A power law distribution in patients' lengths of stay in hospital

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(Dated: July 28, 2006)

Abstract

The distribution of patients' lengths of stay in English hospitals is measured by using routinely collected data from 11 years. It is found to be well approximated by a power law distribution spanning over more than 3 decades. To explain this observation, a theoretical resource allocation model is presented. It is based on iterative long-term scheduling of hospital beds, and its main assumption is that future beds are allocated preferentially. This represents a situation where different parts of the health care system compete for resources, with bargaining powers proportional to current resource levels.

PACS numbers: 87.23.Ge, 87.19.Xx, 89.75.Da

Keywords: health care, power law distributions, preferential attachment, resource competition, rate equation

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I. INTRODUCTION

As a simple indicator of hospital activity, length of stay (LOS) has been used for many purposes on several levels of management. It can be used to monitor quality and appropriateness of resource allocations both internally in hospitals and at a national level [1]. LOS is clearly related to resource consumption and can thus be used for planning and efficient sorting of patients into groups [2].

However, it is well known that the distribution of LOS is highly skewed, which makes many statistical estimates less robust. In a small sample, it can be very difficult to track changes over time since a single outlier can significantly change the mean and variance.

A review of different commonly used distributions for LOS is presented in Ref. [1]. Lognormal, Weibull, and Gamma distributions are compared with empirical data from a number of countries. The results showed that both Weibull and lognormal are useful, and which one to choose depends on the specific country. In contrast to the study presented in this paper, the authors considered specific diagnostic groups separately, here we are only concerned with the aggregation of all diagnoses.

In Ref. [3] it is pointed out that resources allocated can be excessive for a small subset of users and at the same time, others get few resources or even none at all. The skewness in the distribution of LOS reinforces this picture.

There have been some attempts to explain the underlying cause for the skewed distributions. One example is described in Ref. [4], where a model, employing the transitions between different states of severity in the illnesses of patients, is suggested.

The hypothesis brought forward in this paper is that LOS is mainly determined at the resource allocation stage. This would mean that the competition for resources between different groups of patients is the significant underlying cause of the observed distributions. The competition takes place on all levels, between hospitals, between departments and between different types of patients within the same department.

We find that the distribution of LOS is well described by a power law, which is a common observation in both natural and artificial systems and no specific underlying generative model can be inferred from the observation [5–7]. However the association between LOS and resource allocation, makes it possible that a competitive multiplicative process is creating the fat tailed distribution. Such processes are well known and historically trace back to the

Simon model [8]. More generally, it has been shown that exponential growth combined with random survival times are sufficient to produce power law distributions [9]. This broad class of models includes the one proposed in this paper.

This paper is organised as follows: In section II the empirical findings are presented, in section III the model is formulated, analysed and compared to the empirical data, and in the final section the conclusions are discussed.

II. EMPIRICAL FINDINGS

The Hospital Episode Statistics (HES) database contains routinely collected data on patients in the English National Health Service (NHS). The NHS is managed by the Department of Health and is funded by taxpayers and provides care mostly free of charge to the individual patients. For the analysis presented in this paper, an extract of all patient records from 1989-90 to 2002-03 is considered. (The NHS years do not follow calendar years, instead they span between April 1 and March 31.) Some patients admitted before 1989 are present in the data set, but no discharge data exists on patients before 1991-92. Length of Stay (LOS) is calculated by taking, for every episode, the difference between the discharge date (which can also mean the date of transfer or death) and the admission date. The data set contains 130920010 *valid* values for LOS. This is about 95% of the recorded episodes.

Here we consider all patients in the data set with a valid entry for LOS, which means that data from all diagnostic groups are aggregated. In Fig. 1 a double logarithmic histogram for LOS is shown. It is obvious that the distribution of LOS is very fat tailed and that it can be described fairly well as a power law over several decades. This means that $P[LOS = x] \sim x^{-\gamma}$. One of the most reliable ways to find a value for the exponent γ is to use the maximum likelihood estimate [5],

$$\gamma = 1 + \frac{n}{\sum_i^n \ln \frac{x_i}{x_{min}}}, \tag{1}$$

where x_i are the empirically measured LOS, x_{min} is a lower cutoff for the power law, and n is the sample size. LOS-values less than x_{min} are not used in the estimation of the exponent. We choose $x_{min} = 5$ (giving $n = 33956629$) and obtain $\gamma = 2.12$.

