

# EXISTENCE AND STABILITY OF SINGULAR PATTERNS IN A GINZBURG-LANDAU EQUATION COUPLED WITH A MEAN FIELD

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ABSTRACT. We study singular patterns in a particular system of parabolic partial differential equations which consist of a Ginzburg-Landau equation and a mean field equation. We prove existence of the three simplest concentrated periodic stationary patterns (single spikes, double spikes, double transition layers) by composing them of more elementary patterns and solving the corresponding consistency conditions. In the case of spike patterns we prove stability for sufficiently large spatial periods by first showing that the eigenvalues do not tend to zero as the period goes to infinity and then passing in the limit to a nonlocal eigenvalue problem which can be studied explicitly. For the two other patterns we show instability by using the variational characterization of eigenvalues.

## 1. INTRODUCTION

The study of pattern formation in various fields of science leads to the study of systems with a conservation law. Examples, some of which we refer to later, include fluid mechanics as well as many chemical or biological systems. In this paper we consider pattern formation in a particular system of partial differential equations where a Ginzburg-Landau equation is coupled with a mean field.

We consider the following amplitude equations which have been derived by P.C. Matthews and S.M. Cox [5], [8] and arise when expanding the problem in terms of fast and slow (or envelope) variables near a critical set of parameter values that lead to supercritical bifurcation:

$$\begin{cases} A_t = A_{xx} + A - A^3 - AB, & x \in R, t > 0, \\ B_t = \sigma B_{xx} + \mu(A^2)_{xx}, & x \in R, t > 0, \end{cases} \quad (1.1)$$

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where

$$\sigma > 0, \quad \mu \in R.$$

By taking  $\tau = \frac{1}{\sigma}, \mu' = \frac{\mu}{\sigma}$ , equation (1.1) can be rewritten in the form

$$\begin{cases} A_t = A_{xx} + A - A^3 - AB, & x \in R, t > 0, \\ \tau B_t = B_{xx} + \mu' (A^2)_{xx}, & x \in R, t > 0, \end{cases} \quad (1.2)$$

where

$$\tau > 0, \quad \mu' = \frac{\mu}{\sigma} \in R.$$

It is easy to see that the amplitude equation (1.2) is invariant if

$$A \text{ transforms to } -A$$

or if

$$x \text{ transforms to } -x.$$

As a prototype example, equation (1.2) arises in the study of the following PDE

$$\frac{\partial w}{\partial t} = -\frac{\partial^2}{\partial x^2} \left[ r^2 w - s w^2 - w^3 - \left( 1 + \frac{\partial^2}{\partial x^2} \right)^2 w \right], \quad (1.3)$$

where the terms inside the brackets are the same as in the Swift-Hohenberg equation [13] supplemented with a symmetry-breaking quadratic term  $s w^2$ . The symmetry breaking term is necessary for the amplitude equations to become a system as in (1.1). In case  $s = 0$  we would just get the Ginzburg-Landau equation. Note that this system has the following important features:

- It possesses conserved quantities. In a sense, it is a conservation law.
- It is a parabolic equation at lowest order in  $w$ .
- It has the symmetry groups  $x \rightarrow -x$  and  $x \rightarrow x + x_0$  for all  $x_0 \in R$ .
- It arises in the perturbation analysis near a cubic bifurcation point in the supercritical case.
- The Fourier modes  $e^{i0x}$  and  $e^{\pm ix}$  are neutrally stable at the linearized bifurcation point  $r = 0, w = 0$ .

It is shown in [8] how the equation (1.1) arises as an amplitude equation of (1.3). Therefore the ansatz

$$r = \epsilon^2 r_2, \quad T = \epsilon^2 t, \quad X = \epsilon x, \quad (1.4)$$

and

$$\begin{aligned} w(x, t) = & \epsilon A(X, T) e^{ix} + \epsilon A^*(X, T) e^{-ix} + \epsilon^2 \tilde{B}(X, T) \\ & + \epsilon^2 C(X, T) e^{2ix} + \epsilon^2 C^*(X, T) e^{-2ix} + O(\epsilon^2) \end{aligned} \quad (1.5)$$

is made, where the large-scale mode  $B$  has been introduced at order  $\epsilon^2$ .

Substituting (1.5) into (1.3) and solving the system at successive orders of  $\epsilon$ , the following complex equations

$$\begin{cases} A_T = r_2 A + 4A_{XX} - (3 - 2s^2/9)|A|^2 A - 2sAB, \\ B_T = B_{XX} + 2s(|A|^2)_{XX} \end{cases} \quad (1.6)$$

are derived in [8]. If  $s^2 < \frac{27}{2}$ , then it is shown in [8] that the bifurcation is supercritical, and that, assuming that  $A$  is real, then (1.6) can be rescaled to (1.1) with

$$\sigma = \frac{1}{4}, \quad \mu = \frac{s^2}{3 - 2s^2/9}.$$

In the equations (1.2),  $A$  can be complex. Namely, we can write  $A = R \exp(i\theta)$ . The additional phase space  $\theta$  makes analytic analysis very complicated. In this paper, we restrict our attention to the invariant subspace in which  $A$  is **real**. Here we follow the paper [8] where the authors also focus on the case of a real function  $A$ . We hope to return to the general case in a future study.

Amplitude equations of the form (1.2) or conservative models of the form (1.3) have been considered in hydrodynamics. See for instance, [7]. We also refer to [5] and [8], where (1.2) was derived in from nonlinear partial differential equations which arise in thermosolutal convection, rotating convection, or magnetoconvection, respectively. Further, in [4], the equation (1.2) was also derived in the study of secondary stability of a one-dimensional cellular pattern.

Another type of GL equation, where the term  $(|A|^2)_{xx}$  in the  $B$ -equation is replaced by  $\partial_x(|A|^2)$  has been considered by a number of

authors, see [10], [11] and the references therein. There the basic patterns are travelling pulses which arise in convection of binary fluids.

Finally, in [14] a conserved variant of (1.3) has been considered which has the same linear dispersion but different nonlinear behaviour. In this case, the behaviour becomes chaotic.

Equation (1.2) is studied with the following periodic boundary conditions which arise from the expansion (1.4):

$$A(x + L) = A(x), \quad B(x + L) = B(x), \quad (1.7)$$

where  $L$  is the minimal period. Other boundary conditions may be more appropriate for other modelling situations.

We now state our two main results on existence and stability of stationary patterns for system (1.2) with boundary conditions (1.7), which we refer to as Problem (1.2).

We first consider the existence of spikes and fronts.

**Theorem 1.** *There exists an  $\bar{L} > 0$  such that for all  $L > \bar{L}$  the Problem (1.2) admits the following three types of solutions.*

**Type I (Single spike solution).** *Assume that*

$$\mu' > 1, \quad \lim_{L \rightarrow +\infty} L(\mu' - 1) := \frac{2}{\beta_\infty} < 2. \quad (1.8)$$

*Then there exist steady-state solutions of (1.2) with the following asymptotic behaviour*

$$A^\pm(x) \sim \frac{\sqrt{2}c^\pm}{\sqrt{\mu' - 1}} \operatorname{sech}(c^\pm x), \quad B^\pm(x) = -\mu'(A^\pm)^2 + \frac{\mu'}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} (A^\pm)^2 dx, \quad (1.9)$$

*where  $c^- < c^+$  are the two roots of the following algebraic equation:*

$$c^2 - 2\beta_\infty c + 1 = 0. \quad (1.10)$$

**Type II (Double spike solution).** *Assume that*

$$\mu' > 1, \quad \lim_{L \rightarrow +\infty} L(\mu' - 1) = \frac{2}{\beta_\infty} < 4. \quad (1.11)$$

