Output Feedback Robust H_∞ Control with D-stability and Variance Constraints: A Parameterization Approach

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Abstract

In this paper, we study the problem of robust H_∞ controller design for uncertain continuous-time systems with variance and D-stability constraints. The parameter uncertainties are allowed to be unstructured but norm-bounded. The aim of this problem is the design of an output feedback controller such that, for all admissible uncertainties, the closed-loop poles are placed within a specified disk, the H_∞ norm bound constraint on the disturbance rejection attenuation is guaranteed, and the steady-state variance for each state of the closed-loop system is not more than the prespecified individual upper bound, simultaneously. A parametric design method is exploited to solve the problem addressed. Sufficient conditions for the existence of the desired controllers are derived by using the generalized inverse theory. The analytical expression of the set of desired controllers is also presented. It is shown that the obtained results can be readily extended to the dynamic output feedback case and the discrete-time case.

Keywords

Uncertain systems; Output feedback; H_∞ control; Covariance control; Robust control; Regional pole placement.

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I. Introduction

In many stochastic control problems, it is quite common that the performance requirements are naturally expressed in terms of the upper bounds on the steady-state variances, see e.g. [1], [2]. The traditional control design techniques, such as LQG and H_∞ control theories, are very difficult to be directly applied in this kind of design problems, since they do not have a convenient avenue for imposing design objectives stated as the upper bounds on the variance values. For instance, the LQG controllers minimize a linear quadratic performance index which lacks guaranteed variance constraints with respect to individual system states. On the other hand, the covariance control theory (see [1], [3], [4]) has provided a more direct methodology for achieving the individual variance constraints than the LQG control theory. The main idea of the covariance control theory is to choose a state covariance according to different requirements on the system performance and robustness, and then to design a controller so that the specified state covariance is assigned to the closed-loop system.

In the past decade, the covariance control theory has received considerable research attention mainly because of its *multiobjective* flavor to the control design problem, see e.g. [2], [5], [6], [7], [8]. The multiobjective nature of the covariance control theory is based on two facts: 1) there is much remaining design freedom after assigning steady-state covariance (or variance upper bound) to the closed-loop system; and 2) several control design objectives are directly related to the steady-state covariance. So far, in the literature concerning covariance control, in addition to the variance (covariance) constraints, other desired performance requirements, such as stability, multiple-output performance specifications, robustness, H_∞ norm restriction, minimum-energy

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input, have been extensively considered. Also, the dual multiobjective filtering problems with error variance constraints have recently gained initial research interest, see e.g. [9], [10].

However, while the variance-constrained design is primarily concerned with steady-state mean square performance specifications, it says little about the transient behaviour. As is well known, pole location is directly associated with the dynamical characteristics of linear time-invariant systems, and in designing control systems it may be satisfactory in practice that the closed-loop poles are in a specified region. Therefore, in the past decade, the problem of controller design for assigning all closed-loop poles within a desired region (often a disk) has been an active area of research (see e.g. [7], [11]). A linear time-invariant system is said to have the so-called D-stability if it is stable and its poles are all located inside a given disk.

The problem of robust regional pole-assignment subjected to plant parameter perturbations has been well studied. The robust regional stability of a system subjected to uncertainties was analyzed in [12], [13]. Also, in [14], [15], the controller design problem was dealt with for robust regional-pole assignment, but the poles of the nominal system were required to be located in the specified region, and thus these design methods were not suitable for the unstable nominal systems. [16], [17] established the parameterization of robust controllers for regional pole placement for continuous-time systems, but did not take the H_∞ performance requirement into account. [18] investigated the H_∞ control with regional stability constraints, but the system uncertainty was not presented. Up to now, to the best of the author's knowledge, the issue of *variance and D-stability constrained* H_∞ control for linear uncertain systems, which is actually a stochastic multiobjective control problem, has not been fully investigated and remains to be important and challenging.

In this paper, we consider the problem of designing robust H_∞ controllers for linear uncertain continuous-time systems subjected to D-stability and steady-state variance constraints. The goal of this problem is to design an output feedback controller, such that for all admissible parameter perturbations, the closed-loop poles are assigned within a prescribed disk, the steady-state variance of each state is not more than the individual prespecified upper bound, and the H_∞ norm of the transfer function from disturbance inputs to system outputs meets the prespecified upper bound constraint, simultaneously. A purely algebraic parameterization approach is effectively developed to solve the problem addressed. The existence conditions as well as the explicit expression of desired controllers are presented, and an illustrative example is used to demonstrate the applicability of the proposed design procedure.

Notation. The notations in this paper are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem formulation and assumptions

Consider a linear continuous uncertain stochastic system represented by

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + Dw(t),\ x(t_0) = x_0, \tag{1}$$

and the measurement equation

$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, and $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector. A, B, D and C are known constant matrices, and D is assumed to be of full row rank. $w(t) \in \mathbb{R}^{n_w}$ is a zero mean Gaussian white noise process with covariance $I > 0$. x_0 is the unknown random zero-mean initial state with $\mathbb{E}\{x_0^T x_0\} = R_0$ and is uncorrelated with $w(t)$. The matrix ΔA represents parametric perturbations in the system matrix that is of the following form (see e.g. [9], [17]):

$$\Delta A = MFN, \tag{3}$$

where $F \in \mathbb{R}^{i \times j}$, which stands for the norm bounded uncertainty, is an uncertain matrix bounded by

$$FF^T \leq I, \tag{4}$$

and M and N are known constant matrices of appropriate dimensions which specify how the elements of the nominal matrix A are affected by the uncertain parameters in F. ΔA is said to be admissible if both (3) and (4) hold.

