \end{bmatrix} > 0, \quad \forall\, i\in S, \tag{28}$$

where

$$\begin{aligned} \bar{\Pi}_{1i} &:= A_iY_i+Y_iA_i^T+A_{di}Y_i+Y_iA_{di}^T+\Sigma_{j=1}^N\gamma_{ij}Y_iY_j^{-1}Y_i+\bar{\tau}c_eY_i^2+\bar{\tau}Y_i\bar{T}_{11i}Y_i, \\ \bar{\Pi}_{2i} &:= A_i+Y_iA_i^TX_i+A_{di}+Y_iA_{di}^TX_i+\Sigma_{j=1}^N\gamma_{ij}Y_iY_j^{-1}Y_i+\bar{\tau}c_eY_i+\bar{\tau}Y_iT_{12i} \\ &\quad Y_iC_i^TK_{1i}\tilde{G}_i^T+Y_i\tilde{F}_i^T+Y_iC_{di}K_{1i}\tilde{G}_i^T. \end{aligned}$$

From the definitions of P_i and Υ_i, the LMIs in (27)-(28) are equivalent to the following matrix inequalities

$$\begin{bmatrix} \Upsilon_i^T\bar{\Omega}_{1i}\Upsilon_i & 0 & \Upsilon_i^TP_i\bar{B}_i & \Lambda_{1i} & \bar{\tau}\Upsilon_i^T\bar{A}_i^TN^TR & 0 & \Upsilon_i^TP_iN^T & \Lambda_{2i} & \Lambda_{3i} \\ * & \Omega_{2i} & 0 & C_{di}^TK & \bar{\tau}\bar{A}_{di}^TN^TR & 0 & 0 & 0 & 0 \\ * & * & -\gamma^2I & 0 & \bar{\tau}\bar{B}_i^TN^TR & 0 & 0 & 0 & 0 \\ * & * & * & -2I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{\tau}R & \bar{\tau}R & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_{1i}I & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\Upsilon_i^TP_i\Upsilon_i \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \Upsilon_i^TT_i\Upsilon_i & \Upsilon_i^TP_i\bar{A}_{di} \\ * & R \end{bmatrix} > 0, \tag{30}$$

where

$$\begin{aligned} \bar{\Omega}_{1i} &= \Omega_{1i}-N^TE_i^TP_iE_iN \\ \Lambda_{1i} &= \Upsilon_i^TP_i\bar{G}_i+\Upsilon_i^TN^TC_i^TK_i \\ \Lambda_{2i} &= \Upsilon_i^TP_i\bar{A}_{di}N \\ \Lambda_{3i} &= \Upsilon_i^TN^TE_i^TP_i\Upsilon_i. \end{aligned}$$

Finally, pre- and post-multiplying (29) by $\mathrm{diag}(\Upsilon_i^{-T},\underbrace{I,...,I}_{7},\Upsilon_i^{-T})$ and its transpose, (30) by $\mathrm{diag}(\Upsilon_i^{-T},I)$ and its transpose, we can obtain from Theorem 1 and Schur Complement Lemma that, with the given filter parameters in (24), the system (11)-(12) is stochastically stable with disturbance attenuation γ for $\tau\le\bar{\tau}$. ■

Remark 3: The H_∞ filter design problem is solved in Theorem 2 for the addressed delayed stochastic jumping systems with sensor nonlinearities and external nonlinear disturbances. LMI-based sufficient conditions are obtained for the existence of full-order filters that ensure the mean-square exponential stability of the resulting filtering error system and reduce the effect of the disturbance input on the estimated signal to a prescribed level for all admissible time delays and nonlinearities. The feasibility of the filter design problem can be readily checked by the solvability of two sets of LMIs, which can be determined by using the Matlab LMI toolbox in a straightforward way. In the next section, an illustrative example will be provided to show the usefulness of the proposed techniques.

IV. An Illustrative Example

In this section, a simulation example is presented to illustrate the usefulness and flexibility of the filter design method developed in this paper. We are interested in obtaining the upper bound $\bar{\tau}$ of the time delay and designing the H_∞ filter for the stochastic jumping system with nonlinear disturbances and sensor nonlinearities.

