Finite Element Schemes for Elliptic Boundary Value Problems with Rough Coefficients

A Thesis submitted for the degree of Doctor of Philosophy

by

Douglas John Stewart

Department of Mathematics and Statistics, Brunel University.

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Institution: *Brunel University, Uxbridge.*
Department: *Mathematics and Statistics.*
Author: *Douglas John Stewart.*

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ABSTRACT

We consider the task of computing reliable numerical approximations of the solutions of elliptic equations and systems where the coefficients vary discontinuously, rapidly, and by large orders of magnitude. Such problems, which occur in diffusion and in linear elastic deformation of composite materials, have solutions with low regularity with the result that reliable numerical approximations can be found only in approximating spaces, invariably with high dimension, that can accurately represent the large and rapid changes occurring in the solution. The use of the Galerkin approach with such high dimensional approximating spaces often leads to very large scale discrete problems which at best can only be solved using efficient solvers. However, even then, their scale is sometimes so large that the Galerkin approach becomes impractical and alternative methods of approximation must be sought.

In this thesis we adopt two approaches. We propose a new asymptotic method of approximation for problems of diffusion in materials with periodic structure. This approach uses Fourier series expansions and enables one to perform all computations on a periodic cell; this overcomes the difficulty caused by the rapid variation of the coefficients. In the one dimensional case we have constructed problems with discontinuous coefficients and computed the analytical expressions for their solutions and the proposed asymptotic approximations. The rates at which the given asymptotic approximations converge, as the period of the material decreases, are obtained through extensive computational tests which show that these rates are fundamentally dependent on the level of regularity of the right hand sides of the equations. In the two dimensional case we show how one can use the Galerkin method to approximate the solutions of the problems associated with the periodic cell. We construct problems with discontinuous coefficients and perform extensive computational tests which show that the asymptotic properties of the approximations are identical to those observed in the one dimensional case. However, the computational results show that the application of the Galerkin method of approximation introduces a discretization error which can obscure the precise asymptotic rate of convergence for low regularity right hand sides.

For problems of two dimensional linear elasticity we are forced to consider an alternative approach. We use domain decomposition techniques that interface the subdomains with conjugate gradient methods and obtain algorithms which can be efficiently implemented on computers with parallel architectures. We construct the balancing preconditioner, M_h, and show that it has the optimal conditioning property $\kappa(M_h^{-1}S_h) \leq C\,(1+\log(H/h))^2$ where S_h is the discretized Steklov–Poincaré operator, $C > 0$ is a constant which is independent of the magnitude of the material discontinuities, H is the maximum subdomain diameter, and h is the maximum finite element diameter. These properties of the preconditioning operator M_h allow one to use the computational power of a parallel computer to overcome the difficulties caused by the changing form of the solution of the problem. We have implemented this approach for a variety of problems of planar linear elasticity and, using different domain decompositions, approximating spaces, and materials, find that the algorithm is robust and scales with the dimension of the approximating space and the number of subdomains according to the condition number bound above and is unaffected by material discontinuities. In this we have proposed and implemented new inner product expressions which we use to modify the bilinear forms associated with problems over subdomains that have pure traction boundary conditions.

CONTENTS

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II SYMBOLS

II.1. Miscellaneous.

$A \stackrel{\text{def}}{=} B$	A is equal to B by definition.
$\mathbb{R}, \mathbb{C}$	The fields of real and complex numbers.
$\mathbb{F}$	An abstract field of numbers (= $\mathbb{R}$, $\mathbb{C}$).
$\mathcal{Z} \stackrel{\text{def}}{=} \{0, \pm 1, \pm 2, \ldots\}$	The integers.
$\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, \ldots\}$	The natural numbers.
$\mathbb{N}_0 \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$	The non–negative integers.
$\mathbb{N}_0^n \stackrel{\text{def}}{=} \prod_{k=1}^n \mathbb{N}_0$	The set of n–tuples of elements of $\mathbb{N}_0$.
$\underline{1} \in [1, 1, \ldots, 1] \in \mathbb{N}^n$	The unit n–tuple $n \in \mathbb{N}$.
$\underline{e}_r \in \mathbb{N}_0^n$, $(\underline{e}_r)_s \stackrel{\text{def}}{=} \delta_{rs}$, $1 \leq r, s \leq n$	The canonical basis vectors for $\mathbb{R}^n$.
$\mathbb{R}^n \stackrel{\text{def}}{=} \prod_{k=1}^n \mathbb{R}$	Real n–dimensional Euclidean space.
$\mathcal{Z}_n \stackrel{\text{def}}{=} \{0, \pm 1, \ldots, \pm n\}$	
$\mathbb{N}_n \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$	
$\Re[z]$, $\Im[z]$	Real and imaginary parts of $z \in \mathbb{C}$.
$f(\bullet, v)$, $v \in V$	For $v \in V$, the map $f(\bullet, v): U \to W$ where $f: U \times V \to W$.
$\mathcal{N}(A) \stackrel{\text{def}}{=} \{x \in X \mid Ax = 0\}$	The null space of the linear operator $A: X \to Y$.
$\underline{v}_1 \to \underline{v}_2$, $\underline{v}_1$, $\underline{v}_2 \in \mathbb{R}^2$	The straight line connecting $\underline{v}_1$ to $\underline{v}_2$.

$\mathcal{D}(f)$, $\mathcal{R}(f)$	The domain and range of a map $f: X \to Y$.
$\langle L, v \rangle$	The value of the functional $L: V \to \mathbb{F}$ at $v \in V$.
$A^T: \mathcal{BL}(W; \mathbb{F}) \to \mathcal{BL}(V; \mathbb{F})$	The transpose operator of $A: V \to W$ given by $\langle A^T f, v \rangle \stackrel{\text{def}}{=} \langle f, Av \rangle$, $f \in \mathcal{BL}(W; \mathbb{F})$, $v \in V$ where V, W are linear spaces over $\mathbb{F}$.

II.2. Function Spaces.

$C^m(\Omega)$, $m \geq 0$	Space of functions with continuous derivatives of order $\leq m$ (Chap. 1§2).
$C^m(\overline{\Omega})$, $m \geq 0$	Subspace of functions of $C^m(\Omega)$ with uniformly continuous derivatives of order $\leq m$ (Chap. 1§2).
$C_0^m(\Omega)$, $m \geq 0$	Subspace of functions of $C^m(\Omega)$ with compact support in Ω (Chap. 1§2).
$C^{m,\lambda}(\Omega)$, $m \geq 0$, $0 < \lambda \leq 1$	Subspace of functions of $C^m(\overline{\Omega})$ which are Hölder continuous with exponent λ (Chap. 1§2).

$C^m_{per}(\Omega) \stackrel{\text{def}}{=} \{v \in C^n(\mathbb{R}^n) \mid v(\underline{x} + (\underline{b} - \underline{a})\underline{n}) = v(\underline{x}),\ \underline{x} \in \mathbb{R}^n,\ \underline{n} \in \mathcal{Z}^n\}$, $m \geq 0$, $\Omega = \prod_{i=1}^n (a_i, b_i)$.

$\mathcal{L}_p(\Omega)$, $p \geq 1$	Lebesgue space of (equivalence classes of) functions with finite $\|\bullet; \mathcal{L}_p(\Omega)\|$ norm (Chap. 1§2).
$H^s(\Omega)$, $s \in \mathbb{R}$, $s \geq 0$	Sobolev space of (equivalence classes of) weakly differentiable functions in Ω (Chap. 1§2).
$H_0^s(\Omega)$, $s \in \mathbb{R}$, $s \geq 0$	Subspace of $H^s(\Omega)$ obtained as the closure, in the $\|\bullet; H^s(\Omega)\|$ norm topology, of $C_0^\infty(\Omega)$ in $\mathcal{L}_2(\Omega)$.
$H^s(\Gamma)$, $\Gamma \subset \partial\Omega$, $s \in \mathbb{R}$, $s \geq 0$	Sobolev trace space (Chap. 1§2).
$BV(\Omega)$	Spaces of functions of bounded variation over Ω (Chap. 1§2).

II.3. Norms.

$\|x\|_p \stackrel{\text{def}}{=} \left[\sum_{l=1}^\infty \|x_l\|^p\right]^{1/p}$	ℓ_p norm of $x = (x_l)_{l \geq 1} \in \ell_p$.
$\|\underline{x}\|_A \stackrel{\text{def}}{=} \sqrt{\underline{x}^T A \underline{x}}$	Energy norm of $\underline{x} \in \mathbb{R}^n$ w.r.t $A \in \mathbb{R}^{n,n}$ where A is symmetric and positive definite.
$\|M\|_2 \stackrel{\text{def}}{=} \sqrt{\rho(M^H M)}$	The spectral norm of the matrix $M \in \mathbb{C}^{n,n}$.

$\|\bullet; \mathcal{B}\|$	A norm mapping $\mathcal{B} \to \mathbb{R}$ (Chap. 1§2).
$\lvert\bullet; \mathcal{B}\rvert$	A semi–norm mapping $\mathcal{B} \to \mathbb{R}$ (Chap. 1§2).

II.4. Topology.

$B(\underline{x}, \rho, \ell_p) \stackrel{\text{def}}{=} \{\underline{z} \in \mathbb{R}^n \mid \|\underline{x} - \underline{z}\|_p < \rho\}$	The open ℓ_p ball with centre $\underline{x} \in \mathbb{R}^n$ and radius ρ.
$\operatorname{int} \mathcal{O} \stackrel{\text{def}}{=} \{\underline{x} \in \mathcal{O} \mid \exists \rho > 0 \text{ s.t. } B(\underline{x}, \rho, \ell_p) \subset \mathcal{O}\}$	The interior of the set $\mathcal{O} \subset \mathbb{R}^n$.
$\overline{\mathcal{O}} \stackrel{\text{def}}{=} \mathcal{O} \cup \{\underline{x} \in \mathbb{R}^n \mid \exists \{\underline{x}_n\}_{n \geq 1} \subset \mathcal{O} \text{ s.t. } \|\underline{x} - \underline{x}_n\|_2 \to 0 \ (n \to \infty)\}$	The closure of the set $\mathcal{O} \subset \mathbb{R}^n$.
$\partial \mathcal{O} \stackrel{\text{def}}{=} \overline{\mathcal{O}} \backslash \operatorname{int} \mathcal{O} = \overline{\mathcal{O}} \cap \overline{\mathbb{R}^n \backslash \mathcal{O}}$	The boundary of the set $\mathcal{O} \subset \mathbb{R}^n$.
$A \Subset B$	$\overline{A}$ is a compact subset of B.
$\operatorname{dist}(\mathcal{O}, x) \stackrel{\text{def}}{=} \inf\{\|x - y\|_2 \mid y \in \mathcal{O}\}$	Distance between the point $x \in \mathbb{R}^n$ and $\mathcal{O} \subset \mathbb{R}^n$.
$\Pi_i(X) \stackrel{\text{def}}{=} \{x_i \mid (x_1, \dots, x_i, \dots, x_n) \in X\}$	Projection of $X = \prod_{m=1}^n X_m$ onto X_i, $1 \leq i \leq n$.
$f(x\pm) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0+} f(x \pm \varepsilon)$	Left or right hand limit of f at $x \in \mathcal{D}(f)$.
$x_n \rightharpoonup x$ as $n \to \infty$ (in B)	Weak convergence of $\{x_n\} \subset B$ to $x \in B$ where B is a Banach space (Chap. 3§0§1).
$h(\underline{x}) = O(f(\underline{x}))\ (\underline{x} \to \underline{t})$	h has the asymptotic order f as $\underline{x} \to \underline{t}$, i.e., there exist constants $K, \delta > 0$ such that $\lvert h(\underline{x})\rvert \leq K \lvert f(\underline{x})\rvert$ for $\|\underline{x} - \underline{t}\|_2 \leq \delta$.
$h(\underline{x}) = o(f(\underline{x}))\ (\underline{x} \to \underline{t})$	$h(\underline{x})/f(\underline{x}) \to 0$ as $\underline{x} \to \underline{t}$.

II.5. Matrices.

$M^H \stackrel{\text{def}}{=} \overline{M}^T$	The Hermitian transpose of $M \in \mathbb{C}^{n,m}$.
$\sigma(M) \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid \exists \underline{x} \in \mathbb{C}^n \text{ s.t. } M\underline{x} = \lambda \underline{x}\}$	The spectrum of a matrix $M \in \mathbb{C}^{n,n}$, i.e., the set of all eigenvalues of M.
$\rho(M) \stackrel{\text{def}}{=} \max\{\lvert\lambda\rvert \ : \ \lambda \in \sigma(M)\}$	The spectral radius of the matrix $M \in \mathbb{C}^{n,n}$.
$\kappa(A) \stackrel{\text{def}}{=} \|A\|_2 \cdot \|A^{-1}\|_2$	The spectral condition number of $A \in \mathbb{R}^{n,n}$.
$\kappa_S(A) \stackrel{\text{def}}{=} \|A\|_S \cdot \|A^{-1}\|_S$	The energy condition number of $A \in \mathbb{R}^{n,n}$ with respect to the $S \in \mathbb{R}^{n,n}$ where S is symmetric and positive definite.

II.6. Homogenization.

ε	The period of a material with periodic structure.
$\mathcal{P} \stackrel{\text{def}}{=} (0,1)^n$	The canonical periodic cell.
$\mathcal{C} \stackrel{\text{def}}{=} (-1,1)^n$	
$\mathcal{H}_n \stackrel{\text{def}}{=} \{(\varepsilon, t) \in \mathbb{R}^2 \mid \varepsilon t = 2\pi n\}, n \in \mathcal{Z}\backslash\{0\}$	A family of hyperbolae.
$f_{\mathcal{A}}$	The antisymmetric extension of $f:\Omega \to \mathbb{R}$ to $\mathcal{A}$, i.e., $f_{\mathcal{A}}\|_{\Omega} = f$ (Chap. 3§1, 4§1).
$f_{\mathcal{C}}$	The periodic extension to $\mathbb{R}^n$ of the function $f_{\mathcal{A}}$, $\mathcal{D}(f_{\mathcal{A}}) = \mathcal{C}$.
ℓ	The number of terms in a truncated series expansion.

II.7. Domain Decomposition.

$\overline{\Omega} \stackrel{\text{def}}{=} \cup_{i=1}^k \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$	A Non–overlapping decomposition of Ω with simply connected subdomains Ω_i, $1 \leq i \leq k$.
$\Gamma_i \stackrel{\text{def}}{=} \overline{\partial\Omega_i \setminus \partial\Omega}$, $\Gamma \stackrel{\text{def}}{=} \cup_{i=1}^k \Gamma_i$	Subdomain interfaces and global interface.
$\mathcal{G}(\Gamma)$	Geometrical components of the interface polygon Γ, e.g., straight lines and vertices (Chap. 5§3§1).
$\mathcal{V}(\Gamma)$	Vertices of the interface Γ (Chap. 5§3§1).
$H_i \stackrel{\text{def}}{=} \text{diam}(\Omega_i)$	Diameter of subdomain Ω_i, $1 \leq i \leq k$.
$H \stackrel{\text{def}}{=} \max\{H_i \mid 1 \leq i \leq k\}$	
$S_i : (H^{1/2}(\Gamma_i))^2 \to \mathcal{BL}((H^{1/2}(\Gamma_i))^2 ; \mathbb{R})$	The local Steklov–Poincaré operators (Chap. 5§1§2).
$S : (H^{1/2}(\Gamma))^2 \to \mathcal{BL}((H^{1/2}(\Gamma))^2 ; \mathbb{R})$	The Global Steklov–Poincaré operator (Chap. 5§1§2).
$E_i : (H^{1/2}(\Gamma_i))^2 \to (H^1(\Omega_i))^2$	A local Harmonic extension operator (Chap. 5§1§1).
$E : (H^{1/2}(\Gamma))^2 \to (H^1(\Omega))^2$	A global Harmonic extension operator (Chap. 5§1§1).
$R_{\Gamma_i} : (H^{1/2}(\Gamma))^2 \to (H^{1/2}(\Gamma_i))^2$	The interface trace operator (Chap. 5§1§1).
$R_{\Gamma_i,h} : (S^h(\Gamma))^2 \to (S^h(\Gamma_i))^2$	The interface restriction operator.

$S_{i,h}$, S_h	The local and global Schur complement matrices (or the discrete Steklov–Poincaré operators).
$R_{\Gamma_i,h}$, $E_{i,h}$	The discrete restriction and extension matrices (Chap. 5§1§3).

II.8. Finite Element Approximation.

$\mathcal{T}_h(\Omega)$, $h > 0$	An admissible triangulation of Ω (Chap. 2§1).
$h \stackrel{\text{def}}{=} \max\{\text{diam}(\tau) \mid \tau \in \mathcal{T}_h(\Omega)\}$	The diameter of the triangulation $\mathcal{T}_h(\Omega)$ (Chap. 2§1).
$T \stackrel{\text{def}}{=} \{(\xi,\eta) \mid 0 \le \xi + \eta \le 1,\ 0 \le \xi,\eta \le 1\}$	The reference element in a local coordinate system (Chap. 2§2§1).
$\Psi_\tau : T \to \tau$, $\tau \in \mathcal{T}_h(\Omega)$	Affine (isoparametric) transformation that maps the reference element, T, to a global element $\tau \in \mathcal{T}_h(\Omega)$ (Chap. 2§2§1).
$S^h(\Omega)$	The space of continuous piecewise linear functions defined for $\mathcal{T}_h(\Omega)$ (Chap. 2§1).
$S_0^h(\Omega; \partial\Omega_D)$	The subspace of $S^h(\Omega)$ of functions with zero restriction on the boundary $\partial\Omega_D$ (Chap. 2§1).

$S^h(\Gamma) \stackrel{\text{def}}{=} \{v : \Gamma \to \mathbb{C} \mid \exists w \in S^h(\Omega) \text{ such that } v = w|_\Gamma\}$, $\Gamma \subset \partial\Omega$

1 INTRODUCTION

It is an aim in numerical analysis to devise robust computational algorithms which enable one to compute reliable approximations to the solutions of problems of interest and also to analyse the resulting approximation errors. These problems may come from engineering, physics, economics,... and the mathematical models are formulated so that they describe physical or even abstract processes. It is our aim to devise numerical algorithms for systems of elliptic boundary value problems. In particular, we shall treat those problems which arise in the linear elastic deformation of a heterogeneous body, $\Omega = \cup_{r=1}^{\mathcal{K}} \Omega_r \subset\subset \mathbb{R}^2$, i.e., a body composed of different materials in each Ω_r, $1 \leq r \leq \mathcal{K}$ whose characteristics may vary rapidly and may give solutions of different orders of magnitude across Ω. Models of this type lead to *classical* problems of the form: Find $\underline{u} \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ such that

$$-\sum_{i,j,k=1}^{2} \frac{\partial}{\partial x_k}\left[a_{ijkl}(\underline{x})\,\frac{\partial u_i}{\partial x_j}(\underline{x})\right] = f_l(\underline{x}), \quad \underline{x} \in \Omega, \quad 1 \leq l \leq 2 \tag{1.1}$$

$$\underline{u}(\underline{x}) = \underline{u}_D, \quad \underline{x} \in \partial\Omega_D, \quad \sigma(\underline{u}(\underline{x})) \circ \underline{n}(\underline{x}) = \underline{t}(\underline{x}), \quad \underline{x} \in \partial\Omega_N \tag{1.2}$$

where $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$ with $\partial\Omega_N$ an open subset of the boundary $\partial\Omega$ where surface traction forces, $\underline{t}$, apply and $\partial\Omega_D$ a closed subset of the boundary where displacements, $\underline{u}_D$, are imposed; a_{ijkl}, $1 \leq i,j,k,l \leq 2$ define the material properties of the body Ω (differing with each Ω_r, $1 \leq r \leq \mathcal{K}$) and f_l, $1 \leq l \leq 2$ define the body forces acting across Ω. The existence of a solution $\underline{u}$ depends on the regularity of the coefficients a_{ijkl}, $1 \leq i,j,k,l \leq 2$, the body force f, the boundary tractions, $\underline{t}$, the displacements, $\underline{u}_D$, and the boundary $\partial\Omega$, cf. KNOPS & PAYNE (1971). However, we shall take a more general view of the problem and interpret the solution in the weak sense, cf. Section 1.3. This will allow us to work with discontinuous coefficients a_{ijkl}, $1 \leq i,j,k,l \leq 2$ and data for which problem (1.1), (1.2) has no meaning in the above defined space. Furthermore, as a step towards our stated goal, we first study models of steady state diffusion in composite materials over domains $\Omega \subset\subset \mathbb{R}^n$, $n = 1, 2$ because they provide *scalar* elliptic boundary value problems which are simpler to study. Numerical techniques for approximating these simpler problems can correspondingly be generalized to the case of problems of linear elasticity. The classical problems arising from models of diffusion of this type have the form: Find $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[a_{ij}(\underline{x})\,\frac{\partial u}{\partial x_j}(\underline{x})\right] = f(\underline{x}), \quad \underline{x} \in \Omega \tag{1.3}$$

$$u(\underline{x}) = u_D, \quad \underline{x} \in \partial\Omega_D, \quad \sum_{i,j=1}^{2} a_{ij}(\underline{x})\,\frac{\partial u}{\partial x_j}(\underline{x})\,n_i(\underline{x}) = g(\underline{x}), \quad \underline{x} \in \partial\Omega_N \tag{1.4}$$

We shall again allow for discontinuous data in this problem by taking a weaker form with $u \in H^1(\Omega)$ as a weak solution in a Sobolev space setting, cf. Section 1.3.

In fact, we are especially interested in the difficulties which arise when the coefficients a_{ijkl}, $1 \leq i,j,k,l \leq 2$ in (1.1) and a_{ij}, $1 \leq i,j \leq 2$ in (1.2) change rapidly and by many orders of magnitude over Ω, i.e., when the variations $V_\Omega[a_{ijkl}]$, $V_\Omega[a_{ij}]$, $1 \leq i,j,k,l \leq 2$ are *large.* Indeed, we anticipate that the weak solutions, $\underline{u}$, u, which arise for problems of this kind will also have large variations, $V_\Omega[\underline{u}]$, $V_\Omega[u]$, which cannot be accurately approximated unless one employs high dimensional approximating subspaces, $S^h(\Omega) \subset H^1(\Omega)$, $h > 0$, cf. BABUŠKA (1974i). Furthermore, for heterogeneous materials, the coefficients a_{ijkl}, a_{ij}, $1 \leq i,j,k,l \leq 2$ vary discontinuously along the interfaces $\partial\Omega_r \cap \partial\Omega_s$, $1 \leq r,s \leq \mathcal{K}$ between the component materials Ω_r, $1 \leq r \leq \mathcal{K}$ of Ω. This causes the weak solutions to have lower regularity than is the case for a homogeneous body and singularities can arise if the subdomain boundaries $\partial\Omega_r$, $1 \leq r \leq \mathcal{K}$ have vertices, cf. BABUŠKA (1974i), KELLOGG (1972). When features of this type occur the resulting numerical schemes need to reflect the discontinuities, for example by being adaptive, and in general the resulting algebraic systems are simply too large and ill–conditioned for practical solution so that special methods are required.

We now summarise the work of the thesis. In Chapter 1 we briefly introduce some of the mathematical concepts required of the theories of Functional Analysis and Sobolev spaces to construct the weak formulations of problems (1.1), (1.2) and (1.3), (1.4). We make no attempt to be comprehensive and direct the reader to KREYSZIG (1978) and ADAMS (1975) for a more rigorous treatment. In Chapter 2 we introduce some h–version techniques of finite element approximation for elliptic boundary value problems and provide some elements of the theory of approximation; we direct the reader to AZIZ & BABUŠKA (1972) or ODEN & REDDY (1976) for a more comprehensive treatment of these concepts. We should inform the reader that the results provided in Chapters 1 and 2 are frequently employed throughout the remainder of the thesis and, for the complete definition of any symbols in the text which seem unfamiliar, please consult the symbol table on page v.

The behaviour of either analytical or computational approaches for problems of the type (1.1)–(1.2) and (1.3)–(1.4) in $\mathbb{R}^2$, can be difficult to assess for the case of irregular data.

Analytical solutions are rarely available, even for test problems. We emphasize that the assessment is often further complicated by the presence, in $\mathbb{R}^2$, of singular points occurring at corners or edges where different materials interface with one another. In order to avoid some of the difficulties, initially, in Chapter 3 we begin by investigating one dimensional elliptic boundary value problems in which the underlying heterogeneous material, Ω, consists of a periodically repeating cell, $\mathcal{P}_\varepsilon \stackrel{\text{def}}{=} \varepsilon\mathcal{P}$, of diameter $\varepsilon \ll \text{diam}(\Omega)$ comprised of the elemental materials Ω_r, $1 \le r \le \mathcal{K}$. This property of the material is represented in the boundary value problem by a periodic coefficient, a, of period ε, with ε assuming values in the range $(0, \varepsilon_0]$ with ε_0 *small* when the material properties change rapidly. However, problems of this type have been studied in the vast array of literature for problems in $\Omega \subset \mathbb{R}^n$, $n \ge 1$, e.g., convergence in homogenization processes is analysed in TARTAR (1980), the idea of H–convergence is introduced and studied in MURAT & TARTAR (1994), and the notion of two–scale expansions are analysed in ALLAIRE (1992). Indeed, we follow this philosophy and adapt the analysis of BABUŠKA & MORGAN (1991ii) and construct asymptotic approximations u_N^ε, $\varepsilon > 0$, $N \ge 0$ of the solution of the original problem which we now denote u^ε to indicate the different cells. However, general asymptotic treatments of this type do not provide accurate error bounds; generally, the complexities of a general analysis lead to uninformative and pessimistic results. This difficulty has been partially remedied in BAKHVALOV & PANASENKO (1989) where accurate error bounds are included for $\Omega = \mathbb{R}^2$. However, their analysis requires the restrictive conditions a_{ij}, a_{ijkl}, $f \in C^\infty(\mathbb{R}^2)$, $1 \le i, j, k, l \le 2$ and provides little insight into the application of these techniques for more general problems of low regularity which often occur in practice.

In the one dimensional case we obtain an assessment of convergence by employing analytical and computational results to determine the rates of decay,

$$\|u^\varepsilon - u_N^\varepsilon; H^n(\Omega)\| \to 0 \quad (\varepsilon \to 0), \quad N \ge 0, \quad 0 \le n \le 1, \tag{1.5}$$

and to determine how problem regularity affects these. Our results demonstrate that the rate of convergence, $u_N^\varepsilon \to u^\varepsilon$ $(\varepsilon \to 0)$, in the sense of (1.5), occurs at a rate which is independent of the regularity of a but depends primarily on the regularity of f.

In Chapter 4 we generalize this approach to include analogous elliptic boundary value problems in $\mathbb{R}^2$. However, because analytical solutions are no longer available, we find it necessary to include approximating methods and we demonstrate how one can efficiently implement the h–version of finite element approximation for domains $\Omega \subset\subset \mathbb{R}^2$. Indeed, it is apparent from the formulation of our approach that one can quite simply incorporate approximating techniques such as the h, p, or r–adaptive finite element methods into the homogenization process.

The asymptotic approach employed in Chapters 3 and 4 is clearly not suited to problems in which the coefficients, a_{ij}, a_{ijkl}, $1 \le i, j, k, l \le 2$ are non–periodic or ε is *large*, i.e.,

$\varepsilon \notin (0, \varepsilon_0]$. However, if the features of the problem which led us to consider applying asymptotic techniques are still present, e.g., highly heterogeneous materials, coefficients with large variation over Ω, existence of singularities, low regularity, then the need to employ high dimensional approximating spaces, $S^h(\Omega)$, $h > 0$, still exists. However, such spaces lead to large scale systems, i.e., algebraic systems which include many unknown parameters. In Chapter 5 we therefore change our approach to that of domain decomposition and consider ways in which we can exploit the increased computational power provided by modern computers with parallel architecture, in particular, the MIMD – multiple instruction, multiple data – family of machines, cf. BRIGGS & HWANG (1986). Machines of this type possess an array of independent processing *nodes* which are interconnected through a high speed network allowing rapid communication of data. To obtain algorithms which are suitable for implementation on machines of this type we shall work within the framework provided by the theory of domain decomposition using non–overlapping decompositions Ω_i, $1 \le i \le k$ of Ω, i.e.,

$$\overline{\Omega} = \cup_{i=1}^{k} \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j. \tag{1.6}$$

In this we employ extension, restriction, and Steklov–Poincaré operators, cf. AGOSHKOV (1988) and reformulate our problem as a system of boundary value problems, one for each subdomain Ω_i with solution $\underline{u}_{\Omega_i}$, $1 \le i \le k$, coupled by an interface problem on $\Gamma \stackrel{\text{def}}{=} \cup_{i,j=1}^{k} \overline{\Omega}_i \cap \overline{\Omega}_j$ whose solution we denote by $\underline{u}_\Gamma$. However, from our comments above it also follows that the approximating spaces $S^h(\Omega)$, $h > 0$ lead to large scale interface problems and, as is apparent in Section 1 of Chapter 5, it is impractical to construct the interface systems of such large dimension. We therefore turn to iterative solution techniques, in particular, conjugate gradient methods and demonstrate how they can be employed to compute approximations, $\underline{u}_{\Gamma,h}$, $h > 0$, of $\underline{u}_\Gamma$ without explicitly constructing the interface problems. However, a difficulty with iterative techniques of this kind is that, to achieve rapid convergence, they require the discretized Steklov-Poincaré operator, S_h, associated with the interface problem to have a compactly distributed spectrum, $\sigma(S_h)$, though in fact, as the material heterogeneities, the number of subdomains, k, and $\dim(S^h(\Omega))$ grow, the spectrum $\sigma(S_h)$ becomes more sparsely distributed and the rate of convergence slows. This feature of conjugate gradient algorithms can be improved by using a preconditioner; this possibility has been examined in many of the early papers treating domain decomposed interface problems with conjugate gradient type iterative schemes. Indeed, in BJØRSTAD & WIDLUND (1986) a number of preconditioners, P_h, $h > 0$, are constructed which are optimal in the sense that the condition number $\kappa(P_h^{-1} S_h) \stackrel{\text{def}}{=} \|P_h^{-1} S_h\|_2 \, \|S_h^{-1} P_h\|_2$ – a measure of the dispersion of the preconditioned spectrum $\sigma(P_h^{-1} S_h)$ – does not vary with h and the convergence rate is therefore unaffected by the dimension of the approximating space $S^h(\Omega)$, $h > 0$. However, the early papers of this kind deal with relatively simple problems and decompositions $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, i.e., $k = 2$, and, as one should expect, there is little consideration for difficult problems and general decompositions (1.6). Subsequent work by, for example, BRAMBLE, PASCIAK, & SCHATZ (1986), DRYJA & WIDLUND (1991), has led to the construction of preconditioners, P_h, $h > 0$, for rather general problems and decompositions which are optimal in the sense that

$$\kappa(P_h^{-1} S_h) \leq C \left[1 + \log(H/h)\right]^2, \quad H,\, h > 0 \tag{1.7}$$

where $H = \max\{\mathrm{diam}(\Omega_i),\ 1 \leq i \leq k\}$. Although these algorithms are often rather elaborate they do allow one to implement the inverse operator, P_h^{-1}, $h > 0$, efficiently on computers with a parallel architecture because the preconditioner is designed to have a parallel structure that requires little communication between processing nodes. However, the Neumann–Neumann preconditioner, N_h, $h > 0$, studied in LETALLEC & DEROECK (1991), provides a simpler approach which can also be implemented efficiently on a MIMD type computer. The difficulty with this approach is that the preconditioner does not scale well as the number of subdomains, k, increase; this is explained in LETALLEC & DEROECK (1991) where they prove the bound

$$\kappa(N_h^{-1} S_h) \leq \frac{C}{H^2} \left[1 + \log(H/h)\right]^2, \quad H,\, h > 0 \tag{1.8}$$

Following an idea introduced in MANDEL (1993) for scalar elliptic boundary value problems we demonstrate how one can introduce, for problems of heterogeneous linear elasticity, an additional *coarse* problem in the definition of the Neumann–Neumann preconditioner to obtain a new preconditioner, M_h, $h > 0$, which has the optimal spectral property (1.7) and where the constant $C > 0$ is independent of the material heterogeneities. We implement this approach for a variety of problems and compare the computational results with a number of other preconditioners.