When an output feedback control law

$$u(t) = Ky(t) \tag{5}$$

is applied to the system (1)-(2), the closed-loop system is governed by

$$\dot{x}(t) = (A_c + \Delta A)x(t) + Dw(t), \ A_c := A + BKC, \ y(t) = Cx(t). \tag{6}$$

If the closed-loop system (6) is asymptotically stable for all admissible uncertainties, the steady-state covariance, defined by

$$X := \lim_{t \to \infty} E[x(t)x^T(t)],$$

satisfies the following Lyapunov differential equation

$$(A_c + \Delta A)X + X(A_c + \Delta A)^T + DD^T = 0. \tag{7}$$

Furthermore, for the system (6), the closed-loop transfer function $H(s)$ from the disturbance input $w(t)$ to the output $y(t)$ can be written as

$$H(s) = C[sI - (A_c + \Delta A)]^{-1}D. \tag{8}$$

We now consider a disc $D(q,r)$ in the left complex plane with the center at $-q + j0$ $(q > 0)$ and the radius r $(r < q)$ for the continuous systems. To this end, we are in a position to formulate that, the Robust H_∞ Variance and D-stability Constrained Design (RHVDCD) problem under study is to determine the output feedback gain K such that the following performance criteria are simultaneously satisfied for the uncertain system (6):

$(P1)$ The closed-loop poles are constrained to lie within the specified disc, i.e.

$$\sigma(A_c + \Delta A) \subset D(q,r), \tag{9}$$

for all admissible uncertainties.

$(P2)$ The H_∞ norm of the disturbance transfer matrix $H(s)$ from $w(t)$ to $y(t)$ meets the constraint

$$\|H(s)\|_\infty \leq \gamma, \tag{10}$$

where $\|H(s)\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{max}[H(j\omega)]$ and $\sigma_{max}[\cdot]$ denotes the largest singular value of $[\cdot]$; and γ is a given positive constant.

$(P3)$ The steady-state covariance X meets

$$[X]_{ii} \leq \sigma_i^2, \quad i = 1, 2, \cdots, n_x, \tag{11}$$

where $[X]_{ii}$ denotes the variance of the ith state, and σ_i^2 $(i = 1, 2, \cdots, n_x)$ stands for the prespecified steady-state variance constraint on ith state and can be determined by the practical performance indices.

III. Preliminary results

The following result provides a main key for solving the problem RHVDCD. It is shown that the enforcement of the requirements (P1) and (P2) in the problem RHVDCD is related to a modified Riccati equation. Moreover, the solution of this modified Riccati equation gives an upper bound on the actual steady-state covariance matrix X.

Theorem 1: Given a constant $\gamma > 0$ and a disk $D(q,r)$. Assume that the following matrix equation has a positive definite solution $Q > 0$

$$
\begin{aligned}
&(A_c+\Delta A)Q(A_c+\Delta A)^T+(q^2-r^2)Q \\
&+q[(A_c+\Delta A)Q+Q(A_c+\Delta A)^T+\gamma^{-2}QC^TCQ+DD^T+P]=0,
\end{aligned}
\tag{12}
$$

where $P \geq 0$ is an arbitrary matrix. Then, we have the conclusions that: 1) $\sigma(A_c+\Delta A) \subset D(q,r)$; 2) $\|H(s)\|_\infty \leq \gamma$; and 3) the steady-state covariance X exists and satisfies $X \leq Q$.

Proof: Define

$$L := \frac{1}{r}(A+BKC+\Delta A+qI) = \frac{1}{r}(A_c+\Delta A+qI).$$

It is easy to see that the specified D-stability constraint $\sigma(A_c+\Delta A) \subset D(q,r)$ is equivalent to the Schur stability of the matrix L, i.e., the eigenvalues of L are all located inside the unit circle $D(0,1)$. From the discrete-time Lyapunov stability theory, we know that L is a Schur matrix if and only if there exists a positive definite matrix Q meeting $Q-LQL^T>0$. Note that since the matrix D is of full row rank, then $DD^T > 0$, and hence we can rearrange (12) as follows

$$Q-LQL^T = \frac{q}{r^2}(\gamma^{-2}QC^TCQ+DD^T+P) > 0,$$

which implies that the D-stability requirement (P1) is met. Next, (12) can also be rewritten as the following

$$(A_c+\Delta A)Q+Q(A_c+\Delta A)^T+\gamma^{-2}QC^TCQ+DD^T+\Sigma=0, \tag{13}$$

where

$$\Sigma := q^{-1}[(A_c+\Delta A)Q(A_c+\Delta A)^T+(q^2-r^2)Q]+P. \tag{14}$$

Since $\Sigma > 0$, the proof of $\|H(s)\|_\infty \leq \gamma$ can be completed by a standard manipulation of (13); for details see Lemma 1 of [19].