The system data of (1)-(3) are given as follows:

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} -2.5 & 2.5 \\ 0.9 & -0.9 \end{bmatrix}, \qquad \gamma = 1.8.$$

Mode 1:

$$A_1 = \begin{bmatrix} -3.5 & 1 \\ 0 & -2.7 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.21 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.13 & 0 \\ 0 & 0.15 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.13 \\ 0.02 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$C_1 = 0.5I_2, \quad C_{d1} = 0.5I_2, \quad L_1 = [0.3 \ \ 0.7],$$
$$K_{11} = \mathrm{diag}\{0.3, 0.4\}, \quad K_{21} = \mathrm{diag}\{0.6, 0.5\}.$$

Mode 2:

$$A_2 = \begin{bmatrix} -4.3 & 1 \\ 0 & -2.5 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.22 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.31 \end{bmatrix},$$
$$B_{12} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.2 \\ 0.15 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$C_2 = 0.6I_2, \quad C_{d2} = I_2, \quad L_2 = [0.6 \ \ 0.8],$$
$$K_{12} = \mathrm{diag}\{0.4, 0.6\}, \quad K_{22} = \mathrm{diag}\{0.6, 0.9\}.$$

Using Matlab LMI control Toolbox to solve the LMIs in (21) and (22), we obtain the upper bound of time delay as $\bar{\tau} = 2.2520$. Therefore, by Theorem 2, it can be calculated that for all $0 < \tau \leq 2.2520$, there exist the desired H_∞ filters. For demonstration purpose, let us fix $\tau = 1.5$. In this case, by the LMI toolbox, we can calculate that

$$X_1 = \begin{bmatrix} 58.5508 & -10.3863 \\ -10.3863 & 56.9008 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 34.2934 & -6.2039 \\ -6.2039 & 31.9526 \end{bmatrix},$$
$$T_1 = \begin{bmatrix} 45.1367 & -16.5489 & 13.1486 & -6.4879 \\ -16.5489 & 29.2881 & -6.8324 & 25.7375 \\ 13.1486 & -6.8324 & 87.1278 & -28.4811 \\ -6.4879 & 25.7375 & -28.4811 & 54.8590 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 48.3957 & -9.1244 \\ -9.1244 & 44.5760 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 29.6427 & -4.9378 \\ -4.9378 & 25.6274 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 35.0120 & -10.6401 & 14.8960 & -3.7476 \\ -10.6401 & 10.3716 & -2.1487 & 1.4473 \\ 14.8960 & -2.1487 & 73.7385 & -22.6410 \\ -3.7476 & 1.4473 & -22.6410 & 40.0518 \end{bmatrix},$$

$$Q = \begin{bmatrix} 29.9297 & -8.6883 \\ -8.6883 & 26.0809 \end{bmatrix}, \quad R = \begin{bmatrix} 2.2850 & -0.0177 \\ -0.0177 & 2.5803 \end{bmatrix},$$

$$s11 = 132.4881, \quad s12 = 94.5044, \quad s21 = 50.6824, \quad s22 = 65.9064.$$

The filter parameters to be determined are as follows:

$$F_1 = \begin{bmatrix} -5.4748 & 1.8614 \\ 0.3389 & -1.7616 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.1978 & 0.0053 \\ 0.0183 & 0.2870 \end{bmatrix}, \quad H_1 = [0.2951, \quad 0.7075],$$

$$F_2 = \begin{bmatrix} -6.4333 & 1.6877 \\ 0.1598 & -3.2487 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.5151 & 0.0248 \\ 0.0561 & 0.2546 \end{bmatrix}, \quad H_2 = [0.6013, \quad 0.7987].$$

Fig. 1–Fig. 6 are the simulation results for the performance of the designed H_∞ filter, where the sensor nonlinearities are taken as

$$\psi(u) = \frac{K_{1i} + K_{2i}}{2} u + \frac{K_{2i} - K_{1i}}{2} \sin(u),$$

which satisfies (6). It is confirmed from the simulation results that all the expected objectives are well achieved.