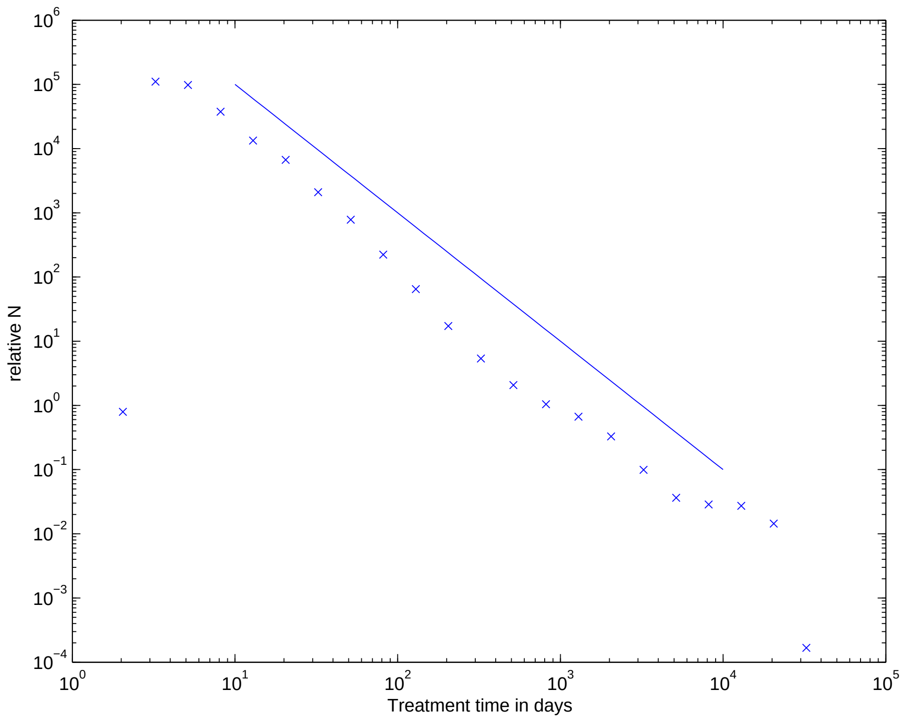

FIG. 1: An exponentially binned histogram, including all measured LOS in the database, 1989-2003. The reference line has a slope of -2. A MLE estimate gives an exponent of -2.12.

III. MODEL

Our model is formulated from the viewpoint of a scheduler, which means that only future hospital episodes are taken into account. Its essential ingredient is that planning is taking place iteratively at the same time as parts of the plan are being realised. We have b beds and a planning window of length T days. This gives a total number of available time slots $S = bT$, where every time slot corresponds to one day of hospital stay for one patient. From now on we do not consider the actual beds, just the fact that we have a pool of S time slots that can be used for planning future hospital episodes. These time slots are allocated between J planned episodes. We assume an arbitrary, initial allocation, where s_j denotes the number of time slots allocated to episode j with $j = 1, 2, \ldots, J$. The initial condition is constrained by the total number of available slots, $\sum_{j=1}^{J} s_j = S$. As time passes in the real world, the planning window is shifted forward which adds new time slots that have to be allocated among s_j. Episodes may also be added and removed. This results in an iterative planning process, where at every iteration one of the following events can take place (see