Next, it follows from (7) and (13) that

$$(A_c+\Delta A)(Q-X)+(Q-X)(A_c+\Delta A)^T+\gamma^{-2}QC^TCQ+\Sigma=0$$

which is, because of the D-stability of the system (6), equivalent to

$$Q-X=\int_0^\infty \exp[(A_c+\Delta A)t](\gamma^{-2}QC^TCQ+\Sigma)\exp[(A_c+\Delta A)^Tt]dt \geq 0.$$

That is, $X \leq Q$. The proof of this theorem is then complete. ■

Remark 1: Theorem 1 shows that the H_∞ disturbance attenuation and the D-stability constraints are automatically enforced when a positive definite solution Q to (12) is known to exist. Furthermore, if the positive definite solution Q satisfies

$$[Q]_{ii} \leq \sigma_i^2, \quad i=1,2,\cdots,n_x, \tag{15}$$

then we will have $[X]_{ii} < [Q]_{ii} \le \sigma_i^2$, $i = 1, 2, \cdots, n$, and therefore our task is accomplished. In what follows, our goal is to study the existence conditions as well as analytical expression of a feedback gain K such that (12) holds.

Lemma 1: [20] Let a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q > 0$ be such that $\varepsilon NQN^T < I$. Then

$$(A_{cq} + \Delta A)Q(A_{cq} + \Delta A)^T \le A_{cq}QA_{cq}^T + A_{cq}QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQA_{cq}^T + \varepsilon^{-1}MM^T, \tag{16}$$

where $A_{cq} := A_c + qI = A + BKC + qI$.

Theorem 2: Let the desired disk $D(q, r)$, the constant $\gamma > 0$ and the output feedback gain K be given. If there exist a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q > 0$ satisfying

$$\varepsilon NQN^T < I \tag{17}$$

$$A_{cq}[Q + QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQ]A_{cq}^T + \varepsilon^{-1}MM^T = r^2Q - q(\gamma^{-2}QC^TCQ + DD^T), \tag{18}$$

then the eigenvalues of the uncertain closed-loop system matrix $A_c + \Delta A$ are located within the desired disk $D(q, r)$, the H_∞ norm of the disturbance transfer matrix $H(s)$ from $w(t)$ to $y(t)$ meets the constraint $\|H(s)\|_\infty \le \gamma$, and the steady-state covariance X satisfies $X \le Q$.

Proof: From Lemma 1, we have

$$\Theta := A_{cq}[Q + QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQ]A_{cq}^T + \varepsilon^{-1}MM^T - (A_{cq} + \Delta A)Q(A_{cq} + \Delta A)^T > 0. \tag{19}$$

Then, by using (19), (18) is equivalent to

$$(A_{cq} + \Delta A)Q(A_{cq} + \Delta A)^T = r^2Q - q(\gamma^{-2}QC^TCQ + DD^T + q^{-1}\Theta). \tag{20}$$

Define $P := q^{-1}\Theta \ge 0$ and note that $A_{cq} = A_c + qI$, (20) can be rewritten as

$$q(A_c + \Delta A)Q + qQ(A_c + \Delta A)^T + (A_c + \Delta A)Q(A_c + \Delta A)^T + (q^2 - r^2)Q + q[\gamma^{-2}QC^TCQ + DD^T + P] = 0, \tag{21}$$

which is the same as (12), then the proof of this theorem follows from Theorem 1 directly. ■

It can be seen from Theorem 2 that, if there exists a controller gain K such that (17)(18) hold for a specified positive definite matrix Q and a positive scalar ε, then the goal of this paper will be achieved. We refer to this problem as the "(Q, ε)-pair assignment" problem, and therefore the addressed RHVDCD problem can be converted into such an auxiliary "(Q, ε)-pair assignment" problem. Now, we can conclude our task as solving the following two alternative problems:

$(A1)$ Find the conditions under which there exists a feedback controller K which satisfies (18) for the specified pair (Q, ε) meeting (17), where $Q > 0$ is positive definite and $\varepsilon > 0$ is a positive scalar. In this case, the pair (Q, ε) is called an *assignable* pair.

$(A2)$ Find the set of all output feedback controllers that can achieve the assignable pair (Q, ε).

Note that Theorem 2 gives sufficient conditions on the RHVDCD problem, and the auxiliary "(Q, ε)-pair assignment" problem is described based on Theorem 2. Therefore, the necessary and sufficient conditions for the existence of solutions to the converted "(Q, ε)-pair assignment" problem, which will be deduced in the next section, are just the sufficient conditions for the solvability of the *original* RHVDCD problem. These sufficient conditions may be conservative which, it is not difficult to find, are produced primarily due to the introduction of the matrix $\Theta \geq 0$ in Theorem 2. Since $\Theta \geq 0$ depends directly on the parameter $\varepsilon > 0$, we can reduce the conservatism by minimizing the matrix $\Theta \geq 0$ over the parameter $\varepsilon > 0$ in a matrix-norm sense. A related detailed research on this issue can be found in [21] and references therein. As will be discussed later, however, when we do not consider the uncertainty and H_∞ index, the RHVDCD problem will be simplified to the problem of circular pole assignment via output feedback, which is equivalent to the (Q, ε)-pair assignment problem.