Then there exist steady-state solutions of (1.2) with the following asymptotic behaviour

$$A^\pm(x) \sim \frac{\sqrt{2}c^\pm}{\sqrt{\mu' - 1}} \left[ -\operatorname{sech} \left( c^\pm \left( x + \frac{L}{2} \right) \right) + \operatorname{sech} (c^\pm x) - \operatorname{sech} \left( c^\pm \left( x - \frac{L}{2} \right) \right) \right], \quad (1.12)$$

$$B^\pm(x) = -\mu' (A^\pm)^2 + \frac{\mu'}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} (A^\pm)^2 dx,$$

where  $c^- < c^+$  are the two roots of the following algebraic equation:

$$c^2 - 4\beta_\infty c + 1 = 0. \quad (1.13)$$

**Type III (Double Front solution).** Assume that

$$\mu' < 1. \quad (1.14)$$

Then there exist (even) steady-state solutions of (1.2) with the following asymptotic behaviour

$$A(x) \sim \frac{c}{\sqrt{1 - \mu'}} \tanh \left( \frac{c}{\sqrt{2}} \left( x - \frac{L}{4} \right) \right) \quad \text{for } 0 < x < \frac{L}{2}, \quad (1.15)$$

$$B(x) = -\mu' A^2 + \frac{\mu'}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} A^2 dx,$$

where  $c$  is the positive root of the following algebraic equation:

$$c^2 - \frac{4\mu'}{L} c - (1 - \mu') = 0. \quad (1.16)$$

Our next theorem classifies the stability of all the three types of solutions given in Theorem (1).

**Theorem 2.** Suppose that  $L \gg 1$  and  $\tau > 0$ . Then for single spike solutions (Type I),  $(A^-, B^-)$  is (linearly) stable, while  $(A^+, B^+)$  is (linearly) unstable. The double spike solutions (Type II) and the double front solutions (Type III) are all (linearly) unstable.

**Remarks:**

1. Theorem 2 confirms the numerical computations in Section 4 of [8].

2. The fronts are as in the usual bistable Allen-Cahn equation. The instability of fronts and double spikes in Theorem 2 is a standard interaction instability. See the proofs in Section 5.

3. We remark that our stability and instability result hold true for **any**  $\tau > 0$ . This is quite a nontrivial fact.

4. Combining Theorem 1 and Theorem 2, we see that stable patterns exist for (1.2) only when  $\mu' > 1$ , that is  $\mu > \sigma$ . Going back to (1.3), this shows that stable patterns exist for the amplitude equation (1.6) only when  $s$  is small. That is when the bifurcation is supercritical. When  $s$  is large, (1.3) will go through a subcritical bifurcation ([12]) and our results show that there are no stable patterns.

5. Roughly speaking, we have proved the existence and stability (instability) of single (double) spike solution in the following parameter regime:

$$1 < \frac{\mu}{\sigma} = \mu' < 1 + \frac{2}{\beta_{+\infty}L}, \quad L \gg 1. \quad (1.17)$$

This agrees with the asymptotic analysis given in [8].

The case when  $L$  is finite remains open and we shall come to this question in a future work.

6. Our results rigorously show that localized solutions (spikes) may be stable when a Ginzburg-Landau equation is coupled to an equation for a mean field, even when the coefficients of the equations are real and when the bifurcation is supercritical. As far as we know, this is the first theoretical result on the stability of such patterns.

Throughout the paper we assume that

$$L \gg 1. \quad (1.18)$$

The organization of this paper is as follows: In Section 2, we prove the existence of the steady states given in Theorem 1 by joining single spikes or fronts and checking their consistency.

In Section 3, we prove preliminaries for the stability analysis and prove a crucial reduction lemma (Lemma 4).

In Section 4, we prove the stability of the single (small) spike solution by reducing the problem to a nonlocal eigenvalue problem which is studied in Lemma 6.

In Section 5, we prove the instability of the other solutions by invoking the variational characterization of eigenvalues.

In Section 6, we discuss some possible extensions.

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## 2. STEADY STATES: PROOF OF THEOREM 1

In this section, we classify all periodic steady states.

Consider steady states of the equations (1.2):

$$\begin{cases} A_{xx} + A - A^3 - AB = 0, & x \in R, \\ B_{xx} + \mu'(A^2)_{xx} = 0, \langle B \rangle = 0, & x \in R, \\ A(x), B(x) \text{ have minimal period } L, \end{cases} \quad (2.1)$$

where  $\langle B \rangle$  is the average of the  $B$  over the minimal period. (Note that by adding a constant to  $B$  we can transform (2.1) back to (1.2).)

By moving the  $x$  variable and changing the origin, we may assume that an interval with minimal period is  $I := [-\frac{L}{2}, \frac{L}{2}]$ .

From the equation for  $B$ , we obtain that

$$B(x) = -\mu' A^2(x) + \mu' \langle A^2 \rangle, \quad \langle A^2 \rangle = \frac{1}{L} \int_I A^2(x) dx. \quad (2.2)$$

Substituting (2.2) into the first equation of (2.1) for  $A$ , we obtain

$$\begin{cases} A_{xx} - aA + bA^3 = 0, & -\frac{L}{2} < x < \frac{L}{2}, \\ A(x) \text{ has minimal period } L, \end{cases} \quad (2.3)$$

where

$$a = \mu' \langle A^2 \rangle - 1, \quad b = \mu' - 1. \quad (2.4)$$

We consider  $a$  as a real parameter first. Since (2.3) is an autonomous equation, it is easy to see that we may assume that  $A$  satisfies the following boundary, symmetry, and positivity conditions:

$$A'(-\frac{L}{2}) = A'(\frac{L}{2}) = 0, A(x) = A(-x), A'(x) > 0 \text{ for } 0 < x < \frac{L}{2}. \quad (2.5)$$

We remark that a periodic solution  $A$  of (2.3) satisfying (2.5) for  $-\frac{L}{2} < x < \frac{L}{2}$  can be extended in a unique way to a periodic function on the real line with minimal period  $L$ .

To describe the asymptotic behaviour of  $A$  as  $L \rightarrow +\infty$ , we introduce two standard limiting equations. The first one is a single spike (also called soliton or bump). Let  $w_\infty$  be the unique solution of the following problem:

$$\begin{cases} w_\infty'' - w_\infty + w_\infty^3 = 0, & w_\infty > 0, \\ w_\infty(0) = \max_{y \in \mathbb{R}} w_\infty(y), & w_\infty(y) \rightarrow 0 \text{ as } |y| \rightarrow \infty. \end{cases} \quad (2.6)$$

By an elementary calculation it follows that  $w_\infty$  is given by

$$w_\infty(y) = \sqrt{2} \operatorname{sech}(y). \quad (2.7)$$

The second one is a ‘‘forward’’ front on  $\mathbb{R}$ . Let  $v_\infty$  be the unique solution of the following problem:

$$\begin{cases} v_\infty'' + v_\infty - v_\infty^3 = 0, & v_\infty' > 0, & y \in \mathbb{R}, \\ v_\infty(0) = 0, & v_\infty(y) \rightarrow \pm 1 \text{ as } y \rightarrow \pm\infty. \end{cases} \quad (2.8)$$

By an elementary calculation it follows that  $v_\infty$  is given by

$$v_\infty(y) = \tanh\left(\frac{y}{\sqrt{2}}\right). \quad (2.9)$$

A ‘‘backward’’ front is then defined by  $v_\infty(-y)$ ,  $y \in \mathbb{R}$ .

Let us introduce the three types of patterns for the solution  $A$  of (2.3) in detail.

### Type I Solution.

Let  $\mu' > 1$ ,  $A(x) > 0$ .