V. Conclusions

In this paper, we have developed a delay-dependent approach to dealing with the stochastic H_∞ filtering problem for a class of Itô type stochastic time-delay jumping systems subject to both the sensor nonlinearities and the exogenous nonlinear disturbances. The time delays are allowed to exist in the system states, the sensor nonlinearities, as well as the external nonlinear disturbances. By using Itô's differential formula and the Lyapunov stability theory, we have proposed a linear matrix inequality method to derive sufficient conditions under which the desired filters exist. These conditions are dependent on the length of the time delay. We have also characterized the expression of the filter parameters, and employed a simulation example to demonstrate the effectiveness of the proposed results. Moreover, we can extend the main results in this paper to more complex and realistic systems, such as systems with polytopic or norm-bounded uncertainties, and systems with general nonlinearities. We will also focus on the real-time applications in network-based communications and bioinformatics. The corresponding results will appear in the near future.

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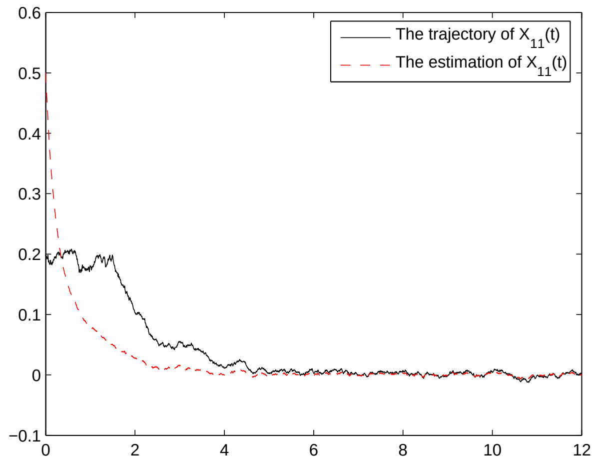