In the next section, the so-called "(Q, ε)-pair assignment" problem will be solved completely based on the generalized inverse theory and the singular value decomposition technique.

IV. Main results and derivation

In this section, we shall first discuss the conditions under which there exists an output feedback controller gain K such that (17)(18) are satisfied, that is, establish the assignability conditions of a specified pair (Q, ε). Then, we shall derive the general expression of a feedback controller gain K that achieves the assignable pair (Q, ε).

The following lemma will be used in the proof of our main results.

Lemma 2: [22] Let $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{m\times p}$ $(m \leq p)$. There exists a matrix V that satisfies simultaneously

$$Y = XV, \quad VV^T = I,$$

if and only if

$$XX^T = YY^T.$$

In this case, a general solution for V can be expressed as

$$V = V_X \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_Y^T, \quad U \in \mathbb{R}^{(n-r_X)\times(p-r_X)}, \quad UU^T = I \tag{22}$$

where V_X and V_Y come from the singular value decomposition of X and Y, respectively,

$$X = U_X \begin{bmatrix} Z_X & 0 \\ 0 & 0 \end{bmatrix} V_X^T = \begin{bmatrix} U_{X1} & U_{X2} \end{bmatrix} \begin{bmatrix} Z_X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{X1}^T \\ V_{X2}^T \end{bmatrix} \tag{23}$$

$$Y = U_Y \begin{bmatrix} Z_Y & 0 \\ 0 & 0 \end{bmatrix} V_Y^T = \begin{bmatrix} U_{Y1} & U_{Y2} \end{bmatrix} \begin{bmatrix} Z_Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{Y1}^T \\ V_{Y2}^T \end{bmatrix} \tag{24}$$

and $r_X = \text{rank}(X)$, $U_X = U_Y$, $Z_X = Z_Y$.

Now, in order to obtain the conditions for the existence of desired controllers, K, with respect to the RHVDCD problem, we can rearrange (18) as follows:

$$\begin{aligned} &A_{cq}\left[Q + QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQ\right]A_{cq}^T \\ &= r^2Q - q(\gamma^{-2}QC^TCQ + DD^T) - \varepsilon^{-1}MM^T. \end{aligned} \tag{25}$$

Considering (25), since its left-hand side is positive semidefinite, we assume that Q and ε satisfy

$$r^2Q - q(\gamma^{-2}QC^TCQ + DD^T) - \varepsilon^{-1}MM^T > 0. \tag{26}$$

We further make the following definitions and square root decompositions

$$\Xi := Q + QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQ, \tag{27}$$

$$\Pi := r^2Q - q(\gamma^{-2}QC^TCQ + DD^T) - \varepsilon^{-1}MM^T, \tag{28}$$

and thus we can rewrite (18) as

$$(A_{cq}\Xi^{1/2})(A_{cq}\Xi^{1/2})^T = (\Pi^{1/2})(\Pi^{1/2})^T. \tag{29}$$

It follows from Lemma 2 that (29) holds if and only if there exists an orthogonal matrix V, $V \in \mathbb{R}^{n_x \times n_x}$, meeting

$$A_{cq}\Xi^{1/2} = \Pi^{1/2}V, \tag{30}$$

or

$$BKC = \Pi^{1/2}V\Xi^{-1/2} - A - qI. \tag{31}$$

Hence, we obtain the following result.

Lemma 3: Suppose that (17)(26) are satisfied. Then, (18) has a solution for K if and only if (31) has a solution for K.

The following result is easily accessible from [23].

Lemma 4: There exists an orthogonal matrix V such that (31) has a solution for K, if and only if there exists an orthogonal matrix V such that

$$(I - BB^+)(\Pi^{1/2}V\Xi^{-1/2} - A - qI) = 0, \tag{32}$$

$$(\Pi^{1/2}V\Xi^{-1/2} - A - qI)(I - C^+C) = 0, \tag{33}$$

where B^+ and C^+ denote the Moore-Penrose inverse of B and C, respectively.

It is easy to see that (32) is equivalent to

$$(I - BB^+)\Pi^{1/2}V = (I - BB^+)(A + qI)\Xi^{1/2}. \tag{34}$$

Note that when (26) holds, we have $\Pi > 0$ and thus $\Pi^{-1/2} > 0$ exists. Furthermore, considering the fact that $I - C^+C$ is symmetric and $V^T = V^{-1}$, we can rearrange (33) as

$$(I - C^+C)(A + qI)^T\Pi^{-1/2}V = (I - C^+C)\Xi^{-1/2}. \tag{35}$$

We now define

$$X = \begin{bmatrix} (I - BB^+)\Pi^{1/2} \\ (I - C^+C)(A + qI)^T\Pi^{-1/2} \end{bmatrix}, \quad Y = \begin{bmatrix} (I - BB^+)(A + qI)\Xi^{1/2} \\ (I - C^+C)\Xi^{-1/2} \end{bmatrix}, \tag{36}$$

then there exists an orthogonal matrix V such that (34)(35) hold if and only if there exists an orthogonal matrix V such that

$$XV = Y, \tag{37}$$

which is, by Lemma 2, equivalent to

$$XX^T = YY^T. \tag{38}$$

Substituting (36) in to (38) yields four equalities, of which two are identities, and the others are as follows:

$$(I - BB^+)[\Pi - (A + qI)\Xi(A + qI)^T](I - BB^+) = 0, \tag{39}$$

$$(I - C^+C)[(A + qI)^T\Pi^{-1}(A + qI) - \Xi^{-1}](I - C^+C) = 0. \tag{40}$$

Now, we can conclude the above results in the following theorem that gives the conditions for the assignability of a given pair (Q, ε).