To summarize: we introduce asymptotic techniques of approximation in Chapter 3 for elliptic problems in $\mathbb{R}$ having discontinuous and periodic data of period ε. We construct asymptotic approximations u_N^ε, $N \geq 0$ of the weak solution u^ε and, using a combination of analytical and computational methods, assess the rates of convergence of the errors $u^\varepsilon - u_N^\varepsilon$, $N \geq 0$ as $\varepsilon \to 0$ in the norm topologies $\|\bullet; H^p(\Omega)\|$, $0 \leq p \leq 1$. In Chapter 4 we describe how finite element techniques of approximation can be combined with our asymptotic approach to compute approximations, $u_{N,h}^\varepsilon$, $N \geq 0$, of the solution, u^ε, for elliptic problems in $\mathbb{R}^2$ when the coefficients, a_{ij}, $1 \leq i, j \leq 2$, are discontinuous and periodic. We apply this approach to a number of problems of varying levels of regularity and assess the corresponding rates of convergence of $u_{N,h}^\varepsilon \to u^\varepsilon$ as $\varepsilon \to 0$ in the norm topologies $\|\bullet; H^p(\Omega)\|$, $0 \leq p \leq 1$. In Chapter 5 we employ domain decomposition techniques to reformulate problems of linear elasticity as systems of coupled problems with each corresponding to either a subdomain or an interface. We describe how one can add a *coarse* problem to the definition of the Neumann–Neumann preconditioner to obtain an iterative solution algorithm for the domain decomposed interface system which is optimal in the sense of (1.7). Finally, we demonstrate the optimality of this approach using a number of computational examples.

1.1. Elements of Functional Analysis.

In Chapters 2, 3, and 4 we use some of the ideas from the theory of functional analysis. A summary of the ideas which we use are assembled below. However, because the theorems are well known we do not, except for the Lax–Milgram Lemma, provide proofs and instead we refer the reader to KREYSZIG (1978) or RIESZ & SZ.-NAGY (1965).

1.1.1. Bounded Linear Operators.

Let X_i, $1 \le i \le 2$ denote normed linear spaces over the field $\mathbb{F}$ $(= \mathbb{R}, \mathbb{C})$ with norms $\|\bullet; X_i\|$, $1 \le i \le 2$ and assume identical linear space operations of addition and scalar multiplication for X_i, $1 \le i \le 2$. If X_i, $1 \le i \le 2$ are function spaces then we call a mapping $A: X_1 \to X_2$ an operator and say that it is antilinear (or conjugate linear) if it satisfies the property

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \overline{\alpha}_1 A x_1 + \overline{\alpha}_2 A x_2, \quad \alpha_i \in \mathbb{F}, \quad x_i \in X_i, \quad 1 \le i \le 2 \tag{1.1.1}$$

We define the norm, $\|A\|$, of an operator $A: X_1 \to X_2$ as follows

$$\|A\| \stackrel{\text{def}}{=} \sup_{x \neq 0} \frac{\|Ax; X_2\|}{\|x; X_1\|} = \sup_{\|x;X_1\|=1} \|Ax; X_2\| \tag{1.1.2}$$

and say that A is bounded if $\|A\| < \infty$. Indeed, we denote the set of all bounded antilinear operators by $\mathcal{BL}(X_1\,;\,X_2)$, i.e.,

$$\mathcal{BL}(X_1\,;\,X_2) \stackrel{\text{def}}{=} \Big\{A \mid A: X_1 \to X_2,\ A \text{ is antilinear and } \|A\| < \infty\Big\} \tag{1.1.3}$$

We observe that if X_2 is a Banach space with respect to the norm $\|\bullet; X_2\|$ then $\mathcal{BL}(X_1\,;\,X_2)$ is also a Banach space with respect to the operator norm defined in relation (1.1.2). If $X_2 = \mathbb{F}$ then the Banach space $\mathcal{BL}(X_1\,;\,\mathbb{F})$ is referred to as the conjugate or dual space of X_1 and its elements are called functionals.

While studying weak formulations of elliptic boundary value problems we will have the need to consider operators $A: X_1 \to X_2$ where, using the notation introduced above, $X_1 = X \times X$, $X_2 = \mathbb{F}$ and X is a Hilbert space with the inner product $(\bullet, \bullet; X)$. For operators of this kind we generalize the notion of antilinearity defined in relation (1.1.1) and say that the mapping $A: X \times X \to \mathbb{F}$ is sesquilinear if the following relations are satisfied

$$\begin{aligned} &\text{(Linear)} && A(\alpha x + \beta y,\, z) = \alpha A(x,\, z) + \beta A(y,\, z) \\ &\text{(Antilinear)} && A(x,\, \alpha y + \beta z) = \overline{\alpha} A(x,\, y) + \overline{\beta} A(x,\, z) \end{aligned} \qquad \forall \alpha, \beta \in \mathbb{F},\ x, y, z \in X \tag{1.1.4}$$

and we also define the norm of a sesquilinear operator $A: X \times X \to \mathbb{F}$ as follows

$$\|A\| \stackrel{\text{def}}{=} \sup\left\{\frac{|A(x,\, y)|}{\|x; X\| \cdot \|y; X\|}\ :\ x, y \in X \backslash \{0\}\right\} \tag{1.1.5}$$

where $\|x; X\| \stackrel{\text{def}}{=} (x, x; X)^{1/2}$, $x \in X$ and say that A is bounded if $\|A\| < \infty$. We denote the collection of all such bounded sesquilinear operators by $\mathcal{BL}(X \times X\,;\,\mathbb{F})$, i.e.,

$$\mathcal{BL}(X \times X\,;\,\mathbb{F}) \stackrel{\text{def}}{=} \Big\{A \mid A: X \times X \to \mathbb{F},\ A \text{ is sesquilinear and } \|A\| < \infty\Big\} \tag{1.1.6}$$

and we observe that this is a Banach space with respect to the norm (1.1.5). We shall call elements of this space bilinear forms if $\mathbb{F} = \mathbb{R}$ and sesquilinear forms if $\mathbb{F} = \mathbb{C}$ to distinguish between problems using real or complex fields. We now define some additional concepts associated with elements $A \in \mathcal{BL}(X \times X\,;\, \mathbb{F})$ which we shall require

$$\text{(Hermitian symmetric)} \qquad A(x,\, y) = \overline{A(y,\, x)}, \quad x,\, y \in X \tag{1.1.7}$$

$$\text{(Non-negative)} \qquad A(x,\, x) \geq 0, \quad x \in X \tag{1.1.8}$$

$$\text{(Positive)} \qquad A(x,\, x) > 0, \quad x \neq 0 \tag{1.1.9}$$

$$\text{(X–elliptic)} \qquad A(x,\, x) \geq \rho \|x; X\|^2, \quad x \in X \tag{1.1.10}$$

where $\rho > 0$ is a constant that is independent of $x \in X$.

To answer questions concerning the existence and uniqueness of weak solutions of elliptic boundary value problems one generally works within the framework provided by the Lax–Milgram Lemma. We now state this theorem and provide a proof of the result.

Lax–Milgram Lemma 1.1. *Let $A \in \mathcal{BL}(\mathcal{H} \times \mathcal{H}\,;\, \mathbb{F})$ be $\mathcal{H}$–elliptic where $\mathcal{H}$ is a Hilbert space over the field $\mathbb{F}$. Then, for any $F \in \mathcal{BL}(\mathcal{H}\,;\, \mathbb{F})$, there exists a unique $u \in \mathcal{H}$ such that*

$$A(u,\, \phi) = \langle F, \phi \rangle, \quad \phi \in \mathcal{H} \tag{1.1.11}$$

The map $\mathcal{R} : u \mapsto F$ defined by (1.1.11) is a linear bijection of $\mathcal{H}$ onto $\mathcal{BL}(\mathcal{H}\,;\, \mathbb{F})$ and

$$\rho \leq \|\mathcal{R}\| \leq \|A\|, \qquad \|A\|^{-1} \leq \|\mathcal{R}^{-1}\| \leq \rho^{-1} \tag{1.1.12}$$

where $\rho > 0$ is the ellipticity constant of A.

Proof If $A \in \mathcal{BL}(\mathcal{H} \times \mathcal{H}\,;\, \mathbb{F})$ then it follows that the norm of A, $\|A\|$, is bounded and satisfies the inequality

$$|A(u,v)| \leq \|A\|\, \|u; \mathcal{H}\|\, \|v; \mathcal{H}\|, \quad u, v \in \mathcal{H} \tag{1.1.13}$$

Therefore $A(u,\, \bullet) \in \mathcal{BL}(\mathcal{H}\,;\, \mathbb{F})$ for any $u \in \mathcal{H}$ and, thus, $\mathcal{R} : \mathcal{H} \to A(\mathcal{H}, \bullet)$ is a well defined linear operator. Furthermore, from the boundedness relation (1.1.13),

$$\|\mathcal{R}u\| \leq \|A\|\, \|u; \mathcal{H}\|, \quad u \in \mathcal{H} \tag{1.1.14}$$

and therefore $\mathcal{R} \in \mathcal{BL}(\mathcal{H}\,;\, \mathcal{BL}(\mathcal{H}\,;\, \mathbb{F}))$. The $\mathcal{H}$–ellipticity of A implies the inequalities

$$\begin{aligned} \rho\, \|v; \mathcal{H}\|^2 &\leq |A(v,v)| \\ &= |\langle \mathcal{R}v, v \rangle| \leq \|\mathcal{R}v\|\, \|v; \mathcal{H}\| \\ \Rightarrow \qquad \rho \|v; \mathcal{H}\| &\leq \|\mathcal{R}v\| \end{aligned} \tag{1.1.15}$$

and, therefore, $\mathcal{R}$ is an injective map with a bounded inverse $\mathcal{R}^{-1}$ on the domain $\mathcal{R}(\mathcal{H})$. It only remains to prove that $\mathcal{R}(\mathcal{H}) = \mathcal{BL}(\mathcal{H}\,;\, \mathbb{F})$. Let $(\mathcal{R}u_n)_{n \geq 1}$ be a convergent sequence in

$\mathcal{BL}(\mathcal{H}\,;\,\mathbb{F}\,)$ then, from (1.1.15), $(u_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}$ which converges to some limit $u \in \mathcal{H}$ because $\mathcal{H}$ is a Hilbert space. However, because $\mathcal{R}$ is continuous, cf. (1.1.14), $\mathcal{R}u$ is the limit of the sequence $(\mathcal{R}u_n)_{n\geq 1}$ and this implies that $\mathcal{R}(\mathcal{H})$ is a closed subspace of $\mathcal{BL}(\mathcal{H}\,;\,\mathbb{F}\,)$. Thus, $\mathcal{BL}(\mathcal{H}\,;\,\mathbb{F}\,) = \mathcal{R}(\mathcal{H}) \oplus \mathcal{R}(\mathcal{H})^{\perp}$ where $\mathcal{R}(\mathcal{H})^{\perp} \stackrel{\text{def}}{=} \{v \in \mathcal{H} \mid \langle f, v\rangle = 0,\ f \in \mathcal{R}(\mathcal{H})\}$. We now show that $\mathcal{R}$ is a surjective map with image $\mathcal{BL}(\mathcal{H}\,;\,\mathbb{F}\,)$ by proving that $\mathcal{R}(\mathcal{H})^{\perp} = \emptyset$. Suppose that there exists a $v_0 \in \mathcal{R}(\mathcal{H})^{\perp}$ with $v_0 \neq 0$ then we have the contradiction

$$0 = \langle \mathcal{R}v_0, v_0\rangle = A(v_0, v_0) \geq \rho\|v_0; \mathcal{H}\|^2 \tag{1.1.16}$$

Finally, the inequalities (1.1.12) follow immediately from (1.1.14) and (1.1.15) and the theorem is proved. ■

We shall employ the Lax–Milgram lemma throughout the thesis to demonstrate the existence and uniqueness of weak solutions of elliptic boundary value problems, in particular, problems (1.1)–(1.2) and (1.3)–(1.4). We note that the property of $\mathcal{H}$–ellipticity is often the most difficult to prove. Indeed, for problems of linear elasticity, we use Korn's inequalities and, for problems of steady state diffusion, we use Poincaré's inequality to establish $\mathcal{H}$–ellipticity for the appropriate a and $\mathcal{H}$. However, we now introduce the function spaces that are required to construct the weak formulations of problems (1.1)–(1.2) and (1.3)–(1.4).

1.2. Function Spaces.

Below, we provide definitions of the function spaces which we shall use and, where necessary, we describe some of their properties. We direct the reader to WLOKA (1987) or HACKBUSCH (1992) for a rigorous treatment of these function spaces.

We begin by specifying the notation which we shall use throughout this section. Let the symbol Ω denote a simply connected bounded open set in $\mathbb{R}^n$, $n = 1, 2$ with closure $\overline{\Omega}$ and boundary $\partial\Omega$. We shall write $\Omega \subset\subset \mathbb{R}^n$ if $\overline{\Omega}$ is a compact subset of $\mathbb{R}^n$, i.e., a bounded and closed subset. If $\alpha \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ then we call α a multi–index of degree $|\alpha| \stackrel{\text{def}}{=} \sum_{i=1}^n \alpha_i$ and, for $D_i \stackrel{\text{def}}{=} \partial/\partial x_i$, $1 \leq i \leq n$, we define the differential operator D^α, $\alpha \in \mathbb{N}_0^n$ of degree $|\alpha|$ according to the relation

$$D^\alpha \stackrel{\text{def}}{=} D_1^{\alpha_1} \cdots D_n^{\alpha_n} \tag{1.2.1}$$

where $D_i^0 \stackrel{\text{def}}{=} I$, $1 \leq i \leq n$ and I is the identity operator. If $\phi: \Omega \to \mathbb{C}$ then we define the *support* of ϕ as

$$\operatorname{supp} \phi = \overline{\{\underline{x} \in \Omega \mid \phi(\underline{x}) \neq 0\}} \tag{1.2.2}$$

We now provide a collection of definitions and lemmas which we shall use to define function spaces of weakly differentiable functions and to introduce the notion of domain regularity. We begin by defining function spaces which consist of functions, ϕ, that can be differentiated in the classical sense and for which the derivatives, $D^\alpha\phi$, are continuous in some sense for

$|\alpha| \leq m$, $m \in \mathbb{N}_0$. Thus, for $m \in \mathbb{N}_0$ we define $C^m(\Omega)$ as

$$C^m(\Omega) \overset{\text{def}}{=} \{\phi : \Omega \to \mathbb{C} \mid D^\alpha \phi \in C^0(\Omega),\ |\alpha| \leq m\} \tag{1.2.3}$$

where $C^0(\Omega)$ is simply the linear space of functions which are continuous over Ω. We then let $C^\infty(\Omega) \overset{\text{def}}{=} \cap_{n=0}^{\infty} C^n(\Omega)$ and define the subspaces $C_0^m(\Omega) \subset C^m(\Omega)$, $m \in \mathbb{N}_0 \cup \{\infty\}$ as follows

$$C_0^m(\Omega) \overset{\text{def}}{=} \{\phi \in C^m(\Omega) \mid \operatorname{supp} \phi \subset\subset \Omega\} \tag{1.2.4}$$

However, because Ω is an open set, the functions $\phi \in C^0(\Omega)$ need not be bounded on Ω and we therefore define $C^0(\overline{\Omega}) \subset C^0(\Omega)$ to be the subspace consisting of all continuous functions whose domain of definition, Ω, can be extended to the boundary, $\partial\Omega$, such that they become uniformly continuous on $\overline{\Omega}$. We now define the function spaces $C^m(\overline{\Omega})$, $m \in \mathbb{N}_0$ as follows

$$C^m(\overline{\Omega}) \overset{\text{def}}{=} \{\phi \in C^m(\Omega) \mid \text{for each } |\alpha| \leq m \text{ there exists a } \psi_\alpha \in C^0(\overline{\Omega}) \text{ such that } D^\alpha \phi = \psi_\alpha|_\Omega\} \tag{1.2.5}$$

and let $C^\infty(\overline{\Omega}) \overset{\text{def}}{=} \cap_{n=0}^{\infty} C^n(\overline{\Omega})$. We observe that the spaces $C^m(\overline{\Omega})$, $m \in \mathbb{N}_0$ are Banach spaces with respect to the norm

$$\|\phi; C^m(\overline{\Omega})\| \overset{\text{def}}{=} \max_{0 \leq |\alpha| \leq m} \sup_{\underline{x} \in \Omega} |D^\alpha \phi(\underline{x})| \tag{1.2.6}$$

The linear spaces of Hölder continuous functions are also required, thus, we let $0 < \lambda \leq 1$, $m \in \mathbb{N}_0$ and define the subspace $C^{m,\lambda}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ as follows

$$\begin{aligned} C^{m,\lambda}(\overline{\Omega}) \overset{\text{def}}{=} \{\phi \in C^m(\overline{\Omega}) \mid & \text{ there exists a constant } C > 0 \text{ such that} \\ & |D^\alpha \phi(\underline{x}_1) - D^\alpha \phi(\underline{x}_2)| \leq C \, \|\underline{x}_1 - \underline{x}_2\|_2^\lambda,\ |\alpha| \leq m,\ \underline{x}_i \in \Omega,\ 1 \leq i \leq 2\} \end{aligned} \tag{1.2.7}$$

which is a Banach space with respect to the norm

$$\|\phi; C^{m,\lambda}(\overline{\Omega})\| \overset{\text{def}}{=} \|\phi; C^m(\overline{\Omega})\| + \max_{0 \leq |\alpha| \leq m} \sup_{\underline{x}, \underline{z} \in \Omega,\ \underline{x} \neq \underline{z}} \frac{|D^\alpha \phi(\underline{x}) - D^\alpha \phi(\underline{z})|}{\|\underline{x} - \underline{z}\|_2^\lambda} \tag{1.2.8}$$

We now assume that Ω is measurable with respect to the Lebesgue measure, μ, and define $\mathcal{L}_p(\Omega)$ to be the linear space of equivalence classes of functions u which are Lebesgue measurable on Ω and satisfy $\|u; \mathcal{L}_p(\Omega)\| < \infty$ where, for $1 \leq p < \infty$,

$$\|u; \mathcal{L}_p(\Omega)\| \overset{\text{def}}{=} \left[\int_\Omega |u(\underline{x})|^p \, d\underline{x} \right]^{1/p} \tag{1.2.9}$$

where $d\underline{x} \overset{\text{def}}{=} d\mu$ and, for $p = \infty$,

$$\|u; \mathcal{L}_\infty(\Omega)\| \overset{\text{def}}{=} \operatorname*{ess\,sup}_{\underline{x} \in \Omega} |u(\underline{x})| = \inf \left\{ \sup_{\underline{x} \in \Omega \setminus \mathcal{O}} |u(\underline{x})| \ :\ \mathcal{O} \subset \Omega,\ \mu(\mathcal{O}) = 0 \right\} \tag{1.2.10}$$

We note that the elements of the equivalence classes of the Lebesgue spaces $\mathcal{L}_p(\Omega)$, $1 \leq p \leq \infty$ are functions that differ only on sets of Lebesgue measure zero. See ADAMS (1975) for a thorough treatment of the Lebesgue spaces $\mathcal{L}_p(\Omega)$.

In order to generalize the classical problems (1.1)–(1.2) and (1.3)–(1.4) we now introduce the notion of the weak derivative which we use to define the Sobolev spaces below: If, for $\alpha \in \mathbb{N}_0^n$, $u \in \mathcal{L}_1^{loc}(\Omega) \stackrel{\text{def}}{=} \{v \mid v \in \mathcal{L}_1(K),\ K \subset\subset \Omega\}$, there exists a $v \in \mathcal{L}_1^{loc}(\Omega)$ satisfying

$$\int_\Omega \varphi(\underline{x})\, v(\underline{x})\, d\underline{x} = (-1)^{|\alpha|} \int_\Omega u(\underline{x})\, D^\alpha \varphi(\underline{x})\, d\underline{x}, \quad \varphi \in C_0^\infty(\Omega) \tag{1.2.11}$$

where $D^\alpha \varphi$ is defined in the classical sense then we call v the weak D^α derivative of u and write $v = D^\alpha u$. If $u \in C^{|\alpha|}(\Omega)$ then we note that the weak and classical derivatives of u, up to those of order $|\alpha|$, coincide except on sets of measure zero, cf. EDMUNDS & EVANS (1989), and the weak derivative is clearly, therefore, an extension of the classical definition of differentiation. For $m \in \mathbb{N}_0$ we now define the Sobolev space of (equivalence classes of) functions $H^m(\Omega)$ as

$$H^m(\Omega) \stackrel{\text{def}}{=} \{u \in \mathcal{L}_2(\Omega) \mid D^\alpha u \in \mathcal{L}_2(\Omega),\ |\alpha| \le m\} \tag{1.2.12}$$

Indeed, these spaces are Hilbert spaces with respect to the inner product

$$(u, v; H^m(\Omega)) \stackrel{\text{def}}{=} \int_\Omega \sum_{|\alpha| \le m} D^\alpha u(\underline{x})\, \overline{D^\alpha v(\underline{x})}\, d\underline{x}, \quad u, v \in H^m(\Omega) \tag{1.2.13}$$

where the complex conjugate is necessary only when considering spaces over the complex field $\mathbb{C}$. We note that the linear subspace $C^\infty(\Omega) \cap H^m(\Omega)$ is dense in $H^m(\Omega)$ in the sense that if $u \in H^m(\Omega)$ then there exists a sequence $\{u_n\}_{n \ge 1} \subset C^\infty(\Omega) \cap H^m(\Omega)$ such that $\|u - u_n; H^m(\Omega)\| \to 0$ $(n \to \infty)$. We shall also consider boundary value problems with homogeneous boundary conditions and we therefore require the spaces $H_0^m(\Omega)$, $m \in \mathbb{N}_0$ defined as

$$\begin{aligned} H_0^m(\Omega) \stackrel{\text{def}}{=} \{v \in \mathcal{L}_2(\Omega) \mid & \text{ there exists a sequence } \{v_n\}_{n \ge 1} \subset C_0^\infty(\Omega) \\ & \text{such that } \|v - v_n; H^m(\Omega)\| \to 0\ (n \to \infty)\} \end{aligned} \tag{1.2.14}$$

For boundary value problems of low regularity we will also require Sobolev spaces of fractional order, $s \in \mathbb{R}\backslash\mathbb{N}$. Thus, for $s > 0$ let $s = m + \lambda$, $m \in \mathbb{N}_0$, $0 < \lambda < 1$ and define the function space $H^s(\Omega)$ as the linear space of (equivalence classes of) functions $v \in \mathcal{L}_2(\Omega)$ for which $\|v; H^s(\Omega)\| < \infty$ where $\|v; H^s(\Omega)\| = (v, v; H^s(\Omega))^{1/2}$, $(v, v; H^s(\Omega)) \stackrel{\text{def}}{=} (v, v; H^m(\Omega)) + (v, v; H^\lambda(\Omega))$ and

$$(u, v; H^\lambda(\Omega)) \stackrel{\text{def}}{=} \sum_{|\alpha| \le m} \iint_{\Omega \times \Omega} \frac{[D^\alpha u(\underline{x}) - D^\alpha u(\underline{z})]\, \overline{[D^\alpha v(\underline{x}) - D^\alpha v(\underline{z})]}}{\|\underline{x} - \underline{z}\|_2^{n+2\lambda}}\, d\underline{x}\, d\underline{z} \Big] \tag{1.2.15}$$

The density properties observed above for the integer ordered spaces $H^m(\Omega)$, $H_0^m(\Omega)$ are also valid here, i.e., $C^\infty(\Omega) \cap H^s(\Omega)$ and $C_0^\infty(\Omega)$ are dense in $H^s(\Omega)$ and $H_0^s(\Omega)$ with respect to the norm topology $\|\bullet; H^s(\Omega)\|$.

When studying boundary value problems we often find it necessary to consider function spaces of elements which are defined on the boundary, $\partial\Omega$, of the domain Ω. The regularity or smoothness of the domain, Ω, is crucial in the definition of these spaces and we therefore formalize the notion of domain regularity in definition 1.2 below.

Definition 1.2. *(Domain Regularity). Let $\Omega \subset\subset \mathbb{R}^n$. Then we shall write $\Omega \in C^{m,\lambda}$ with $m \in \mathbb{N}_0$, $0 < \lambda \leq 1$ if, for every $\underline{x} \in \partial\Omega$, there exists a neighbourhood $\mathcal{O}_{\underline{x}} \subset \mathbb{R}^n$ and a bijective map $\varphi_{\underline{x}}: \mathcal{O}_{\underline{x}} \to \mathcal{S}$ where $\mathcal{S} \stackrel{\text{def}}{=} B(0,1,\ell_2)$ satisfying*

$$\varphi_{\underline{x}} \in C^{k,\lambda}(\overline{\mathcal{O}}_{\underline{x}}), \quad \varphi_{\underline{x}}^{-1} \in C^{k,\lambda}(\overline{\mathcal{S}}) \tag{1.2.16}$$

$$\varphi_{\underline{x}}(\mathcal{O}_{\underline{x}} \cap \partial\Omega) = \{(\xi_1, \cdots, \xi_n) \in \mathcal{S} \mid \xi_n = 0\} \tag{1.2.17}$$

$$\varphi_{\underline{x}}(\mathcal{O}_{\underline{x}} \cap \Omega) = \{(\xi_1, \cdots, \xi_n) \in \mathcal{S} \mid \xi_n > 0\} \tag{1.2.18}$$

$$\varphi_{\underline{x}}(\mathcal{O}_{\underline{x}} \cap \Omega^c) = \{(\xi_1, \cdots, \xi_n) \in \mathcal{S} \mid \xi_n < 0\} \tag{1.2.19}$$

where (1.2.16) is understood in terms of the components of $\varphi_{\underline{x}} = (\varphi_{\underline{x}}^{(1)}, \cdots, \varphi_{\underline{x}}^{(n)})$ and $\Omega^c \stackrel{\text{def}}{=} \mathbb{R}^n \backslash \Omega$ is the complement of Ω in $\mathbb{R}^n$. ■

For the problems in which we are interested Ω is a polygonal domain with vertices, which we denote $\mathcal{V}_i \in \partial\Omega$, $1 \leq i \leq \mathcal{V}$, lying on the boundary $\partial\Omega$. We assume that the interior domain angle at each vertex, θ_i, $1 \leq i \leq \mathcal{V}$, satisfies the inequality $0 < \theta_i < 2\pi$: this eliminates domains with cuts. If a vertex, say $\mathcal{V}_r$, $r \in \{1, \cdots, \mathcal{V}\}$, is located at the origin, O, then, within a neighbourhood $\mathcal{O}_{\mathcal{V}_r}$ of $\mathcal{V}_r$, the arms of the vertex are the lines Γ_i, $1 \leq i \leq 2$ where

$$\Gamma_1 = \{(x_1, \alpha x_1) \mid 0 \leq x_1 \leq \chi_1\} \tag{1.2.20}$$

$$\Gamma_2 = \{(x_1, \beta x_1) \mid -\chi_2 \leq x_1 \leq 0\} \tag{1.2.21}$$

The bijective maps $\varphi_{\mathcal{V}_r}$, $\varphi_{\mathcal{V}_r}^{-1}$ corresponding to the vertex point $\mathcal{V}_r$ defined in Definition 1.2 are, for $\underline{x} \in \mathcal{O}_{\mathcal{V}_r} = \varphi_{\mathcal{V}_r}^{-1}(\mathcal{S})$, $\underline{\xi} \in \mathcal{S}$,

$$\varphi_{\mathcal{V}_r}(\underline{x}) = \begin{cases} (x_1,\, x_2 - \alpha\, x_1), & \text{if } 0 \leq x_1 \leq \chi_1 \\ (x_1,\, x_2 - \beta\, x_1), & \text{if } -\chi_2 \leq x_1 < 0 \end{cases}, \quad \varphi_{\mathcal{V}_r}^{-1}(\underline{\xi}) = \begin{cases} (\chi_1\xi_1,\, \alpha\,\chi_1\xi_1 + \xi_2), & \text{if } \xi_1 \geq 0 \\ (\chi_2\xi_1,\, \beta\,\chi_2\xi_1 + \xi_2), & \text{if } \xi_1 < 0 \end{cases} \tag{1.2.22}$$

Clearly, $\varphi_{\mathcal{V}_r}$ is continuous and piecewise linear on the bounded domain $\overline{\mathcal{O}}_{\mathcal{V}_r}$ and is therefore Lipschitz continuous although it is not continuously differentiable. Thus, $\varphi_{\mathcal{V}_r} \in C^{0,1}(\overline{\mathcal{O}}_{\mathcal{V}_r})$ and $\Omega \in C^{0,1}$.

The following lemmas are required to define the Sobolev spaces of functions whose domain of definition is a subset of the boundary $\partial\Omega$: they provide some important properties of the boundary of a domain and they also define what is meant by a chart of $\partial\Omega$ and a partition of unity of Ω.