Since  $b = \mu' - 1 > 0$ , in order that (2.3) has a solution, we must have  $a > 0$ .

We rescale  $A$  as follows

$$A(x) = \sqrt{\frac{a}{b}} H_l(y), \quad (2.10)$$

where

$$l = \sqrt{a}L, y = \sqrt{a}x. \quad (2.11)$$

Then  $H_l(y)$  is the unique solution of the following ODE (a rescaled version of equation (2.3) on the real line with minimal period  $l = \sqrt{a}L$ ):

$$H_l'' - H_l + H_l^3 = 0, H_l > 0, H_l(-y) = H_l(y), \quad (2.12)$$

satisfying

$$H_l'(-\frac{l}{2}) = H_l'(\frac{l}{2}) = 0, H_l(y) = H_l(-y), H_l'(y) > 0 \text{ for } 0 < y < \frac{l}{2}. \quad (2.13)$$

In this case, we see that for  $l \gg 1$ ,

$$H_l(y) = w_\infty(y) + O(e^{-l}), \text{ where } w_\infty(y) = \sqrt{2} \operatorname{sech}(y). \quad (2.14)$$

By a translation, this corresponds to the so-called  $dn$  function in Section 3.2 of [8].

We now return to check the consistency of our earlier calculations in (2.4).

Substituting (2.10) into (2.4) and by simple computations, we arrive at

$$c^2 - 2\beta_L^1 c + 1 = 0, \quad (2.15)$$

where

$$c = \sqrt{a}, \quad \beta_L^1 = \frac{\mu'}{2L(\mu' - 1)} \int_{-cL/2}^{cL/2} H_{cL}^2(y) dy. \quad (2.16)$$

Since  $L \gg 1$ , we have

$$\beta_L^1 = \frac{2\mu'}{L(\mu' - 1)} (1 + O(e^{-cL})),$$

since

$$\int_{\mathbb{R}} w_\infty^2(y) dy = 4.$$

Equation (2.15) has a solution if

$$\beta_\infty = \lim_{L \rightarrow +\infty} \frac{2}{L(\mu' - 1)} = \lim_{L \rightarrow +\infty} \beta_L^1 > 1. \quad (2.17)$$

Condition (2.17) is equivalent to (1.8). Note that (2.17) forces  $\mu' \rightarrow 1$ .

Under the condition (2.17), equation (2.15) has two roots:

$$c_\pm = \beta_L^1 \pm \sqrt{(\beta_L^1)^2 - 1}. \quad (2.18)$$

Thus we have obtained two single spike solutions:

$$A^\pm = \frac{c^\pm}{\sqrt{\mu' - 1}} H_{c^\pm L}(c^\pm x), \quad B^\pm(x) = -\mu'(A^\pm)^2 + \mu' \langle (A^\pm)^2 \rangle. \quad (2.19)$$

We will call  $(A^+, B^+)$  the single (large) spike solution and  $(A^-, B^-)$  the single (small) spike solution.

This finishes the proof of Type I Solutions.

### Type II Solutions.

Assume that  $\mu' > 1$  and that  $A(x)$  changes sign.

Similar to Type I Solutions, we rescale  $A(x)$  as in (2.10) and let  $l = \sqrt{a}L, y = \sqrt{a}x$ .

Then  $H_l(y)$  is the unique solution of the following ODE:

$$H_l'' - H_l + H_l^3 = 0, \quad H_l\left(\frac{l}{4}\right) = 0, \quad (2.20)$$

and

$$H_l'\left(-\frac{l}{2}\right) = H_l'\left(\frac{l}{2}\right) = 0, \quad H_l(y) = H_l(-y), \quad H_l'(y) > 0 \text{ for } 0 < y < \frac{l}{2}. \quad (2.21)$$

In this case,  $H_l$  looks like the superposition of two half solitons at the boundaries, which are both positive, and an interior soliton, which is negative.

Note that then as  $l \gg 1$ ,

$$H_l(y) = w_\infty\left(y + \frac{l}{2}\right) - w_\infty(y) + w_\infty\left(y - \frac{l}{2}\right) + O(e^{-l}), \quad -\frac{l}{2} < y < \frac{l}{2}. \quad (2.22)$$

This corresponds to the so-called  $cn$  function in Section 3.2 of [8].

It is easy to see that the consistency condition (2.4) implies

$$\begin{aligned} a &= \mu' \langle A^2 \rangle - 1 \\ &= \frac{\sqrt{a}\mu'}{L(\mu' - 1)} \left( 2 \int_{-\infty}^{\infty} w_\infty^2(y) dy + O(e^{-\sqrt{a}L}) \right) - 1. \end{aligned}$$

Therefore for  $L \gg 1$  a  $cn$  solution exists if and only if the quadratic equation

$$c^2 - 2\beta_L^2 c + 1 = 0$$

has a solution, where

$$\beta_L^2 = \frac{\mu'}{L(\mu' - 1)} \int_{-\infty}^{\infty} w_{\infty}^2(y) dy (1 + O(e^{-\sqrt{a}L})) = \frac{4\mu'}{L(\mu' - 1)} (1 + O(e^{-\sqrt{a}L})).$$

This is the case if  $2\beta_{\infty} = \lim_{L \rightarrow +\infty} \beta_L^2 > 1$ , or equivalently (1.11).

Similarly as for Type I Solutions, under condition (2.17), there are two roots for  $c$ . Correspondingly we obtain two solutions  $(A^{\pm}, B^{\pm})$ .  $(A^+, B^+)$  is called the double (large) spike solution and  $(A^-, B^-)$  is called the double (small) spike solution.

This finishes the existence of Type II Solutions.

### Type III Solutions.

Let  $\mu' < 1$ .

In this case, since  $b < 0$ ,  $a < 0$ , we rescale  $A$  as follows

$$A(x) = \sqrt{\frac{a}{b}} H_{\sqrt{-a}L}(\sqrt{-a}x), \quad (2.23)$$

where  $H_l, l = \sqrt{-a}L$  solves

$$H_l'' + H_l - H_l^3 = 0$$

with the following boundary and symmetry conditions:

$$H_l'(-\frac{l}{2}) = H_l'(\frac{l}{2}) = 0, \quad H_l(y) = H_l(-y), \quad H_l'(y) > 0 \text{ for } 0 < y < \frac{l}{2}. \quad (2.24)$$

In this case,  $H_l$  looks like a "backward" front connected to a "forward" front. More precisely, we need to introduce a front  $v_l$  on a bounded interval. This is the unique solution of the problem

$$\begin{cases} v_l'' + v_l - v_l^3 = 0, & v_l' > 0, & -\frac{l}{2} < y < \frac{l}{2}, \\ v_l(0) = 0, & v_l'(-\frac{l}{2}) = v_l'(\frac{l}{2}) = 0. \end{cases} \quad (2.25)$$

Then we have

$$H_l(y) = \begin{cases} v_{l/2}(-\frac{l}{4} - y), & -\frac{l}{2} < y \leq 0, \\ v_{l/2}(\frac{l}{4} - y), & 0 < y < \frac{l}{2}. \end{cases} \quad (2.26)$$

This corresponds to the so-called  $sn$  function in Section 3.2 of [8].