Theorem 3: Consider the uncertain linear system (1)(2). Given the desired disc $D(q, r)$ and the H_∞ norm upper bound γ. Suppose that a positive definite matrix $Q > 0$ and a positive scalar $\varepsilon > 0$ satisfy

$$\varepsilon N Q N^T < I, \tag{41}$$

$$r^2 Q - q(\gamma^{-2} Q C^T C Q + D D^T) - \varepsilon^{-1} M M^T > 0. \tag{42}$$

Then, the specified pair (Q, ε) is assignable if and only if (39)(40) holds.

In what follows, we will introduce the solution to the auxiliary "(Q, ε)-pair assignment" problem. First, take the following singular value decompositions

$$X = \begin{bmatrix} (I - BB^+)\Pi^{1/2} \\ (I - C^+C)(A + qI)^T \Pi^{-1/2} \end{bmatrix} = U_X \begin{bmatrix} Z_X & 0 \\ 0 & 0 \end{bmatrix} V_X^T, \tag{43}$$

$$Y = \begin{bmatrix} (I - BB^+)(A + qI)\Xi^{1/2} \\ (I - C^+C)\Xi^{-1/2} \end{bmatrix} = U_Y \begin{bmatrix} Z_Y & 0 \\ 0 & 0 \end{bmatrix} V_Y^T. \tag{44}$$

It follows from Theorem 3 and [23] that, if the pair (Q, ε) satisfying (41)(42) is assignable, then a general solution of (31) is

$$K = B^+(\Pi^{1/2} V \Xi^{-1/2} - A - qI)C^+ + Z - B^+BZCC^+, \tag{45}$$

where $Z \in \mathbb{R}^{n_u \times n_x}$ is arbitrary, V is any orthogonal matrix satisfying $Y = XV$ and can be expressed, by Lemma 2, as

$$V = V_X \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_Y^T, \quad U \in \mathbb{R}^{(n_x - r_X)\times(n_x - r_X)}, \tag{46}$$

where the matrix U is arbitrary orthogonal.

Substituting (46) into (45) leads to the following theorem that characterizes the output feedback gains associated with assignable pair (Q, ε).

Theorem 4: Let the pair (Q, ε) satisfying (41)(42) be assignable, then the set of all output feedback gains that assign this pair is parameterized as

$$K = B^+(\Pi^{1/2} V_X \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_Y^T \Xi^{-1/2} - A - qI)C^+ + Z - B^+BZCC^+, \tag{47}$$

where Ξ and Π are defined in (27) and (28), respectively; V_X and V_Y are defined in (43) and (44), respectively; and $Z \in \mathbb{R}^{n_u \times n_x}$ is arbitrary, $U \in \mathbb{R}^{(n_x - n_X)\times(n_x - n_X)}$ is arbitrary orthogonal, $r_X = \operatorname{rank} X$.

Finally, the following theorem, which gives a solution to the addressed RHVDCD problem, is easily accessible as a summary of the results obtained in this section.

Theorem 5: Given the desired disc D(q,r), the H_∞ norm upper bound γ, and the prespecified steady-state variance constraints σ_i^2 $(i = 1, 2, \cdots, n_x)$. Assume that a specified pair (Q, ε) satisfying (41)(42) is assignable and $[Q]_{ii} \leq \sigma_i^2$ $(i = 1, 2, \cdots, n_x)$. Then the expression (47) gives solutions to the RHVDCD problem addressed in this paper.

Remark 2: It is easy to find out that, when the uncertainties are absent (i.e. $M = 0$, $N = 0$) and there are no constraints on the H_∞ norm of the disturbance transfer function (i.e. $\gamma = \infty$, $D = 0$), the condition in Theorem 1 will be both sufficient and necessary (see also [24]), and thus Theorem 4 actually parameterizes all output feedback controllers which place the closed-loop poles within a specified disk for continuous-time systems. This means, Theorem 4 generalizes partial results of [11], [24].

Remark 3: It can be seen that there exists much freedom in the design of desired controllers due to the non-uniqueness in choosing Q, ε, U, Z. This design flexibility can be used to achieve other performance requirements, such as reliability against sensor failures, implementation accuracies and gain reduction, etc., which still require further investigation.

Remark 4: It is not difficult to generalize our main results to the corresponding discrete-time systems. In this case, the disk $D(q,r)$ will be understood to lie inside the unit circle with center at the origin, centered at $(q,0)$ with radius r, where $r<1$ and $|q|+r<1$. Such a disk is often treated as a desired pole region for linear discrete-time systems, see [25] and the references therein. And then what we should do is only to simply modify the key equation (12) in order to enforce the desired robustness, D-stability and H_∞ disturbance rejection attenuation properties. Considering (12), instead of the term "$\gamma^{-2}QC^TCQ$" for the purpose of implementing continuous-time H_∞ performance index, the well-known Riccati-equation based discrete-time H_∞ control approach (see for example [26]) can be exploited, so as to construct an alternative matrix equation whose positive solution guarantees the simultaneous realization of all desired performance requirements on the D-stability and H_∞ index for uncertain discrete-time systems. After that, the design steps are the same as those presented in this paper.