Lemma 1.3. *Let $\Omega \in C^{m,1}$ be a bounded open subset in $\mathbb{R}^n$. Then there exists a $\mathcal{B} \in \mathbb{N}$, bounded open subsets $\mathcal{O}_i$, $0 \leq i \leq \mathcal{B}$ with $\mathcal{O}_0 \subset\subset \Omega$, and, for $\Gamma_i \stackrel{\text{def}}{=} \mathcal{O}_i \cap \partial\Omega$, $1 \leq i \leq \mathcal{B}$, bijective maps $\alpha_i : \Gamma_i \to \alpha_i(\Gamma_i)$, $1 \leq i \leq \mathcal{B}$ where $\alpha_i(\Gamma_i) \subset \mathbb{R}^{n-1}$, $1 \leq i \leq \mathcal{B}$ such that*

$$\overline{\Omega} \subset \cup_{i=0}^{\mathcal{B}} \mathcal{O}_i, \qquad \partial\Omega = \cup_{i=1}^{\mathcal{B}} \Gamma_i, \qquad \alpha_i \circ \alpha_j^{-1} \in C^{m,1}(\overline{\alpha_j(\Gamma_i \cap \Gamma_j)}) \tag{1.2.23}$$

Furthermore, there exist maps $\varphi_i : \mathcal{O}_i \to \mathcal{S}$, $1 \leq i \leq \mathcal{B}$ which satisfy properties (1.2.16)–(1.2.19) with $\lambda = 1$. The pairs $\mathcal{C}_i \stackrel{\text{def}}{=} (\Gamma_i, \alpha_i)$, $1 \leq i \leq \mathcal{B}$ are called the charts of $\partial\Omega$. ■

Lemma 1.4. *Let $\mathcal{O}_i$, $0 \le i \le \mathcal{B}$ be defined as in Lemma 1.3. Then there exist functions $\sigma_i \in C_0^\infty(\mathbb{R}^n)$, $1 \le i \le \mathcal{B}$ satisfying* $\operatorname{supp}\sigma_i \subset \mathcal{O}_i$, $0 \le i \le \mathcal{B}$ *with*

$$\sum_{i=0}^{\mathcal{B}} \sigma_i^2(\underline{x}) = 1, \quad \underline{x} \in \overline{\Omega} \tag{1.2.24}$$

The functions σ_i, $0 \le i \le \mathcal{B}$ are said to form a partition of unity of $\overline{\Omega}$ subordinate to the covering $\mathcal{O}_i$, $0 \le i \le \mathcal{B}$. ■

We can now define the Sobolev spaces of functions which are defined on the boundary, $\partial\Omega$, of Ω: Let $\Omega \in C^{m,1}$ then there exist charts $(\Gamma_i,\, \alpha_i)$, $1 \le i \le \mathcal{B}$, an open covering $\mathcal{O}_i$, $0 \le i \le \mathcal{B}$ of $\overline{\Omega}$, and a partition of unity σ_i, $0 \le i \le \mathcal{B}$ subordinate to $\mathcal{O}_i$, $0 \le i \le \mathcal{B}$ satisfying Lemmas 1.3 and 1.4. For $s \le m+1$ we define the Sobolev space $H^s(\partial\Omega)$ as

$$H^s(\partial\Omega) \stackrel{\text{def}}{=} \{u\colon \partial\Omega \to \mathbb{C} \mid (\sigma_i u) \circ \alpha_i^{-1} \in H_0^s(\alpha_i(\Gamma_i)),\ 1 \le i \le \mathcal{B}\} \tag{1.2.25}$$

and, with respect to the inner product $(\bullet\,,\bullet; H^s(\partial\Omega))$ where

$$(u,v; H^s(\partial\Omega)) \stackrel{\text{def}}{=} \sum_{i=1}^{\mathcal{B}} ((\sigma_i u) \circ \alpha_i^{-1},\ (\sigma_i v) \circ \alpha_i^{-1}; H^s(\alpha_i(\Gamma_i))), \quad u,v \in H^s(\partial\Omega), \tag{1.2.26}$$

$H^s(\partial\Omega)$ is a Hilbert space. However, because $\operatorname{supp}((\sigma_i u) \circ \alpha_i^{-1}) \subset \alpha_i(\Gamma_i)$, $1 \le i \le \mathcal{B}$, the definition (1.2.25) is unchanged if we replace $H_0^s(\alpha_i(\Gamma_i))$ by $H_0^s(\mathbb{R}^{n-1})$ and use any bounded extension of α_i^{-1} from $\alpha_i(\Gamma_i)$ to $\mathbb{R}^{n-1}$. An important property of these spaces is that they do not depend on the open covering $\mathcal{O}_i$, $0 \le i \le \mathcal{B}$ of Ω, the charts $(\Gamma_i,\, \alpha_i)$, $1 \le i \le \mathcal{B}$, or the partition of unity σ_i, $0 \le i \le \mathcal{B}$. Thus, if one uses a different open covering $\mathcal{Q}_i$, $0 \le i \le \mathcal{M}$ of Ω, different charts $(\Upsilon_i,\, \beta_i)$, $1 \le i \le \mathcal{M}$ of $\partial\Omega$, and a different partition of unity τ_i, $0 \le i \le \mathcal{M}$ of Ω which is subordinate to the covering $\mathcal{Q}_i$, $0 \le i \le \mathcal{M}$, then these quantities also lead to the identical space $H^s(\partial\Omega)$ defined in relation (1.2.25). However, using these quantities, the inner product

$$(u,v; H^s(\partial\Omega)) \stackrel{\text{def}}{=} \sum_{i=1}^{\mathcal{M}} ((\tau_i u) \circ \beta_i^{-1},\ (\tau_i v) \circ \beta_i^{-1}; H^s(\beta_i(\Upsilon_i))), \quad u,v \in H^s(\partial\Omega), \tag{1.2.27}$$

will then differ from that defined in (1.2.26) although the norm that this inner product induces will be equivalent to the norm induced by the inner product (1.2.26), cf. HACKBUSCH (1992).

In our study of elliptic problems with mixed boundary conditions we will often find it necessary to consider spaces of functions which are defined on a subset $\Gamma \subset \partial\Omega$. For $\Omega \in C^{m,1}$ we assume that $\Gamma \cap \Gamma_i, 1 \le i \le \mathcal{B}$, cf. Lemma 1.3, is given by an equation of the form

$$\Gamma \cap \Gamma_i = \Big\{(x_1, \cdots, x_{n-1}, \psi_i(x_1, \cdots, x_{n-1})) \mid x_j \in \alpha_i(\Gamma_i),\ 1 \le j \le n-1\Big\} \tag{1.2.28}$$

Then, for $s \ge 0$, we define the Sobolev space $H^s(\Gamma)$ as follows

$$H^s(\Gamma) \stackrel{\text{def}}{=} \{u\colon \Gamma \to \mathbb{C} \mid \text{ there exists a } v \in H^s(\partial\Omega) \text{ such that } u = v|_\Gamma\}, \quad s \ge 0 \tag{1.2.29}$$

In our study of domain decomposition algorithms we are interested only in the case $0 < s < 1$ and therefore, following GRISVARD (1985), we define the norm $\|\bullet; H^s(\Gamma)\|$, $0 < s < 1$ as

$$\|u; H^s(\Gamma)\|^2 \stackrel{\text{def}}{=} \int_\Gamma |u(\underline{x})|^2 \, d\sigma(\underline{x}) + \iint_{\Gamma\times\Gamma} \frac{|u(\underline{x}) - u(\underline{z})|^2}{\|\underline{x} - \underline{z}\|_2^{n-1+2s}} \, d\sigma(\underline{x}) \, d\sigma(\underline{z}), \quad u \in H^s(\Gamma) \tag{1.2.30}$$

where σ is the surface element defined according to the relation, cf. WOLKA (1987),

$$\sigma(\underline{x}) = \left[1 + \sum_{j=1}^{n-1} |\partial\psi_i(\underline{x})/\partial x_j|^2\right]^{1/2} dx_1 \cdots dx_{n-1}, \quad \underline{x} \in \Gamma \cap \Gamma_i. \tag{1.2.31}$$

Clearly, for polygonal domains ψ_i, $1 \leq i \leq \mathcal{B}$ is piecewise linear and the derivatives $\partial\psi_i/\partial x_j$, $1 \leq i \leq \mathcal{B}$, $1 \leq j \leq n-1$ are defined everywhere except at the vertices of Γ. We note that, if $\Gamma = \partial\Omega$ then the spaces (1.2.25) and (1.2.29) are identical and the norm defined in relation (1.2.30) is equivalent to the norm induced by the inner product defined in relation (1.2.26), cf. GRISVARD (1985)

In formulating boundary value problems it is necessary to specify some condition which the solution must satisfy on the boundary, $\partial\Omega$, of the domain Ω. For problems understood in the classical sense the solutions, u, belong to $C^0(\overline{\Omega})$ and their boundary values can be obtained simply by taking the restriction $u|_{\partial\Omega}$. However, for functions $u \in H^s(\Omega)$, $s \geq 0$ with $\Omega \in C^{m,1}$, $m \geq 0$ the boundary, $\partial\Omega$, has zero Lebesgue measure, i.e., $\mu(\partial\Omega) = 0$ and it therefore makes no sense to consider the restriction to $\partial\Omega$ of functions in such spaces. Thus, for $\Omega \in C^{m,1}$, $m \geq 0$, we employ the trace operator which is defined to be the surjective map $\mathrm{Tr} \in \mathcal{BL}(H^s(\Omega)\,;\, H^{s-1/2}(\partial\Omega))$, $m+1 \geq s > 1/2$ which satisfies $\mathrm{Tr}\,(u) = u|_{\partial\Omega}$, $u \in C^0(\overline{\Omega})$ and has a right inverse $\mathrm{Tr}^{-1} \in \mathcal{BL}(H^{s-1/2}(\partial\Omega)\,;\, H^s(\Omega))$, i.e., $\mathrm{Tr} \circ \mathrm{Tr}^{-1} = I$, cf. GRISVARD (1985). We note that, for $\Omega \in C^{0,1}$, there is the identity

$$H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \mid \mathrm{Tr}\,(v) = 0\} \tag{1.2.32}$$

and, for $\Gamma \subset \partial\Omega$, we define the closed subspace $H_0^1(\Omega;\Gamma) \subset H^1(\Omega)$ as

$$H_0^1(\Omega;\Gamma) \stackrel{\text{def}}{=} \{v \in H^1(\Omega) \mid \mathrm{Tr}\,(v)|_\Gamma = 0\} \tag{1.2.33}$$

In our study of asymptotic methods in Chapter 3 we consider functions $u : \Omega \to \mathbb{R}$, $\Omega \Subset \mathbb{R}^n$, $1 \leq n \leq 2$ which we say have bounded variation if $V_\Omega(u) < \infty$ where, for $\Omega = (a,b)$,

$$V_\Omega(u) \stackrel{\text{def}}{=} \sup\left\{ \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| \ : \ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \right\} \tag{1.2.34}$$

If a function $u : \Omega \to \mathbb{R}$ has bounded variation then it is bounded and can be written as the difference of two positive non-decreasing functions φ, ψ, i.e., $u = \varphi - \psi$, cf. SMIRNOV (1964). If $u : \Omega \to \mathbb{C}$ then we say that u has bounded variation if, and only if, $V_\Omega(\Re[u])$, $V_\Omega(\Im[u]) < \infty$. We now define the space of functions of bounded variation over Ω as

$$BV(\Omega) \stackrel{\text{def}}{=} \{u : \Omega \to \mathbb{C} \mid V_\Omega(u) < \infty\} \tag{1.2.35}$$

For $\Omega = (a,b) \times (c,d)$ we define the variation, $V_\Omega(u)$, of a map $u : \Omega \to \mathbb{R}$ as follows, cf. SMIRNOV (1964),

$$V_\Omega(u) \stackrel{\text{def}}{=} \sup \left\{ \sum_{i,j=1}^{m,n} |V_{\Omega_{ij}}(u)| \; : \; \{\Omega_{ij}\}_{i,j=1}^{m,n} \text{ is a subdivision of } \Omega \right\} \tag{1.2.36}$$

where, for $1 \le i \le m$, $1 \le j \le n$,

$$V_{\Omega_{ij}}(u) \stackrel{\text{def}}{=} u(x_i, y_j) - u(x_{i-1}, y_j) - u(x_i, y_{j-1}) + u(x_{i-1}, y_{j-1}) \tag{1.2.37}$$

and $\{\Omega_{ij}\}_{i,j=1}^{m,n}$ is a subdivision of Ω if $\Omega_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$ where

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_m = b, \qquad c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$

Using this definition of variation, we again define the function space $BV(\Omega)$ according to (1.2.35). We note if the map $u : \Omega \to \mathbb{R}$ has bounded variation then there exist non–negative non–decreasing functions φ_i, ψ_i, $1 \le i \le 2$ such that $u = \varphi_1 - \psi_1 - \varphi_2 + \psi_2$, cf. SMIRNOV (1964).

In the case of functions $\underline{u} : \Omega \to \mathbb{C}^2$, i.e., $\underline{u} = [u_1, u_2]$, we use the notation $\underline{u} \in (\mathcal{H})^2$ if $u_i \in \mathcal{H}$, $1 \le i \le 2$. If $\mathcal{H}$ is a normed linear space with norm $\|\bullet; \mathcal{H}\|$ then we define the norm $\|\bullet; (\mathcal{H})^2\|$ according to the relation

$$\|\underline{u}; (\mathcal{H})^2\| \stackrel{\text{def}}{=} \left[\sum_{i=1}^{2} \|u_i; \mathcal{H}\|^2 \right]^{1/2}, \quad \underline{u} \in (\mathcal{H})^2. \tag{1.2.38}$$

Indeed, we shall use (1.2.38) to define norms for the Hilbert spaces $(H^s(\Omega))^2$, $(H^s(\Gamma))^2$, $\Gamma \subset \partial\Omega$, $s \ge 0$ in Chapter 5.

1.3. Weak Formulations of Elliptic Boundary Value Problems.

We now aim to reformulate problems (1.1)–(1.2) and (1.3)–(1.4) in a Sobolev space setting rather than the classical setting of the $(C^2(\Omega) \cap C^1(\overline{\Omega}))^n$, $1 \le n \le 2$ spaces used in the introduction. This will allow us to study problems with discontinuous data over polygonal domains, Ω, which, we should point out, are often excluded in the classical theory because it typically requires conditions such as $\Omega \in C^{m,\lambda}$, $m \ge 2$, $0 < \lambda < 1$ or, for problem (1.1)–(1.2) with $\partial\Omega_D = \partial\Omega$, $u_D = 0$, a_{ij}, $f \in C^{m-2,\lambda}(\Omega)$, $1 \le i,j \le 2$.

We begin with problem (1.3)–(1.4) and assume that the coefficients a_{ij}, $1 \le i,j \le 2$ are symmetric and uniformly elliptic, i.e., there exists a constant $\rho > 0$ such that

$$\sum_{i,j=1}^{2} \xi_i \, a_{ij}(\underline{x}) \, \xi_j \ge \rho \sum_{i=1}^{2} \xi_i^2, \quad (\xi_1, \xi_2) \in \mathbb{R}^2, \quad \underline{x} \in \Omega. \tag{1.3.1}$$

We also assume that Ω, a_{ij}, $1 \le i,j \le 2$, f, u_D, g are sufficiently smooth to ensure the existence of a unique classical solution, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then, for $\varphi \in V \stackrel{\text{def}}{=} \{v \in$

$C^\infty(\overline{\Omega}) \mid v|_{\partial\Omega_D} = 0\}$ we multiply (1.3) by φ and use the divergence theorem to deduce the equation

$$\int_\Omega \sum_{i,j=1}^{2} a_{ij}(\underline{x}) \frac{\partial u}{\partial x_j}(\underline{x}) \frac{\partial \varphi}{\partial x_i}(\underline{x})\, d\underline{x} = \int_\Omega f(\underline{x})\,\varphi(\underline{x})\, d\underline{x} + \int_{\partial\Omega_N} g(\underline{x})\,\varphi(\underline{x})\, d\sigma(\underline{x}) \tag{1.3.2}$$

where we have used boundary condition (1.4) and the property $v|_{\partial\Omega_D} = 0$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies boundary conditions (1.4) and equation (1.3.2) then, applying the divergence theorem to (1.3.2), it follows that

$$\int_\Omega \left(\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[a_{ij}(\underline{x}) \frac{\partial u}{\partial x_j}(\underline{x}) \right] - f(\underline{x}) \right) \varphi(\underline{x})\, d\underline{x} = 0, \quad \varphi \in V \tag{1.3.3}$$

This implies that u is a classical solution, i.e., it satisfies equations (1.3) and (1.4). Thus, with respect to classical solutions, problems (1.3)–(1.4) and (1.3.2) are equivalent. We can now generalize the elliptic boundary value problem (1.3)–(1.4) to include domains $\Omega \in C^{0,1}$; right hand sides $f \in \mathcal{L}_2(\Omega)$; symmetric coefficients $a_{ij} \in \mathcal{L}_\infty(\Omega)$, $1 \le i, j \le 2$ which are uniformly elliptic almost everywhere in Ω; boundary conditions $u_D \in H^{1/2}(\partial\Omega_D)$, $g \in \mathcal{L}_2(\partial\Omega_N)$. We do this by interpreting derivatives in the weak sense, cf. (1.2.11), and defining $u \in H^1(\Omega)$ to be the weak solution of problem (1.3)–(1.4) if it satisfies $\mathrm{Tr}\,(u)|_{\partial\Omega_D} = u_D$ and

$$a(u,v) = F(v), \quad v \in H_0^1(\Omega; \partial\Omega_D) \tag{1.3.4}$$

where, for $u, v \in H_0^1(\Omega; \partial\Omega_D)$,

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \sum_{i,j=1}^{2} a_{ij}(\underline{x}) \frac{\partial u}{\partial x_j}(\underline{x}) \frac{\partial v}{\partial x_i}(\underline{x})\, d\underline{x},\ F(v) \stackrel{\text{def}}{=} \int_\Omega f(\underline{x})\, v(\underline{x})\, d\underline{x} + \int_{\partial\Omega_N} g(\underline{x})\, \mathrm{Tr}\,(v(\underline{x}))\, d\sigma(\underline{x}) \tag{1.3.5}$$

We now assume that $\sigma(\partial\Omega_D) > 0$ and show that problem (1.3.4) is solvable by demonstrating that a, F satisfy the conditions of the Lax–Milgram Lemma. The continuity of the linear operator F follows from the Cauchy–Schwarz inequality, i.e., for $v \in H_0^1(\Omega; \partial\Omega_D)$,

$$\begin{aligned} |F(v)| &\le \left| \int_\Omega f(\underline{x})\, v(\underline{x})\, d\underline{x} \right| + \left| \int_{\partial\Omega_N} g(\underline{x})\, \mathrm{Tr}\,(v(\underline{x}))\, d\sigma(\underline{x}) \right| \\ &\le \|f; \mathcal{L}_2(\Omega)\|\, \|v; \mathcal{L}_2(\Omega)\| + \|g; \mathcal{L}_2(\partial\Omega_N)\|\, \|\mathrm{Tr}\,(v); \mathcal{L}_2(\partial\Omega_N)\| \end{aligned} \tag{1.3.6}$$

and, from the continuity of the trace operator $\mathrm{Tr} \in \mathcal{BL}(H^1(\Omega)\,;\, H^{1/2}(\partial\Omega))$, it is clear that

$$\|\mathrm{Tr}\,(v); \mathcal{L}_2(\partial\Omega_N)\| \le \|\mathrm{Tr}\,(v); \mathcal{L}_2(\partial\Omega)\| \le \|\mathrm{Tr}\,(v); H^{1/2}(\partial\Omega)\| \le \|\mathrm{Tr}\,\|\, \|v; H^1(\Omega)\| \tag{1.3.7}$$

and it then follows that $F \in \mathcal{BL}(H^1(\Omega)\,;\, \mathbb{R})$ where $\|\mathrm{Tr}\,\|$ is the operator norm of Tr, i.e.,

$$\|\mathrm{Tr}\,\| = \sup \left\{ \frac{\|\mathrm{Tr}\,(v); H^{1/2}(\partial\Omega)\|}{\|v; H^1(\Omega)\|} \ :\ v \in H^1(\Omega)\backslash\{0\} \right\} \tag{1.3.8}$$

We use the boundedness of the coefficients $a_{ij} \in \mathcal{L}_\infty(\Omega)$, $1 \le i,j \le 2$ and the Cauchy–Schwarz inequality to prove the continuity of the linear operator a as follows, for $u, v \in H_0^1(\Omega; \partial\Omega_D)$,

$$\begin{aligned} |a(u,v)| &\le \sum_{i,j=1}^{2} \|a_{ij}; \mathcal{L}_\infty(\Omega)\| \left[\int_\Omega \left|\frac{\partial u}{\partial x_i}(\underline{x})\right|^2 d\underline{x}\right]^{1/2} \left[\int_\Omega \left|\frac{\partial u}{\partial x_j}(\underline{x})\right|^2 d\underline{x}\right]^{1/2} \\ &\le C_1 \|u; H^1(\Omega)\| \, \|v; H^1(\Omega)\| \end{aligned} \tag{1.3.9}$$

where the constant $C_1 > 0$ depends on the coefficients a_{ij}, $1 \le i,j \le 2$. The $H_0^1(\Omega; \partial\Omega_D)$-ellipticity of a follows from the ellipticity relation (1.2.1) and Poincaré's inequality, i.e.,

$$a(v,v) \ge \rho \int_\Omega \sum_{i=1}^{2} \left|\frac{\partial v}{\partial x_i}(\underline{x})\right|^2 d\underline{x} \ge C_2 \|v; H^1(\Omega)\|^2, \quad v \in H_0^1(\Omega; \partial\Omega_D) \tag{1.3.10}$$

where the constant $C_2 > 0$ depends on ρ. Thus, the conditions of the Lax–Milgram Lemma are satisfied and therefore there exists a unique solution $u \in H_0^1(\Omega; \partial\Omega_D)$ of problem (1.2.4). For the homogeneous Dirichlet problem ($\partial\Omega_D = \partial\Omega$, $u_D = 0$) it is known, cf. HACKBUSCH (1992), that if Ω is convex, $a_{ij} \in C^{0,1}(\overline{\Omega})$, $1 \le i,j \le 2$, and $f \in \mathcal{L}_2(\Omega)$ then $u \in H^2(\Omega) \cap H_0^1(\Omega)$. The problems which we study, however, do not have continuous coefficients and so we expect the solutions to have lower regularity, i.e., $u \in H^{1+\lambda}(\Omega)$, $0 < \lambda < 1$. For an analysis of the regularity of the solution, u, in the case of scalar elliptic problems with discontinuous coefficients, we direct the reader to KELLOGG (1971) & (1972).

We now reformulate the classical linear elasticity problem (1.1)–(1.2) following the same steps used in the reformulation (1.3.4) of problem (1.3)–(1.4). For a rigorous treatment of the theory of elasticity we direct the reader to either MARSDEN & HUGHES (1987) or SPENCER (1980). We will restrict ourselves to problems of isotropic linear elasticity, i.e., problems for which the coefficients a_{ijkl}, $1 \le i,j,k,l \le 2$ are given by the equations

$$\begin{array}{llll} a_{1111}(\underline{x}) = \lambda(\underline{x}) + 2\mu(\underline{x}), & a_{1112}(\underline{x}) = 0, & a_{1121}(\underline{x}) = 0, & a_{1122}(\underline{x}) = \lambda(\underline{x}) \\ a_{1211}(\underline{x}) = 0, & a_{1212}(\underline{x}) = \mu(\underline{x}), & a_{1221}(\underline{x}) = \mu(\underline{x}), & a_{1222}(\underline{x}) = 0 \\ a_{2111}(\underline{x}) = 0, & a_{2112}(\underline{x}) = \mu(\underline{x}), & a_{2121}(\underline{x}) = \mu(\underline{x}), & a_{2122}(\underline{x}) = 0 \\ a_{2211}(\underline{x}) = \lambda(\underline{x}), & a_{2212}(\underline{x}) = 0, & a_{2221}(\underline{x}) = 0, & a_{2222}(\underline{x}) = \lambda(\underline{x}) + 2\mu(\underline{x}) \end{array} \tag{1.3.11}$$

where λ and μ are the Lamé and shear modulii functions defined according to the relations

$$\lambda(\underline{x}) \stackrel{\text{def}}{=} \frac{\nu E(\underline{x})}{1-\nu^2}, \quad \mu(\underline{x}) \stackrel{\text{def}}{=} \frac{E(\underline{x})}{2(1+\nu)}, \quad \underline{x} \in \Omega \tag{1.3.12}$$

where $\nu \in (0, 1/2)$ is Poisson's ratio and E is Young's Modulus of elasticity, cf. KNOPS & PAYNE (1971). We shall say that the coefficients a_{ijkl}, $1 \le i,j,k,l \le 2$ are uniformly elliptic if there is a constant $\rho > 0$ such that, for $\underline{x} \in \Omega$,

$$\sum_{i,j,k,l=1}^{2} \xi_{ij} \, a_{ijkl}(\underline{x}) \, \xi_{kl} \ge \rho \sum_{i,j=1}^{2} \xi_{ij}^2, \qquad \xi_{ij} = \xi_{ji}, \quad \xi_{ij} \in \mathbb{R}, \ 1 \le i,j \le 2. \tag{1.3.13}$$

However, it is known, cf. KNOPS & PAYNE (1971), that the coefficients are uniformly elliptic if, and only if,

$$\lambda(\underline{x}) + 2\mu(\underline{x}) > 0, \quad \mu(\underline{x}) > 0, \quad \underline{x} \in \Omega \tag{1.3.14}$$

Thus, assume that Ω, λ, μ, ν, f are such that a unique solution, $\underline{u} \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$, of problem (1.1)–(1.2) exists; multiply (1.2) by $v_l \in V$, $1 \le l \le 2$; integrate the resulting equation over Ω; use the divergence theorem to deduce the identity

$$a(\underline{u}, \underline{v}) = F(\underline{v}) \tag{1.3.15}$$

where, for $\underline{u}$, $\underline{v} \in V^2$,

$$a(\underline{u}, \underline{v}) \stackrel{\text{def}}{=} \int_\Omega \sum_{i,j,k,l=1}^{2} a_{ijkl} \frac{\partial u_i}{\partial x_j}(\underline{x}) \frac{\partial v_k}{\partial x_l}(\underline{x})\, d\underline{x}, \; F(\underline{v}) \stackrel{\text{def}}{=} \int_\Omega f(\underline{x}) \cdot \underline{v}(\underline{x})\, d\underline{x} + \int_{\partial\Omega_N} \underline{t}(\underline{x}) \cdot \underline{v}(\underline{x})\, d\sigma(\underline{x}) \tag{1.3.16}$$

We can now generalize the elliptic boundary value problem (1.1)–(1.2) to include domains $\Omega \in C^{0,1}$; right hand sides $f \in (\mathcal{L}_2(\Omega))^2$; Lamé and Shear moduli $\lambda, \mu \in \mathcal{L}_\infty(\Omega)$ which satisfy inequalities (1.3.14) almost everywhere in Ω; boundary conditions $\underline{u}_D \in (H^{1/2}(\partial\Omega_D))^2$, $\underline{t} \in (\mathcal{L}_2(\partial\Omega_N))^2$. We do this, once again, by interpreting derivatives in the weak sense, cf. (1.2.11), and defining $\underline{u} \in (H^1(\Omega))^2$ to be the weak solution if it satisfies $\mathrm{Tr}\,(\underline{u})|_{\partial\Omega_D} = \underline{u}_D$ and

$$a(\underline{u}, \underline{v}) = F(\underline{v}), \quad \underline{v} \in (H_0^1(\Omega; \partial\Omega_D))^2 \tag{1.3.17}$$

where a, F are defined in relation (1.3.16). We assume that $\sigma(\partial\Omega_D) > 0$ and use the Lax–Milgram Lemma to show that the weak problem (1.3.17) has a unique solution $\underline{u} \in (H^1(\Omega))^2$. We do this by demonstrating that a, F satisfy the conditions of the Lax–Milgram Lemma. If $f \in (\mathcal{L}_2(\Omega))^2$ and $\underline{t} \in (\mathcal{L}_2(\partial\Omega_N))^2$ then the Cauchy–Schwarz inequality implies that $F \in \mathcal{BL}((H^1(\Omega))^2; \mathbb{R})$ and if $\lambda, \mu \in \mathcal{L}_\infty(\Omega)$ then the Cauchy–Schwarz inequality also implies that $a \in \mathcal{BL}((H^1(\Omega))^2 \times (H^1(\Omega))^2; \mathbb{R})$. The $(H_0^1(\Omega; \partial\Omega_D))^2$–ellipticity of the bilinear form a follows from Korn's inequality, cf. BRENNER & RIDGWAY SCOTT (1994),

$$\int_\Omega \sum_{i,j=1}^{2} |\varepsilon_{ij}(\underline{v})|^2\, d\underline{x} \ge C\, \|\underline{v}; (H^1(\Omega))^2\|, \quad \underline{v} \in (H^1(\Omega))^2 \tag{1.3.18}$$

where $C > 0$ is a constant independent of $\underline{v}$, $\sigma(\partial\Omega_D) > 0$, and $\varepsilon_{ij}(\underline{v}) \stackrel{\text{def}}{=} (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$, $1 \le i,j \le 2$. If λ, μ, $\underline{u}_D$, $\underline{t}$, f_l, $1 \le l \le 2$, and Ω are sufficiently smooth then the weak solution, $\underline{u}$, will belong to $(H^2(\Omega))^2$. However, for problems with discontinuous Lamé functions λ, μ we anticipate that the weak solution, $\underline{u}$, will possess the lower level of regularity $\underline{u} \in (H^{1+\alpha}(\Omega))^2$ for some $\alpha \in (0, 1]$.

We note that the level of regularity of the solutions u, $\underline{u}$ of problems (1.3.4) and (1.3.17) will play an important role in determining how rapidly the approximation errors

$$\|u - u_h; H^n(\Omega)\|, \quad \|\underline{u} - \underline{u}_h; (H^n(\Omega))^2\|, \quad 0 \le n \le 1 \tag{1.3.19}$$

converge to zero as the discretization parameter $h \to 0$ where u_h, $\underline{u}_h$ are finite element approximations of u, $\underline{u}$ respectively, cf. Chapter 2.