The consistency condition (2.4) becomes

$$\begin{aligned} a &= \mu' < A^2 > -1 \\ &= \frac{2\mu'}{L(1 - \mu')} \sqrt{-a} \int_0^{\frac{l}{2}} v_{l/2}^2(y) dy - 1 \end{aligned}$$

$$= \frac{2\sqrt{-a}\mu'}{L(1-\mu')} \left( \frac{l}{2} - 2 + O(e^{-\sqrt{-a}L}) \right) - 1.$$

Therefore for  $L \gg 1$  the  $sn$  solution exists if and only if the quadratic equation

$$c^2 - \frac{4\mu'}{L}c - (1 - \mu') = 0$$

has a positive solution, where  $c^2 = -a$  and  $c > 0$ . This quadratic equation always has two solutions one of which is positive and given by

$$c = \frac{2\mu'}{L} + \sqrt{\left(\frac{2\mu'}{L}\right)^2 + (1 - \mu')}.$$

Thus we have solved equation (2.1) with  $L \gg 1$  and the boundary conditions (2.5) in all three cases. This proves Theorem 1.

### 3. PRELIMINARIES FOR THE STABILITY ANALYSIS: A REDUCTION

In this section, we study some preliminary properties of the linearized eigenvalue problem. We show that the eigenvalues must be real. Moreover, we reduce the system of eigenvalue equations to a single eigenvalue equation.

To study the linearized stability of (1.2), we perturb  $(A(x), B(x))$  as follows:

$$A_\epsilon(x, t) = A(x) + \epsilon\phi(x)e^{\lambda_L t}, B_\epsilon(x, t) = B(x) + \epsilon\psi(x)e^{\lambda_L t}, \quad (3.1)$$

where  $\lambda_L \in \mathcal{C}$  – the set of complex numbers.

Since we have assumed that (1.2) is invariant under the transformations  $-x \rightarrow x$  and  $x \rightarrow x + L$ . we may suppose the perturbation  $(\phi(x), \psi(x))$  possesses the same symmetry. Thus we may assume that

$$\phi, \psi \in \mathcal{X}_L,$$

where

$$\mathcal{X}_L = \left\{ \phi \in H^1\left(-\frac{L}{2}, \frac{L}{2}\right) \mid \phi(-x) = \phi(x), \phi'\left(-\frac{L}{2}\right) = \phi'\left(\frac{L}{2}\right) = 0 \right\}. \quad (3.2)$$

Here  $H^1\left(-\frac{L}{2}, \frac{L}{2}\right)$  is the usual Sobolev space of measurable functions which as well as their first derivatives are square Lebesgue integrable functions in  $\left(-\frac{L}{2}, \frac{L}{2}\right)$ .

Substituting (3.1) into (1.2) and considering the leading order part, we obtain the following eigenvalue problem:

$$\begin{cases} \phi_{xx} + (1 - B)\phi - 3A^2\phi - A\psi = \lambda_L\phi, & -\frac{L}{2} < x < \frac{L}{2}, \\ \psi_{xx} + 2\mu'(A\phi)_{xx} = \tau\lambda_L\psi, & -\frac{L}{2} < x < \frac{L}{2}, \\ \lambda_L \in \mathcal{C}, \phi, \psi \in \mathcal{X}_L, \\ \langle \psi \rangle = 0. \end{cases} \quad (3.3)$$

We shall prove that the single (small) spike solution of Type I is stable for all  $\tau > 0$  and all the other solutions of Type I, II, or III are unstable for all  $\tau > 0$ .

Let

$$\psi = -2\mu' A\phi + 2\mu' \langle A\phi \rangle + \tau\lambda_L\hat{\psi}, \quad (3.4)$$

where

$$\langle \hat{\psi} \rangle = 0.$$

Equation (3.4) together with (3.3) implies

$$\hat{\psi}_{xx} - \tau\lambda_L\hat{\psi} = -2\mu' A\phi + 2\mu' \langle A\phi \rangle. \quad (3.5)$$

Substituting (2.2) and (3.4) into the first equation of (3.3), we obtain that

$$\phi_{xx} - a\phi + 3bA^2\phi - 2\mu' \langle A\phi \rangle A - \tau\lambda_L A\hat{\psi} = \lambda_L\phi, \quad -\frac{L}{2} < x < \frac{L}{2}, \quad (3.6)$$

where  $a$  and  $b$  are given by (2.4). If  $\tau = 0$ , then (3.6) becomes

$$\begin{cases} \phi_{xx} - a\phi + 3bA^2\phi - 2\mu' \langle A\phi \rangle A = \lambda_L\phi, & -\frac{L}{2} < x < \frac{L}{2}, \\ \phi \in \mathcal{X}_L. \end{cases} \quad (3.7)$$

Our main result in this section is the following reduction lemma. It will be proved by variational techniques. Note that parts (a) and (b) are relatively trivial. Part (c) follows by the application of the intermediate value theorem to a suitably defined function.

**Lemma 3.** (a) *All eigenvalues of (3.3) are real.*

(b) *If all eigenvalues of (3.7) are negative, then all eigenvalues of (3.3) are negative.*

(c) *If problem (3.7) has a positive eigenvalue, then problem (3.3) also has a positive eigenvalue.*

Lemma 3 implies that the (in)stability of (3.3) is equivalent to the (in)stability of (3.7).

The following lemma proves (a) of Lemma 3 and is an easy consequence of integration by parts.

**Lemma 4.** *The eigenvalues  $\lambda_L$  of (3.3) are real.*

**Proof.**

Multiplying (3.6) by  $\bar{\phi}$  – the conjugate function of  $\phi$  – and integrating over  $I$ , we obtain

$$\begin{aligned} \lambda_L \int_I |\phi|^2 dx &= - \int_I [|\phi_x|^2 + a|\phi|^2 - 3bA^2|\phi|^2] dx - \frac{2\mu'}{L} \int_I (A\phi)^2 dx \\ &\quad - \tau \lambda_L \int_I A\hat{\psi}\bar{\phi} dx. \end{aligned} \quad (3.8)$$

Multiplying the conjugate of (3.5) by  $\hat{\psi}$  and integrating over  $R$  we get

$$\int_I A\bar{\phi}\hat{\psi} dx = \frac{1}{2\mu'} \int_I |\hat{\psi}_x|^2 dx + \frac{\tau\bar{\lambda}_L}{2\mu'} \int_I |\hat{\psi}|^2 dx. \quad (3.9)$$

Substituting (3.9) into (3.8) gives

$$\begin{aligned} \lambda_L \int_I |\phi|^2 dx + \int_I [|\phi_x|^2 + a|\phi|^2 - 3bA^2|\phi|^2] dx + \frac{2\mu'}{L} \int_I (A\phi)^2 dx \\ + \frac{\tau\lambda_L}{2\mu'} \int_I |\hat{\psi}_x|^2 dx + \frac{\tau^2|\lambda_L|^2}{2\mu'} \int_I |\hat{\psi}|^2 dx = 0. \end{aligned} \quad (3.10)$$

Taking the imaginary part of (3.10) we obtain

$$\lambda_i \left( \int_I |\phi|^2 dx + \frac{\tau}{2\mu'} \int_I |\hat{\psi}_x|^2 dx \right) = 0, \quad (3.11)$$

where  $\lambda_L = \lambda_r + \sqrt{-1}\lambda_i$ .