Fig. 1. The trajectory and estimation of x_{11}

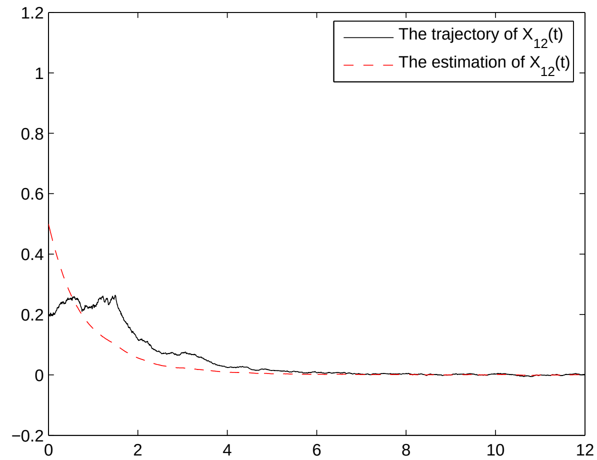

Fig. 2. The trajectory and estimation of x_{12}

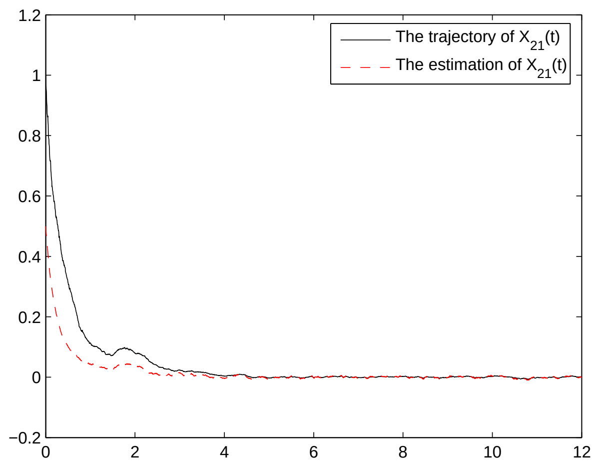

Fig. 3. The trajectory and estimation of x_{21}

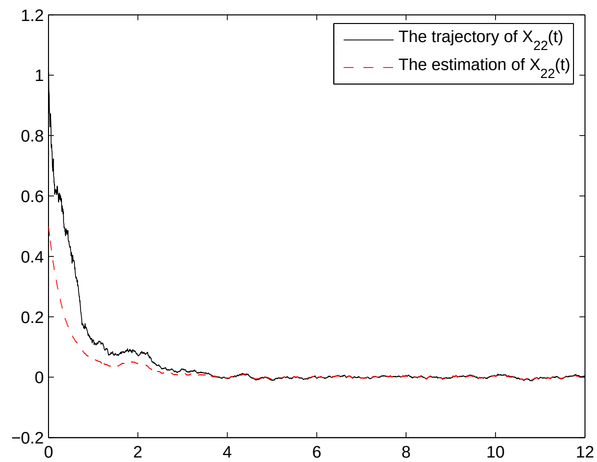

Fig. 4. The trajectory and estimation of x_{22}

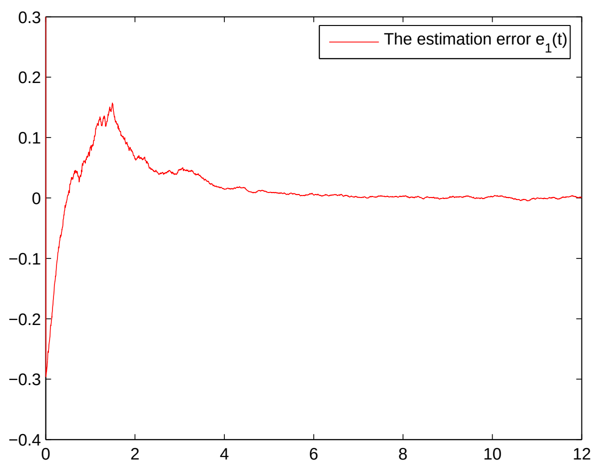

Fig. 5. The estimation error e_1

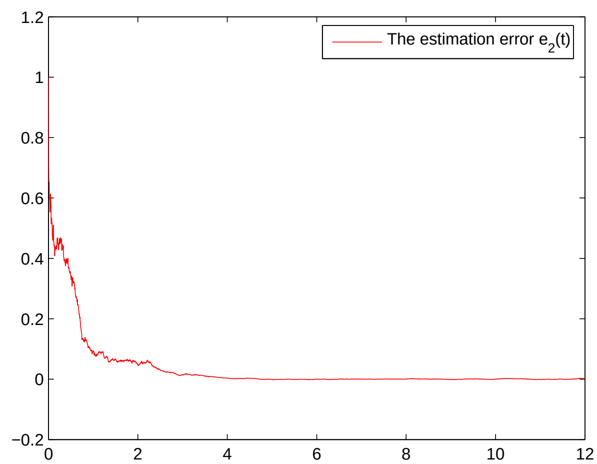

Fig. 6. The estimation error e_2

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Appendix

The Proof of Theorem 1

Proof: Recall the Newton-Leibniz formula and (14), we can write

$$\xi_\tau = \xi(t) - \int_{t-\tau}^{t} d\xi(s) = \xi(t) - \int_{t-\tau}^{t} \eta(s)ds - \int_{t-\tau}^{t} \bar{E}_i N\xi(s)d\omega(s). \tag{31}$$

It is easy to know from (31) that the following system is equivalent to (11)-(12):

$$\begin{aligned}
d\xi(t) &= \Big[(\bar{A}_i + \bar{A}_{di}N)\xi(t) - \bar{A}_{di}N\int_{t-\tau}^{t}\eta(s)ds - \bar{A}_{di}N\int_{t-\tau}^{t}\bar{E}_iN\xi(s)d\omega(s) \\
&\quad +\bar{B}_iv(t) + N^Tf(x(t),x(t-\tau),i) + \bar{G}_i\psi_s(u)\Big]dt + \bar{E}_iN\xi(t)d\omega(t), \qquad (32)\\
e(t) &= \bar{L}_i\xi(t), \qquad (33)\\
\xi(t) &= \rho(t),\ r(t) = r(0), \quad \forall\, t\in[-2\tau,0], \qquad (34)
\end{aligned}$$

where $\rho(t)\in L^p_{\mathcal{F}_0}([-2\tau,0];\mathbb{R}^{2n})$ is the initial function. Hence, we only need to show that the system (32)-(34) is stochastically stable with the disturbance attenuation level γ.