Remark 5: In designing practical systems, we usually wish to construct an assignable pair (Q,ε) satisfying (41)(42) from the assignability conditions (39)(40), and then obtain the desired controller from (47) easily. The equations (39)(40) are, in fact, the generalized algebraic Riccati equations that also appeared in [27] with similar forms, thus they can be solved by using the same parametric method provided by [27]. Also, for relatively lower-order models, the addressed generalized algebraic Riccati equations can be treated by exploiting the local numerical searching method over the parameters Q, ε.

Remark 6: It should be noted that, for relatively higher-order models, considerable studies are still needed to investigate the global convergence of the numerical searching algorithm. That is, developing an efficient general computational algorithm remains an important issue for further research, at least in a theoretical sense, while for relatively lower-order models with small parameter perturbations the numerical experience is promising.

V. A numerical example

In this section, a numerical example is provided to show the usefulness and applicability of the present approach. We consider an uncertain linear continuous-time system described by

$$\dot{x}(t) = (A+\Delta A)x(t) + Bu(t) + Dw(t), \quad \Delta = MFN, \tag{48}$$

$$y(t) = Cx(t), \quad u(t) = Ky(t), \tag{49}$$

where

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.18 & 0 \\ 0 & 0 & 0.16 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad M = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is clear that the open-loop system is unstable. We assume that the robust constraints on the D-stability, H_∞ norm of the disturbance transfer function, and the steady-state variance are, respectively,

$$\sigma(A+BKC+\Delta A) \subset D(q,r) = D(3,2), \quad \|H(s)\| \le \gamma = 0.9, \tag{50}$$

$$[X]_{11} \le \sigma_1^2 = 4.5, \quad [X]_{22} \le \sigma_2^2 = 4, \quad [X]_{33} \le \sigma_3^2 = 1.5. \tag{51}$$

Now, it is supposed that the positive definite matrix Q has the form $Q = \mathrm{diag}\{q_{11}, q_{22}, q_{33}\}$. Note that (39) reduces to the following

$$4q_{11} - 3(0.9^{-2} \cdot 0.25q_{11}^2 + 0.25^2) - 0.25\varepsilon^{-1} = [0.25(\varepsilon^{-1} - 0.25q_{11})^{-1}]q_{11}^2 + q_{11}. \tag{52}$$

Thus, subject to the constraints (51), we can choose $q_{11} = 4$, $\varepsilon = 0.9998$. Next, the constraint (40) implies that $q_{22} = 3.887$. To meet the inequalities (41) and (42), we can set $q_{33} = 1$, and then the assignable pair (Q, ε) is determined.

Using the results provided in the previous section, we can obtain the important matrices as follows

$$\Xi = (1.0e+004)\begin{bmatrix} 2.000 & 0 & 0 \\ 0 & 0.0004 & 0 \\ 0 & 0 & 0.0001 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 15.5624 & 0 & 0 \\ 0 & 1.4611 & 0 \\ 0 & 0 & 2.9973 \end{bmatrix},$$
$$X = \begin{bmatrix} 3.9449 & 0 & 0 & 0.2535 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad Y = \begin{bmatrix} 141.4214 & 0 & 0 & 0.0071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Furthermore, in the expression (47), the arbitrary matrix Z can be set to be zero matrix, and the orthogonal matrix U can be selected as

$$U_1 = I_2, \quad U_2 = \mathrm{diag}\{1, -1\}, \quad U_3 = \mathrm{diag}\{-1, 1\}, \quad U_4 = -I_2,$$

respectively. Then, four desired output feedback controller gains are obtained from (47) as the following

$$\text{Case 1}: K = \begin{bmatrix} 1.2262 & -2.0000 \\ 0 & -4.5375 \end{bmatrix}, \quad \text{Case 2}: K = \begin{bmatrix} 1.2262 & -2.0000 \\ 0 & -11.4625 \end{bmatrix},$$
$$\text{Case 3}: K = \begin{bmatrix} -1.2262 & -2.0000 \\ 0 & -4.5375 \end{bmatrix}, \quad \text{Case 4}: K = \begin{bmatrix} -1.2262 & -2.0000 \\ 0 & -11.4625 \end{bmatrix}.$$

In these four cases, assume now that the uncertain matrix is of the form $F = \sin t I_3$. The state responses of the closed-loop system to initial conditions are shown, respectively, in Figures 1-4. The simulation results verify that the desired goal has been achieved.

VI. Conclusions

This paper has introduced a parametric approach to the robust H_∞ output feedback stochastic control problem for linear continuous uncertain systems with D-stability and steady-state variance constraints. First, we have derived some Riccati-like matrix equations whose positive definite solutions give the upper bounds for the actual steady-state variance, and therefore indicate the simultaneous enforcement of all desired performance requirements. Then, based on these Riccati-like matrix equations, the generalized inverse theory and singular value decomposition technique have been exploited to obtain the existence conditions and solutions of expected controllers. It is not difficult to extend the results of this paper to the dynamic output feedback case and the discrete-time case. Further study will concentrate on utilizing the remaining design freedom to achieve other performance constraints.