2 Finite Element Approximation Theory For Elliptic Boundary Value Problems

We recall that the weak problems (1.3.4), (1.3.17) are formulated in terms of the infinite dimensional Sobolev spaces $H^1(\Omega)$ and that practical analytical expressions for the weak solutions u, $\underline{u}$ of these problems are rarely available. Thus, we aim to show how one can use finite element techniques to construct approximating subspaces $S^h(\Omega)$, $h > 0$ of the Sobolev space $H^1(\Omega)$ and obtain practical algorithms which allow one to compute approximations $u_h \in S^h(\Omega)$, $\underline{u}_h \in (S^h(\Omega))^2$ of the respective weak solutions $u \in H^1(\Omega)$, $\underline{u} \in (H^1(\Omega))^2$. We demonstrate how the approximations are computed using the Galerkin approach and, taking into account the solution regularity, we provide some error estimates for the approximations. We make no attempt to be comprehensive and direct the reader to any of the texts Aziz & Babuška (1972), Oden & Reddy (1976), Ciarlet (1978) for a rigorous treatment of finite element methods.

2.1. Finite Element Approximating Spaces.

We assume that $\Omega \subset\subset \mathbb{R}^2$ is a polygonal domain and say that $\mathcal{T}_h(\Omega) \stackrel{\text{def}}{=} \{\tau_i \mid \tau_i \subset \Omega,\ 1 \le i \le \nu\}$ is an *admissible* triangulation of Ω if the following conditions are satisfied: (1) if $\tau \in \mathcal{T}_h(\Omega)$ then τ is an open triangle, i.e., $\tau = \text{int}(\tau)$, (2) $\tau_i \cap \tau_j = \emptyset \ \Leftrightarrow\ i \ne j$, (3) $\overline{\Omega} = \cup_{i=1}^{\nu} \overline{\tau}_i$, (4) if $i \ne j$ then $\overline{\tau}_i \cap \overline{\tau}_j$ is either null or a common side of the elements τ_i, τ_j, (5) $\max\{\text{diam}(\tau) \mid \tau \in \mathcal{T}_h(\Omega)\} = h$.

Let $\mathcal{T}_h(\Omega)$ be an admissible triangulation of Ω then a point $\underline{x} \in \overline{\Omega}$ is said to be a node of $\mathcal{T}_h(\Omega)$ if $\underline{x}$ is a vertex of some finite element $\tau \in \mathcal{T}_h(\Omega)$. We define the approximating space $S^h(\Omega)$ of piecewise linear functions, over the field $\mathbb{F}$, for the triangulation $\mathcal{T}_h(\Omega)$ as follows

$$S^h(\Omega) \stackrel{\text{def}}{=} \{v \in C^0(\overline{\Omega}) \mid \text{for } \tau \in \mathcal{T}_h(\Omega) \text{ there exist } a_i \in \mathbb{F},\ 1 \le i \le 3 \\ \text{such that } v(\underline{x}) = a_1 x_1 + a_2 x_2 + a_3,\ \underline{x} \in \overline{\tau}\} \tag{2.1.1}$$

where, clearly, each function $v \in S^h(\Omega)$ is uniquely determined by its values at each node of

$\mathcal{T}_h(\Omega)$. The approximating space $S^h(\Omega) \subset H^1(\Omega)$ is said to be *conforming* if $S^h(\Omega) \subset H^1(\Omega)$, we demonstrate the validity of this inclusion relation as follows: Let $u \in S^h(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ then, for $|\alpha| = 1$,

$$\int_\Omega u(\underline{x})\, D^\alpha \phi(\underline{x})\, d\underline{x} = \sum_{i=1}^{\nu} \int_{\tau_i} u(\underline{x})\, D^\alpha \phi(\underline{x})\, d\underline{x}$$

$$= \sum_{i=1}^{\nu} \left[\int_{\partial\tau_i} u(\underline{x})\, \phi(\underline{x})\, n_\alpha^{(i)}(\underline{x})\, d\sigma(\underline{x}) - \int_{\tau_i} D^\alpha u(\underline{x})\, \phi(\underline{x})\, d\underline{x} \right] \tag{2.1.2}$$

$$= -\sum_{i=1}^{\nu} \int_{\tau_i} D^\alpha u(\underline{x})\, \phi(\underline{x})\, d\underline{x} = -\int_\Omega D^\alpha u(\underline{x})\, \phi(\underline{x})\, d\underline{x} \tag{2.1.3}$$

where $\underline{n}^{(i)} = [n_1^{(i)}, n_2^{(i)}]$ is the unit outward normal vector to the boundary $\partial\tau_i$, $1 \le i \le \nu$ and all derivatives are understood in the classical sense. We obtain (2.1.3) from (2.1.2) using the continuity of u, ϕ, the property $\text{supp}(\phi) \subset \Omega$, and observing that $\underline{n}^{(i)}(\underline{x}) = -\underline{n}^{(j)}(\underline{x})$, $\underline{x} \in \partial\tau_i \cap \partial\tau_j$. Thus, $D^\alpha u \in \mathcal{L}_\infty(\Omega)$ is a piecewise constant function defined almost everywhere in Ω and the inclusion $S^h(\Omega) \subset H^1(\Omega)$ follows. For an admissible triangulation $\mathcal{T}_h(\Omega)$ we say that $S^h(\Omega)$ is the corresponding conforming subspace.

For $n \stackrel{\text{def}}{=} \dim(S^h(\Omega))$ let $\underline{x}_i$, $1 \le i \le n$ denote the nodal points of $\mathcal{T}_h(\Omega)$ and define the basis $\mathcal{B}(S^h(\Omega)) \stackrel{\text{def}}{=} \{\phi_i\}_{i=1}^n$ of $S^h(\Omega)$ where ϕ_i, $1 \le i \le n$ are the functions with the properties

$$\phi_i(\underline{x}_j) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \ne j \end{cases}, \quad 1 \le i, j \le n \tag{2.1.4}$$

In the case of elliptic problems with mixed boundary conditions on $\partial\Omega$ we assume that the endpoints of $\partial\Omega_D$ are nodes of the triangulation $\mathcal{T}_h(\Omega)$ and define the subspace $S_0^h(\Omega; \partial\Omega_D) \subset H_0^1(\Omega; \partial\Omega_D)$ ($\Omega \in C^{0,1}$) as

$$S_0^h(\Omega; \partial\Omega_D) \stackrel{\text{def}}{=} \{v \in S^h(\Omega) \mid v|_{\partial\Omega_D} = 0\} \tag{2.1.5}$$

For $m \stackrel{\text{def}}{=} \dim(S_0^h(\Omega; \partial\Omega_D))$ let $\underline{x}_i \in \overline{\Omega} \backslash \partial\Omega_D$, $1 \le i \le m$, $\underline{x}_i \in \partial\Omega_D$, $m < i \le n$ denote the nodal points of $\mathcal{T}_h(\Omega)$ and define the basis $\mathcal{B}(S_0^h(\Omega; \partial\Omega_D)) \stackrel{\text{def}}{=} \{\phi_i\}_{i=1}^m$ of the subspace $S_0^h(\Omega; \partial\Omega_D)$ where ϕ_i, $1 \le i \le m$ are the functions which satisfy

$$\phi_i(\underline{x}_j) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \ne j \end{cases}, \quad 1 \le i \le m,\ 1 \le j \le n \tag{2.1.6}$$

The use of the parameter h as an index in the symbol $\mathcal{T}_h(\Omega)$ is ambiguous because there are many different admissible triangulations of Ω with identical h. We restrict our attention to the families of uniform triangulations of Ω, cf. ODEN & REDDY (1976), i.e., $\{\mathcal{T}_h(\Omega) \mid h > 0\}$ is a family of uniform triangulations of Ω if, for $h_\tau \stackrel{\text{def}}{=} \text{diam}(\tau)$, $\tau \in \mathcal{T}_h(\Omega)$,

$$h / \min\{h_\tau \mid \tau \in \mathcal{T}_h(\Omega)\} = 1, \quad h > 0 \tag{2.1.7}$$

We note that it is often necessary when attempting to approximate solutions of singular problems to consider families of quasi-uniform triangulations, i.e., families of triangulations $\{\mathcal{T}_h(\Omega) \mid h > 0\}$ which satisfy

$$h / \min\{\rho_\tau \mid \tau \in \mathcal{T}_h(\Omega)\} \le C, \quad h > 0 \tag{2.1.8}$$

where $C > 0$ is a constant independent of h and ρ_τ is the maximum diameter of any circle which can be inscribed in $\tau \in \mathcal{T}_h(\Omega)$. In Section 2.2 we introduce the Galerkin approach and obtain approximations $u_h \in S_0^h(\Omega; \partial\Omega_D)$, $\underline{u}_h \in (S_0^h(\Omega; \partial\Omega_D))^2$ of the respective weak solutions $u \in H_0^1(\Omega; \partial\Omega_D)$, $\underline{u} \in (H_0^1(\Omega; \partial\Omega_D))^2$ of the weak problems (1.3.4), (1.3.17). We will determine upper bounds for the approximation errors $\|u - u_h; H^n(\Omega)\|$, $\|\underline{u} - \underline{u}_h; (H^n(\Omega))^2\|$, $0 \leq n \leq 1$ using the following result from approximation theory, cf. HACKBUSCH (1992).

Theorem 2.1. *Let $\mathcal{T}_h(\Omega)$, $h > 0$ be an admissible triangulation of Ω then, for $u \in H^{1+\lambda}(\Omega) \cap H_0^1(\Omega; \partial\Omega_D)$, $0 \leq \lambda \leq 1$,*

$$\inf\{\, \|u - v_h; H^1(\Omega)\| \ : \ v_h \in S_0^h(\Omega; \partial\Omega_D)\} \leq C(\theta)\, h^\lambda\, \|u; H^{1+\lambda}(\Omega)\| \qquad (2.1.9)$$

where θ is the smallest interior angle of any $\tau \in \mathcal{T}_h(\Omega)$. ∎

For the case of problems with piecewise smooth coefficients which vary discontinuously along a polygonal curve $\Gamma \subset \Omega$ we construct admissible triangulations $\mathcal{T}_h(\Omega)$, $h > 0$ which have the property that $\tau \cap \Gamma = \emptyset$, $\tau \in \mathcal{T}_h(\Omega)$. We do this because the solution has a higher level of regularity over a neighbourhood $\mathcal{O}$ when it excludes regions of discontinuity and, in this way, we obtain more accurate approximations than would otherwise be the case. For example, if $u \in H^{1+\lambda}(\Omega) \cap H^2(\tau)$, $0 < \lambda < 1$, $\tau \in \mathcal{T}_h(\Omega)$ then it follows from the theory of approximation, cf. HACKBUSCH (1992), that

$$\inf\{\, \|u - v_h; H^n(\Omega)\| \ : \ v_h \in S^h(\Omega)\} \leq C\, h^{2-n} \left[\sum_{\tau \in \mathcal{T}_h(\Omega)} \|u; H^2(\tau)\|^2 \right]^{1/2}, \quad 0 \leq n \leq 1 \quad (2.1.10)$$

where $C > 0$ is a constant independent of h. However, if there exists a $\tau \in \mathcal{T}_h(\Omega)$ such that $\tau \cap \Gamma \neq \emptyset$ then $\|\nabla(u - v_h)\|_2 = O(1)$ $(h \to 0)$, $v_h \in S^h(\Omega)$ and the optimal $\|\bullet; H^1(\Omega)\|$ approximation order is reduced from $O(h)$ to $O(h^{1/2})$ as $(h \to 0)$, i.e.,

$$\inf\{\, \|u - v_h; H^1(\Omega)\| \ : \ v_h \in S^h(\Omega)\} = O(h^{1/2}) \ (h \to 0)$$

We note that the discontinuities along Γ can lead to solutions with singular points, cf. KELLOGG (1971), which often result in lower orders of approximation than is suggested by (2.1.10). For a rigorous treatment of approximation in Sobolev spaces we direct the reader to AZIZ & BABUŠKA (1972) .

2.2. Galerkin Approximations.

We now introduce the Galerkin approach to approximation for the weak problems (1.3.4) and (1.3.17). We demonstrate how the finite element spaces defined in Section 2.1 can be used to construct approximations of the weak solutions and we establish upper bounds for the errors which this process introduces.

For the case of scalar problems let V denote an infinite dimensional subspace of $H^1(\Omega)$, e.g., $H_0^1(\Omega)$, and define u as the solution of the weak problem: Find $u \in V$ such that

$$a(u,v) = F(v), \quad v \in V \tag{2.2.1}$$

where $a \in \mathcal{BL}(V \times V\,;\,\mathbb{C}\,)$ is a V–elliptic sesquilinear form and $F \in \mathcal{BL}(V\,;\,\mathbb{C}\,)$. We let V_h denote a finite element subspace of V, cf. Section 2.1, corresponding to an admissible triangulation, $\mathcal{T}_h(\Omega)$, of Ω and define the Galerkin approximation $u_h \in V_h$ of $u \in V$ as the solution of the problem: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h), \quad v_h \in V_h \tag{2.2.2}$$

Because $V_h \subset V$ the Lax–Milgram Lemma shows that problem (2.2.2) is well defined, i.e., it has a unique solution $u_h \in V_h$. To compute the solution, $u_h \in V_h$, of problem (2.2.2) we require a basis $\mathcal{B}(V_h)$ of V_h. We use the basis $\mathcal{B}(V_h) = \{\phi_i\}_{i=1}^m$ where ϕ_i, $1 \le i \le m$ are the functions which satisfy the nodal interpolation conditions (2.1.6). Clearly, problem (2.2.2) is equivalent to the problem: Find $u_h \in V_h$ such that

$$a(u_h, \phi_i) = F(\phi_i), \quad 1 \le i \le m. \tag{2.2.3}$$

Furthermore, this problem can be formulated as a system of algebraic equations: Find $\underline{u}_h \in \mathbb{C}^m$ such that

$$A_h\,\underline{u}_h = \underline{F}_h, \quad A \in \mathbb{C}^{m,m},\ \underline{F}_h \in \mathbb{C}^m \tag{2.2.4}$$

where $(A_h)_{ij} = a(\phi_j, \phi_i)$, $(\underline{F}_h)_j = F(\phi_j)$, $1 \le i,j \le m$. Indeed, defining the bijective linear operator $M\colon \mathbb{C}^m \to V_h$ according to the relation

$$M\underline{u} \stackrel{\text{def}}{=} \sum_{i=1}^m u_i\,\phi_i, \quad \underline{u} \in \mathbb{C}^m, \tag{2.2.5}$$

it is apparent that problem (2.2.3) is equivalent to the linear system (2.2.4) in the sense that the solutions satisfy $M\underline{u}_h = u_h$. In the case of linear elasticity we note the following differences: The Hilbert space V is a subspace of $(H^1(\Omega))^2$; the system (2.2.4) has dimension $2m$ (rather than m) with

$$(A_h)_{ij} = \begin{bmatrix} a(\underline{e}_1\phi_j, \underline{e}_1\phi_i) & a(\underline{e}_1\phi_j, \underline{e}_2\phi_i) \\ a(\underline{e}_2\phi_j, \underline{e}_1\phi_i) & a(\underline{e}_2\phi_j, \underline{e}_2\phi_i) \end{bmatrix}, \quad (\underline{F}_h)_j = \begin{bmatrix} F(\underline{e}_1\phi_j) \\ F(\underline{e}_2\phi_j) \end{bmatrix}, \quad 1 \le i,j \le m; \tag{2.2.6}$$

the linear operator $M\colon \mathbb{C}^{2m} \to V_h$ is defined as

$$M\underline{u} \stackrel{\text{def}}{=} \sum_{i=1}^m \underline{u}_i\,\phi_i, \quad \underline{u} \in \mathbb{C}^{2m}, \qquad \underline{u}_i \stackrel{\text{def}}{=} \begin{bmatrix} u_{2i-1} \\ u_{2i} \end{bmatrix}, \quad 1 \le i \le m. \tag{2.2.7}$$

Clearly, from the definition of A_h, it follows that $A_h = A_h^H$ and

$$a(u_h, v_h) = \underline{v}_h^H A_h \underline{u}_h, \quad u_h, v_h \in V_h$$

where $u_h = M\underline{u}_h$, $v_h = M\underline{v}_h$ and $\underline{v}_h^H \stackrel{\text{def}}{=} \bar{\underline{v}}_h^T$ (conjugate transpose). We will sometimes use the engineering terminology and call the system matrix A_h the stiffness matrix and the system right hand side $\underline{F}_h$ the load vector.

2.2.1. Computation of the Stiffness Matrices.

We now describe how the stiffness matrices are computed for problems (1.3.4), (1.3.17). We begin with scalar problems and observe that, for u_h, $v_h \in V_h$,

$$a(u_h, v_h) = \sum_{\tau \in \mathcal{T}_h(\Omega)} a_\tau(u_h, v_h), \quad F(v_h) = \sum_{\tau \in \mathcal{T}_h(\Omega)} F_\tau(v_h) \tag{2.2.8}$$

where the subscript τ in (2.2.8) indicates that the integrals which define the operators a, F are restricted to the triangle τ of the triangulation $\mathcal{T}_h(\Omega)$, cf. (1.3.5), (1.3.16). For each $\tau \in \mathcal{T}_h(\Omega)$ let $\underline{x}_\tau^{(i)}$, $1 \le i \le 3$ be local node labels for the triangle τ which are also labelled $\underline{x}_{n_i}$, $1 \le i \le 3$, cf. Section 2.1, where $n_i \stackrel{\text{def}}{=} G_\tau(i)$, $1 \le i \le 3$ and $G_\tau : \{1,2,3\} \to \{1, \cdots, m\}$ is the globalization map which maps the local node numbers, $\{1,2,3\}$, to their global values, $\{1, \cdots, m\}$. Then we define the boolean matrices $\Lambda_\tau \in \mathbb{R}^{m,3}$, $\tau \in \mathcal{T}_h(\Omega)$ according to the relation

$$(\Lambda_\tau)_{p,q} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } G_\tau(q) = p \\ 0, & \text{if } G_\tau(q) \ne p \end{cases}, \quad 1 \le p \le m, \quad 1 \le q \le 3 \tag{2.2.9}$$

The decomposition (2.2.8) and definition (2.2.9) permit one to express A_h, $\underline{F}_h$ as follows

$$A_h = \sum_{\tau \in \mathcal{T}_h(\Omega)} \Lambda_\tau A_{\tau,h} \Lambda_\tau^T, \quad \underline{F}_h = \sum_{\tau \in \mathcal{T}_h(\Omega)} \Lambda_\tau \underline{F}_{\tau,h} \tag{2.2.10}$$

where, for $\tau \in \mathcal{T}_h(\Omega)$, $A_{\tau,h} \in \mathbb{R}^{3,3}$, $\underline{F}_{\tau,h} \in \mathbb{R}^3$ are defined as follows, for $n_i \stackrel{\text{def}}{=} G_\tau(i)$, $1 \le i \le 3$,

$$(A_{\tau,h})_{ij} \stackrel{\text{def}}{=} a_\tau(\phi_{n_j}, \phi_{n_i}), \quad (\underline{F}_{\tau,h})_j \stackrel{\text{def}}{=} F_\tau(\phi_{n_j}), \quad 1 \le i,j \le 3 \tag{2.2.11}$$

For additional properties of the boolean matrices Λ_τ, $\tau \in \mathcal{T}_h(\Omega)$ we direct the reader to ODEN & REDDY (1976). For problems of linear elasticity we observe the following differences. The boolean matrices $\Lambda_\tau \in \mathbb{R}^{2m,6}, \tau \in \mathcal{T}_h(\Omega)$ are defined blockwise as

$$(\Lambda_\tau)_{p,q} \stackrel{\text{def}}{=} \begin{cases} I, & \text{if } G_\tau(q) = p \\ 0, & \text{if } G_\tau(q) \ne p \end{cases}, \quad 1 \le p \le m, \quad 1 \le q \le 3 \tag{2.2.12}$$

where $I \in \mathbb{R}^{2,2}$ is the identity matrix and $0 \in \mathbb{R}^{2,2}$ is the zero matrix and, for $\tau \in \mathcal{T}_h(\Omega)$, $A_{\tau,h} \in \mathbb{R}^{6,6}$, $\underline{F}_{\tau,h} \in \mathbb{R}^6$ are defined blockwise as

$$(A_{\tau,h})_{ij} = \begin{bmatrix} a_\tau(\underline{e}_1\phi_{n_j}, \underline{e}_1\phi_{n_i}) & a_\tau(\underline{e}_1\phi_{n_j}, \underline{e}_2\phi_{n_i}) \\ a_\tau(\underline{e}_2\phi_{n_j}, \underline{e}_1\phi_{n_i}) & a_\tau(\underline{e}_2\phi_{n_j}, \underline{e}_2\phi_{n_i}) \end{bmatrix}, \quad (\underline{F}_{\tau,h})_j = \begin{bmatrix} F_\tau(\underline{e}_1\phi_{n_j}) \\ F_\tau(\underline{e}_2\phi_{n_j}) \end{bmatrix}, \quad 1 \le i,j \le 3 \tag{2.2.13}$$

We determine the values of $a_\tau(\phi_{n_j}, \phi_{n_i})$, $a_\tau(\underline{e}_r\phi_{n_j}, \underline{e}_s\phi_{n_i})$, $F_\tau(\phi_{n_j})$, $F_\tau(\underline{e}_r\phi_j)$, $1 \le r,s \le 2$, $1 \le i,j \le 3$ used above by employing an affine map $\Psi_\tau : T \to \tau$, $T \stackrel{\text{def}}{=} \{(\xi,\eta) \mid 0 \le \xi + \eta \le 1,\ 0 \le \xi,\eta \le 1\}$ to transform integrals over elements $\tau \in \mathcal{T}_h(\Omega)$ to integrals over T. Thus, if $\tau \in \mathcal{T}_h(\Omega)$ is a triangle with nodes $\underline{x}_\tau^{(i)}$, $1 \le i \le 3$ then we define Ψ_τ as

$$\Psi_\tau(\underline{t}) \stackrel{\text{def}}{=} \underline{x}_\tau^{(1)}\psi_1(\underline{t}) + \underline{x}_\tau^{(2)}\psi_2(\underline{t}) + \underline{x}_\tau^{(3)}\psi_3(\underline{t}), \quad \underline{t} \in T \tag{2.2.14}$$

where $\psi_1(\underline{t}) \stackrel{\text{def}}{=} 1 - t_1 - t_2$, $\psi_2(\underline{t}) \stackrel{\text{def}}{=} t_1$, $\psi_3(\underline{t}) \stackrel{\text{def}}{=} t_2$ and use Ψ_τ to transform integrals as follows

$$\int_\tau b(\underline{x}) \frac{\partial u}{\partial x_i}(\underline{x}) \frac{\partial v}{\partial x_j}(\underline{x})\, d\underline{x} = \int_T b(\Psi_\tau(\underline{t}))\, \hat{u}_i(\underline{t})\, \hat{v}_j(\underline{t})\, |J(\Psi_\tau(\underline{t}))|\, d\underline{t}, \quad 1 \leq i,j \leq 2. \tag{2.2.15}$$

where $|J(\Psi_\tau(\underline{t}))|$ denotes the determinant of the Jacobian of $\Psi_\tau(\underline{t}) = [\Psi_{\tau,1}(\underline{t}), \Psi_{\tau,2}(\underline{t})]$, $\underline{t} \in T$, i.e., $J_{ij}(\Psi_\tau(\underline{t})) = \Psi_{\tau,i}(\underline{t})/\partial t_j$, $1 \leq i,j \leq 2$,

$$|J(\Psi_\tau(\underline{t}))| = \frac{\partial \Psi_{\tau,1}}{\partial t_1}(\underline{t}) \frac{\partial \Psi_{\tau,2}}{\partial t_2}(\underline{t}) - \frac{\partial \Psi_{\tau,2}}{\partial t_1}(\underline{t}) \frac{\partial \Psi_{\tau,1}}{\partial t_2}(\underline{t}), \quad \underline{t} \in T, \tag{2.2.16}$$

and the functions $\hat{u}_i$, $\hat{v}_j$, $1 \leq i,j \leq 2$ are determined from the following relation which, for $v \in C^1(\overline{\tau})$, shows how derivatives change under the transformation Ψ_τ

$$\begin{bmatrix} \dfrac{\partial v}{\partial x_1}(\underline{x}) \\ \dfrac{\partial v}{\partial x_2}(\underline{x}) \end{bmatrix} = \frac{1}{|J(\Psi_\tau(\underline{t}))|} \begin{bmatrix} \dfrac{\partial \Psi_{\tau,2}}{\partial t_2}(\underline{t}) & -\dfrac{\partial \Psi_{\tau,2}}{\partial t_1}(\underline{t}) \\ -\dfrac{\partial \Psi_{\tau,1}}{\partial t_2}(\underline{t}) & \dfrac{\partial \Psi_{\tau,1}}{\partial t_1}(\underline{t}) \end{bmatrix} \begin{bmatrix} \dfrac{\partial \tilde{v}}{\partial t_1}(\underline{t}) \\ \dfrac{\partial \tilde{v}}{\partial t_2}(\underline{t}) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{v}_1(\underline{t}) \\ \hat{v}_2(\underline{t}) \end{bmatrix}, \tag{2.2.17}$$

where $\tilde{v}(\underline{t}) \stackrel{\text{def}}{=} v(\Psi_\tau(\underline{t}))$, $\underline{t} \in T$. We note that the transformation (2.2.14) has a constant Jacobian matrix $J(\Psi_\tau(\underline{t})) \in \mathbb{R}^{2,2}$, e.g., $J_{ij}(\Psi_\tau(\underline{t})) = x_{\tau,i}^{(j+1)} - x_{\tau,i}^{(1)}$, $1 \leq i,j \leq 2$. Thus, we determine $A_{\tau,h}$, $\underline{F}_{\tau,h}$, $\tau \in \mathcal{T}_h(\Omega)$, cf. (2.2.11), (2.2.13), using the affine transformation Ψ_τ, cf. (2.2.14), which allows us to perform all computations over the reference element T.

2.2.2. Analysis of the Galerkin Approximation Errors.

We provide a short description of how one combines the results from the theory of approximation in Sobolev spaces with the lemmas of Céa and Aubin–Nitsche to obtain a priori error bounds on the Galerkin approximations, u_h, $\underline{u}_h$, $h > 0$, of the weak solution u, $\underline{u}$. The results which we obtain are abstract in the sense that they demonstrate that the Galerkin approximations converge to the weak solutions in the Sobolev norm topologies as $h \to 0$ but they do not provide estimates of the actual errors.

We begin with the important Lemma of Céa which we use to demonstrate convergence of the Galerkin approximations in the $H^1(\Omega)$ norm topology.

Theorem 2.2. *(Céa's Lemma) Let V_h be a finite element subspace of V corresponding to an admissible triangulation $\mathcal{T}_h(\Omega)$, $h > 0$ of Ω. If $u \in V$ is the weak solution of (2.2.1) and $u_h \in V_h$ is the Galerkin approximation of $u \in V$, i.e., it is the solution of (2.2.2) then*

$$\|u - u_h; H^1(\Omega)\| \leq C \inf\{\, \|u - v_h; H^1(\Omega)\| \;:\; v_h \in V_h\} \tag{2.2.18}$$

where $C > 0$ is a constant independent of $h > 0$. ∎

Proof It is apparent from relations (2.2.1) and (2.2.2) that

$$a(u - u_h, v_h) = 0, \quad v_h \in V_h. \tag{2.2.19}$$

Thus, using orthogonality property (2.2.19) and the continuity and V–ellipticity of the sesquilinear form $a: V \times V \to \mathbb{C}$ we obtain the inequalities, for $v_h \in V_h$,

$$\rho \, \|u - u_h; H^1(\Omega)\|^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \qquad (2.2.20)$$

$$\leq \|a\| \, \|u - u_h; H^1(\Omega)\| \, \|u - v_h; H^1(\Omega)\| \qquad (2.2.21)$$

$$\Rightarrow \qquad \|u - u_h; H^1(\Omega)\| \leq C \, \|u - v_h; H^1(\Omega)\| \qquad (2.2.22)$$

where $C = \|a\|/\rho$. ■

The importance of Céa's Lemma is now clear: If $u \in H^{1+\lambda}(\Omega)$, $0 < \lambda \leq 1$ then Theorem 2.1 and inequality (2.2.18) imply the upper bound

$$\|u - u_h; H^1(\Omega)\| \leq C \, h^\lambda \, \|u; H^{1+\lambda}(\Omega)\|, \quad h > 0 \qquad (2.2.23)$$

where $C > 0$ depends on θ, $\|a\|$, ρ, cf. Theorem 2.1. We point out that for problems of linear elasticity the above results are valid if one replaces u, u_h, $H^1(\Omega)$ with, respectively, $\underline{u}$, $\underline{u}_h$, $(H^1(\Omega))^2$.

It is sometimes necessary to obtain upper bounds for the error in the $\mathcal{L}_2(\Omega)$ norm topology. We demonstrate how one can use the approach of Aubin–Nitsche to determine a bound of this type from results which are already available. Thus, let $u \in V$ be the weak solution and $u_h \in V_h$ its Galerkin approximation and, for $f \in \mathcal{L}_2(\Omega)$, define $Af \in V$ as the unique solution of the weak problem, cf. Lax–Milgram Lemma,

$$a(v, Af) = (f, v; \mathcal{L}_2(\Omega)), \quad v \in V \qquad (2.2.24)$$

However, noting that $u - u_h \in V$ we let $v = u - u_h$ in (2.2.24) and obtain the identity

$$a(u - u_h, Af) = (f, u - u_h; \mathcal{L}_2(\Omega)) \qquad (2.2.25)$$

The orthogonality relation (2.2.19) and identity (2.2.25) then imply

$$(f, u - u_h; \mathcal{L}_2(\Omega)) = a(u - u_h, Af - v_h), \quad v_h \in V_h \qquad (2.2.26)$$

and we use the continuity of a to deduce the inequality

$$|(f, u - u_h; \mathcal{L}_2(\Omega))| \leq \|a\| \, \|u - u_h; H^1(\Omega)\| \, \inf\{ \|Af - v_h; H^1(\Omega)\| \; : \; v_h \in V_h\} \qquad (2.2.27)$$

Indeed, (2.2.27) and the identity

$$\|u - u_h; \mathcal{L}_2(\Omega)\| = \sup\left\{ |(f, u - u_h)|/\|f; \mathcal{L}_2(\Omega)\| \; : \; f \in \mathcal{L}_2(\Omega)\right\} \qquad (2.2.28)$$

then imply the inequality

$$\|u - u_h; \mathcal{L}_2(\Omega)\| \leq \|a\| \, \|u - u_h; H^1(\Omega)\|$$
$$\sup\left\{ \inf\{ \|Af - v_h; H^1(\Omega)\| \; : \; v_h \in V_h\}/\|f; \mathcal{L}_2(\Omega)\| \; : \; f \in \mathcal{L}_2(\Omega)\right\} \qquad (2.2.29)$$

However, if $A \in \mathcal{BL}(\mathcal{L}_2(\Omega)\,;\, H^{1+\lambda}(\Omega)\,)$, $0 < \lambda \leq 1$ then Theorems 2.1 and 2.2 imply, for $f \in \mathcal{L}_2(\Omega)$,

$$\inf\{\, \|Af - v_h; H^1(\Omega)\| \;:\; v_h \in V_h\}/\|f; \mathcal{L}_2(\Omega)\| \leq C(\theta)\, h^\lambda \frac{\|Af; H^{1+\lambda}(\Omega)\|}{\|f; \mathcal{L}_2(\Omega)\|} \leq C(\theta)\, h^\lambda \|A\| \tag{2.2.30}$$

where $C(\theta) > 0$ and θ is the minimum interior angle of any triangle $\tau \in \mathcal{T}_h(\Omega)$. It now follows from the error bound (2.2.23) and inequality (2.2.30) that there exists a constant $C > 0$ which is independent of u, h, u_h such that

$$\|u - u_h; \mathcal{L}_2(\Omega)\| \leq C\, h^{2\lambda}\, \|u; H^{1+\lambda}(\Omega)\| \tag{2.2.31}$$

The sequence of steps leading to the upper bound (2.2.31) are to due to Aubin and Nitsche, cf. CIARLET (1978), and require that the linear operator $A\colon \mathcal{L}_2(\Omega) \to H^{1+\lambda}(\Omega)$ be bounded. For problems with smooth boundaries and coefficients it is known that $A \in \mathcal{BL}(\mathcal{L}_2(\Omega)\,;\, H^2(\Omega)\,)$, however, for general abstract problems of lower regularity this remains an open question. We will assume that A is bounded for the problems which we consider. Furthermore, we point out that the above steps can be generalized to include problems of linear elasticity in the same way that we modified the steps of the proof of Céa's result for problems of this kind.