Now, let $P_i>0$, $Q>0$, $R>0$, $c_e=\max_{i\in S}\|\bar{E}_i\|^2$ and define the following Lyapunov-Krasovskii function candidate for the system (32):

$$\begin{aligned}
V(x(t),t,i) &= \xi^T(t)P_i\xi(t) + \int_{t-\tau}^{t}\xi^T(s)N^TQN\xi(s)ds + \int_{t-\tau}^{t}\int_{s}^{t}\eta^T(\beta)N^TRN\eta(\beta)d\beta ds \\
&\quad + \int_{t-\tau}^{t}\int_{s}^{t}c_e\xi^T(\beta)N^TN\xi(\beta)d\beta ds. \qquad (35)
\end{aligned}$$

It can be derived by Itô's differential formula [14] that

$$dV(\xi(t),t,i) = \mathcal{L}V(\xi(t),t,i)dt + 2\xi^T(t)P_i\bar{E}_iN\xi(t)d\omega(t), \qquad (36)$$

where

$$\begin{aligned}
\mathcal{L}V(\xi(t),t,i) &= \xi^T(t)[(\bar{A}_i+\bar{A}_{di}N)^TP_i + P_i(\bar{A}_i+\bar{A}_{di}N) + \sum_{j=1}^{N}\gamma_{ij}P_j + N^TQN]\xi(t) \\
&\quad -2\xi^T(t)P_i\bar{A}_{di}N\left(\int_{t-\tau}^{t}\eta(s)ds + \int_{t-\tau}^{t}\bar{E}_iN\xi(s)d\omega(s)\right) - \xi_\tau^TN^TQN\xi_\tau \\
&\quad +2\xi^T(t)P_i\bar{B}_iv(t) + 2\xi^T(t)P_iN^Tf(x(t),x(t-\tau),i) + 2\xi^T(t)P_i\bar{G}_i\psi_s(u) \\
&\quad +\xi^T(t)N^T\bar{E}_i^TP_i\bar{E}_iN\xi(t) + \tau\eta^T(t)N^TRN\eta(t) + \tau c_e\xi^T(t)N^TN\xi(t) + \tau\xi^T(t)T_i\xi(t) \\
&\quad -\int_{t-\tau}^{t}\left(\eta^T(s)N^TRN\eta(s) + c_e\xi^T(s)N^TN\xi(s) + \xi^T(t)T_i\xi(t)\right)ds \qquad (37)
\end{aligned}$$

Noting (5) and Lemma 2, we have

$$\begin{aligned}
&2\xi(t)^TP_iN^Tf(x(t),x(t-\tau),i) \\
\le\ & \varepsilon_{1i}^{-1}\xi^T(t)P_iN^TNP_i\xi(t) + \varepsilon_{1i}f^T(x(t),x(t-\tau),i)f(x(t),x(t-\tau),i) \\
\le\ & \varepsilon_{1i}^{-1}\xi^T(t)P_iN^TNP_i\xi(t) + \varepsilon_{1i}(|M_{1i}x(t)| + |M_{2i}x(t-\tau)|)^2 \\
\le\ & \varepsilon_{1i}^{-1}\xi^T(t)P_iN^TNP_i\xi(t) + 2\varepsilon_{1i}(\xi^T(t)N^TM_{1i}^TM_{1i}N\xi(t) + \xi_\tau^TN^TM_{2i}^TM_{2i}N\xi_\tau). \qquad (38)
\end{aligned}$$

Again, from Lemma 2, we obtain

$$-2\xi^T(t)P_i\bar{A}_{di}N\int_{t-\tau}^{t}\bar{E}_iN\xi(s)d\omega(s) \le \xi^T(t)P_i\bar{A}_{di}NN^T\bar{A}_{di}^TP_i\xi(t) + |\int_{t-\tau}^{t}\bar{E}_iN\xi(s)d\omega(s)|^2. \qquad (39)$$

Note that in deriving (39), we have fixed the scalar parameter ε as 1, which is to maintain the simplicity of the Lyapunov function.

Moreover,

$$\mathbb{E}|\int_{t-\tau}^{t} \bar{E}_i N\xi(s)d\omega(s)|^2 \leq \int_{t-\tau}^{t} \mathbb{E}|\bar{E}_i N\xi(s)|^2 ds. \tag{40}$$