Equation (3.11) implies

$$\lambda_i = 0 \quad (3.12)$$

and therefore  $\lambda$  is real.  $\square$

Now we prove (b) and (c) part of Lemma 3. We use variational techniques. To this end, we need to introduce two quadratic forms:

Let

$$L[\phi] = \int_I (|\phi_x|^2 + a\phi^2 - 3bA^2\phi^2) dx + \frac{2\mu'}{L} \left( \int_I (A\phi) dx \right)^2, \quad \phi \in \mathcal{X}_L \quad (3.13)$$

and

$$\mathcal{L}_\lambda[\phi] = L[\phi] + \frac{\tau\lambda}{2\mu'} \int_I (|\hat{\psi}_x|^2 + \tau\lambda|\hat{\psi}|^2) dx, \quad (3.14)$$

where  $\hat{\psi}$  is the unique solution of the problem

$$\begin{cases} \hat{\psi}'' - \tau\lambda\hat{\psi} = -2\mu' A\phi + 2\mu' \langle A\phi \rangle, \\ \hat{\psi} \in \mathcal{X}_L, \langle \hat{\psi} \rangle = 0. \end{cases} \quad (3.15)$$

Observe that for  $\tau \geq 0$  and  $\lambda \geq 0$

$$\mathcal{L}_0[\phi] = L[\phi], \quad L[\phi] \leq \mathcal{L}_\lambda[\phi]. \quad (3.16)$$

**Proof of (b) and (c) of Lemma 3:**

(b). To prove (b), we note that if all eigenvalues of (3.7) are negative, then the quadratic form  $L[\phi]$  is positive definite, which by (3.16) implies that  $\mathcal{L}_\lambda$  is positive definite if  $\lambda \geq 0$ . Let  $\lambda_L \geq 0$  be an eigenvalue of (3.3), then by (3.10), we obtain that

$$\lambda_L \int_I |\phi|^2 dx + \mathcal{L}_{\lambda_L}[\phi] = 0 \quad (3.17)$$

which is clearly impossible if  $\lambda_L \geq 0$ . Thus we have shown that all eigenvalues of (3.3) must be negative.

(c). Suppose (3.7) has a positive eigenvalue. Then the eigenvalue problem

$$-\mu_L = \min_{\phi \in \mathcal{X}_L, \int_I \phi^2 = 1} L[\phi] \quad (3.18)$$

has a positive value  $\mu_L > 0$ . We now claim that (3.3) admits a positive eigenvalue.

Fixing  $\lambda \in [0, +\infty)$ , let us consider another eigenvalue problem

$$-\mu(\lambda) = \min_{\phi \in \mathcal{X}_L, \int_I \phi^2 = 1} \mathcal{L}_\lambda[\phi]. \quad (3.19)$$

A minimizer  $\phi$  of (3.19) satisfies the equation

$$\phi_{xx} - a\phi + 3bA^2\phi - 2\mu' \langle A\phi \rangle A - A\hat{\psi} = \mu(\lambda)\phi, \quad \phi \in \mathcal{X}_L, \quad (3.20)$$

where  $\hat{\psi}$  is given by (3.15).

By (3.16),  $-\mu(\lambda) \geq -\mu_L$ . Hence  $\mu(\lambda) \leq \mu_L$ . Moreover, since  $\hat{\psi}$  is continuous with respect to  $\lambda$  in  $[0, +\infty)$ , we see that  $\mu(\lambda)$  is also continuous in  $[0, +\infty)$ .

Let us consider the following algebraic equation

$$h(\lambda) := \mu(\lambda) - \lambda = 0, \quad \lambda \in [0, +\infty). \quad (3.21)$$

By our assumption,  $h(0) = \mu(0) = \mu_L > 0$ . On the other hand, for  $\lambda > 2\mu_L$ ,  $h(\lambda) \leq \mu_L - \lambda < -\mu_L < 0$ . By the mean-value theorem, there exists a  $\lambda_L \in (0, \mu_L)$  such that  $h(\lambda_L) = 0$ .

Substituting  $\mu(\lambda_L) = \lambda_L$  into (3.20), we see that  $\lambda_L$  is an eigenvalue of problem (3.3).

Part (c) of Lemma 3 is thus proved. □

#### 4. STABILITY OF SINGLE (SMALL) SPIKE SOLUTION OF TYPE I

In this section, we prove the stability of the single (small) spike solution of Type I. Let  $A(x), B(x)$  be the single (small) spike solution of Type I obtained in Section 2. Then, as  $L \rightarrow +\infty$ , we have

$$A(x) = \sqrt{\frac{a}{\mu' - 1}} w_\infty(\sqrt{a}x) + O(e^{-aL}), \quad B(x) = -\mu' A^2 + \mu' \langle A^2 \rangle,$$

where  $c = \sqrt{a}$  satisfies as  $L \rightarrow \infty$

$$c = c_- = \beta_L^1 - \sqrt{(\beta_L^1)^2 - 1}, \quad \beta_L^1 = \frac{2\mu'}{L(\mu' - 1)}(1 + O(e^{-L})). \quad (4.1)$$

By Lemma 3, to prove the stability, we just need to consider the positive definiteness of  $L[\phi]$ , defined by (3.13). By the rescaling (2.10) and (2.11), we see that  $L[\phi]$  is transformed to

$$\begin{aligned} L[\phi] &= \\ &= \sqrt{a} \left( \int_{\tilde{I}} (|\tilde{\phi}_y|^2 + |\tilde{\phi}|^2 - 3H_{\tilde{I}}^2 \tilde{\phi}^2) dy + \frac{2\mu'}{L(1 - \mu')\sqrt{a}} \left( \int_{\tilde{I}} H_{\tilde{I}} \tilde{\phi} dy \right)^2 \right), \end{aligned} \quad (4.2)$$

where we recall  $l = \sqrt{a}L, y = \sqrt{a}x, \tilde{\phi}(y) = \phi(x)$ . Formally, as  $L \rightarrow +\infty$ , we obtain the following quadratic form in  $H^1(R)$ :

$$L_\infty[\phi] = \sqrt{a} \int_R (|\phi_y|^2 + |\phi|^2 - 3w_\infty^2 \phi^2) dy + \beta \left( \int_R w_\infty \phi dy \right)^2, \quad (4.3)$$

where

$$\phi \in H^1(R), \quad \phi(-y) = \phi(y),$$

and

$$\beta = \lim_{L \rightarrow +\infty} \frac{2}{L(\mu' - 1)\sqrt{a}}. \quad (4.4)$$

The study of (4.3) is equivalent to the study of the following nonlocal eigenvalue problem:

$$L_\beta \phi := L_0 \phi - \beta \left( \int_R w_\infty \phi dy \right) w_\infty = \lambda_0 \phi, \quad \phi \in \mathcal{X}_\infty, \quad (4.5)$$

where

$$L_0 \phi := \phi'' - \phi + 3w_\infty^2 \phi, \quad \phi \in H^1(R), \quad (4.6)$$

and

$$\mathcal{X}_\infty = \{\phi \in H^1(R) | \phi(-y) = \phi(y)\}. \quad (4.7)$$

We first collect some properties associated with  $w_\infty$ .

**Lemma 5.** (a)  $\text{Ker}(L_0) = \{cw'_\infty(y) | c \in R\}$ .

(b)  $L_0$  has a unique (principal) positive eigenvalue  $\nu_1 > 0$ . The associated eigenfunction  $\phi_0(y)$  can be chosen to be positive and even.

(c)  $L_0(\frac{1}{2}(w_\infty + yw'_\infty(y))) = w_\infty(y)$ .

**Proof:**

For the proof of (a), please see Lemma 4.1 of [15], where a more general result in  $R^N$  is proved.