3 Homogenization of One Dimensional Elliptic Boundary Value Problems

3.0. Introduction.

The general effects of rough coefficients in elliptic problems and systems, particularly the difficulties they cause, have been discussed in chapter 2 and, as has been stated there, we seek to produce robust numerical schemes which are effective for solving multi–dimensional problems, ultimately of linear elasticity, where material properties change repeatedly and rapidly due to the presence of composite materials. As a first step towards this end we limit our attention in this chapter to rough scalar problems with a single function u as the solution. Moreover, for reasons given earlier we also limit consideration to problems in one space dimension.

A feature of problems of this type is that the coefficients and the solutions depend on a problem defined parameter, $\varepsilon > 0$, which is, generally, significantly smaller than the diameter of the domain of the problem, Ω. Indeed, we consider the particular circumstance in which the coefficients are periodic with the period defined by the parameter ε and introduce an asymptotic approach which is motivated by a concept called *homogenization*. Thus, if the abstract problem: Find $u^\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega a^\varepsilon(x)\, Du^\varepsilon(x)\, Dv(x)\, dx = \int_\Omega f(x) v(x)\, dx, \quad v \in H_0^1(\Omega) \tag{R}$$

is impractical for numerical approximation and if there is a homogenization principle, i.e., in some sense, $a^\varepsilon \to a_0$, $u^\varepsilon \to u_0$ $(\varepsilon \to 0)$ (cf. Section 3.0.1) where $u_0 \in H_0^1(\Omega)$ satisfies the *Homogenized* equation

$$\int_\Omega a_0(x)\, Du_0(x)\, Dv(x)\, dx = \int_\Omega f(x) v(x)\, dx, \quad v \in H_0^1(\Omega), \tag{H}$$

then one should employ (H) as a basis for the approximation of u^ε rather than attempting to approximate the solution of (R) directly. This assumes, of course, that the solution, u_0, of the homogenized problem (H) can be approximated more efficiently and accurately than the solution, u^ε, of (R). This is often the case however, because the homogenized coefficient, a_0, is constant and the solution u_0 generally has a higher level of regularity than u^ε.

The difficulties with rough coefficients are reduced by studying model one dimensional prototype differential equations because, in this case, the computations can be performed analytically for problems exhibiting a variety of levels of regularity. We introduce our asymptotic approach in Section 3.3 and in Sections 3.4, 3.6–3.8 we determine how problem regularity affects this approach through a number of examples in which analytical and computational results and graphical illustrations are provided.

3.0.1. Motivation for the Asymptotic approach.

The asymptotic properties of the mathematical model, as $\varepsilon \to 0$, where ε is the period of the medium, are fundamental to the concept of homogenization. Thus, let us first consider the following abstract problem, stated in the classical form, over the domain $\Omega \stackrel{\text{def}}{=} (0,1)$ with mixed boundary conditions: Find $u^\varepsilon \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$-\frac{\partial}{\partial x}\left[a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\right] + b^\varepsilon(x)\,u^\varepsilon(x) = f(x), \quad x \in \Omega \tag{3.0.1}$$

$$a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\bigg|_{x=0} = u^\varepsilon(1) = 0 \tag{3.0.2}$$

where $f \in C^0(\Omega)$, $a^\varepsilon \in C^1(\overline{\Omega})$, $b^\varepsilon \in C^0(\Omega)$ and, for $x \in \Omega$,

$$0 < \alpha \le a^\varepsilon(x) \le \beta < \infty \tag{3.0.3}$$

$$0 \le b^\varepsilon(x) \le \beta < \infty \tag{3.0.4}$$

By rewriting relations (3.0.1), (3.0.2) in the weak form, cf. Chapter 2, and assuming that relations (3.0.3), (3.0.4) hold for almost all $x \in \Omega$, we generalize this problem to include functions $f \in \mathcal{L}_2(\Omega)$, a^ε, $b^\varepsilon \in \mathcal{L}_\infty(\Omega)$ as follows: multiply (3.0.1) by a test function $v \in H^{1;0}(\Omega) \stackrel{\text{def}}{=} \{v \in H^1(\Omega) \mid v(1) = 0\}$ and then integrate the resulting equation by parts to obtain

$$\int_\Omega a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\,\frac{\partial v}{\partial x}(x)\,dx + \int_\Omega b^\varepsilon(x)\,u^\varepsilon(x)\,v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx, \quad v \in H^{1;0}(\Omega) \tag{3.0.5}$$

where, as a consequence of the boundary conditions (3.0.2) and the definition of $H^{1;0}(\Omega)$, we have observed that the following boundary term vanishes:

$$-\,a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\,v(x)\bigg|_{x=0}^{x=1} \tag{3.0.6}$$

The weak formulation of problem (3.0.1), (3.0.2) is then: Find $u^\varepsilon \in H^{1;0}(\Omega)$ such that (3.0.5) holds for all $v \in H^{1;0}(\Omega)$. Because this problem satisfies all the conditions of the Lax–Milgram lemma it is evident that a unique solution $u^\varepsilon \in H^{1;0}(\Omega)$ exists.

If, conversely, we begin with the weak formulation (3.0.5) and $a^\varepsilon \in C^1(\overline{\Omega})$, $b^\varepsilon, f \in C^0(\Omega)$, and $u^\varepsilon \in H^{1;0}(\Omega) \cap C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies (3.0.5) then integrating relation (3.0.5) by parts we deduce

$$\int_\Omega \left\{ -\frac{\partial}{\partial x}\left[a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\right] + b^\varepsilon(x)\,u^\varepsilon(x) - f(x) \right\} v(x)\,dx + a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\,v(x)\Big|_{x=0}^{x=1} = 0 \quad (3.0.7)$$

Now consider the test functions $v_n \in C^\infty(\overline{\Omega}) \cap H^{1;0}(\Omega)$, $n \in \mathbb{N}$ defined as follows

$$v_n(x) \stackrel{\text{def}}{=} \begin{cases} e^{-(1/n-x)^{-1}}/e^{-n}, & \text{if } 0 \le x < 1/n \\ 0, & \text{if } 1/n \le x \le 1 \end{cases} \quad (3.0.8)$$

Clearly, $v_n(0) = 1$, $v_n(1) = 0$ for all $n \in \mathbb{N}$, $\|v_n; \mathcal{L}_2(\Omega)\| \to 0$ $(n \to \infty)$, and

$$\left| \int_\Omega \left\{ -\frac{\partial}{\partial x}\left[a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\right] + b^\varepsilon(x)\,u^\varepsilon(x) - f(x) \right\} v_n(x)\,dx \right| \to 0 \quad (n \to \infty) \quad (3.0.9)$$

$$a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\,v_n(x)\Big|_{x=0}^{x=1} = a^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\Big|_{x=0} \quad (3.0.10)$$

Thus, relations (3.0.10) and (3.0.7) imply that u^ε satisfies the boundary conditions (3.0.2). It then follows from (3.0.7) that u^ε also satisfies the differential equation (3.0.1). Thus, the weak formulation (3.0.5) and the abstract formulation (3.0.1), (3.0.2) are, therefore, equivalent with regard to classical solutions, i.e., if there is a unique solution $u^\varepsilon \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of one formulation of the problem then it also uniquely satisfies the other.

It is well known, cf. BABUŠKA (1974i), that for $f \in \mathcal{L}_2(\Omega)$ the solution $u^\varepsilon \in H^{1;0}(\Omega) \subset H^1(\Omega)$ can be bounded in the $H^1(\Omega)$ norm topology, e.g.,

$$\|u^\varepsilon; H^1(\Omega)\| \le C(\alpha,\beta)\,\|f; \mathcal{L}_2(\Omega)\|, \quad 0 < \varepsilon \le 1. \quad (3.0.11)$$

where $C(\alpha,\beta) > 0$ is independent of f and ε. It follows, cf. BABUŠKA (1974i), that there exists a monotonically decreasing sequence $\{\varepsilon_n\}_{n\ge1} \subset (0,1]$ and an element $u_0 \in H^{1;0}(\Omega)$, called the *homogenized* solution, such that, for $0 < \rho \le 1$ and $f \in \mathcal{BL}(H^1(\Omega); \mathbb{R})$,

$$\|u^{\varepsilon_n} - u_0; H^{1-\rho}(\Omega)\| \to 0 \quad (n \to \infty) \quad (3.0.12)$$

$$|\langle f, u^{\varepsilon_n}\rangle - \langle f, u_0\rangle| \to 0 \quad (n \to \infty) \quad (3.0.13)$$

For a homogenization principle to exist one asks – Does u_0 satisfy a boundary value problem of the same type as u^ε ? Indeed, there are a number of theorems which establish precisely this property, i.e., u_0 is the solution of an elliptic problem, analogous to (3.0.1), which is independent of ε. The following is typical of such theorems, see, for example, MURAT & TARTAR (1994), BABUŠKA (1974i), and ALLAIRE (1992).

Theorem 3.0.1. *Let a^ε, b^ε satisfy conditions (3.0.3), (3.0.4). Further, let $1/a^\varepsilon \rightharpoonup 1/a_0$, $b^\varepsilon \rightharpoonup b_0$ $(\varepsilon \to 0)$ in $\mathcal{L}_2(\Omega)$. Then u^ε converges to u_0 as in (3.0.12), (3.0.13) where $u_0 \in H^{1;0}(\Omega)$ satisfies*

$$\int_\Omega a_0\,\frac{\partial u_0}{\partial x}(x)\,\frac{\partial v}{\partial x}(x)\,dx + \int_\Omega b_0\,u_0(x)\,v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx, \quad v \in H^{1;0}(\Omega)$$

In addition $a^\varepsilon\,\partial u^\varepsilon/\partial x \to a_0\,\partial u_0/\partial x$ $(\varepsilon \to 0)$ *in* $\mathcal{L}_2(\Omega)$. ■

The properties of u^ε, described above, motivate the consideration of *asymptotic expansions* as a form of representation for u^ε. Although, the homogenization concept defined earlier is primarily concerned with the utility of the leading term, u_0, in such representations, it will be seen that the inclusion of additional terms can provide more accurate approximations of u^ε in the $\mathcal{L}_2(\Omega)$ and $H^1(\Omega)$ norm topologies. Thus, the homogenization approach is subsequently assumed to encompass also the higher order asymptotics.

We take the following cell boundary value problem as our prototype for illustrating the practical/computational difficulties caused by the irregular data. The coefficients are chosen to model the presence of heterogeneous materials – this introduces irregularities (indeed, in higher dimensions, singularities) – and the parameters $\varepsilon = 1/r$ (cell size), n, a_1, a_2, b_1, b_2 control the variation of material properties within the medium.

$$-\frac{\partial}{\partial x}\left[a_n^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\right] + b_n^\varepsilon(x)\,u^\varepsilon(x) = f(x), \quad x \in \bigcup_{i=0}^{2nr-1}(x_i,\, x_{i+1}) \tag{3.0.14}$$

$$[u^\varepsilon(x)]_{x_i} = \left[\frac{\partial}{\partial x}\Big(a_n^\varepsilon(x)u^\varepsilon(x)\Big)\right]_{x_i} = 0, \quad 1 \le i \le 2nr-1 \tag{3.0.15}$$

$$\frac{\partial}{\partial x}\Big(a_n^\varepsilon(x)u^\varepsilon(x)\Big)\Big|_{x=0} = u^\varepsilon(1) = 0 \tag{3.0.16}$$

where $a_n^\varepsilon(x) \stackrel{\text{def}}{=} a^n(x/\varepsilon)\,,\ b_n^\varepsilon(x) \stackrel{\text{def}}{=} b^n(x/\varepsilon)$,

$$x_i = \xi_m^l \stackrel{\text{def}}{=} (l + m/2n)\,\varepsilon, \quad i = l+m, \quad 0 \le l \le r,\ 0 \le m \le 2n$$

$$[v(x)]_{x_i} \stackrel{\text{def}}{=} \lim_{\delta\to 0+} v(x_i+\delta) - \lim_{\delta\to 0+} v(x_i-\delta), \quad 1 \le i \le 2nr-1$$

and the functions a^n, b^n are 1–periodic and are defined below, $0 \le i \le n-1$,

$$a^n(x) \stackrel{\text{def}}{=} \begin{cases} a_1, & \frac{2i}{2n} \le x < \frac{2i+1}{2n} \\ a_2, & \frac{2i+1}{2n} \le x < \frac{2i+2}{2n} \end{cases} \qquad b^n(x) \stackrel{\text{def}}{=} \begin{cases} b_1, & \frac{2i}{2n} \le x < \frac{2i+1}{2n} \\ b_2, & \frac{2i+1}{2n} \le x < \frac{2i+2}{2n} \end{cases} \tag{3.0.17}$$

Evidently, $r \in \mathbb{N}$ denotes the number of periodic cells in $\Omega = (0,1)$ while $2n \in \mathbb{N}$ is the number of transition points generated by a typical cell, see Figures 3.0.1a,b. Increasing the parameters r or n will cause the functions a_n^ε, b_n^ε to oscillate more rapidly while varying a_1, a_2, b_1, b_2 alters the magnitude of the discontinuities.

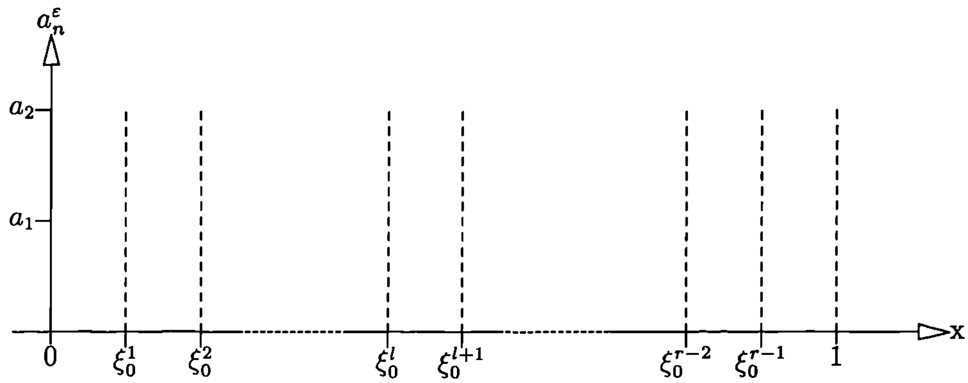


Figure 3.0.1a: Overall problem domain, Ω: $\xi_0^l = l\varepsilon$, $0 \le l \le r$.

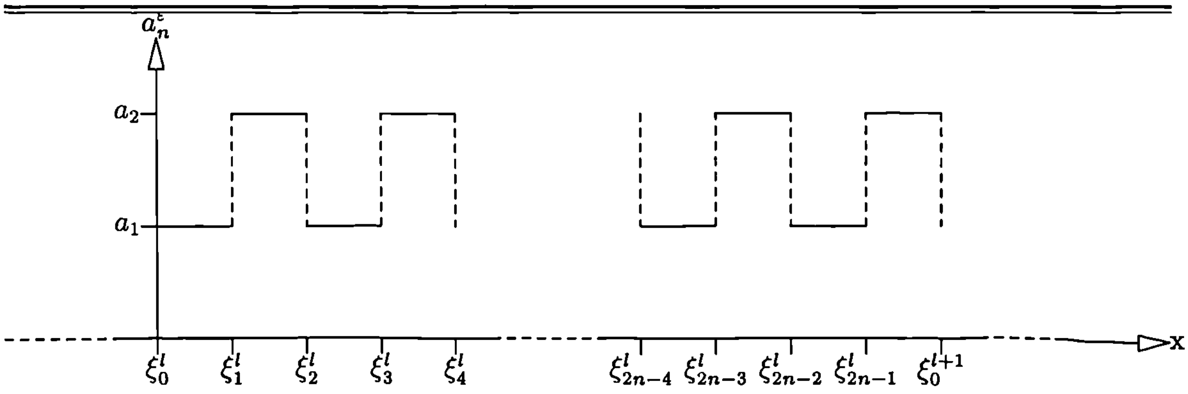


Figure 3.0.1b: Graph of a_n^ε : $\xi_m^l = (l + m/2n)\,\varepsilon,\ 0 \le m \le 2n,\ 0 \le l \le r$.

It is assumed that constants $\alpha_i \in \mathbb{R}$ exist, which are independent of ε, such that, for $i = 1, 2$

$$\begin{aligned} &0 < \alpha_1 \le a_i \le \alpha_2 < \infty \\ &0 \le b_i \le \alpha_2 < \infty \end{aligned} \tag{3.0.18}$$

The weak formulation of the boundary value problem (3.0.14)–(3.0.16), obtained by multiplying (3.0.14) by $v \in H^{1;0}(\Omega)$, integrating by parts over Ω, applying the boundary conditions (3.0.16), and observing the transition conditions (3.0.15) is: Find $u^\varepsilon \in H^{1;0}(\Omega) \stackrel{\text{def}}{=} \{v \in H^1(\Omega) \mid v(1) = 0\}$ such that

$$\int_\Omega a_n^\varepsilon(x)\,\frac{\partial u^\varepsilon}{\partial x}(x)\,\frac{\partial v}{\partial x}(x)\,dx + \int_\Omega b_n^\varepsilon(x)\,u^\varepsilon(x)\,v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx, \quad v \in H^{1;0}(\Omega) \tag{3.0.19}$$

If one employs, as described in chapter 2, an isoparametric piecewise linear finite element approximation, $S^h(\Omega) \subset H^{1;0}(\Omega)$, on a uniform triangulation with each finite element corresponding to a single periodic cell, i.e., $h = \varepsilon$, then, with such an arrangement, it is known that one obtains the algebraic system of equations, cf. BABUŠKA (1974i),

$$A_h\,\underline{u}_h^\varepsilon = \underline{F}_h \tag{3.0.20}$$

where $A_h = A + P_n \in \mathbb{R}^{r,r}$ is the stiffness matrix, $\underline{F}_h \in \mathbb{R}^r$ is the load vector, and $A \in \mathbb{R}^{r,r}$ is obtained from the identical finite element discretization of the weak problem: Find $\overline{u} \in H^{1;0}(\Omega)$ such that

$$\int_\Omega \overline{a}\,\frac{\partial \overline{u}}{\partial x}(x)\,\frac{\partial v}{\partial x}(x)\,dx + \int_\Omega \overline{b}\,\overline{u}(x)\,v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx, \quad v \in H^{1;0}(\Omega) \tag{3.0.21}$$

where $\overline{a} = (a_1 + a_2)/2$, $\overline{b} = (b_1 + b_2)/2$, and the matrix $P_n \in \mathbb{R}^{r,r}$ has the property $(P_n)_{l,m} \to 0$ $(n \to \infty)$ for $1 \le l, m \le r$. We denote the finite element approximation of u^ε by $u_h^\varepsilon = \sum_{l=0}^r (\underline{u}_h^\varepsilon)_l\,\psi_l$ and, similarly, $\overline{u}_h = \sum_{l=0}^r (\underline{\overline{u}}_h)_l\,\psi_l$, $S^h(\Omega) = \text{span}\{\psi_l\}_{l=0}^r$ denotes the finite element approximation of $\overline{u} \in H^{1;0}(\Omega)$. It follows from the identities $(I + A^{-1}P_n)\,\underline{u}_h^\varepsilon = A^{-1}\,\underline{F}_h = \underline{\overline{u}}_h$ and the upper bounds $\|A^{-1}\|_2$, $\|(A + P_n)^{-1}\|_2 \le C_1(r)$, $\|\underline{F}_h; \ell_2(r)\| \le C_2\,r$ as

$n \to \infty$ that

$$
\begin{aligned}
\|\overline{\underline{u}}_h - \underline{u}_h^\varepsilon; \ell_2(r)\| &= \|A^{-1}P_n\underline{u}_h^\varepsilon; \ell_2(r)\| \\
&\le \|A^{-1}\|_2 \, \|P_n\|_2 \, \|\underline{u}_h^\varepsilon; \ell_2(r)\| \\
&\le \|A^{-1}\|_2 \, \|P_n\|_2 \, \|(A+P_n)^{-1}\|_2 \, \|\underline{F}_h; \ell_2(r)\| \\
&\le C(r, f, \Omega) \, \|P_n\|_2 \\
&\to 0 \quad (n \to \infty,\ \varepsilon = 1/r \text{ fixed})
\end{aligned}
\tag{3.0.22}
$$

In order to obtain (3.0.22) we have observed that the spectral norm $\|A^{-1}\|_2$, which is independent of n, remains bounded as $n \to \infty$. The continuous dependence of the spectrum, $\sigma(A + P_n)$, on the coefficients, $(P_n)_{l,m}$, cf. HORN & JOHNSON (1985), leads to the observation that $\|(A + P_n)^{-1}\|_2 = \lambda_{min}(A + P_n)/\lambda_{max}(A + P_n) \to \|A^{-1}\|_2 = \lambda_{min}(A)/\lambda_{max}(A)$ as $n \to \infty$. Thus, we can choose a common upper bound, $C_1(r)$, for the spectral norms $\|A^{-1}\|_2$, $\|(A + P_n)^{-1}\|_2$. The upper bound for $\|\underline{F}_h; \ell_2(r)\|$ follows immediately from the Cauchy–Schwarz inequality, e.g.,

$$
\begin{aligned}
\|\underline{F}_h; \ell_2(r)\| &= \Big(\sum_{l=1}^{r} |f_l|^2\Big)^{1/2} = \Big(\sum_{l=1}^{r} \Big|\int_\Omega f(x)\,\psi_l(x)\,dx\Big|^2\Big)^{1/2} \\
&\le \sum_{l=1}^{r} \|f; \mathcal{L}_2(\Omega)\| \, \|\psi_l; \mathcal{L}_2(\Omega)\| \\
&\le C_3(f, \Omega)\, r
\end{aligned}
$$

Consequently, from the continuity of the norm function $\|\bullet; \mathcal{L}_2(\Omega)\|$, it is clear that

$$
\|\overline{u} - u_h^\varepsilon; \mathcal{L}_2(\Omega)\| \to \|\overline{u} - \overline{u}_h; \mathcal{L}_2(\Omega)\| \quad (n \to \infty,\ r \text{ fixed}) \tag{3.0.23}
$$

Thus, the finite element approximations of u^ε, obtained from the subspaces $S^h(\Omega) \subset H^{1;0}(\Omega)$, which *do not* model the fine scale variation of the coefficients, converge, as $n \to \infty$, to the finite element approximation, $\overline{u}_h$, of the weak solution, $\overline{u}$, of problem (3.0.21). However, for ε, or equivalently, r, fixed and n increasing it is known that, in $\mathcal{L}_2(\Omega)$,

$$
\frac{1}{a_n^\varepsilon} \rightharpoonup \frac{1}{a_0} = \frac{1}{2}\Big[\frac{1}{a_1} + \frac{1}{a_2}\Big] \ne \overline{a}, \qquad b_n^\varepsilon \rightharpoonup \frac{1}{2}(b_1 + b_2) = \overline{b} \tag{3.0.24}
$$

$$
\|u^\varepsilon - u_0; \mathcal{L}_2(\Omega)\| \to 0 \quad (n \to \infty,\ r \text{ fixed}) \tag{3.0.25}
$$

where u_0 is then the solution of the weak problem: Find $u_0 \in H^{1;0}(\Omega)$ such that

$$
\int_\Omega a_0 \frac{\partial u_0}{\partial x}(x)\,\frac{\partial v}{\partial x}(x)\,dx + \int_\Omega \overline{b}\,u_0(x)\,v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx, \quad v \in H^{1;0}(\Omega) \tag{3.0.26}
$$

So, introducing the finite element approach has in effect, cf. theorem 3.0.1, lead to a numerical approximation of the problem (3.0.21) rather than problem (3.0.19) when n is large. However, if $a_1 \ne a_2$ then $a_0 \ne \overline{a}$ and, from the identities,

$$
\frac{\overline{a}}{a_0} = \frac{1}{2}\,(a_1 + a_2)(a_1^{-1} + a_2^{-1}) = \frac{1}{4}\frac{(\lambda+1)^2}{\lambda}, \quad \lambda \overset{\text{def}}{=} a_1/a_2
$$

$$
|\overline{a} - a_0| = \left|\frac{a_1 + a_2}{2} - \frac{2}{a_1^{-1} + a_2^{-1}}\right| = \frac{(a_2 - a_1)^2}{2(a_1 + a_2)} \ge \frac{1}{2}\,|a_2 - a_1|\,\frac{|\lambda - 1|}{\lambda + 1}
$$

it is clear that the difference, $|\overline{a} - a_0|$, increases proportionately with the magnitude of the discontinuities, $|a_2 - a_1|$. Furthermore, the quotient $\overline{a}/a_0$ grows unboundedly as $\lambda = a_1/a_2 \to 0, \infty$. Thus, if the jumps $|a_2 - a_1|$ are large or the quotient $\lambda = a_1/a_2 \gg 1, \ll 1$, then the problems (3.0.21) and (3.0.19) are significantly different and, consequently, so are the respective weak solutions $\overline{u}$, u_0. Therefore we expect the approximation u_h^ε of u^ε to be extremely poor when n is large. Indeed, in BABUŠKA (1974i) it is shown that the error, $\|u^\varepsilon - u_h^\varepsilon; \mathcal{L}_2(\Omega)\|$, will exceed 70% of $\|u^\varepsilon; \mathcal{L}_2(\Omega)\|$ when $\lambda = a_1/a_2 \geq 10$. The rapid variations of the coefficients a^ε and b^ε of the problem cannot be practicably accounted for by simply employing successively higher dimensional subspaces of $H^{1;0}(\Omega)$, such a requirement would rapidly exhaust the resources of most modern computers.

The difficulties illustrated by the simple analysis above demonstrate the need to consider an alternative approach which is practical and respects the large, rapid changes in the coefficients of the problem. In section 3.2 we will consider the application of asymptotic techniques which exploit the rapid variations of the periodic data. The approximation properties of such methods are well understood for regular problems. However, their behaviour is an open question in the context of problems with data possessing low regularity. In the following sections, homogenization techniques are applied to problems with low regularity data and the results are explained.

3.1. The Model One Dimensional Problem.

Let $u^\varepsilon \in H_0^1(\Omega)$ be a weak solution of the classical problem

$$\begin{gathered} -\frac{\partial}{\partial x}\left[a(x/\varepsilon)\frac{\partial u^\varepsilon}{\partial x}(x)\right] = f(x), \quad x \in \Omega = (0,\, 1) \\ u^\varepsilon(0) = u^\varepsilon(1) = 0 \end{gathered} \tag{3.1.1}$$

where $a \in \mathcal{L}_\infty(\Omega)$ is a 1–periodic function which is continuous at the points $n \in \mathcal{Z}$ and satisfies $0 < \alpha_1 \leq a(y) \leq \alpha_2 < \infty$, for $0 < y \leq 1$, and $f \in \mathcal{L}_2(\Omega)$ and $\varepsilon > 0$ is a parameter which corresponds to the period of the medium being modelled.

Application of the Lax–Milgram lemma to the weak form of (3.1.1), interpreted in a Sobolev space setting, establishes the existence of a unique solution $u^\varepsilon \in H_0^1(\Omega)$ which, furthermore, satisfies the regularity estimate

$$\|u^\varepsilon; H^1(\Omega)\| \leq C\,\|f; \mathcal{L}_2(\Omega)\| \tag{3.1.2}$$

where $C = C(f) > 0$ is independent of u^ε. However, this problem is also obtained as the restriction to Ω of the related problem

$$-\frac{\partial}{\partial x}\left[a(x/\varepsilon)\frac{\partial u^\varepsilon}{\partial x}(x)\right] = f_C(x), \quad -\infty < x < \infty \tag{3.1.3}$$

where f_C is then the periodic extension to $\mathbb{R}$ of the function

$$f_A(x) \stackrel{\text{def}}{=} \begin{cases} -f(-x), & \text{if } -1 \leq x \leq 0 \\ f(x), & \text{if } 0 < x \leq 1 \end{cases} \tag{3.1.4}$$

Thus, $f_{\mathcal{C}}$ can be represented with a Fourier series expansion

$$f_{\mathcal{C}}(x) \stackrel{\text{def}}{=} \sum_{n \in \mathcal{Z}\backslash\{0\}} a_n \, e^{in\pi x}, \quad x \in \mathbb{R} \tag{3.1.5}$$

where

$$a_n \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathcal{C}} f_{\mathcal{A}}(x) \, e^{in\pi x} \, dx, \quad \mathcal{C} \stackrel{\text{def}}{=} (-1, 1) \tag{3.1.6}$$

Thus, following the analysis of BABUŠKA & MORGAN (1991), one can write the solution of (3.1.1) in the form

$$u^\varepsilon(x) = \sum_{n \in \mathcal{Z}\backslash\{0\}} a_n \, e^{it_n x} \, \phi(x/\varepsilon, \varepsilon, t_n) \tag{3.1.7}$$

where $t_n = n\pi$ and a_n, $n \in \mathcal{Z}\backslash\{0\}$, are the Fourier coefficients of $f_{\mathcal{C}}$ and $x \mapsto \phi(x, \varepsilon, t)$ is a complex-valued, 1-periodic function that satisfies the periodic boundary value problem, for $\varepsilon > 0$, $|t| > 0$,

$$\begin{aligned} -\frac{\partial}{\partial x}\left[a(x)\frac{\partial}{\partial x}\left(e^{it\varepsilon x}\phi(x,\varepsilon,t)\right)\right] &= \varepsilon^2 e^{it\varepsilon x}, \quad 0 < x < 1 \\ \phi(0,\varepsilon,t) &= \phi(1,\varepsilon,t) \\ a(x)\,\frac{\partial \phi}{\partial x}(x,\varepsilon,t)\bigg|_{x=0} &= a(x)\,\frac{\partial \phi}{\partial x}(x,\varepsilon,t)\bigg|_{x=1} \end{aligned} \tag{3.1.8}$$

The differential equation (3.1.8) is evidently defined within the standard periodic cell $\mathcal{P} \stackrel{\text{def}}{=} (0,\ 1)$ and, therefore, if one determines ϕ, either analytically or approximately, the corresponding expression for u^ε is provided by (3.1.7). Thus, instead of analysing the global problem (3.1.1) one can, alternatively, examine a related problem within the periodic cell, $\mathcal{P}$. However, before considering techniques of approximation, the properties of the weak formulation of problem (3.1.8) and the respective weak solution, ϕ, will be studied.