From (13) and Lemma 3, it is not difficult to see that

$$\begin{aligned}
\tau\eta^T(t)N^T RN\eta(t) &= [\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + N^T f(x(t), x(t-\tau), i) + \bar{G}_i\psi_s(u)]^T N^T(\tau R)N \\
&\quad \cdot[\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + N^T f(x(t), x(t-\tau), i) + \bar{G}_i\psi_s(u)] \\
&\leq [\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + \bar{G}_i\psi_s(u)]^T N^T((\tau R)^{-1} - \varepsilon_{2i}^{-1}NN^TNN^T)^{-1}N \\
&\quad \cdot[\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + \bar{G}_i\psi_s(u)] \\
&\quad +\varepsilon_{2i}f^T(x(t), x(t-\tau), i)f(x(t), x(t-\tau), i) \\
&\leq [\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + \bar{G}_i\psi_s(u)]^T N^T((\tau R)^{-1} - \varepsilon_{2i}^{-1}NN^TNN^T)^{-1}N \\
&\quad \cdot[\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + \bar{G}_i\psi_s(u)] \\
&\quad +2\varepsilon_{2i}(\xi^T(t)N^T M_{1i}^T M_{1i}N\xi(t) + \xi_\tau^T N^T M_{2i}^T M_{2i}N\xi_\tau).
\end{aligned} \tag{41}$$

Substituting (38)-(41) into (37) and taking the mathematical expectation on both sides, we have

$$\begin{aligned}
\mathbb{E}\mathcal{L}V(\xi(t), t, i) &\leq \mathbb{E}\{\xi^T(t)[(\bar{A}_i + \bar{A}_{di}N)^T P_i + P_i(\bar{A}_i + \bar{A}_{di}N) + \sum_{i=1}^{N}\gamma_{ij}P_j + N^T QN \\
&\quad +N^T\bar{E}_i^T P_i\bar{E}_i N + 2(\varepsilon_{1i} + \varepsilon_{2i})N^T M_{1i}^T M_{1i}N + P_i\bar{A}_{di}NN^T\bar{A}_{di}^T P_i \\
&\quad +\varepsilon_{1i}^{-1}P_i N^T NP_i + \tau c_e N^T N + \tau T_i]\xi(t) - \xi_\tau^T[2(\varepsilon_{1i} + \varepsilon_{2i})N^T M_{2i}^T M_{2i}N \\
&\quad +N^T QN]\xi_\tau + 2\xi^T(t)P_i\bar{B}_i v(t) + 2\xi^T(t)P_i\bar{G}_i\psi_s(u) - 2\psi_s^T(u)\psi_s(u) \\
&\quad +2\psi_s(u)K_i C_i N\xi(t) + 2\psi_s(u)K_i C_{di}N\xi_\tau \\
&\quad +[\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + \bar{G}_i\psi_s(u)]^T N^T((\tau R)^{-1} \\
&\quad -\varepsilon_{2i}^{-1}NN^TNN^T)^{-1}N[\bar{A}_i\xi(t) + \bar{A}_{di}N\xi_\tau + \bar{B}_i v(t) + \bar{G}_i\psi_s(u)] \\
&\quad -\int_{t-\tau}^{t}(\eta^T(s)N^T RN\eta(s) + 2\xi^T(t)P_i\bar{A}_{di}N\eta(s) + \xi^T(t)T_i\xi(t))ds\} \\
&\leq \mathbb{E}\{\bar{\xi}^T(t)\Omega_i\bar{\xi}(t)\} - \int_{t-\tau}^{t}\mathbb{E}\{\bar{\xi}^T(t,s)\Gamma_i\bar{\xi}(t,s)\}ds
\end{aligned} \tag{42}$$

where

$$\bar{\xi}(t) = [\xi^T(t) \quad \xi_\tau^T N^T \quad \psi_s^T(u)]^T,\ \bar{\xi}(t,s) = [\xi^T(t) \quad \eta^T(s)N^T]^T$$

$$\Omega_i := \begin{bmatrix} \Omega_{11i} & 0 & P_i\bar{G}_i + N^T C_i^T K_i \\ * & \Omega_{2i} & C_{di}^T K_i \\ * & * & -2I \end{bmatrix} + \begin{bmatrix} \bar{A}_i^T N^T \\ \bar{A}_{di}^T N^T \\ 0 \end{bmatrix}\Phi_i^{-1}[N\bar{A}_i\ N\bar{A}_{di}\ 0] \tag{43}$$

$$\Gamma_i = \begin{bmatrix} T_i & P_i\bar{A}_{di} \\ \bar{A}_{di}^T P_i & R \end{bmatrix}, \tag{44}$$