The proof of (b) follows by the variational characterization of the eigenvalues:

$$-\nu_1 = \inf_{\phi \in H^1(R), \phi \neq 0} \frac{\int_R [(\phi')^2 + \phi^2 - 3w_\infty^2 \phi^2] dy}{\int_R \phi^2 dy} < 0 \quad (4.8)$$

since by the last inequality for  $\phi = w_\infty$

$$-\nu_1 \leq -2 \frac{\int_R w_\infty^4 dy}{\int_R w_\infty^2 dy} < 0.$$

The fact that  $\nu_1$  is the unique positive eigenvalue follows from Lemma 1.2 of [16]. By the variational characterization (4.8) of  $\nu_1$ , we see that the corresponding eigenfunction can be chosen to be positive. Since  $w_\infty$  is even, this eigenfunction can also be chosen to be even. (Suppose  $\phi$  is not even. Then we write

$$\phi(y) = \begin{cases} \phi_1(y), & y \geq 0, \\ \phi_2(y), & y < 0. \end{cases}$$

If

$$\frac{\int_{\{y>0\}} [(\phi')^2 + \phi^2 - 3w_\infty^2 \phi^2] dy}{\int_{\{y>0\}} \phi^2 dy} < \frac{\int_{\{y<0\}} [(\phi')^2 + \phi^2 - 3w_\infty^2 \phi^2] dy}{\int_{\{y<0\}} \phi^2 dy} \quad (4.9)$$

then we may choose

$$\phi_{\text{even}}(y) = \begin{cases} \phi_1(y), & y \geq 0, \\ \phi_1(-y), & y < 0. \end{cases}$$

Then

$$\frac{\int_{\mathbb{R}} [(\phi'_{\text{even}})^2 + \phi_{\text{even}}^2 - 3w_\infty^2 \phi_{\text{even}}^2] dy}{\int_{\mathbb{R}} \phi_{\text{even}}^2 dy} < \frac{\int_{\mathbb{R}} [(\phi')^2 + \phi^2 - 3w_\infty^2 \phi^2] dy}{\int_{\mathbb{R}} \phi^2 dy}$$

and thus the function  $\phi$  is not the eigenfunction to the principal eigenvalue. This is a contradiction. Accordingly, we arrive at a contradiction if in (4.9) we have the reverse inequality. If we have equality in (4.9), then we can construct an even eigenfunction  $\phi_{\text{even}}$  from the eigenfunction  $\phi$  in the same way as above with the same eigenvalue. Since the principal eigenfunction is unique (up to constant factors) and since an even eigenfunction exists, the eigenfunction has to be even. Note that in general eigenvalue problems with even coefficients may have eigenfunction which are not even. Our argument works in this example for the eigenfunction to the principal eigenvalue.)

To prove (c), we note that if  $u$  satisfies  $u'' + f(u) = 0$ , then  $yu'(y)$  satisfies  $(yu')'' + f'(u)(yu') = -2f(u)$ . By simple computations, we see that

$$L_0 w_\infty = 2w_\infty^3, \quad L_0(yw'_\infty(y)) = -2(-w_\infty + w_\infty^3).$$

Hence

$$L_0 \left( \frac{1}{2}(w_\infty + yw'_\infty) \right) = w_\infty, \quad L_0^{-1} w_\infty = \frac{1}{2}(w_\infty + yw'_\infty).$$

□

We are now ready to study  $L_\beta$ .

Since  $L_\beta$  is a self-adjoint operator, the eigenvalues of  $L_\beta$  must be real.

The following is our key lemma.

**Lemma 6.** (a) *The eigenvalue problem (4.5) has an eigenfunction  $\phi \in \mathcal{X}_\infty$  with a positive eigenvalue if and only if  $\beta < 1$ . Moreover, for  $0 < \beta < 1$ , this positive eigenvalue is simple and isolated.*

(b) *If  $\beta = 1$ , then the eigenvalue problem (4.5) has a zero eigenvalue with eigenfunction  $\phi_0 = w_\infty + yw'_\infty(y)$ .*

(c) *If  $\beta > 1$ , then there exists  $c_0 > 0$  such that*

$$\int_R (|\phi_y|^2 + \phi^2 - 3w_\infty^2\phi^2)dy + \beta \left( \int_R w_\infty\phi dy \right)^2 \geq c_0 \int_R \phi^2 dy \quad (4.10)$$

for all  $\phi \in \mathcal{X}_\infty$ .

**Proof:**

By Lemma 5,  $\nu_1$  is the only positive eigenvalue of  $L_0$  and the corresponding eigenfunction  $\phi_0$  is positive and belongs to  $\mathcal{X}_\infty$ . For fixed  $\lambda_0 > 0$ ,  $\lambda_0 \neq \nu_1$ ,  $(L_0 - \lambda_0)^{-1}$  exists in  $\mathcal{X}_\infty$ .

For  $\beta > 0$  and  $\lambda_0 > 0$  we may rewrite (4.5) as

$$\phi = \beta \left( \int_R w_\infty\phi dy \right) [(L_0 - \lambda_0)^{-1}w_\infty]. \quad (4.11)$$

Assume first that (4.11) holds. Multiplying (4.11) by  $w_\infty$  and integrating over  $R$  gives

$$\int_R w_\infty\phi dy = \beta \left( \int_R w_\infty\phi dy \right) \int_R [(L_0 - \lambda_0)^{-1}w_\infty]w_\infty dy. \quad (4.12)$$

Now we use the fact that

$$\int_R w_\infty\phi dy \neq 0,$$

which follows by contradiction as follows: Suppose that

$$\int_R w_\infty\phi dy = 0.$$

Then (4.5) implies that

$$L_0\phi = \lambda_0\phi, \quad \lambda_0 > 0, \quad \phi \in \mathcal{X}_\infty.$$

By the properties of  $\phi$  given in Lemma 5 (b), however, it follows that

$$\int_R w_\infty \phi \, dy \neq 0$$

This is a contradiction. Therefore (4.12) implies

$$\rho(\lambda_0) := \beta \int_R ((L_0 - \lambda_0)^{-1} w_\infty) w_\infty \, dy - 1 = 0, \quad \lambda_0 > 0. \quad (4.13)$$

On the other hand, suppose that (4.13) holds. For the positive root  $\lambda_0$  of (4.13) we define  $\phi$  by

$$\phi = (L_0 - \lambda_0)^{-1} w_\infty.$$

By (4.13) we have

$$\beta \left( \int_R w_\infty \phi \, dy \right) = \beta \int_R ((L_0 - \lambda_0)^{-1} w_\infty) w_\infty \, dy = 1$$

and therefore  $\lambda_0 \neq 0$  and  $\phi > 0$  solve (4.11).

Thus for  $\beta > 0$  problem (4.5) has a positive eigenvalue if and only if the algebraic equation (4.13) has a positive root.

We now discuss (4.13). It is easy to see that  $\rho(\lambda) < 0$  for  $\lambda > \nu_1$ .

Thus we only need to consider  $\lambda \in (0, \nu_1)$ . In this case,

$$\rho'(\lambda) = \beta \int_R (w_\infty (L_0 - \lambda)^{-2} w_\infty) \, dy = \beta \int_R ((L_0 - \lambda)^{-1} w_\infty)^2 \, dy > 0.$$

On the other hand, as  $\lambda \rightarrow \nu_1^-$ ,  $\rho(\lambda) \rightarrow \infty$ . Thus (4.13) has a positive real root if and only if  $\rho(0) < 0$ .

It remains to compute  $\rho(0)$ .

By (c) of Lemma 5, we have

$$\begin{aligned} \rho(0) &= \beta \int_R w_\infty \frac{1}{2} (w_\infty + y w'_\infty) \, dy - 1 = \frac{\beta}{2} \int_R \left[ w_\infty^2 - y \frac{1}{2} (w_\infty^2)' \right] \, dy - 1 \\ &= \frac{\beta}{4} \int_R w_\infty^2 \, dy - 1 = \beta - 1. \end{aligned}$$

Therefore  $\rho(0) < 0$  if and only if  $\beta < 1$ .

Moreover, since  $\rho'(\lambda) > 0$ , we see that the positive eigenvalue, if it exists, is unique.

This proves part (a) of the lemma.

Part (b) of the lemma follows from (c) of Lemma 5.