3.1.1. Properties of the Cell Problem.

The weak formulation of the cell problem (3.1.8) is derived by multiplying equation (3.1.8) by the function $e^{-it\varepsilon x}\,\overline{v(x)}$, $v \in H^1_{per}(\mathcal{P}) \stackrel{\text{def}}{=} \{v \in H^1(\Omega) \mid v(0) = v(1)\}$ and then integrating by parts to obtain the problem : Find $\phi(\bullet, \varepsilon, t) \in H^1_{per}(\mathcal{P})$ such that

$$\int_{\mathcal{P}} a(x)\,\frac{\partial}{\partial x}\left(e^{it\varepsilon x}\,\phi(x,\varepsilon,t)\right)\frac{\partial}{\partial x}\left(e^{-it\varepsilon x}\,\overline{v(x)}\right) dx = \varepsilon^2 \int_{\mathcal{P}} \overline{v(x)}\, dx\,, \quad v \in H^1_{per}(\mathcal{P}) \tag{3.1.9}$$

where it has been observed that the boundary terms

$$\left(it\varepsilon\, a(x)\,\phi(x,\varepsilon,t) + a(x)\,\frac{\partial\phi}{\partial x}(x,\varepsilon,t)\right)\overline{v(x)}\,\bigg|_{x=0}^{x=1} \tag{3.1.10}$$

vanish as a consequence of the continuity hypothesis for a and the boundary condition provided in (3.1.8) for $\phi(\bullet, \varepsilon, t)$. Observe that $\overline{v(x)} = \Re[v(x)] - i\,\Im[v(x)]$ is the complex conjugate of $v(x) \in \mathbb{C}$. Clearly, the sesquilinear form for this problem is defined as follows, for

$u,\ v \in H^1_{per}(\mathcal{P})$,

$$\begin{aligned}
\Phi(\varepsilon,t)[u,\ v] &\stackrel{\text{def}}{=} \int_{\mathcal{P}} a(x)\,\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\,u(x)\Big)\,\overline{\frac{\partial}{\partial x}\Big(e^{-it\varepsilon x}\,v(x)\Big)}\,dx \\
&= \int_{\mathcal{P}} a(x)\,\frac{\partial u}{\partial x}(x)\,\overline{\frac{\partial v}{\partial x}(x)}\,dx + \varepsilon\, it \int_{\mathcal{P}} a(x)\,\Big(u(x)\,\overline{\frac{\partial v}{\partial x}(x)} - \frac{\partial u}{\partial x}(x)\,\overline{v(x)}\Big)\,dx \\
&\quad + \varepsilon^2\, t^2 \int_{\mathcal{P}} a(x)\,u(x)\,\overline{v(x)}\,dx \\
&\stackrel{\text{def}}{=} \Phi_0[u,v] + \varepsilon\,\Phi_1(t)[u,v] + \varepsilon^2\,\Phi_2(t)[u,v]
\end{aligned}$$

The sesquilinear form is clearly Hermitian symmetric, that is, $\Phi(\varepsilon,t)[u,\ v] = \overline{\Phi(\varepsilon,t)[v,\ u]}$, $u,\ v \in H^1_{per}(\mathcal{P})$. Furthermore, that $\Phi(\varepsilon,t)$ is continuous over $H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P})$ follows from the inequalities

$$|\Phi_0[u,v]| \leq \alpha_2\,|(Du,Dv;\mathcal{L}_2(\mathcal{P}))| \leq \alpha_2\,\|Du;\mathcal{L}_2(\mathcal{P})\|\,\|Dv;\mathcal{L}_2(\mathcal{P})\| \tag{3.1.11}$$

$$\leq \alpha_2\,\|u;H^1(\mathcal{P})\|\,\|v;H^1(\mathcal{P})\| \tag{3.1.12}$$

$$|\Phi_1(t)[u,v]| \leq \alpha_2\,|t|\Big(|(u,Dv;\mathcal{L}_2(\mathcal{P}))| + |(Du,v;\mathcal{L}_2(\mathcal{P}))|\Big)$$

$$\leq \alpha_2\,|t|\Big(\|u;\mathcal{L}_2(\mathcal{P})\|\,\|Dv;\mathcal{L}_2(\mathcal{P})\| + \|Du;\mathcal{L}_2(\mathcal{P})\|\,\|v;\mathcal{L}_2(\mathcal{P})\|\Big) \tag{3.1.13}$$

$$\leq 2\alpha_2\,|t|\,\|u;H^1(\mathcal{P})\|\,\|v;H^1(\mathcal{P})\| \tag{3.1.14}$$

$$|\Phi_2(t)[u,v]| \leq \alpha_2\,t^2|(u,v;\mathcal{L}_2(\mathcal{P}))| \tag{3.1.15}$$

$$\leq \alpha_2\,t^2\|u;\mathcal{L}_2(\mathcal{P})\|\,\|v;\mathcal{L}_2(\mathcal{P})\|$$

$$\leq \alpha_2\,t^2\|u;H^1(\mathcal{P})\|\,\|v;H^1(\mathcal{P})\| \tag{3.1.16}$$

$$\Rightarrow \qquad |\Phi(\varepsilon,t)[u,\ v]| \leq |\Phi_0[u,v]| + \varepsilon\,|\Phi_1(t)[u,v]| + \varepsilon^2\,|\Phi_2(t)[u,v]|$$

$$\leq C(\varepsilon,\ t)\,\|u;H^1(\mathcal{P})\|\,\|v;H^1(\mathcal{P})\| \tag{3.1.17}$$

where $C(\varepsilon,\ t) = \alpha_2(1+2\varepsilon\,|t|+t^2\varepsilon^2) > 0$. Thus, the sesquilinear mappings Φ_0, $\Phi_1(t)$, $\Phi_2(t)$, and $\Phi(\varepsilon,t) \in \mathcal{BL}(H^1_{per}(\mathcal{P})\times H^1_{per}(\mathcal{P})\,;\,\mathbb{C}\,)$ with $\|\Phi(\varepsilon,t)\| \leq C(\varepsilon,\ t)$ and Φ_0 is positive semi–definite over $H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P})$, i.e., $\Phi_0[v,v] \geq 0$, $v \in H^1_{per}(\mathcal{P})$. In fact, from (3.1.11), $\Phi_0[1,v] = \Phi_0[u,1] = \Phi_0[1,1] = 0$ and, furthermore, from (3.1.13), $\Phi_1(t)[1,1] = 0$. In arriving at the following $H^1_{per}(\mathcal{P})$–Ellipticity property of $\Phi(\varepsilon,t)$ we have employed Lemma 2 of BABUŠKA & MORGAN (1991ii):

$$\begin{aligned}
|\Phi(\varepsilon,t)[v,\ v]| &= \int_{\mathcal{P}} a(x)\,|D(e^{it\varepsilon x}\,\overline{v(x)})|^2\,dx \\
&\geq \alpha_1 \int_{\mathcal{P}} |D(e^{it\varepsilon x}\,\overline{v(x)})|^2\,dx \\
&\geq C\alpha_1(1+|t|)^{-1}\,\|v;H^1_{per}(\mathcal{P})\|
\end{aligned}$$

where $C > 0$ is a constant independent from ε. Thus, the Lax–Milgram lemma proves that there exists a unique solution $\phi(\bullet,\varepsilon,t)$, $\varepsilon > 0$, $|t| > 0$ of (3.1.9) in $H^1_{per}(\mathcal{P})$. Furthermore, with $v \in H^1_{per}(\mathcal{P})$, we observe that

$$\begin{aligned}\Phi(\varepsilon,t)[\phi(\bullet,\varepsilon,t),v] &= \varepsilon^2 \int_{\mathcal{P}} \overline{v(x)}\,dx \\ &= \Phi(\varepsilon,-t)[\phi(\bullet,\varepsilon,-t),v]\end{aligned}$$

However, it follows from this relation and the definition of $\Phi(\varepsilon,t)$ that

$$\begin{aligned}\overline{\Phi(\varepsilon,-t)[\phi(\bullet,\varepsilon,-t),v]} &= \Phi(\varepsilon,t)[\overline{\phi(\bullet,\varepsilon,-t)},\overline{v}] \\ &= \varepsilon^2 \int_{\mathcal{P}} v(x)\,dx, \quad v \in H^1_{per}(\mathcal{P})\end{aligned}$$

and, therefore,

$$\Phi(\varepsilon,t)[\phi(\bullet,\varepsilon,t) - \overline{\phi(\bullet,\varepsilon,-t)},v] = 0, \quad v \in H^1_{per}(\mathcal{P})$$

Thus, with $v = \phi(\bullet,\varepsilon,t) - \overline{\phi(\bullet,\varepsilon,-t)}$ in this relation we deduce that

$$\Rightarrow \quad \overline{\phi(x,\varepsilon,-t)} = \phi(x,\varepsilon,t), \quad x \in \mathcal{P},\ \varepsilon > 0,\ |t| > 0$$

Furthermore, if it occurs that a is symmetric about the origin then, exploiting periodicity and employing a sequence of elementary transformations for the defining integral of the sesquilinear form $\Phi(\varepsilon,t)$, $\varepsilon > 0$, $|t| > 0$, we deduce the following equations, for $v \in H^1_{per}(\mathcal{P})$,

$$\begin{aligned}&\int_{\mathcal{P}} a(-x)\,\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\,\overline{\phi(-x,\varepsilon,t)}\Big)\,\frac{\partial}{\partial x}\Big(e^{-it\varepsilon x}\,v(-x)\Big)\,dx = \varepsilon^2 \int_{\mathcal{P}} \overline{v(-x)}\,dx \\ &\qquad\Rightarrow \quad \Phi(\varepsilon,t)[\overline{\psi(\bullet,\varepsilon,t)},v] = \varepsilon^2 \int_{\mathcal{P}} \overline{v(x)}\,dx\end{aligned}$$

where $\psi(x,\varepsilon,t) \stackrel{\text{def}}{=} \phi(-x,\varepsilon,t)$, $x \in \mathbb{R}$, $\varepsilon > 0$, $|t| > 0$. However, from these relations we now deduce the following conjugate symmetry properties of ϕ

$$\begin{aligned}&\Phi(\varepsilon,t)[\phi(\bullet,\varepsilon,t) - \overline{\psi(\bullet,\varepsilon,t)},v] = 0, \quad v \in H^1_{per}(\mathcal{P}) \\ \Rightarrow \quad &\phi(x,\varepsilon,t) = \overline{\phi(-x,\varepsilon,t)} \\ \text{(Periodicity)} \quad &\qquad = \overline{\phi(1-x,\varepsilon,t)}, \quad x \in \mathbb{R},\ \varepsilon > 0,\ |t| > 0\end{aligned}$$

Consequently, if a is symmetric about the origin then $\phi(\bullet,\varepsilon,t)$ is conjugate symmetric about both the origin and $x = 1/2$ for $\varepsilon > 0$, $|t| > 0$. Now consider the circumstance in which a in (3.1.9) is a piecewise C^1 function, i.e., suppose that, with $\overline{\mathcal{P}} = \cup_{l=1}^{m} \overline{\mathcal{P}}_l$, $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$, $i \neq j$, there exist functions $a_l \in C^1(\overline{\mathcal{P}}_l)$, $1 \le l \le m$ such that

$$a(x) = a_l(x), \quad x \in \mathcal{P}_l, \quad 1 \le l \le m \tag{3.1.18}$$

where $a \notin C^0(\mathcal{P})$ and $\mathcal{P}_l = (x_{l-1}, x_l)$. The weak solution, $\phi(\bullet,\varepsilon,t)$, of problem (3.1.9) is then also piecewise defined, i.e., $\phi(x,\varepsilon,t) = \phi_l(x,\varepsilon,t)$, $x \in \mathcal{P}_l$, $1 \le l \le m$ with $\phi_l(\bullet,\varepsilon,t) \in C^2(\mathcal{P}_l) \cap C^1(\overline{\mathcal{P}}_l)$ and the piecewise components ϕ_l of ϕ satisfy the following ordinary differential equations, for $1 \le l \le m-1$, $\varepsilon > 0$, $|t| > 0$,

$$-\frac{\partial}{\partial x}\left[a_l(x)\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\phi_l(x,\varepsilon,t)\Big)\right] = \varepsilon^2 e^{it\varepsilon x}, \quad x \in \mathcal{P}_l \tag{3.1.19}$$

with interface transition conditions, for $1 \le l \le m-1$,

$$\phi_l(x_l, \varepsilon, t) = \phi_{l+1}(x_l, \varepsilon, t) \tag{3.1.20}$$

$$a_l(x)\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\phi_l(x,\varepsilon,t)\Big)\Big|_{x=x_l} = a_{l+1}(x)\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\phi_{l+1}(x,\varepsilon,t)\Big)\Big|_{x=x_l} \tag{3.1.21}$$

and periodic boundary conditions at $x = 0,\ 1$

$$\phi_1(0, \varepsilon, t) = \phi_m(1, \varepsilon, t) \tag{3.1.22}$$

$$a_1(x)\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\phi_1(x,\varepsilon,t)\Big)\Big|_{x=0} = a_m(x)\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\phi_m(x,\varepsilon,t)\Big)\Big|_{x=1} \tag{3.1.23}$$

It is assumed, without loss of generality, that $a(0+) = a(1-)$ and, therefore, the boundary condition (3.1.23) simplifies as follows

$$\frac{\partial\phi_1}{\partial x}(x,\varepsilon,t)\Big|_{x=0} = \frac{\partial\phi_m}{\partial x}(x,\varepsilon,t)\Big|_{x=1}$$

However, if this assumption is invalid then one considers the related problem of the form (3.1.1) with coefficient $\tilde{a}(x) \overset{\text{def}}{=} a(x+\alpha)$ and right hand side $\tilde{f}(x) = f(x + \alpha/\varepsilon)$ where α is chosen such that $\tilde{a}(0+) = \tilde{a}(1-)$. The solution of this related problem is thus $\tilde{u}^\varepsilon(x) = u^\varepsilon(x + \alpha/\varepsilon) - u^\varepsilon(\alpha/\varepsilon)$, $x \in \mathbb{R}$. The general solution, ϕ, is synthesized from the components ϕ_l which we have determined have the form

$$\phi_l(x,\varepsilon,t) = \frac{i\varepsilon}{t}e^{-it\varepsilon x}\int_{x_{l-1}}^{x}\frac{e^{it\varepsilon z}}{a(z)}\,dz + c_l(\varepsilon,\, t)\,e^{-it\varepsilon x}\int_{x_{l-1}}^{x}\frac{1}{a(z)}dz + d_l(\varepsilon,\, t)\,e^{-it\varepsilon x} \tag{3.1.24}$$

where the arbitrary functions c_l, d_l are determined from the transition conditions specified in (3.1.20), (3.1.21). If $a \in C^0(\mathcal{P})$ but $a \notin C^n(\mathcal{P})$, $n \geq 1$ then we observe that the transition conditions (3.1.20), (3.1.21) imply the continuity $\partial\phi(\bullet,\varepsilon,t)/\partial x \in C^0(\mathcal{P})$. If, however, $a \in C^1(\mathcal{P})$ then the transition conditions (3.1.20), (3.1.21) are redundant and ϕ is obtained directly in the form

$$\phi(x,\varepsilon,t) = \frac{i\varepsilon}{t}e^{-it\varepsilon x}\int_0^x\frac{e^{it\varepsilon z}}{a(z)}\,dz + c(\varepsilon,\, t)\,e^{-it\varepsilon x}\int_0^x\frac{1}{a(z)}dz + d(\varepsilon,\, t)\,e^{-it\varepsilon x} \tag{3.1.25}$$

where the arbitrary functions c, d are then determined solely from the boundary conditions specified in relations (3.1.22), (3.1.23).

If one includes in equation (3.1.19) the additional term $a_0(x)\,e^{itx}\phi(x,\varepsilon,t)$ where $a_0(x) \geq \gamma > 0$, $x \in \mathcal{P}$ and $a_0 \in \mathcal{L}_\infty(\mathcal{P})$ is 1-periodic, then the weak solution, ϕ, of the resulting problem exhibits the important property of holomorphism within a neighbourhood, $(\varepsilon, t) \in \widehat{G}$, of $\mathbb{R}^2$. This property is established in BABUŠKA AND MORGAN (1991i) which, thus, establishes that one can justifiably represent the function $\phi(x,\bullet,\bullet)$, $x \in \mathcal{P}$ as a convergent power series within the neighbourhood $\widehat{G}$. Similarly, to provide a theoretical basis for the power series representations subsequently employed for $\phi(x,\bullet,\bullet) \in \mathcal{P}$, which is the weak solution of problem (3.1.9), we propose the following Theorem, which is supported by the computational results provided in Sections 3.2.1 and 3.2.2.

Conjecture 3.1.1. *A neighbourhood* $\widehat{G} \subset \mathbb{C}^2$ *of* $\widehat{V} \stackrel{\text{def}}{=} \{(\varepsilon, t) \in \mathbb{R}^2 : |\varepsilon t| < 2\pi,\ t \neq 0\}$ *can be found such that for each* $(\varepsilon, t) \in \widehat{G}$*, there exists a function* $\phi(\bullet, \varepsilon, t) \in H^1_{per}(\mathcal{P})$ *that satisfies, uniquely for* $(\varepsilon, t) \in \widehat{G}$*, the weak problem*

$$\Phi(\varepsilon, t)[\phi(\bullet, \varepsilon, t), v] = \varepsilon^2 \int_{\mathcal{P}} \overline{v(x)}\, dx, \quad v \in H^1_{per}(\mathcal{P})$$

Furthermore, the mapping $(\varepsilon, t) \in \widehat{G} \mapsto \phi(\bullet, \varepsilon, t) \in H^1_{per}(\mathcal{P})$ *is holomorphic, i.e., there exist functions* $\phi_n(\bullet, t) \in H^1_{per}(\mathcal{P}), n \geq 0$ *such that for each point* $(\varepsilon, t) \in \widehat{G}$ *one can write*

$$\phi(x, \varepsilon, t) = \sum_{n=0}^{\infty} \phi_n(x, t)\, \varepsilon^n, \quad x \in \mathcal{P} \tag{3.1.26}$$

which is convergent in $H^1_{per}(\mathcal{P})$*, i.e.,*

$$\left\|\phi(\bullet, \varepsilon, t) - \phi_N(\bullet, \varepsilon, t); H^1(\mathcal{P})\right\| \to 0 \quad (N \to \infty)$$

where

$$\phi_N(x, \varepsilon, t) \stackrel{\text{def}}{=} \sum_{n=0}^{N} \phi_n(x, t)\, \varepsilon^n$$

for $N \geq 0$. ■

This property provides the basis for the asymptotic approach developed in Section 3.2 when the data are piecewise regular, cf., (3.1.19)–(3.1.23). The methods thus developed are then used to obtain asymptotic approximations for a number of sample problems of varying levels of regularity, thereby illustrating the behaviour conjectured above.

3.2. Homogenization: Expansions in powers of ε.

It has been observed in Conjecture 3.1.1, that with respect to $H^1_{per}(\mathcal{P})$, $\phi(x, \bullet, t)$ is holomorphic. Consequently one can employ the expansion

$$\phi(x, \varepsilon, t) = \phi_0(x, t) + \varepsilon\, \phi_1(x, t) + \varepsilon^2\, \phi_2(x, t) + \cdots, \quad (\varepsilon, t) \in \widehat{G} \tag{3.2.1}$$

where $\phi_n(\bullet, t) \in H^1_{per}(\mathcal{P})$, $n \in \mathbb{N}_0$. To determine the functions ϕ_n, we substitute the expansion (3.2.1) of ϕ into the weak formulation (3.1.9), then, equate the coefficients of identical ε^n terms, $n \in \mathbb{N}_0$. This process will generate a sequence of equations in $H^1_{per}(\mathcal{P})$ with ϕ_n, $n \in \mathbb{N}_0$ as the unknowns. Thus, substitution of (3.2.1) into (3.1.9) produces, for $v \in H^1_{per}(\mathcal{P})$,

$$\sum_{n=0}^{\infty} \varepsilon^n \Big(\Phi_0[\phi_n(\bullet, t), v] + \varepsilon\, \Phi_1(t)[\phi_n(\bullet, t), v] + \varepsilon^2\, \Phi_2(t)[\phi_n(\bullet, t), v]\Big) = \varepsilon^2 \int_{\mathcal{P}} \overline{v(x)}\, dx\,, \tag{3.2.2}$$

where the linearity and continuity of $\Phi(\varepsilon, t)$ have been employed to extract the sum from the sesquilinear mappings Φ_0, $\Phi_1(t)$, $\Phi_2(t)$. Comparing the coefficients of ε^n, $n = 0, 1, \ldots$ one obtains the relations, for $v \in H^1_{per}(\mathcal{P})$,

$$\begin{aligned}
\varepsilon^0:&\quad \Phi_0[\phi_0(\bullet,t),v] = 0 &(3.2.3)\\
\varepsilon^1:&\quad \Phi_0[\phi_1(\bullet,t),v] = -\,\Phi_1(t)[\phi_0(\bullet,t),v] &(3.2.4)\\
\varepsilon^2:&\quad \Phi_0[\phi_2(\bullet,t),v] = \int_{\mathcal{P}} \overline{v(x)}\,dx - \Phi_1(t)[\phi_1(\bullet,t),v] - \Phi_2(t)[\phi_0(\bullet,t),v] &(3.2.5)\\
\vdots&\qquad\qquad \vdots\\
\varepsilon^n:&\quad \Phi_0[\phi_n(\bullet,t),v] = -\,\Phi_1(t)[\phi_{n-1}(\bullet,t),v] - \Phi_2(t)[\phi_{n-2}(\bullet,t),v] &(3.2.6)
\end{aligned}$$

Now, write the above equations as follows, for $k = 0, 1, \ldots,$

$$\Phi_0[\phi_k(\bullet,t),v] = F_k(\phi_0,\cdots,\phi_{k-1};v), \quad \phi_k(\bullet,t),\, v \in H^1_{per}(\mathcal{P}) \tag{3.2.7}$$

Then, from the properties of Φ_0 observed in Section 3.1.1, it is clear that problems (3.2.3)–(3.2.6) are solvable if, and only if, $F_k(\phi_0,\cdots,\phi_{k-1};1) = 0$ for $k \geq 0$. Furthermore, the semi-positive definiteness of Φ_0 over $H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P})$ implies that, if $\phi_k(\bullet,t)$ is a solution of (3.2.7) then so is $\phi_k(\bullet,t) + c_k(t)$ where c_k is an arbitrary mapping $c_k : \mathbb{R}\backslash\{0\} \to \mathbb{C}$. However, the solvability requirements $F_k(\phi_0,\cdots,\phi_{k-1};1) = 0$, $k \geq 0$ uniquely determine the arbitrary functions c_k. Indeed, this is demonstrated in the following theorem, cf. BABUŠKA & MORGAN (1991ii), which also illustrates that the functions $\phi_k(\bullet,t)$ can be determined in a systematic fashion.

Theorem 3.2.1. *For each $k \geq 1$ define $\chi_k(\bullet,t) \in H^1_{per,0}(\mathcal{P}) \stackrel{\text{def}}{=} \{v \in H^1_{per}(\mathcal{P}) \mid \int_{\mathcal{P}} v(x)\,dx = 0\}$ to be the solution of*

$$\Phi_0[\chi_k(\bullet,t),v] = \begin{cases} -\,\Phi_1(t)[1,v], & k=1\\ -\,\Phi_1(t)[\chi_1(\bullet,t),v] - \Phi_2(t)[1,v], & k=2\\ -\,\Phi_1(t)[\chi_{k-1}(\bullet,t),v] - \Phi_2(t)[\chi_{k-2}(\bullet,t),v], & k\geq 3\end{cases} \tag{3.2.8}$$

for all $v \in H^1_{per,0}(\mathcal{P})$, and for each $k \geq 0$ define $g_k(t) \in \mathbb{C}$ by

$$g_k(t) \stackrel{\text{def}}{=} \begin{cases} \Big[\Phi_1(t)[\chi_1(\bullet,t),1] + \Phi_2(t)[1,1]\Big]^{-1}, & k=0\\ -\,g_0(t)\, \sum_{i=0}^{k-1} g_i(t)\Big[\Phi_1(t)[\chi_{k+1-i}(\bullet,t),1] + \Phi_2(t)[\chi_{k-i}(\bullet,t),1]\Big], & k\geq 1\end{cases} \tag{3.2.9}$$

Then the coefficient of ε^k in (3.2.1) is given by

$$\phi_k(\bullet,t) = \begin{cases} g_0(t), & k=0\\ \sum_{i=0}^{k-1} g_i(t)\,\chi_{k-i}(\bullet,t) + g_k(t), & k\geq 1\end{cases} \tag{3.2.10}$$

where χ_k, $k \geq 0$ are the functions defined in (3.2.8). ■

Thus, examining, for example, expressions (3.2.8) for $k = 1$ and (3.2.9) for $k = 0$, the leading term, g_0, is evidently

$$g_0(t) = \frac{1}{A\,t^2}, \quad A = \int_{\mathcal{P}} a(x)\left[1 + \frac{\partial \chi_1}{\partial x}(x)\right] dx \tag{3.2.11}$$

where A is commonly referred to as the homogenized coefficient and $\chi_1 \in H^1_{per,0}(\mathcal{P})$ is the solution of the weak problem

$$\Phi_0[\chi_1, v] = -\int_{\mathcal{P}} a(x)\, \overline{\frac{\partial v}{\partial x}(x)}\, dx, \quad v \in H^1_{per,0}(\mathcal{P}) \tag{3.2.12}$$

and $\chi_1(x,t) \stackrel{\text{def}}{=} it\, g_0(t)\, \chi_1(x)$, see Theorem 3.2.1. We observe that, although Theorem 3.2.1 clearly provides a systematic process for the construction of the functions $\phi_k(\bullet, t) \in H^1_{per}(\mathcal{P})$, the expansion (3.2.1) is constructed using direct knowledge of the function ϕ rather than employing the above process for the specific sample problems provided in Sections 3.2.1, 3.2.2. Now we define the asymptotic approximations ϕ_N, $u^\varepsilon_{N,\ell}$ according to the expressions

$$\phi_N(x,\varepsilon,t) \stackrel{\text{def}}{=} \sum_{m=0}^{N} \varepsilon^m\, \phi_m(x,t), \quad \phi_0(x,t) = g_0(t), \quad x \in \mathcal{P},\ t \neq 0 \tag{3.2.13}$$

$$u^\varepsilon_{N,\ell}(x) \stackrel{\text{def}}{=} \sum_{n \in Z_\ell \setminus \{0\}} a_n\, e^{in\pi x}\, \phi_N(x/\varepsilon, \varepsilon, n\pi), \quad x \in \Omega \tag{3.2.14}$$

Because ϕ_0 and, thus, $u^\varepsilon_{0,\ell}$ do not depend on ε we subsequently denote $u^\varepsilon_{0,\ell}$ by $u_{0,\ell}$. We observe here that for the *homogenized* problem (H), discussed in the introduction, i.e.,

$$-A\, \frac{\partial^2 u_0}{\partial x^2}(x) = f_C(x), \quad -\infty < x < \infty \tag{3.2.15}$$

$\phi = g_0$ and $u_{0,\infty} = u_0 \in H^2(\Omega)$ is the solution. The utility of the asymptotic approximations, (3.2.14), is established in the following theorem, cf. BAKHVALOV & PANASENKO (1989), which is restricted to the context of elliptic boundary value problems of the type (3.1.1) with high regularity, i.e., $f_C \in C^\infty(\mathbb{R})$, $a_l \in C^\infty(\Omega_l)$, $1 \le l \le m$. In the statement of the following theorem we employ the notation $u^\varepsilon_N \stackrel{\text{def}}{=} u^\varepsilon_{N,\infty}$.

Theorem 3.2.2. *The asymptotic approximation u^ε_N exhibits the following properties, for $l \geq 1$,*

$$-\frac{\partial}{\partial x}\left[a(x/\varepsilon)\, \frac{\partial u^\varepsilon_N}{\partial x}(x)\right] = f_C + \varepsilon^{N-1}\, \theta_1(x,\varepsilon), \quad x/\varepsilon \notin \bigcup_{l=1}^{\infty} \{x_l\} \tag{3.2.16}$$

$$[u^\varepsilon_N]_{x_l/\varepsilon} = 0 \tag{3.2.17}$$

$$\left[a(x)\, \frac{\partial u^\varepsilon_N}{\partial x}(x)\right]_{x_l/\varepsilon} = \varepsilon^N\, \theta_2(x,\varepsilon) \tag{3.2.18}$$

$$\int_C u^\varepsilon_N(x)\, dx = \theta_3(\varepsilon) \tag{3.2.19}$$

where $|\theta_1(x,\varepsilon)| \leq c_1$, $|\theta_2(x,\varepsilon)| \leq c_2$, $|\theta_3(x,\varepsilon)| \leq c_3(\alpha)\, \varepsilon^\alpha$ for any α, and the positive constants α, c_1, c_2, $c_3(\alpha)$ are independent of ε. Then the function u^ε_N is 2–periodic and has the approximation property

$$\| u^\varepsilon - u^\varepsilon_N; H^1(\mathcal{C}) \| \leq C\, \varepsilon^N \tag{3.2.20}$$

where $\mathcal{C} = (-1,1)$ and $C > 0$ is a constant independent of ε. ■

Theorem 3.2.2 will be used later, in sections 3.4 and 3.7, to justify the computational results obtained. We observe that conditions (3.2.17), (3.2.18) become redundant if $a \in C^\infty_{per}(\mathcal{P})$. Before applying the homogenization (3.2.1) to problems of low regularity, the behaviour of such techniques will be investigated computationally for specific problems with smooth data.