where Ω_{2i} is defined in (20) and

$$\Omega_{11i} := \Omega_{1i} + \varepsilon_{1i}^{-1}P_i N^T NP_i + P_i\bar{A}_{di}NN^T\bar{A}_{di}^T P_i - \bar{L}_i^T\bar{L}_i \tag{45}$$

$$\Phi_i := (\tau R)^{-1} - \varepsilon_{2i}^{-1}NN^TNN^T. \tag{46}$$

By Schur Complement, we can obtain from (17) and (18) that, for $\tau \le \bar{\tau}$,

$$\Omega_i < 0,\ \Gamma_i > 0,\ \forall\, i \in S. \tag{47}$$

Based on the inequality (42), the mean-square exponential stability of the system (32) can be proved as follows. Define

$$\lambda_P = \max_{i\in S} \lambda_{\max}(P_i),\ \lambda_p = \min_{i\in S} \lambda_{\min}(P_i),\ \lambda_\Omega = \min_{i\in S}(-\lambda_{\max}(\Omega_i)).$$

It follows from (42) that

$$\mathbb{E}\mathcal{L}V(\xi(t),t,i) \le -\lambda_\Omega \mathbb{E}|\bar{\xi}(t)|^2 \le -\lambda_\Omega \mathbb{E}|\xi(t)|^2. \tag{48}$$

From the definition of $\eta(t)$ and (35), there exist positive scalars $\delta_1,\ \delta_2$ such that

$$\lambda_p \mathbb{E}|\xi(t)|^2 \le \mathbb{E}V(\xi(t),t,i) \le \lambda_P \mathbb{E}|\xi(t)|^2 + \int_{t-2\tau}^{t} \delta_1 \mathbb{E}|\xi(s)|^2 ds, \tag{49}$$

and

$$\mathbb{E}V(\xi(0),0,r(0)) \le \delta_2 \mathbb{E}\|\rho\|^2. \tag{50}$$

Let δ be a root to the inequality

$$\delta(\lambda_P + 2\tau\delta_1 e^{2\delta\tau}) \le \lambda_\Omega. \tag{51}$$

To prove the mean-square exponentially stability, we modify the Lyapunov function candidate (35) as

$$V_1(\xi(t),t,i) = e^{\delta t} V(\xi(t),t,i), \tag{52}$$

and then, by Dynkin's formula [14], we obtain that for each $r(t) = i,\ i \in S,\ t > 0$

$$\mathbb{E}V_1(\xi(t),t,i) = \mathbb{E}V_1(\xi(0),0,r(0)) + \mathbb{E}\int_0^t e^{\delta s}\left[\delta V(\xi(s),s,r(s)) + \mathcal{L}V(\xi(s),s,r(s))\right] ds. \tag{53}$$

It then follows from (48), (49) that

$$\begin{aligned}\mathbb{E}V_1(\xi(t),t,i) \ \ \le\ \ & \delta_2 \mathbb{E}\|\rho\|^2 + \mathbb{E}\int_0^t e^{\delta s}\delta\left(\lambda_P|\xi(s)|^2 + \int_{s-2\tau}^{s}\delta_1|\xi(\beta)|^2 d\beta\right) ds \\ & -\lambda_\Omega \mathbb{E}\int_0^t e^{\delta s}|\xi(s)|^2 ds.\end{aligned} \tag{54}$$

Noticing the definition of δ and the fact of

$$\begin{aligned}&\int_0^t e^{\delta s}\int_{s-2\tau}^{s}(\delta_1|\xi(\beta)|^2 d\beta ds \le \int_{-2\tau}^{t}\delta_1|\xi(\beta)|^2\int_{\beta}^{\beta+2\tau} e^{\delta s} ds d\beta \\ \le\ \ & 2\tau e^{2\delta\tau}\int_{-2\tau}^{t}\delta_1|\xi(\beta)|^2 e^{\delta\beta} d\beta \le 2\tau e^{2\delta\tau}\left(\int_0^t \delta_1|\xi(s)|^2 e^{\delta s} ds + \int_{-2\tau}^{0}(\delta_1|\xi(s)|^2 e^{\delta s} ds\right).\end{aligned} \tag{55}$$

Finally, it follows from (49), (54) and (55) that

$$e^{\delta t}\lambda_p \mathbb{E}|\xi(t)|^2 \le (\delta_2 + 2\tau\delta\delta_1 e^{2\delta\tau})\mathbb{E}\|\rho\|^2,$$

or

$$\lim_{t\to\infty}\sup \frac{1}{t}\log(\mathbb{E}|\xi(t,\rho)|^2) \le -\delta,$$

which indicates that, for $\tau \le \bar{\tau}$, the trivial solution of (31) is exponentially stable in the mean square.