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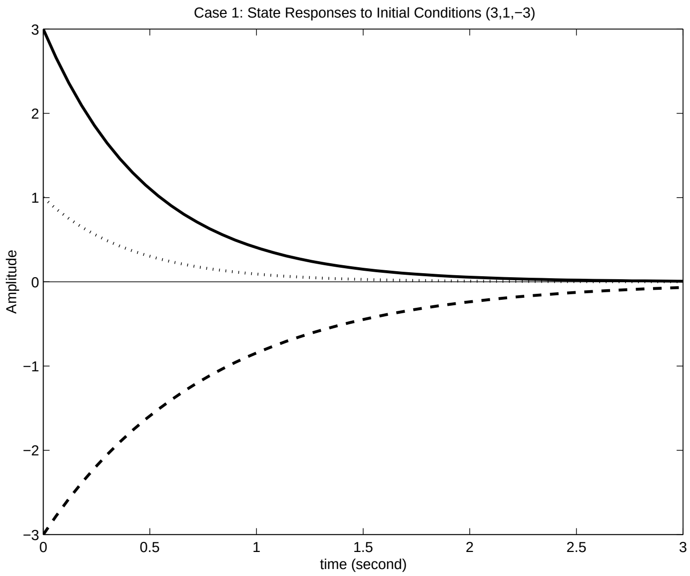

Fig. 1. x_1 (solid), x_2 (point), x_3 (dashed).

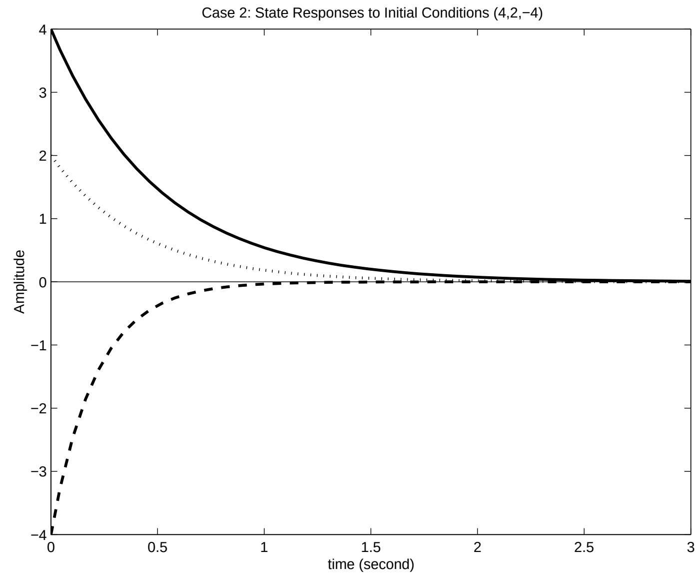

Fig. 2. x_1 (solid), x_2 (point), x_3 (dashed).

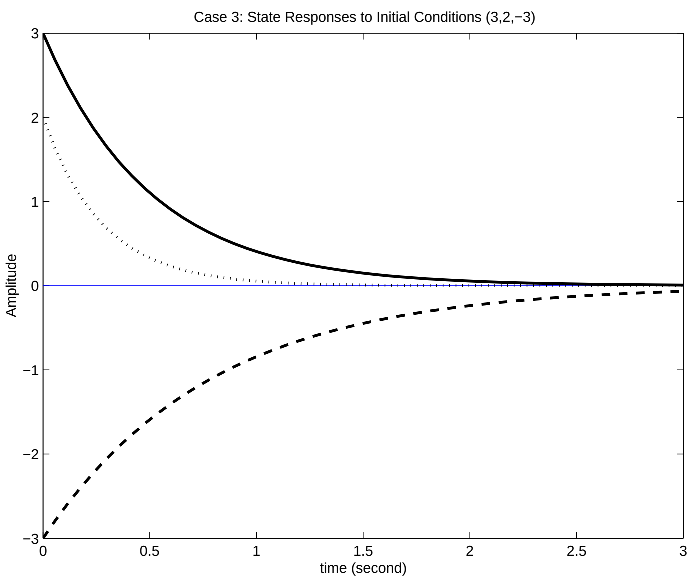

Fig. 3. x_1 (solid), x_2 (point), x_3 (dashed).

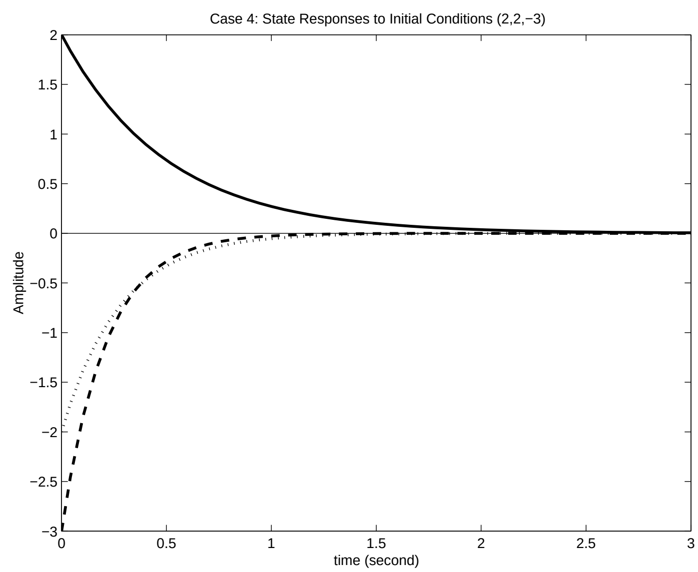

Fig. 4. x_1 (solid), x_2 (point), x_3 (dashed).