To prove part (c), let

$$\lambda_0 = \min_{\phi \in \mathcal{X}_\infty} \frac{\int_R (|\phi_y|^2 + \phi^2 - 3w_\infty^2 \phi^2) \, dy + \beta (\int_R w_\infty \phi \, dy)^2}{\int_R \phi^2 \, dy}$$

$$= \min_{\phi \in \mathcal{X}_\infty} \frac{\int_R (-L_\beta \phi) \phi \, dy}{\int_R \phi^2 \, dy}.$$

By (a), if  $\beta > 1$ ,  $L_\beta$  has no positive real eigenvalues. Thus  $\lambda_0 \geq 0$ . We have to exclude the case when  $\lambda_0 = 0$ . Suppose  $\lambda_0 = 0$ , then we have a  $\phi_0 \in \mathcal{X}_L$  such that

$$\phi_0'' - \phi_0 + 3w_\infty^2 \phi_0 - \beta_0 \left( \int_R w_\infty \phi_0 \, dy \right) w_\infty = 0, \quad \int_R \phi_0^2 \, dy = 1. \quad (4.14)$$

Using (c) of Lemma 5, we see that

$$\tilde{\phi}_0'' - \tilde{\phi}_0 + 3w_\infty^2 \tilde{\phi}_0 = 0,$$

where

$$\tilde{\phi}_0 = \phi_0 - \beta \left( \int_R w_\infty \phi_0 \, dy \right) \left( \frac{1}{2} (w_\infty + y w_\infty') \right).$$

By (a) of Lemma 5, we have that

$$\tilde{\phi}_0 = \phi_0 - \beta \left( \int_R w_\infty \phi_0 \, dy \right) \left( \frac{1}{2} (w_\infty + y w_\infty') \right) = c w_\infty'$$

for some constant  $c$ .

Since  $\phi_0 \in \mathcal{X}_\infty$ , it follows that  $\phi(-y) = \phi(y)$ . Thus  $c = 0$  and

$$\phi_0 = \beta \left( \int_R w_\infty \phi_0 \, dy \right) \left( \frac{1}{2} (w_\infty + y w_\infty') \right). \quad (4.15)$$

Multiplying (4.15) by  $w_\infty$  and integrating over  $R$ , we have that

$$\left( 1 - \beta \int_R w_\infty \frac{1}{2} (w_\infty + y w_\infty') \, dy \right) \int_R w_\infty \phi_0 \, dy = 0.$$

Since  $\beta > 1$  (recall that  $\int_R w_\infty \frac{1}{2} (w_\infty + y w_\infty') \, dy = 1$ ), we have  $\int_R w_\infty \phi_0 \, dy = 0$  and hence  $\phi_0 = 0$ . This is a contradiction.

Therefore  $\lambda_0 > 0$  and hence part (c) of the lemma is proved.  $\square$

As a corollary of (c) of Lemma 6, we obtain

**Corollary 7.** *Let  $A$  be the single (small) spike solution of Type I. Then there exists  $c_0 > 0$  such that for  $L$  sufficiently large and  $\phi \in \mathcal{X}_L$ , we have*

$$L[\phi] \geq c_0 \int_I \phi^2 \, dx, \quad (4.16)$$

where  $L[\phi]$  is defined by (3.13).

**Proof:** Note that

$$\begin{aligned} \lim_{L \rightarrow +\infty} \beta_L &= \lim_{L \rightarrow +\infty} \frac{2}{c_- L(\mu' - 1)} \\ &= \lim_{L \rightarrow +\infty} \frac{\beta_L^1}{\beta_L^1 - \sqrt{(\beta_L^1)^2 - 1}} \\ &= (\beta_\infty)^2 + \sqrt{(\beta_\infty)^2((\beta_\infty)^2 - 1)} > 1 \end{aligned}$$

by the assumption (1.8).

Since  $H_l = w_\infty(y) + O(e^{-l})$ , (4.16) follows from (4.10) of Lemma 6 and (4.2).  $\square$

From Lemma 3 and Corollary 7, we see that for  $L$  sufficiently large, the single (small) spike solution of Type I is linearly stable for any  $\tau > 0$ .

## 5. INSTABILITY OF OTHER SOLUTIONS

In this section will show that the other solutions (i.e., the single (large) spike solution of Type I, the double spike layer of Type II, or the double transition layer solution of Type III, respectively) are linearly unstable.

By the reduction lemma (Lemma 3), we just need to consider problem (3.7). To show instability, all we need is to show that the following minimization problem admits a negative value for a certain test function:

$$-\mu_L = \min_{\phi \in \mathcal{X}_L, \int_I \phi^2 = 1} \left[ \int_I (|\phi_x|^2 + a\phi^2 - 3bA^2\phi^2) dx + \frac{2\mu'}{L} \left( \int_I A\phi dx \right)^2 \right] < 0. \quad (5.1)$$

Inequality (5.1) is equivalent to

$$-\mu_L = \min_{\phi \in H^1(I), \int_I \phi^2 = 1} \left[ \int_I (|\phi_x|^2 + a\phi^2 - 3bA^2\phi^2) dx + \frac{2\mu'}{L} \left( \int_I A\phi dx \right)^2 \right] < 0. \quad (5.2)$$

Recall that  $I = [-\frac{l}{2}, \frac{l}{2}]$ ,  $l = \sqrt{a}L$ ,  $\tilde{I} = [-\frac{l}{2}, \frac{l}{2}]$ .

Thus it is enough to find a  $\phi \in H^1(I)$  such that

$$\int_I (|\phi_x|^2 + a\phi^2 - 3bA^2\phi^2) dx + \frac{2\mu'}{L} \left( \int_I A\phi dx \right)^2 < 0.$$

We now consider the three solution types separately, with  $L \gg 1$ .

### 1. Single (large) spike solution of Type I

Let  $(A(x), B(x)) = (A^+, B^+)$  be the single (large) spike solution of Type I.

It is easy to see that

$$-\mu_L = \min_{\phi \in H^1(\bar{I}), \int_{\bar{I}} \phi^2 = 1} \left[ \int_{\bar{I}} (|\phi_y|^2 + \phi^2 - 3H_l^2 \phi^2) dy + \beta_L \left( \int_{\bar{I}} H_l \phi dy \right)^2 \right],$$

where

$$\beta_L = \frac{2\mu'}{c_+ L(\mu' - 1)}.$$

Observe that as  $L \rightarrow +\infty$ ,

$$\beta_L \rightarrow \beta = (\beta_\infty)^2 - \sqrt{(\beta_\infty)^2((\beta_\infty)^2 - 1)} < 1. \quad (5.3)$$

By Lemma 6, for  $\beta \in (0, 1)$ , there exists a unique principal eigenvalue  $\lambda_0 > 0$  and a corresponding eigenfunction  $\phi_0(y) \in \mathcal{X}_\infty$  for the following eigenvalue problem

$$\phi_0'' - \phi_0 + 3w_\infty^2 \phi_0 - \beta \left( \int_R w_\infty \phi_0 dy \right) w_\infty = \lambda_0 \phi_0.$$

Now since  $\phi_0(y) = O(e^{-l})$  for  $|y| \geq l$ , a simple computation shows that  $\phi_0(c_+x)$  makes (5.2) negative.

### 2. Double Spike Layer Solutions of Type II.

By Lemma 5 the eigenvalue problem

$$\phi_0'' - \phi_0 + 3w_\infty^2 \phi_0 = \lambda_0 \phi_0, \quad \phi_0 \in H^2(R)$$

has an eigenvalue  $\lambda_0 > 0$  with a corresponding eigenfunction  $\phi_0$ .