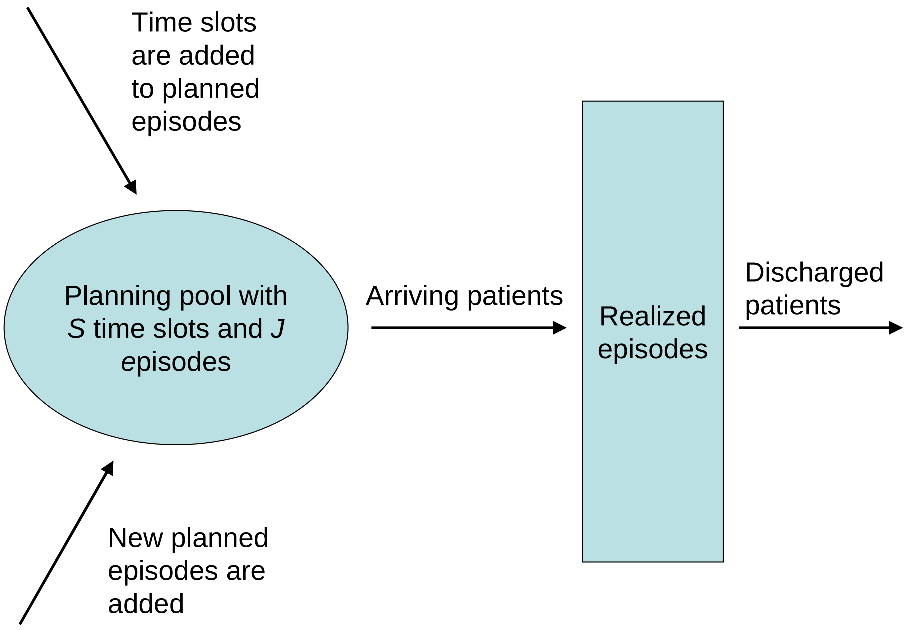

FIG. 2: A schematic overview of the model structure. The important dynamics take place at the planning stage. When a planned episode is realised due to an arrival of a suitable patient at a hospital it is *removed* from the planning pool. As a consequence of these removals, new episodes are added to the planning pool. Planned episodes are also constantly competing to be lengthened by attracting added time slots. What takes place after realisation (arrival of an actual patient) is not taken into account in the model. This means that it is assumed that patients on average stay as long as originally planned.

also figures 2 and 3):

1. With probability p, a new planned episode is created, which means that $S \leftarrow S+1$ and $J \leftarrow J+1$. One time slot is allocated to this episode, $s_J \leftarrow 1$.

2. Otherwise, an episode is chosen randomly with a preferential probability, $\Pi_j = \frac{s_j}{\sum_{k=1}^{J} s_k}$. An additional time slot is allocated to this planned episode, $s_j \leftarrow s_j+1$ and $S \leftarrow S+1$.

At the same iteration, with probability p, an actual patient arrives at the hospital, which means that one of the planned episodes is realised for this patient. This episode is chosen randomly with a uniform probability, $\Pi_j = \frac{1}{J}$. After realisation, the episode is removed from the planning window and cannot be used by subsequent arrivals. In the model, this is equivalent to taking $S \leftarrow S - s_j$, $J \leftarrow J-1$ and doing a re-labelling $n_k \leftarrow n_{k+1}$ for $k = j, j+1, \ldots, J$.

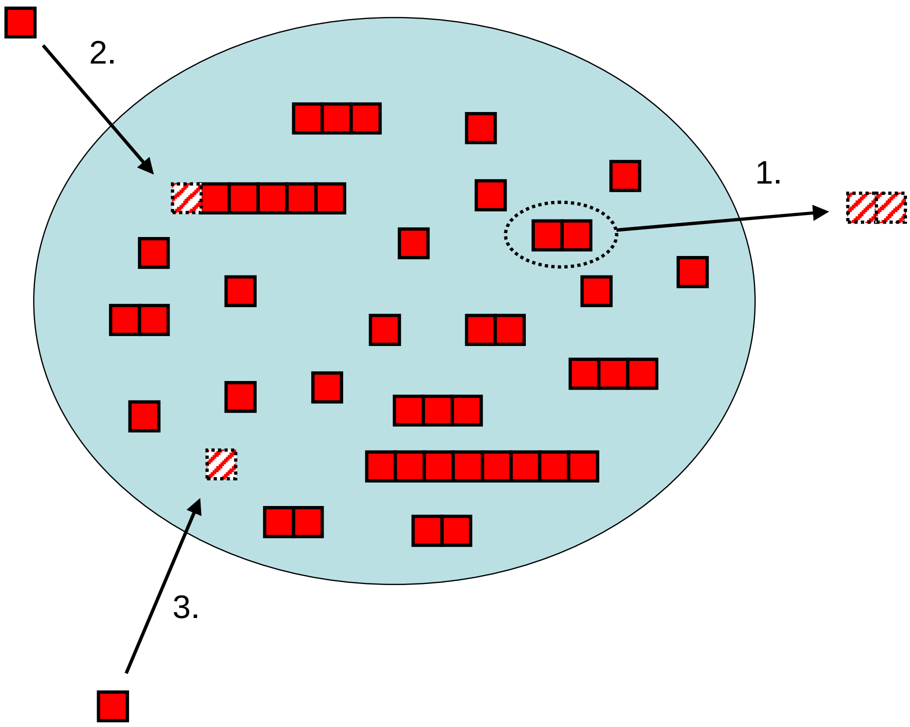

FIG. 3: An example of the flow in and out of the planning pool. 1. Random episodes are realised and *removed* from the pool. 2. New time slots are attached to planned episodes. Longer episodes have a greater chance of attracting these time slots (preferential attachment). 3. New episodes consisting of one time slot are added.