In the sequel, we shall deal with the H_∞ performance of the the system (32)-(34). Assume zero initial condition, i.e., $\xi(t) = 0$ for $t \in [-2\tau, 0]$, and define

$$J(t) = \mathbb{E}\left\{\int_0^t [e^T(s)e(s) - \gamma^2 v^T(s)v(s)]ds\right\}. \tag{56}$$

It follows from Dynkin's formula [14] and fact $\xi(0) = 0$ that

$$\mathbb{E}\{V(\xi(t), t, r(t))\} = \mathbb{E}\left\{\int_0^t \mathcal{L}V(\xi(s), s, r(s))ds\right\}. \tag{57}$$

From (56) and (57), it is easy to see that

$$\begin{aligned} J(t) &= \mathbb{E}\left\{\int_0^t [e^T(s)e(s) - \gamma^2 v^T(s)v(s) + \mathcal{L}V(\xi(s), s, r(s))]ds\right\} - \mathbb{E}\{V(\xi(t), t, r(t))\} \\ &\le \mathbb{E}\left\{\int_0^t [e^T(s)e(s) - \gamma^2 v^T(s)v(s) + \mathcal{L}V(\xi(s), s, r(s))]ds\right\}. \end{aligned} \tag{58}$$

Next, let

$$\bar{\xi}(s, v) = [\xi^T(s) \quad \xi_\tau^T N^T \quad \psi_s^T(u) \quad v^T(s)]^T, \;\; \bar{\xi}(s, \beta) = [\xi^T(s) \quad \eta^T(\beta)N^T]^T, \tag{59}$$

and then, it follows from (42) that

$$\begin{aligned} &\mathbb{E}\{e^T(s)e(s) - \gamma^2 v^T(s)v(s) + \mathcal{L}V(x(s), s, i)\} \\ \le\; &\mathbb{E}\{\bar{\xi}^T(s, v)\Pi_i\bar{\xi}(s, v)\} - \int_{s-\tau}^s \mathbb{E}\{\bar{\xi}^T(s, \beta)\Gamma_i\bar{\xi}(s, \beta)\}d\beta, \end{aligned} \tag{60}$$

where

$$\begin{aligned} \Pi_i := &\begin{bmatrix} \Omega_{11i} + \bar{L}_i^T\bar{L}_i & 0 & P_i\bar{B}_i & P_i\bar{G}_i + N^T C_i^T K_i \\ * & \Omega_{2i} & 0 & C_{di}^T K_i \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -2I \end{bmatrix} \\ &+ \begin{bmatrix} \bar{A}_i^T N^T \\ \bar{A}_{di}^T N^T \\ \bar{B}_i^T N^T \\ 0 \end{bmatrix} \Phi_i^{-1}[N\bar{A}_i \; N\bar{A}_{di} \; N\bar{B}_i \; 0] \end{aligned} \tag{61}$$

and Ω_{2i}, Γ_i, Ω_{11i} and Φ_i are defined in (20), (44), (45) and (46), respectively.

By the Schur Complement and the conditions (17)-(18), for $\tau \le \bar{\tau}$, it follows that

$$\Pi_i < 0, \quad \Gamma_i > 0 \quad \forall\, i \in S, \tag{62}$$

and we can obtain from (58), (60) and (62) that, for all $t > 0$, $J(t) < 0$. Therefore, we arrive at

$$\mathbb{E}\left\{\int_0^t [z^T(s)z(s)\right\} ds \le \gamma^2 \left\{\int_0^t v^T(s)v(s)ds\right\},$$

which implies that

$$\|z(t)\|_{E_2} < \gamma\|v(t)\|_2. \tag{63}$$

From the Definition 2, it is concluded that the filtering error system (11)-(12) is stochastically stable with a disturbance attenuation level $\gamma > 0$ for $\tau \le \bar{\tau}$. The proof is now complete. ■