We now set

$$\phi(x) = \phi_0\left(\sqrt{a}\left(x + \frac{L}{2}\right)\right) + \phi_0\left(\sqrt{a}\left(x - \frac{L}{2}\right)\right) + \phi_0(\sqrt{a}x), \quad x \in I.$$

Then we calculate

$$\begin{aligned} \int_I A\phi dx &= 2 \int_{\frac{L}{4}}^{\frac{L}{2}} A\phi_0\left(\sqrt{a}\left(x - \frac{L}{2}\right)\right) dx + \int_{-\frac{L}{4}}^{\frac{L}{4}} A\phi_0(\sqrt{a}x) dx + O(e^{-l/2}) \\ &= \int_R w_\infty \phi_0 dy - \int_R w_\infty \phi_0 dy + O(e^{-l/2}) \\ &= O(e^{-l/2}). \end{aligned}$$

Recalling that

$$\tilde{\phi}(x) = \phi(\sqrt{a}x),$$

this implies

$$\int_{\tilde{I}} H_l \tilde{\phi} dy = O(e^{-l/2}).$$

Hence

$$\begin{aligned} \frac{1}{\sqrt{a}} L[\phi] &\leq \int_{\tilde{I}} [(\tilde{\phi}')^2 + \tilde{\phi}^2 - 3H_l^2 \tilde{\phi}^2] dy + \frac{2\mu'}{L(\mu' - 1)\sqrt{a}} \left( \int_{\tilde{I}} H_l \tilde{\phi} dy \right)^2 \\ &\leq \int_{\tilde{I}} [|\tilde{\phi}'|^2 + \tilde{\phi}^2 - 3H_l^2 \tilde{\phi}^2] dy + O(e^{-l/4}) \\ &= 2 \int_R [|\phi'_0|^2 + \phi_0^2 - 3w^2 \phi_0^2] dy + O(e^{-l/4}) \\ &= -2\lambda_0 \int_R \phi_0^2 dy + O(e^{-l/4}) < 0. \end{aligned} \quad (5.4)$$

Therefore we have  $\mu_L > 0$ . (In fact we have  $\mu_L \geq \lambda_0 + O(e^{-l/4})$ .) We have shown that the double spike solution of Type II is unstable.

### 3. Double front solutions of Type III.

We now consider the double front solution, the so-called *sn* solution of Type III, and we will show that it is unstable.

In this case, we choose our function  $\phi(x)$  so that

$$\phi'(0) = \phi'\left(\frac{L}{2}\right) = 0, \quad \phi(x) = \phi\left(\frac{L}{2} - x\right) \text{ for } 0 < x < \frac{L}{2}. \quad (5.5)$$

The last equality says that  $\phi(x - \frac{L}{4})$  is an even function in  $[-\frac{L}{4}, \frac{L}{4}]$ . We then extend  $\phi(x)$  evenly to  $[-\frac{L}{2}, 0]$ . In this case

$$\int_I A\phi dx = 2 \int_0^{L/2} A\phi dx = 0$$

and

$$\begin{aligned} \sqrt{a} L[\phi] &= \sqrt{a} \int_I (|\phi_x|^2 + a\phi^2 - 3bA^2\phi^2) dx = 2 \int_0^{\frac{L}{2}} (|\phi_x|^2 + a\phi^2 - 3bA^2\phi^2) dx \\ &= 2\sqrt{a} \int_{-\frac{L}{4}}^{\frac{L}{4}} (|\tilde{\phi}_y|^2 - \tilde{\phi}^2 + 3v_{l/2}^2 \tilde{\phi}^2) dy, \end{aligned}$$

where  $v_{l/2}$  is defined by (2.25) and

$$\tilde{\phi}(y) = \phi(x), \quad y = \sqrt{a}\left(x - \frac{L}{4}\right). \quad (5.6)$$

Let  $\varphi_l$  be such that

$$\begin{cases} \varphi_l'' - 2\varphi_l = 0, & -\frac{l}{4} < y < \frac{l}{4}, \\ \varphi_l(-y) = \varphi_l(y), \varphi_l(\frac{l}{4}) = -v_{l/2}(\frac{l}{4}) = v_{l/2}(\frac{l}{4})(1 - (v_l(\frac{l}{4}))^2). \end{cases} \quad (5.7)$$

Set

$$\tilde{\phi}(y) = v'_{l/2}(y) + \varphi_l(y).$$

It is easy to check that  $\phi(x)$ , defined by (5.6), satisfies (5.5).

We compute:

$$\begin{aligned} \tilde{\phi}'' + \tilde{\phi} - 3v_l^2\tilde{\phi} &= v_{l/2}''' + v'_{l/2} - 3v_{l/2}^2v'_{l/2} + \varphi_l'' + \varphi_l - 3v_l^2\varphi_l \\ &= 0 + 3(1 - v_l^2)\varphi_l, \\ L[\phi] &= 2 \int_{-\frac{l}{4}}^{\frac{l}{4}} (|\tilde{\phi}_y|^2 - \tilde{\phi}^2 + 3v_{l/2}^2\tilde{\phi}^2) dy \\ &= -6 \int_{-\frac{l}{4}}^{\frac{l}{4}} (1 - v_l^2)\varphi_l\tilde{\phi} dy < 0, \end{aligned} \quad (5.8)$$

since

$$1 - v_l^2 > 0, \quad \varphi_l > 0, \quad \tilde{\phi} > 0. \quad (5.9)$$

By (5.1) this shows that the double front solution is unstable. In fact, the behaviour of the double front solution is very similar to that of two-layer solutions of the Cahn-Hilliard equation. See for example [1],[2], [3], [6], [9].

## 6. EXTENSIONS

The results of this paper can be extended in several ways.

We first consider the following so-called ABC system in [5] arising from two-dimensional rotating convection and magnetoconvection:

$$\begin{cases} A_t = A_{xx} + A - A^3 - AB, & x \in R, t > 0, \\ B_t = \sigma B_{xx} + \mu(A^2)_{xx}, & x \in R, t > 0 \\ C_t = \tau C_{xx} + \nu(A^2)_{xx}, & x \in R, t > 0, \end{cases} \quad (6.1)$$

where

$$\sigma > 0, \tau > 0.$$

In [5], it is shown that all rolls are unstable if

$$\frac{\mu}{\sigma} + \frac{\nu}{\tau} > 1. \quad (6.2)$$

The steady-state of the ABC system (6.1) takes the same form as (2.3), except that we replace  $\mu'$  by  $\frac{\mu}{\sigma} + \frac{\nu}{\tau}$ .

Similar proofs as in Theorems 1 and (2) show that if (6.2) holds, then there exists a stable single (small) spike solution, an unstable single (large) spike solution, and an unstable double spike solution. If  $\frac{\mu}{\sigma} + \frac{\nu}{\tau} < 1$ , then there exists an unstable double front solution.

Second, we can consider the following model which includes higher order terms (equations (5.1) and (5.4) of [8]):

$$\begin{cases} A_t = A_{xx} + A - A^3 - AB, & x \in R, t > 0, \\ B_t = \sigma B_{xx} + \mu(A^2)_{xx} + \delta(BA^2)_{xx}, & x \in R, t > 0, \end{cases} \quad (6.3)$$

where

$$\sigma > 0, \quad \delta > 0.$$

After rescaling, the steady state problem for  $\delta$  small can be approximated by the following model equation

$$\begin{cases} w''_{\infty,\delta} - w_{\infty,\delta} + w_{\infty,\delta}^3 - \delta w_{\infty,\delta}^5 = 0, \\ w_{\infty,\delta} > 0, w_{\infty,\delta}(y) \rightarrow 0 \text{ as } |y| \rightarrow +\infty. \end{cases} \quad (6.4)$$

As  $\delta \rightarrow 0$ ,  $w_{\infty,\delta} \rightarrow w_\infty = \sqrt{2} \operatorname{sech}(y)$ . Our results extend without any difficulty to the case of  $\delta \ll 1$ .

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