3.2.1. Smooth Problems: Homogenization and the Classical Taylor Series.

It will be demonstrated below that the homogenization described in section 3.2 coincides precisely with a classical Taylor series expansion of $\phi(x, \bullet, t)$ when $a \in C^{\infty}_{per}(\mathcal{P})$ and that, even in this favourable circumstance, ϕ can have an infinite number of singularities which are not isolated and, therefore, in the classical context, cannot be represented in the neighbourhood of any such singular point by even the more general Laurent series expansion. Thus assume that $a \in C^{\infty}_{per}(\mathcal{P})$ and consider the equations (3.1.22) and (3.1.23) that one obtains for the determination of the arbitrary functions c, d with $(\varepsilon, t) \notin \mathcal{H}_n$ where $\mathcal{H}_n$ is the hyperbola $\varepsilon t = 2\pi n$, $n \in \mathcal{Z}$

$$c(\varepsilon, t)\, B(1) + d(\varepsilon, t)\,(1 - e^{i\varepsilon t}) = \frac{\varepsilon}{it} A(1, \varepsilon, t) \qquad (3.2.21)$$

$$c(\varepsilon, t)\Big(B(1) + \frac{1}{a(0)}\frac{i}{\varepsilon t}(1 - e^{it\varepsilon})\Big) + d(\varepsilon, t)\,(1 - e^{it\varepsilon}) = \frac{\varepsilon}{it} A(1, \varepsilon, t) \qquad (3.2.22)$$

and the mappings A, B, $\partial\phi/\partial x$ are specified below

$$\frac{\partial \phi}{\partial x}(x, \varepsilon, t) = \Big(\varepsilon^2 A(x, \varepsilon, t) + c(\varepsilon, t)\Big(\frac{1}{a(x)} - it\varepsilon\, B(x)\Big) - it\varepsilon\, d(\varepsilon, t)\Big)\, e^{-it\varepsilon x} + \frac{i\varepsilon}{t}\frac{1}{a(x)} \qquad (3.2.23)$$

$$A(x, \varepsilon, t) = \int_0^x \frac{e^{it\varepsilon z}}{a(z)}\, dz, \qquad B(x) = \int_0^x \frac{1}{a(z)}\, dz \qquad (3.2.24)$$

Thus, solving the equations (3.2.21) and (3.2.22), the functions c, d are determined by the following expressions.

$$c(\varepsilon, t) = 0, \qquad d(\varepsilon, t) = \frac{i\varepsilon}{t}\frac{1}{e^{it\varepsilon} - 1} A(1, \varepsilon, t) \qquad (3.2.25)$$

Then, substituting the values (3.2.25) for the arbitrary constants into the general solution, (3.1.25), one obtains the following identity

$$\phi(x, \varepsilon, t) = \frac{i\varepsilon}{t} e^{-i\varepsilon t x} \int_0^x \frac{e^{i\varepsilon t z}}{a(z)}\, dz + \frac{i\varepsilon}{t}\frac{e^{-it\varepsilon x}}{e^{i\varepsilon t} - 1} \int_0^1 \frac{e^{it\varepsilon z}}{a(z)}\, dz \qquad (3.2.26)$$

The solution $\phi(x, \bullet, \bullet)$ is then defined everywhere in the $(\varepsilon,\, t)$–plane except on the hyperbolae $\mathcal{H}_n$, $n \in \mathcal{Z}\backslash\{0\}$ where, generally, $|\phi(x, \varepsilon,\, t)| \to \infty$ as $\operatorname{dist}((\varepsilon,\, t), \mathcal{H}_n) \to 0$. Furthermore, substituting the Fourier series representation of the 1–periodic function $1/a$, i.e.,

$$1/a(x) = \sum_{m \in \mathcal{Z}} c_m\, e^{2\pi m x i}, \quad x \in \mathcal{P}$$

into relation (3.2.26) for ϕ, one obtains the relation

$$\phi(x, \varepsilon, t) = \frac{i\varepsilon}{t} e^{-i\varepsilon t x} \int_0^x \frac{e^{i\varepsilon t z}}{a(z)}\, dz + c_0\, \frac{e^{-it\varepsilon x}}{t^2} + \frac{\varepsilon\, e^{-it\varepsilon x}}{t} \sum_{m \neq 0} \frac{c_m}{\varepsilon t + 2\pi m}$$

Thus, with $\mathcal{O} \stackrel{\text{def}}{=} \mathbb{R}^2\backslash\mathcal{H}$, $\mathcal{H} \stackrel{\text{def}}{=} \cup_{n \in \mathcal{Z}\backslash\{0\}}\mathcal{H}_n$ it follows that $\phi(x, \bullet, \bullet) \in C^{\infty}(\mathcal{O})$ and therefore one can employ the representation, for $x \in \mathcal{P}$, $|t| > 0$, $\varepsilon \in B(0, r_t) \stackrel{\text{def}}{=} \{\varepsilon \in \mathbb{R}\colon |\varepsilon| < r_t\}$,

$$\begin{aligned}\phi(x, \varepsilon, t) &= \sum_{n=0}^{N-1} \frac{\varepsilon^n}{n!}\frac{\partial^n \phi}{\partial \varepsilon^n}(x, \varepsilon, t)\Big|_{\varepsilon=0} + \frac{\varepsilon^N}{N!}\frac{\partial^N \phi}{\partial \varepsilon^N}(x, \xi(\varepsilon), t), \quad \xi(\varepsilon) \in B(0, r_t) \qquad (3.2.27)\\ &\stackrel{\text{def}}{=} T_N(x, \varepsilon, t) + R_N(x, \varepsilon, t)\end{aligned}$$

where $r_t < \text{dist}((0,t),H)$ and the remainder, $R_N(x,\varepsilon,t)$, is written in the classical differential form. We observe that, because $\phi(x,\bullet,\bullet) \in C^\infty(\mathcal{O})$ and $\phi(\bullet,\varepsilon,t) \in C^\infty_{per}(\mathcal{P})$, it is clear that the N^{th} partial sum of the series, $T_N(\bullet,\varepsilon,t)$, belongs to $C^\infty_{per}(\mathcal{P})$ and, from the defining relations (3.2.3)–(3.2.6) and the smoothness of the coefficient function a, it is evident that $\phi_n(\bullet,t)$, $\phi_N(\bullet,\varepsilon,t) \in C^\infty_{per}(\mathcal{P})$, where n, $N \in \mathbb{N}$. It is demonstrated next that, in a neighbourhood of $\varepsilon = 0$, the classical Taylor series expansion, (3.2.27), coincides with the asymptotic expansion, (3.2.1), obtained from the homogenization described in section 3.2, in the sense that both converge to the identical function in the $H^1(\mathcal{P})$ norm topology. The property of holomorphism proposed in Conjecture 3.1.1 implies that

$$\frac{\partial^m \phi}{\partial \varepsilon^m}(x,\varepsilon,t) = \sum_{n=m}^{\infty} m!\,\varepsilon^{n-m}\phi(x,t), \quad m \in \mathbb{N} \tag{3.2.28}$$

with convergence, again, in terms of the $H^1(\mathcal{P})$ topology, i.e.,

$$\left\| \frac{\partial^m \phi}{\partial \varepsilon^m}(\bullet,\varepsilon,t) - \frac{\partial \phi_N}{\partial \varepsilon^m}(\bullet,\varepsilon,t); H^1(\mathcal{P}) \right\| \to 0 \quad (N \to \infty)$$

where ϕ_N is defined in theorem 3.2.1. This is established as follows: Let $(\varepsilon_0,t) \in \widehat{G}$ (cf. Conjecture 3.1.1), $\varepsilon_0 \neq 0$, then representation (3.2.13) converges in $H^1(\mathcal{P})$ for $|\varepsilon| < r_t$ where $r_t < |\varepsilon_0|$. This is immediate from the following inequality, the Weierstrass test, and the ratio test

$$\|\varepsilon^n \phi_n(\bullet,t); H^1(\mathcal{P})\| = \left|\frac{\varepsilon}{\varepsilon_0}\right|^n \|\varepsilon_0^n \phi_n(\bullet,t); H^1(\mathcal{P})\| \tag{3.2.29}$$

$$\leq M\,\alpha^n, \quad \alpha = \frac{r_t}{|\varepsilon_0|} < 1 \tag{3.2.30}$$

where $M > 0$ is a constant satisfying $\|\varepsilon_0^n \phi_n(\bullet,t); H^1(\mathcal{P})\| \leq M$, $n \geq 0$. Indeed, the convergence of the series (3.2.13) in $H^1(\mathcal{P})$ guarantees the existence of such a constant, M. However, it is then evident that

$$\|n\,\varepsilon^{n-1}\phi_n(\bullet,t); H^1(\mathcal{P})\| = n\left|\frac{\varepsilon}{\varepsilon_0}\right|^{n-1} \|\varepsilon_0^{n-1}\phi_n(\bullet,t); H^1(\mathcal{P})\|$$

$$\leq \frac{n}{|\varepsilon_0|} M\,\alpha^{n-1} \tag{3.2.31}$$

where, from the ratio test, the upper bounds of both (3.2.30) and (3.2.31) yield convergent series. Thus, the Weierstrass test shows that the termwise derivative of (3.2.13) converges in $H^1(\mathcal{P})$ whenever the power series (3.2.13) does. Let $(\varepsilon,t) \in \widehat{G}$ be an arbitrary point such that $|\varepsilon| < r_t$ and let $\rho > 0$ be any value such that $|\varepsilon| < \rho < r_t$. If $h \in \mathbb{C}$ is an arbitrary value, for which $|h| < \rho - |\varepsilon| = \delta\ (\delta > 0)$, then $|\varepsilon + h| < \rho$ and, formally,

$$\frac{\phi(x,\varepsilon+h,t) - \phi(x,\varepsilon,t)}{h} = \sum_{n=1}^{\infty} \beta_n(h)\,\phi_n(x,t) \tag{3.2.32}$$

where

$$\beta_n(h) = \frac{(\varepsilon+h)^n - \varepsilon}{h} \tag{3.2.33}$$

$$= (\varepsilon+h)^{n-1} + (\varepsilon+h)^{n-2}\varepsilon + \ldots + \varepsilon^{n-1}, \quad n \geq 1 \tag{3.2.34}$$

$$\to n\,\varepsilon^{n-1}\ (h \to 0) \tag{3.2.35}$$

Thus the functions β_n, $n \geq 1$ are continuous within the domain $|h| < \delta$. However, it follows from (3.2.35) that $|\beta_n(h)| < n\,\rho^{n-1}$ and, therefore,

$$\|\beta_n(h)\,\phi_n(\bullet,t); H^1(\mathcal{P})\| < \frac{n}{|\varepsilon_0|^n} M\,\rho^{n-1} \tag{3.2.36}$$

Therefore, by the Weierstrass and ratio tests, the sum, (3.2.32), of continuous functions $h \mapsto \beta_n(h)\,\phi_n(x,t)$ converges uniformly with respect to h, $|h| < \delta$ in $H^1(\mathcal{P})$ and, therefore,

$$\frac{\partial\phi}{\partial\varepsilon}(x,\varepsilon,t) = \lim_{h\to 0} \frac{\phi(x,\varepsilon+h,t) - \phi(x,\varepsilon,t)}{h} \tag{3.2.37}$$

$$= \sum_{n=1}^{\infty} \beta_n(0)\,\phi_n(x,t) \tag{3.2.38}$$

$$= \sum_{n=1}^{\infty} n\,\varepsilon^{n-1}\phi_n(x,t) \tag{3.2.39}$$

Clearly, this argument can then be repeated for derivatives with respect to ε of any order, $m \geq 1$, and thus, with $\varepsilon = 0$, leads to the following identity

$$\left.\frac{\partial^m\phi}{\partial\varepsilon^m}(x,\varepsilon,t)\right|_{\varepsilon=0} = m!\,\phi_m(x,t) \tag{3.2.40}$$

Consequently, the asymptotic expansion (3.2.1) becomes

$$\phi(x,\varepsilon,t) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left.\frac{\partial^n\phi}{\partial\varepsilon^n}(x,\varepsilon,t)\right|_{\varepsilon=0} \tag{3.2.41}$$

which is, evidently, the Taylor series expansion of $\phi(x,\bullet,t)$. Now, if $(\varepsilon,t) \in \mathcal{H}_n$ for some $n \in \mathcal{Z}\backslash\{0\}$ then equations (3.2.21), (3.2.22) become linearly dependent and yield the solution

$$c(\varepsilon,\,t) = \frac{\varepsilon}{i\,t}\frac{A(1,\varepsilon,t)}{B(1)}, \qquad d \text{ an arbitrary function of } \varepsilon, t \tag{3.2.42}$$

Thus, for $(\varepsilon,t) \in \mathcal{H}_n$, $n \in \mathcal{Z}\backslash\{0\}$ the solution ϕ is determined, up to the function d, by the relation

$$\phi(x,\varepsilon,t) = \frac{i\,\varepsilon}{t} e^{-i\varepsilon t x} \int_0^x \frac{e^{i\varepsilon t z}}{a(z)}\,dz + \frac{\varepsilon}{i\,t}\frac{A(1,\varepsilon,t)}{B(1)}\,e^{-i\varepsilon t x} \int_0^x \frac{1}{a(z)}\,dz \tag{3.2.43}$$

However, it has already been demonstrated in Section 3.1.1 that the solution, ϕ, of the weak problem (3.1.9) possesses the following property, for $x \in \mathcal{P}$, $\varepsilon > 0$, $|t| > 0$,

$$\phi(x,\varepsilon,t) = \overline{\phi(x,\varepsilon,-t)} \tag{3.2.44}$$

$$\Rightarrow \qquad \Re(\phi(x,\varepsilon,t)) = \Re(\phi(x,\varepsilon,-t)), \quad \Im(\phi(x,\varepsilon,t)) = -\,\Im(\phi(x,\varepsilon,-t)) \tag{3.2.45}$$

Property (3.2.44) then implies, in the context of the current problem, that

$$d(\varepsilon,t)=\overline{d(\varepsilon,-t)},\qquad (\varepsilon,\,t)\in\mathcal{H}_n,\ n\in\mathcal{Z}\backslash\{0\} \tag{3.2.46}$$

If it occurs that $(\varepsilon,\pm t_k)\in\mathcal{H}_{\pm n_k}$, $k\in I(\varepsilon)\subset\mathbb{N}$ where $I(\varepsilon)$ is an index set (varying with ε) then the weak solution of (3.1.3), u^ε, so obtained can be written

$$\begin{aligned}u^\varepsilon(x)&=\sum_{n\in\mathcal{Z}\backslash\{0\}}a_n\,e^{n\pi xi}\phi(x/\varepsilon,\varepsilon,n\pi)\\&\quad+\sum_{k\in I(\varepsilon)}\Big[a_{-k}\,e^{-k\pi xi}\,d(\varepsilon,\,-k\pi)\,e^{k\pi xi}+a_k\,e^{k\pi xi}\,d(\varepsilon,\,k\pi)\,e^{-k\pi xi}\Big]\\&=\sum_{n\in\mathcal{Z}\backslash\{0\}}a_n\,e^{n\pi xi}\phi(x/\varepsilon,\varepsilon,n\pi)+\sum_{k\in I(\varepsilon)}a_k\Big[d(\varepsilon,\,k\pi)-d(\varepsilon,\,-k\pi)\Big]\end{aligned} \tag{3.2.47}$$

where it has been observed that, because of the antisymmetry of the function f_C, $a_n=-a_{-n}$, $n\in\mathcal{Z}\backslash\{0\}$ and, depending on the nature of the point (ε,t_k), ϕ is given by either of the relations (3.2.26) or (3.2.43). If the coefficient a is symmetric about $x=1/2$ then the boundary condition $u^\varepsilon(0)=0$; expression (3.2.43); the property $\Im[\phi(0,\varepsilon,k\pi)]=0$, $k\in\mathcal{Z}\backslash\{0\}$ (this follows from the *infinite series form* of (3.2.26) obtained by expanding $1/a$ as a 1–periodic Fourier series) imply the following identities

$$\begin{aligned}u^\varepsilon(0)&=\sum_{k\in\mathbb{N}\backslash I(\varepsilon)}a_k\Big[\phi(0,\varepsilon,\,k\pi)-\phi(0,\varepsilon,\,-k\pi)\Big]+\sum_{k\in I(\varepsilon)}a_k\Big[d(\varepsilon,\,k\pi)-d(\varepsilon,\,-k\pi)\Big]\\&=\sum_{k\in\mathbb{N}\backslash I(\varepsilon)}2i\,a_k\,\Im[\phi(0,\varepsilon,k\pi)]+\sum_{k\in I(\varepsilon)}2i\,a_k\,\Im[d(\varepsilon,k\pi)]\\&=\sum_{k\in I(\varepsilon)}2i\,a_k\,\Im[d(\varepsilon,k\pi)]\\&=0\end{aligned} \tag{3.2.48}$$

However, the function d and the coefficients a_k, $k\in I(\varepsilon)$ are independent from one another; this indicates that, for $(\varepsilon,t)\in\mathcal{H}$, $d(\varepsilon,t)\in\mathbb{R}$ or, equivalently, $d(\varepsilon,t)=d(\varepsilon,-t)$. Of course, the symmetry properties of f_C and ϕ imply that u^ε can be rewritten in the following fashion

$$u^\varepsilon(x)=\sum_{n=1}^{\infty}b_n\,\Im\Big[e^{n\pi xi}\,\phi(x/\varepsilon,\varepsilon,n\pi)\Big] \tag{3.2.49}$$

where $b_n=2i\,a_n$, $n\in\mathbb{N}$ are the Fourier coefficients of a sine series expansion of f_C.

Thus, if one maintains the proviso that the relation (3.2.48) is satisfied, then the choice of the arbitrary constant, $d(\varepsilon,t)$, is inconsequential insofar as it has no influence upon the solution u^ε. Finally, if $(\varepsilon,t)\in\mathcal{H}_n$, i.e., $t=2\pi n/\varepsilon$, $n\in\mathcal{Z}\backslash\{0\}$, then, selecting $d(\varepsilon,t)=0$, the cell function ϕ is given by (3.2.44) and becomes a quadratic in ε along the hyperbola $\mathcal{H}_n$, i.e.,

$$\phi(x,\varepsilon,t)=\frac{i\varepsilon^2}{2\pi n}e^{-2n\pi xi}\int_0^x\frac{e^{2\pi nzi}}{a(z)}dz-\frac{i\varepsilon^2}{2\pi n}\frac{A(1,\varepsilon,t)}{B(1)}e^{-2n\pi xi}\int_0^x\frac{1}{a(z)}dz,\quad(\varepsilon,t)\in\mathcal{H}_n \tag{3.2.50}$$

One can then employ Taylor series expansions along the hyperbolae, $\mathcal{H}_n$, which are equivalent to the asymptotic approximations derived from the power series (3.2.1), i.e., the homogenization. However, the form (3.2.50) of ϕ, for $(\varepsilon, t) \in \mathcal{H}_n$, suggests that (3.2.1) is then, simply, a finite polynomial. The application of these results to a boundary value problem of infinitely high regularity are illustrated in Section 3.4

3.3. Computational aspects of the asymptotic approximations $u^{\epsilon}_{N,\ell}, N, \ell \in \mathbb{N}$.

We now want to make some comments regarding the computational aspects of our approach. We focus, in particular, on the role of convergence, as ℓ (Fourier series truncation as in (3.2.14)) and N (Taylor series truncation as in (3.2.27)) tend, respectively, to infinity and how this affects the application of the asymptotic approximations $u^{\epsilon}_{N,\ell}$, N, $\ell \in \mathbb{N}$.

We demonstrate in Theorem 3.3.1, below, how the formulae provided in Theorem 3.2.1 for the terms, $\phi_n(\bullet, t) \in H^1_{per}(\mathcal{P})$, $n \geq 0$, $|t| > 0$, of the homogenization (3.2.1), can be rewritten in an alternative form in which the functional dependence on the variables x, t of these terms is *separated*. We show that this property is important because the homogenization (3.2.1) can then be determined more efficiently by solving problems, cf. (3.3.3), which are analogous to the t–dependent formulations (3.2.8) but which do not depend on the unbounded variable t. Thus, we show how the expansion (3.2.1) can be constructed more efficiently when the computations are based on Theorem 3.3.1 rather than Theorerm 3.2.1. The details of this alternative representation for ϕ_n, $n \geq 0$ are provided below:

Theorem 3.3.1. *The functions $\phi_k(\bullet, t) \in H^1_{per}(\mathcal{P})$, $t \neq 0$, $n \geq 1$, defined in relation (3.2.10) of Theorem 3.2.1, can also be expressed in the form*

$$\phi_n(x,t) = (it)^n g_0(t) \left[\sum_{j=0}^{n-1} \kappa_j \, \chi_{n-j}(x) + \kappa_n \right], \quad x \in \mathcal{P},\, t \neq 0 \tag{3.3.1}$$

where $\kappa_0 \stackrel{\text{def}}{=} 1$ and the constants κ_n, $n \geq 1$ are given by the relation

$$\kappa_n = -t^2 g_0(t) \sum_{j=0}^{n-1} \kappa_j \left[-\Phi_1[\chi_{n+1-j}, 1] + \Phi_2[\chi_{n-j}, 1] \right] \tag{3.3.2}$$

Furthermore, $\chi_0 \stackrel{\text{def}}{=} 1$ and $\chi_n \in H^1_{per,0}(\mathcal{P})$, $n \geq 1$ is defined as the solution, over the field $\mathbb{R}$, of the problem

$$\Phi_0[\chi_n, v] = \Theta^{(n)}(v), \quad v \in H^1_{per,0}(\mathcal{P}) \tag{3.3.3}$$

where $\Theta^{(n)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P})\,;\,\mathbb{R})$, $n \geq 1$ is defined in relations (3.3.4), (3.3.5).

Proof Define $\chi_n \in H^1_{per,0}(\mathcal{P})$, $n \geq 1$ as the solution of problem (3.3.3) where, for $v \in H^1_{per,0}(\mathcal{P})$,

$$\text{if } n = 1, \quad \Theta^{(n)}(v) \stackrel{\text{def}}{=} -\Phi_1[1, v] \tag{3.3.4}$$

$$\text{if } n \geq 2, \quad \Theta^{(n)}(v) \stackrel{\text{def}}{=} -\Phi_1[\chi_{n-1}, v] + \Phi_2[\chi_{n-2}, v] \tag{3.3.5}$$

where $\chi_{-1} = 0$, $\chi_0 = 1$ and, for u, $v \in H^1_{per,0}(\mathcal{P})$,

$$\Phi_1[u,v] \stackrel{\text{def}}{=} \int_{\mathcal{P}} a(x)\Big(u(x)\,\frac{\partial v}{\partial x}(x) - \frac{\partial u}{\partial x}(x)\,v(x)\Big)\,dx \tag{3.3.6}$$

$$\Phi_2[u,v] \stackrel{\text{def}}{=} \int_{\mathcal{P}} a(x)\,u(x)\,v(x)\,dx \tag{3.3.7}$$

We now substitute expression (3.3.1) into relation (3.2.4) and, employing the functions $\chi_k \in H^1_{per,0}(\mathcal{P})$, $k \geq 1$ defined in problem (3.3.3), we deduce the following equations

$$\begin{aligned}\Phi_0[\phi_1(\bullet,t),v] &= -\Phi_1(\underline{t})[g_0(\underline{t}),v]\\ &= -g_0(\underline{t})\,it\,\Phi_1[1,v]\\ &= g_0(\underline{t})\,it\,\Phi_0[\chi_1,v]\end{aligned} \tag{3.3.8}$$

Thus, $\phi_1(\bullet,t) \in H^1_{per}(\mathcal{P})$ can be written in the form (3.3.1). Now substitute relation (3.3.1) for ϕ_2 into (3.2.5) thereby obtaining the equation, for $v \in H^1_{per}(\mathcal{P})$,

$$\Phi_0[\phi_2(\bullet,t),v] = \int_{\mathcal{P}} \overline{v(x)}\,d\underline{x} - g_0(t)\Big[it\,\Phi_1(t)[\chi_1,v] + it\,\kappa_1\,\Phi_1(t)[1,v] + \Phi_2(t)[1,v]\Big] \tag{3.3.9}$$

Let $v = 1$ in this equation and note that $\Phi_0[v,1] = \Phi_1(t)[1,1] = 0$, $v \in H^1_{per}(\mathcal{P})$. The following identity for g_0 is thus obtained

$$g_0(t) = t^{-2}\Big[-\Phi_1[\chi_1,1] + \Phi_2[1,1]\Big]^{-1} \tag{3.3.10}$$

However, if $v \in H^1_{per,0}(\mathcal{P})$ then relation (3.3.9) becomes

$$\begin{aligned}\Phi_0[\phi_2(\bullet,t),v] &= -g_0(t)\Big[it\,\Phi_1(t)[\chi_1,v] + it\,\kappa_1\,\Phi_1(t)[1,v] + \Phi_2(t)[1,v]\Big]\\ &= (it)^2 g_0(t)\Big[-\Phi_1[\chi_1,v] + \Phi_2[1,v]\Big] - (it)^2 g_0(t)\kappa_1\,\Phi_1[1,v]\\ &= (it)^2 g_0(t)\sum_{j=0}^{1}\kappa_j\,\Phi_0[\chi_{2-j},v]\end{aligned} \tag{3.3.11}$$

Comparing relations (3.3.11) and (3.3.1), it is now evident that ϕ_1, ϕ_2 have the form specified in (3.3.1) where κ_1, κ_2 are constants, which we have yet to demonstrate, are determined by (3.3.2). We now assume inductively that, for some $k \geq 3$, $\kappa_n \in \mathbb{R}$, $\phi_n(\bullet,t) \in H^1_{per}(\mathcal{P})$, $n \leq k-3$ are given by (3.3.2), (3.3.1) respectively and $\phi_n(\bullet,\underline{t}) \in H^1_{per}(\mathcal{P})$, $n \geq k-2$ has the *form* (3.3.1) but the constants κ_n, $n \geq k-2$ are unknown. Thus, substitution of (3.3.1) into (3.2.6) yields

$$\begin{aligned}\Phi_0[\phi_n(\bullet,t),v] = &-(it)^{n-1}g_0(t)\left[\sum_{j=0}^{n-2}\kappa_j\,\Phi_1(t)[\chi_{n-1-j},v] + \kappa_{n-1}\,\Phi_1(t)[1,v]\right]\\ &-(it)^{n-2}g_0(t)\left[\sum_{j=0}^{n-3}\kappa_j\,\Phi_2(t)[\chi_{n-2-j},v] + \kappa_{n-2}\,\Phi_2(t)[1,v]\right]\end{aligned} \tag{3.3.12}$$

Setting $v = 1$ in (3.3.12) yields the equation

$$-t^2\sum_{j=0}^{n-3}\kappa_j\Big[-\Phi_1[\chi_{n-1-j},1] + \Phi_2[\chi_{n-2-j},1]\Big] - \frac{\kappa_{n-2}}{g_0(t)} = 0 \tag{3.3.13}$$

Thus, solving (3.3.13) for κ_{n-2} and shifting the index $n \to n+2$ we obtain relation (3.3.2) for κ_n, $n \geq 1$. However, with $v \in H^1_{per,0}(\mathcal{P})$ equation (3.3.12) becomes

$$\begin{aligned}\Phi_0[\phi_n(\bullet,\underline{t}),v] &= (it)^n g_0(t)\sum_{j=0}^{n-1}\kappa_j\Big[-\Phi_1[\chi_{n-1-j},v]+\Phi_2[\chi_{n-2-j},v]\Big]\\ &= (it)^n g_0(t)\sum_{j=0}^{n-1}\kappa_j\,\Phi_0[\chi_{n-j},v] \qquad (3.3.14)\end{aligned}$$

Thus, comparing relations (3.3.14) and (3.3.1), it is now evident that $\phi_n(\bullet,\underline{t}) \in H^1_{per}(\mathcal{P})$, $\underline{t} \neq 0$ is uniquely determined by expressions (3.3.1), (3.3.2) and satisfies (3.2.3)–(3.2.6). ■

If we substitute the expression (3.3.1) for ϕ_k into the definition (3.2.14) of the asymptotic approximation $u^{\varepsilon}_{N,\ell}$, $N \geq 0$, $\ell \in \mathbb{N}$ then we observe that the following relation arises

$$\begin{aligned}u^{\varepsilon}_{N,\ell}(x) &= \sum_{n\in\mathcal{Z}_\ell\backslash\{0\}} a_n\,e^{n\pi xi}\phi_0(n\pi) + \varepsilon\sum_{n\in\mathcal{Z}_\ell\backslash\{0\}} a_n\,e^{n\pi xi}\phi_1(x/\varepsilon,n\pi) +\\ &\quad + \varepsilon^2\sum_{n\in\mathcal{Z}_\ell\backslash\{0\}} a_n\,e^{n\pi xi}\phi_2(x/\varepsilon,n\pi) + \ldots + \varepsilon^N\sum_{n\in\mathcal{Z}_\ell\backslash\{0\}} a_n\,e^{n\pi xi}\phi_N(x/\varepsilon,n\pi)\\ &= u_{0,\ell}(x) + \varepsilon\left[\mathcal{X}_1(x/\varepsilon)\frac{\partial u_{0,\ell}}{\partial x}(x) + G_{1,\ell}(x)\right] + \varepsilon^2\left[\mathcal{X}_2(x/\varepsilon)\frac{\partial^2 u_{0,\ell}}{\partial x^2}(x) + G_{2,\ell}(x)\right]\\ &\quad + \ldots + \varepsilon^N\left[\mathcal{X}_N(x/\varepsilon)\frac{\partial^N u_{0,\ell}}{\partial x^N}(x) + G_{N,\ell}(x)\right] \qquad (3.3.15)\end{aligned}$$

where, clearly,

$$u_{0,\ell}(x) \stackrel{\text{def}}{=} \sum_{n\in\mathcal{Z}_\ell\backslash\{0\}} a_n\,e^{n\pi xi}\phi_0(n\pi),\ G_{k,\ell}(x) \stackrel{\text{def}}{=} \sum_{n\in\mathcal{Z}_\ell\backslash\{0\}} a_n\,e^{n\pi xi}g_k(n\pi),\ \mathcal{X}_n(x) \stackrel{\text{def}}{=} \sum_{j=0}^{n-1}\kappa_j\,\chi_{n-j}(x) + \kappa_n$$

and, as commented above, u_0 $(= u_{0,\infty})$ is the solution of the *homogenized* problem

$$-A\,\frac{\partial^2 u_0}{\partial x^2}(x) = f_C(x), \quad -\infty < x < \infty \qquad (3.3.16)$$

where A is the *homogenized* coefficient defined in relation (3.2.11) and we assume the level of regularity $f_C \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$. The coefficients, $a_n(f_C)$, $n \in \mathcal{Z}\backslash\{0\}$, of the Fourier expansion of f_C will then satisfy the asymptotic relation $\sum_{n\in\mathcal{Z}\backslash\{0\}}|a_n(f_C)|^2 < \infty$, cf. Theorem 15.14 of CHAMPENEY (1987). It now follows from Theorem 3.3.1 that $g_k(t) = O(|t|^{k-2})\,(|t| \to \infty)$ and, therefore, $\phi_k(\bullet,t) = O(|t|^{k-2})\,(|t| \to \infty)$. However, from these asymptotic relations, we can now deduce the convergence behaviour, as $\ell \to \infty$, of the functions $G_{k,\ell}$, $k \geq 1$ and $\partial^k u_{0,\ell}/\partial x^k$, $m \geq 0$, as follows