The reason for having the same probability for a patient arriving and for a new planned episode being created is to keep the number of planned episodes, J, constant on average. S will not be constant in this model, but it will find an equilibrium level as shown in Eq. (3) below.

When a time slot is allocated to an existing planned episode according to event 2 above, the target episode is chosen with a preferential probability, i.e. proportional to the length of the episode. This represents an assumption that bargaining power is proportional to current allocation of resources.

If we denote the number of episodes with k time slots as N_k, then we get a rate equation

$$\frac{dN_k}{dt} = \frac{1-p}{S}[(k-1)N_{k-1} - kN_k] - p\frac{N_k}{J} + p\delta_{k,1}. \quad (2)$$

Note that the time t in this equation is abstract and represents iterations in the planning process. To find a stationary solution for N_k, we first observe that

$$\frac{dS}{dt} = \sum_k k \frac{dN_k}{dt} = 1 - p\frac{S}{J} \tag{3}$$

which means that we must have $J = pS$. We can now solve

$$0 = \frac{1-p}{S}[(k-1)N_{k-1} - kN_k] - \frac{N_k}{S} + p\delta_{k,1} \tag{4}$$

by recursion to find

$$N_k = \frac{k-1}{k + \frac{1}{1-p}} N_{k-1}, \text{ for } k > 1 \tag{5}$$

and $N_1 = S\frac{p}{2-p}$. This yields an exact solution

$$N_k = S\frac{p}{2-p}\Gamma\left(\frac{1}{1-p} + 2\right)\frac{\Gamma(k)}{\Gamma\left(1 + \frac{1}{1-p} + k\right)} \tag{6}$$

and asymptotics give, for large k,

$$N_k \sim k^{-\frac{2-p}{1-p}}. \tag{7}$$

To find the actual LOS of patients in the model we must consider realised patients. These correspond to a uniform sampling of the planned episodes, with some random noise representing deviations. Hence, if we disregard the noise, the distribution of LOS is proportional to N_k in Eq. (7). This means that the model will give an average LOS of $\frac{S}{pS} = \frac{1}{p}$. If the LOS is purely power law distributed we also have an approximate average stay of

$$< k >= (1-\gamma)\int_1^\infty k k^{-\gamma} dk = \frac{\gamma - 1}{\gamma - 2}. \tag{8}$$

This gives also another way of predicting the exponent γ,

$$\gamma = \frac{2 < k > -1}{< k > -1} = \frac{2-p}{1-p}. \tag{9}$$

To check the validity of this we can estimate p, from the number of available beds and the total number of patients. To get the number of available beds (excluding those reserved for day only patients) we use NHS data [10] to obtain an estimate of the total number of available bed nights, n_b, during the relevant years (1991-92 to 2002-03) of the data set. The estimate is $n_b = 2435860 \times 365 = 2.13 \times 10^{10}$. The total number of patients in the data set used for the empirical results in Sec. II with $LOS > 0$ (staying at least one night) is $n = 82047768$. We get an estimation of the arrival rate, $p_{est} = \frac{n}{n_b} = 0.092$. Fitting this into Eq. (9) gives $\gamma = 2.10$, in good agreement with the MLE estimate in the previous section.

IV. DISCUSSION

When all different diagnoses and hospitals are taken into account, and aggregated statistics are produced, the distribution of length of stay becomes very broad. It is well approximated by a power law spanning over more than 3 decades. It is well known that there exist a large number of models generating power laws. Here, a model based on a preferential scheduling process is proposed. The micro mechanisms have not been verified, but the model is falsifiable. To test if preferential allocation is present in the real scheduling process, we would need access to low level data on planned capacity in different hospital departments. It would also be necessary to study how local planned capacities change over time.

If the model is a good description of the actual micro mechanisms, then there are a number of interesting conclusions to be made. First, the resource allocation is iteratively based on previous allocations. This could clearly result in a sub-optimal allocation of resources. Second, a preferential allocation tends to favour parts of the health care system that already possess large amounts of resources. It cannot be ruled out that this could be caused mainly by the better opportunities for large departments or influential patient groups to lobby for resources.

Regardless of whether or not the proposed mechanisms are actually present in the real system, the fact that the distribution of LOS is very wide is important to note in itself. It makes statistical estimations of average and standard deviation of the LOS very sensitive to sample size and clearly undesirable as performance indicators. A MLE estimate for a power law exponent would be a much better option for tracking performance, if a single number indicator is needed.

Acknowledgments

This work was funded by the Leverhulme Trust. We also wish to thank T.P. Young for useful discussions.

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