References

[1] Skelton, R. E. and Iwasaki, T., 1993, Liapunov and covariance controllers, *Int. J. Control*, Vol. 57, 519-536.

[2] Skelton, R. E., Iwasaki, T. and Grigoriadis, K.M., 1997, *A unified algebraic approach to linear control design*, Taylor & Francis, Basingstoke.

[3] Hotz, A. and Skelton, R. E., 1987, Covariance control theory, *Int. J. Control*, Vol. 46, 13-32.

[4] Collins, E. G., Jr. and Skelton, R. E., 1987, A theory of state covariance assignment for discrete systems, *IEEE Trans. Automat. Control*, Vol. 32, 35-41.

[5] Skelton, R. E., Xu, J. H. and Yasuda, K., 1994, On the freedom in covariance control, *Int. J. Control*, Vol. 59, 1567-1578.

[6] Grigoriadis, K. M. and Skelton, R. E., 1997, Minimum-energy covariance controllers, *Automatica*, Vol. 33, 569-578.

[7] Wang, Z., Chen, X. and Guo Z., 1995, Controller design for continuous systems with variance and circular pole constraints, *Int. J. Systems Sci.*, Vol. 26, 1249-1256.

[8] Wang, Z. and Unbehauen, H., 2000, Robust H_∞ control for systems with time-varying parameter uncertainty and variance constraints, *Cybernetics and Systems: An International Journal*, Vol. 31, 175-191.

[9] Wang, Z. and Huang, B., 2000, Robust H_2/H_∞ filtering for linear systems with error variance constraints, *IEEE Trans. Signal Processing*, Vol. 48, 2463-2467.

[10] Wang, Z., Huang, B. and Huo, P., 2001, Sampled-data filtering with error covariance assignment, *IEEE Trans. Signal Processing*, Vol. 49, 666-670.

[11] Haddad, W. M. and Bernstein, D. S., 1992, Controller design with regional pole constraints, *IEEE Trans. Automat. Control*, Vol. 37, 54-69.

[12] Chou, J. H., 1991, Pole-assignment robustness in a specified disk, *Systems & Control Letters*, Vol. 16, 41-44.

[13] Horng, H. Y., Chou, J. H. and Horng, I. R., 1993, Robustness of eigenvalue clustering in various regions of the complex plane for perturbed systems, *Int. J. Control*, Vol. 57, pp. 1469-1483.

[14] Juang, Y. T. and Chen, K. H., 1989, Robust pole-assignment of linear dynamic systems, *Control - Theory and Advanced Technology*, Vol. 5, 67-74.

[15] Tsay, S. C., Fong, I. K. and Kuo, T. S., 1990, D-stability analysis for discrete optimal regulator, *Control - Theory and Advanced Technology*, Vol. 6, 237-246.

[16] Yaz, E., Skelton, R. E. and Grigoriadis, E., 1993, Robust regional pole assignment with output feedback, *Proc. IEEE Conf. on Decision and Control*, 3009-3013.

[17] Wang, Z. and and Burnham, K. J., 2002, LMI approach to output feedback control for linear uncertain systems with D-stability constraints, *J. Optimization Theory and Applications*, Vol. 113, No. 2, 357-372.

[18] Yedavalli, R. K. and Liu, Y., 1995, H_∞ control with regional stability constraints, *Automatica*, Vol. 31, 611-615.

[19] Willems, J. C., 1971, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Automat. Control*, Vol. 16, 621-634.

[20] Wang, Y., Xie, L. and de Souza, C. E., 1992, Robust control of a class of uncertain nonlinear systems, *Systems & Control Lett.*, Vol. 19, 139-149.

[21] Xie, L., 1996, Output feedback H_∞ control of systems with parameter uncertainty, *Int. J. Control*, Vol. 63, 741-750

[22] Glover, K., 1984, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, *Int. J. Control*, Vol. 39, 1115-1193.

[23] Ben-Israel, A. and Greville, T. N. E., 1974, Generalized inverses: theory and applications, New York: Wiley.

[24] Furuta K. and Kim S. B., 1987, Pole assignment in a specified disk, *IEEE Trans. Automat. Control*, Vol. 32, 423-427.

[25] Wang, Z., 2000, Robust state estimation for perturbed systems with error variance and circular pole constraints: the discrete-time case, *Int. J. Control*, Vol. 73, 303-311.

[26] Haddad, W. M., Bernstein, D. S. and Mustafa, D., 1991, Mixed-norm H_2/H_∞ regulation and estimation: the discrete-time case, *Systems & Control Lett.*, Vol. 16, 235-248.

[27] Chang, W. J. and Chung, H. Y., A study of H_∞ norm and variance-constrained design using dynamic output feedback for linear discrete systems, *Int. J. Control*, Vol. 57, 473-484.