(1) The sum $G_{1,\ell}$ converges uniformly, as $\ell \to \infty$, to the limit function $G_{1,\infty}$. This follows immediately from the asymptotic inequality $|a_n(f_C)\,e^{n\pi xi}g_1(n\pi)| \leq C\,|n^{-1}a_n(f_C)|$, $x \in \mathcal{C}$, $n \in \mathcal{Z}\backslash\{0\}$ and, from Hölder's inequality,

$$\sum_{n\in\mathcal{Z}\backslash\{0\}} |n^{-1}\,a_n(f_C)| \leq 2\|\{n^{-1}\}_{n\geq 1};\ell_2(\mathbb{N})\|\cdot\|\{a_n(f_C)\}_{n\geq 1};\ell_2(\mathbb{N})\| < \infty.$$

Now, we consider the well defined function h obtained from the following series expression

$$h(x) = \sum_{n \in \mathcal{Z}\setminus\{0\}} a_n \, n\pi i \, e^{n\pi x i} g_1(n\pi), \quad x \in \mathbb{R}$$

The asymptotic realtion $n\, g_1(n\pi) = O(1)$ $(|n| \to \infty)$ implies the existence of a positive constant $K > 0$, independent of n, such that, given $f_C \in \mathcal{L}_2^{loc}(\mathbb{R})$ and Theorem 15.11 of CHAMPENEY (1987),

$$\sum_{n \in \mathcal{Z}\setminus\{0\}} |a_n \, n\pi i \, g_1(n\pi)|^2 < K \sum_{n \in \mathcal{Z}\setminus\{0\}} |a_n|^2 < \infty$$

However, according to Theorem 15.10 of CHAMPENEY (1987), $h \in \mathcal{L}_2^{loc}(\mathbb{R})$ and, furthermore, it then follows that $G_{1,\infty}$ can be expressed as an indefinite integral of h, i.e.,

$$G_{1,\infty}(x) = \int_0^x h(z)\, dz + \sum_{n \in \mathcal{Z}\setminus\{0\}} a_n \, g_1(n\pi)$$

where $\sum_{n \in \mathcal{Z}\setminus\{0\}} a_n \, g_1(n\pi)$ is a constant. Thus, from Theorem 15.18 of CHAMPENEY (1987), it is correct and valid to write

$$\frac{\partial}{\partial x}\left[\sum_{n \in \mathcal{Z}\setminus\{0\}} a_n \, e^{n\pi x i} g_1(n\pi)\right] = \sum_{n \in \mathcal{Z}\setminus\{0\}} \frac{\partial}{\partial x}\left[a_n \, e^{n\pi x i} g_1(n\pi)\right], \quad x \in \mathcal{C}$$

(2) If the Fourier coefficients, $a_n(f_C)$, satisfy $\sum_{n \in \mathcal{Z}\setminus\{0\}} |a_n(f_C)|^p < \infty \Rightarrow 1 < p \leq 2$ then the sum, $G_{2,\ell}$, must converge non–uniformly to some discontinuous, locally integrable 2–periodic function. However, uniform convergence is a *necessary* condition for the valid termwise differentiation of a series of uniformly continuous functions, thus, for almost all $x \in \mathcal{C}$,

$$\frac{\partial}{\partial x}\left[\sum_{n \in \mathcal{Z}\setminus\{0\}} a_n \, e^{n\pi x i} g_2(n\pi)\right] \neq \sum_{n \in \mathcal{Z}\setminus\{0\}} \frac{\partial}{\partial x}\left[a_n \, e^{n\pi x i} g_2(n\pi)\right] \qquad \text{(Pointwise limit)}$$

(3) The sums $G_{k,\ell}$, $k \geq 3$ are divergent as $\ell \to \infty$ – unless $g_k = 0$, $k \geq 3$ – because the general term, $F_n(x) = a_n \, e^{n\pi x i} g_k(n\pi)$, has the property $|F_n(x)| \not\to 0$ $(|n| \to \infty)$ for all $x \in \mathcal{C}$.

(4) From the observation that $\partial^m u_{0,\ell}(x)/\partial x^m = \sum_{n \in \mathcal{Z}_\ell\setminus\{0\}} a_n \, (n\pi i)^m e^{n\pi x i} \phi_0(n\pi)$ it is evident that, employing the same arguments used in (1) above, the sum of the derivatives of order m converges uniformly, as $\ell \to \infty$, to the corresponding derivative of $u_{0,\infty}$ provided $0 \leq m \leq 1$. However, as m increases to 2 the type of convergence weakens to the non–uniform pointwise variety and for $m \geq 3$ the sequence of partial sums of derivatives diverge.

Thus, for $f_C \in H^0(\mathcal{C})\setminus H^1(\mathcal{C})$, the approximations $u^\epsilon_{N,\ell}$ provided by relation (3.3.7) are well defined for $0 \leq N \leq 2$. *However, the termwise derivative of the partial sums* $u^\epsilon_{N,\ell}$, $\ell \in \mathbb{N}$ *provide valid approximations of the derivative of the limit functions* $u^\epsilon_N \stackrel{\text{def}}{=} u^\epsilon_{N,\infty}$ *only for* $0 \leq N \leq 1$. Although it is clear that the partial sums which define these asymptotic approximations, $u^\epsilon_{N,\ell}$, $0 \leq N \leq 2$, $\ell \in \mathbb{N}$, converge, with the type of convergence specified in paragraphs (1) – (4) above, they are derived from a representation of $\phi(x,\bullet,\bullet)$, $x \in \mathcal{P}$ which is valid only within a neighbourhood $\widehat{G} \subset \mathbb{C}^2$ of $\widehat{V} = \{(\varepsilon,t) \in \mathbb{R}^2 : |\varepsilon t| < 2\pi,\ |t| > 0\}$, cf. Conjecture 3.1.1. Therefore, based on the properties of ϕ furnished by Conjecture 3.1.1, we propose the following higher order *asymptotic approximations* $\tilde{u}^{\varepsilon}_{N,M,\ell}$, $N \geq 2$, $1 \leq M \leq 2$, $\ell \in \mathbb{N}$

$$\tilde{u}^{\varepsilon}_{N,M,\ell}(x) \overset{\text{def}}{=} \sum_{n \in \mathcal{Z}_{\tau(\varepsilon)} \setminus \{0\}} a_n\, e^{n\pi x i} \phi_N(x/\varepsilon, \varepsilon, n\pi) + \sum_{n \in \mathcal{Z}_\ell \setminus \mathcal{Z}_{\tau(\varepsilon)}} a_n\, e^{n\pi x i} \phi_M(x/\varepsilon, \varepsilon, n\pi) \tag{3.3.17}$$

where $\tau(\varepsilon) \overset{\text{def}}{=} \max\{n \in \mathbb{N} \mid n < 2/\varepsilon\}$. It is apparent from the definition of the approximations $\tilde{u}^{\varepsilon}_{N,M,\ell}$ that the type of convergence, as $\ell \to \infty$, is dictated by the choice of M. Indeed, the comments regarding u^{ε}_M above provide the necessary information to deduce how the approximations $\tilde{u}^{\varepsilon}_{N,M,\ell}$ converge as $\ell \to \infty$.

3.4. Sample problem: Smooth Data, $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathbb{R})$.

Let $a(x) = 1/(1 + \cos(2\pi x)/2)$, cf. Figure 3.4.0, $f(x) = \sin(\pi x)$ then the boundary value problem (3.1.1) becomes: Find $u^\varepsilon \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that

$$-\frac{\partial}{\partial x}\Big(\frac{1}{1 + \frac{1}{2}\cos(2\pi x/\varepsilon)}\frac{\partial u^\varepsilon}{\partial x}(x)\Big) = \sin(\pi x), \quad x \in \Omega = (0,1) \tag{3.4.1}$$

$$u^\varepsilon(0) = u^\varepsilon(1) = 0 \tag{3.4.2}$$

where $\alpha_1 = 2/3$, $\alpha_2 = 2$ (cf. (3.1.1)). Because f is 2–periodic and antisymmetric the extension f_C described in relations (3.1.4) and (3.1.5) is automatic, i.e., $f_C(x) = f(x)$, $x \in \mathbb{R}$ and therefore problem (3.1.3) is as above but with $\mathbb{R}$ replacing Ω and with the boundary conditions (3.4.2) omitted. The cell problem (3.1.19)–(3.1.23) then becomes

$$-\frac{\partial}{\partial x}\Big(\frac{1}{1 + \frac{1}{2}\cos(2\pi x)}\frac{\partial}{\partial x}\Big(e^{it\varepsilon x}\,\phi(x,\varepsilon,t)\Big)\Big) = \varepsilon^2 e^{it\varepsilon x}, \quad 0 < x < 1,\ \varepsilon > 0,\ |t| > 0 \tag{3.4.3}$$

$$\phi(0,\varepsilon,t) = \phi(1,\varepsilon,t) \tag{3.4.4}$$

$$\frac{\partial\phi}{\partial x}(x,\varepsilon,t)\Big|_{x=0} = \frac{\partial\phi}{\partial x}(x,\varepsilon,t)\Big|_{x=1} \tag{3.4.5}$$

The equations (3.4.4) and (3.4.5) are linearly independent everywhere in $\mathcal{O} = \mathbb{R}^2 \setminus (\mathcal{H}_{-1} \cup \mathcal{H}_1)$ and, solving this problem in $\mathcal{O}$, one obtains

$$\phi(x,\varepsilon,t) = \frac{-8\pi^2 + \varepsilon^2 t^2(2 + \cos(2\pi x)) - 2i\varepsilon\pi t \sin(2\pi x)}{2t^2(\varepsilon^2 t^2 - 4\pi^2)} \tag{3.4.6}$$

which is then, evidently, singular only on the hyperbolae $\mathcal{H}_{\pm 1}$ where ϕ is then specified as follows

$$\begin{aligned}\phi(x,\varepsilon,t) &= \phi(x,\varepsilon,n\pi) \\ &= \frac{\varepsilon^2}{64\pi^2}(16(1 - e^{-i\varepsilon t x}) + 2(e^{i\varepsilon t x} - e^{-i\varepsilon t x}) + e^{-i2\varepsilon t x} - 1), \quad n = \pm 1\end{aligned} \tag{3.4.7}$$

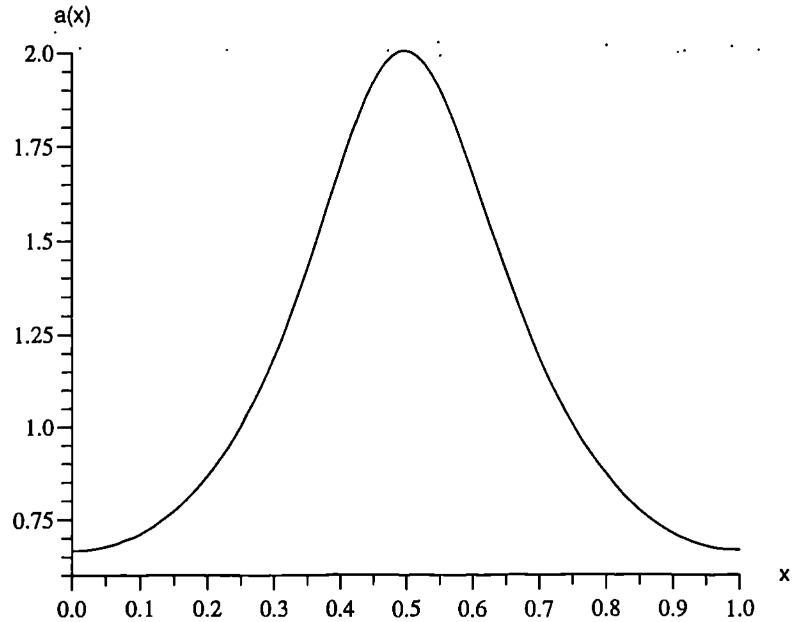


Fig. 3.4.0. a(x) = 1/(1 + cos(2 pi x)/2), 0 < x < 1.

where, in this instance, condition (3.2.46) is explicitly satisfied by the choice $d(\varepsilon, t) = 0$. However, for this problem, $\varepsilon < \mu(\Omega) = 1$ and t is restricted to the circle $\mathcal{C}_\pi = \{t \in \mathbb{R} \,:\, |t| = \pi\}$, consequently, ϕ is analytic within the domain of the cell problem $x \in \mathcal{P}$, $0 < \varepsilon < 1$, $t \in \mathcal{C}_\pi$. Thus, observing that $\varepsilon = 1/n_\varepsilon \le 1$, $n_\varepsilon \in \mathbb{N}$ and $a_1 = -a_{-1} = 1/2i$, $a_n = 0$, $n \neq \pm 1$, the analytical solution, u^ε, is

$$u^\varepsilon(x) = a_{-1}\, e^{-i\pi x}\phi(x/\varepsilon, \varepsilon, -\pi) + a_1\, e^{i\pi x}\phi(x/\varepsilon, \varepsilon, \pi) \tag{3.4.8}$$

$$= \frac{\sin(2/\varepsilon - 1)\pi x}{4\pi^2(2/\varepsilon - 1)} + \frac{\sin(\pi x)}{\pi^2} + \frac{\sin(2/\varepsilon + 1)\pi x}{4\pi^2(2/\varepsilon + 1)} \tag{3.4.9}$$

However, employing simple trigonometric identities and power series expansions for $|\varepsilon| < 2$ the solution, u^ε, is rewritten in the following form

$$\begin{aligned} u^\varepsilon(x) = \frac{\sin(\pi x)}{\pi^2} &+ \frac{\varepsilon}{4\pi^2} \sin(2\pi x/\varepsilon)\cos(\pi x)\Big(1 + \frac{\varepsilon^2}{2^2} + \frac{\varepsilon^4}{2^4} + \frac{\varepsilon^6}{2^6} + \dots\Big) \\ &- \frac{\varepsilon^2}{8\pi^2} \cos(2\pi x/\varepsilon)\sin(\pi x)\Big(1 + \frac{\varepsilon^2}{2^2} + \frac{\varepsilon^4}{2^4} + \frac{\varepsilon^6}{2^6} + \dots\Big) \end{aligned} \tag{3.4.10}$$

It is evident from relation (3.4.6) that the function $\phi(x, \bullet, \bullet)$, $x \in \mathcal{P}$ belongs to $C^\infty(\mathcal{O})$. Thus, computing the Taylor series expansion up to 6^{th} order asymptotic terms, one obtains, for $(\varepsilon, t) \in B(0, 2\sqrt{\pi}, \ell_2)$, the expression

$$\begin{aligned} \phi(x, \varepsilon, t) =& \frac{1}{t^2} + \varepsilon\, \frac{i\, \sin(2\pi x)}{4\pi t} - \varepsilon^2\, \frac{\cos(2\pi x)}{8\pi^2} + \varepsilon^3\, \frac{it\sin(2\pi x)}{16\pi^3} \\ &- \varepsilon^4\, \frac{t^2\cos(2\pi x)}{32\pi^4} + \varepsilon^5\, \frac{it^3\sin(2\pi x)}{64\pi^5} - \varepsilon^6\, \frac{t^4\cos(2\pi x)}{128\pi^6} + O(\varepsilon^7), \quad x \in \mathcal{P} \end{aligned} \tag{3.4.11}$$

However, we can now confirm, for this problem, that expansion (3.4.11) and (3.2.1) are identical. We compute the solution, $\chi_1(\bullet, t) \in H^1_{per,0}(\mathcal{P})$, of problem (3.2.8), $k = 1$, to be $\chi_1(x, t) = it\,\chi_1(x)$ where $\chi_1(x) = \sin(2\pi x)/4\pi$ is the solution of problem (3.2.12) and, from (3.2.9), (3.2.11), the homogenized coefficient is therefore given by

$$\begin{aligned} A &= 1/(\Phi_1(t)[\chi_1(\bullet, t), 1] + \Phi_2(t)[1, 1]) \\ &= \int_{\mathcal{P}} a(x)\left(1 + \frac{\partial \chi_1}{\partial x}(x)\right) dx = 1 \end{aligned} \tag{3.4.12}$$

Thus, from relations (3.2.10), (3.2.11),

$$\phi_0(t) = \frac{1}{t^2}, \quad |t| > 0 \tag{3.4.13}$$

Furthermore, solving problems (3.2.8) for $\chi_k(\bullet, t) \in H^1_{per,0}(\mathcal{P})$, $k \geq 1$ we determine

$$\chi_k(x, t) = \frac{(it)^k}{2^{2k}\pi^{2k-1}} \frac{d^{k-1}\sin(2\pi x)}{dx^{k-1}}, \quad x \in \mathcal{P}, \; |t| > 0 \tag{3.4.14}$$

Now, noting the above expression for $\chi_k(\bullet, t)$, $k \geq 1$ we calculate

$$\begin{aligned} &\Phi_1(t)[\chi_k(\bullet, t), 1] + \Phi_2(t)[\chi_{k-1}(\bullet, t), 1] = \\ &= -it \int_{\mathcal{P}} a(x) \frac{(it)^k}{2^{2k}\pi^{2k-1}} \frac{d^k \sin(2\pi x)}{dx^k} \overline{v(x)}\, dx + t^2 \int_{\mathcal{P}} a(x) \frac{(it)^{k-1}}{2^{2(k-1)}\pi^{2(k-1)-1}} \frac{d^{k-2}\sin(2\pi x)}{dx^{k-2}} \overline{v(x)}\, dx \\ &= -\frac{(it)^{k+1}}{2^{2k}\pi^{2k-1}} \left[\int_{\mathcal{P}} a(x) \frac{d^k \sin(2\pi x)}{dx^k} \overline{v(x)}\, dx - \int_{\mathcal{P}} a(x) \frac{d^k \sin(2\pi x)}{dx^k} \overline{v(x)}\, dx \right] \\ &= 0 \end{aligned} \tag{3.4.15}$$

Thus, observing formulae (3.2.9), we deduce that $g_k = 0$, $k \geq 1$ and, therefore, from (3.2.10), the terms, ϕ_k, $k \geq 1$, of the homogenization (3.2.1) are given as follows

$$\phi_k(x, t) = g_0(t) \frac{(it)^k}{2^{2k}\pi^{2k-1}} \frac{d^{k-1}\sin(2\pi x)}{dx^{k-1}}, \quad x \in \mathcal{P}, \; |t| > 0, \; k \geq 1 \tag{3.4.16}$$

It is now evident that the functions in (3.4.16) coincide with the corresponding terms of the Taylor series expansion (3.4.11). This demonstrates, for this problem, the equality of the expansions (3.2.1) and (3.2.27) as proven generally in Section 3.2.1. Indeed, within the open ball $B(0, 2\sqrt{\pi}, \ell_2)$, the power series expansion (3.2.11) of $\phi(x, \bullet, \bullet)$ is unique and, therefore, we expect this result. For $0 \leq N \leq 2$, we now employ the approximations

$$\begin{aligned} \phi_N(x, \varepsilon, t) &= \sum_{n=0}^{N} \varepsilon^n \phi_n(x, t) \\ &= \frac{1}{t^2} + \varepsilon \frac{i}{4\pi t} \sin(2\pi x)\, T_{m_1}(\varepsilon, t) - \varepsilon^2 \frac{1}{8\pi^2} \cos(2\pi x)\, T_{m_2}(\varepsilon, t) \end{aligned} \tag{3.4.17}$$

$$\begin{aligned} u^\varepsilon_N(x) &= \sum_{n \in \mathcal{Z}\setminus\{0\}} a_n\, e^{in\pi x} \phi_N(x/\varepsilon, \varepsilon, n\pi) \\ &= \frac{\sin(\pi x)}{\pi^2} + \frac{\varepsilon}{4\pi^2} \sin(2\pi x/\varepsilon)\cos(\pi x)\, T_{m_1}(\varepsilon, \pi) - \frac{\varepsilon^2}{8\pi^2} \cos(2\pi x/\varepsilon)\sin(\pi x)\, T_{m_2}(\varepsilon, \pi) \end{aligned} \tag{3.4.18}$$

where

$$T_m(\varepsilon, t) \stackrel{\text{def}}{=} 1 + \frac{\varepsilon^2 t^2}{2^2 \pi^2} + \frac{\varepsilon^4 t^4}{2^4 \pi^4} + \ldots + \frac{\varepsilon^{2m} t^{2m}}{(2\pi)^{2m}} \tag{3.4.19}$$

and (1) $m_1 = m_2 = m - 1$ if $N = 2m$, (2) $m_1 = m, m_2 = m - 1$ if $N = 2m + 1$. The following relation for the homogenization error is simply deduced from expressions (3.1.7) for u^ε and (3.2.14) for $u^\varepsilon_{N,\ell}$, $0 \le N \le 2$,

$$(u^\varepsilon_\ell - u^\varepsilon_{N,\ell})(x) = \sum_{n \in \mathbb{N}_\ell} 2i\, \Im\Big[a_n\, e^{n\pi x i}\, (\phi - \phi_N)(x/\varepsilon, \varepsilon, n\pi)\Big], \quad x \in \Omega, \quad \varepsilon > 0 \tag{3.4.20}$$

With this expression, we have computed the homogenization errors in both $\mathcal{L}_2(\Omega)$ norm and $H^1(\Omega)$ semi-norm topologies with the analytical expressions for ϕ, ϕ_N, $0 \le N \le 2$, determined above, used to compute the errors $\phi - \phi_N$. The integrals are approximated numerically by splitting each integral over Ω into a sum of integrals over subdomains $\Omega_i \subset \Omega$, $i \in \mathbb{N}$ and then applying to each of these integrals the 5-point Gauss-Legendre quadrature formula

$$\int_{-1}^{1} \gamma(x)\, dx = \sum_{k=1}^{5} H_k\, \gamma(x_k) + E_5(\gamma) \tag{3.4.21}$$

where the quadrature points, x_k, $1 \le k \le 5$, are determined as the roots of the Legendre polynomial $P_5(x) = (63x^5 - 70x^3 + 15x)/8$, i.e.,

$$x_k = 0, \pm \left[\frac{35 \pm \sqrt{280}}{63}\right]^{1/2}, \quad 1 \le k \le 5 \tag{3.4.22}$$

and the quadrature weights, H_k, $1 \le k \le 5$, are defined by the identity

$$H_k = \frac{(1 - x_k^2)}{18[P_6(x_k)]^2}, \quad 1 \le k \le 5 \tag{3.4.23}$$

where P_6 is the Legendre polynomial of degree 6 and, for $\gamma \in C^6(-1, 1)$, the quadrature error is $E_5(\gamma) = 13\, \gamma^{(6)}(\xi)/756 \cdot 6!$, $-1 < \xi < 1$, cf. HILDEBRAND (1987), pages 414–420.

Table 3.4.1: $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathcal{C})$

Cell Size, ε	$\Vert u^\varepsilon - u_0; \mathcal{L}_2(\Omega)\Vert$	$\vert u^\varepsilon - u_0; H^1(\Omega)\vert$
0.5	6.96263411(−3)	7.95774914(−2)
0.25	3.24157818(−3)	7.95774914(−2)
0.125	1.59245348(−3)	7.95774914(−2)
0.0625	7.92732513(−4)	7.95774914(−2)
0.03125	3.95930946(−4)	7.95774914(−2)
0.015625	1.97911105(−4)	7.95774914(−2)
	$O(\varepsilon)$	$O(1)$

Table 3.4.2: $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(C)$

Cell Size, ε	$\|u^\varepsilon - u_1^\varepsilon; \mathcal{L}_2(\Omega)\|$	$\|u^\varepsilon - u_1^\varepsilon; H^1(\Omega)\|$
0.5	1.74065853(−3)	1.98943729(−2)
0.25	4.05197273(−4)	9.94718643(−3)
0.125	9.95283422(−5)	4.97359322(−3)
0.0625	2.47728910(−5)	2.48679661(−3)
0.03125	6.18642103(−6)	1.24339830(−3)
0.015625	1.54618051(−7)	7.95774914(−4)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Table 3.4.3: $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(C)$

Cell Size, ε	$\|u^\varepsilon - u_2^\varepsilon; \mathcal{L}_2(\Omega)\|$	$\|u^\varepsilon - u_2^\varepsilon; H^1(\Omega)\|$
0.5	4.35164632(−4)	4.97359322(−3)
0.25	5.06496591(−5)	1.24339830(−3)
0.125	6.22052139(−6)	3.10849576(−4)
0.0625	7.74152850(−7)	7.77123940(−5)
0.03125	9.66628300(−8)	1.94280985(−5)
0.015625	1.20795400(−8)	4.85702462(−6)
	$O(\varepsilon^3)$	$O(\varepsilon^2)$

The graphs illustrated in Figures 3.4.1 – 3.4.6 clearly reveal the high accuracy of the asymptotic approximations, ϕ_N, $0 \leq N \leq 2$, of ϕ. Indeed, it is difficult to distinguish between the various approximations and the weak solution, ϕ, of problem (3.1.9). Thus, although graphical in nature, the figures demonstrate the utility of the low order asymptotic functions, ϕ_N, $0 \leq N \leq 2$, which provide accurate approximations of ϕ. However, we observe the disparity, characterized by a *spike*, between the asymptotic approximations and ϕ at the discrete points $t = \pm 2\pi/\varepsilon$ where ϕ becomes singular and ϕ_N, $0 \leq N \leq 2$ do not.

The results illustrated in the tables 3.4.1 – 3.4.3 clearly fulfill the error estimates provided by theorem 3.2.2., i.e.,

$$\|u^\varepsilon - u_N^\varepsilon; H^1(\Omega)\| \leq C_1\, \varepsilon^N, \quad N = 0, 1, 2, \ldots$$

Furthermore, they also suggest the following $\mathcal{L}_2(\Omega)$ error estimates, for $N = 0, 1, \ldots,$

$$\|u^\varepsilon - u_N^\varepsilon; \mathcal{L}_2(\Omega)\| \leq C_2\, \varepsilon^{N+1} \tag{3.4.24}$$

where C_1, $C_2 > 0$ are constants independent of ε. Further, the results imply that one will benefit from the inclusion of additional asymptotic terms in the expansion (3.2.1) or, equivalently, (3.2.14), with approximations of ever greater accuracy in both $\mathcal{L}_2(\Omega)$ and $H^1(\Omega)$ norms. Indeed, tables 3.4.1 – 3.4.3 illustrate precisely the successive improvements obtained by including higher order asymptotics where, in this instance, the coefficients are smooth.

Figure 3.4.1

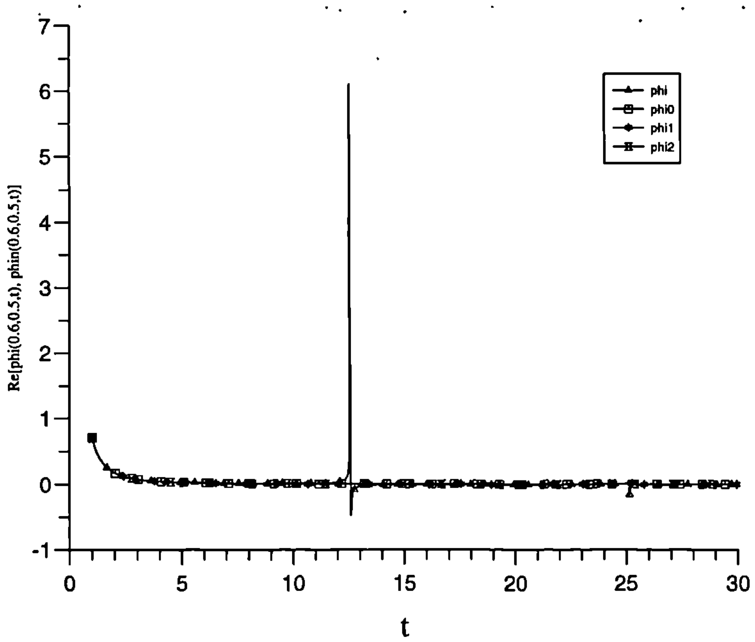
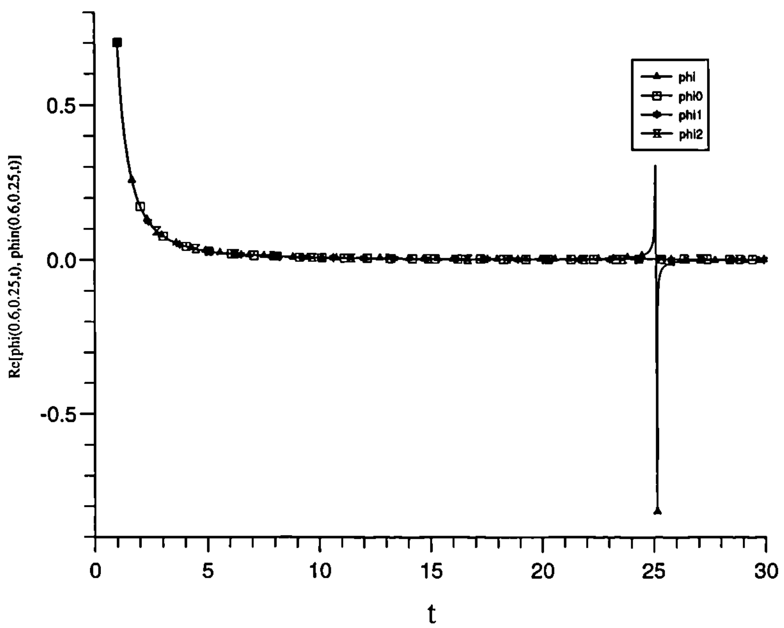


Figure 3.4.2



Graphs of the real or imaginary parts of $\phi(0.6, \varepsilon, t)$, $\phi_N(0.6, \varepsilon, t)$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3$, $0 \leq N \leq 2$, and $1 \leq t \leq 30$. The curves are distinguished by the symbols, e.g., $\triangle \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$.

Figure 3.4.3

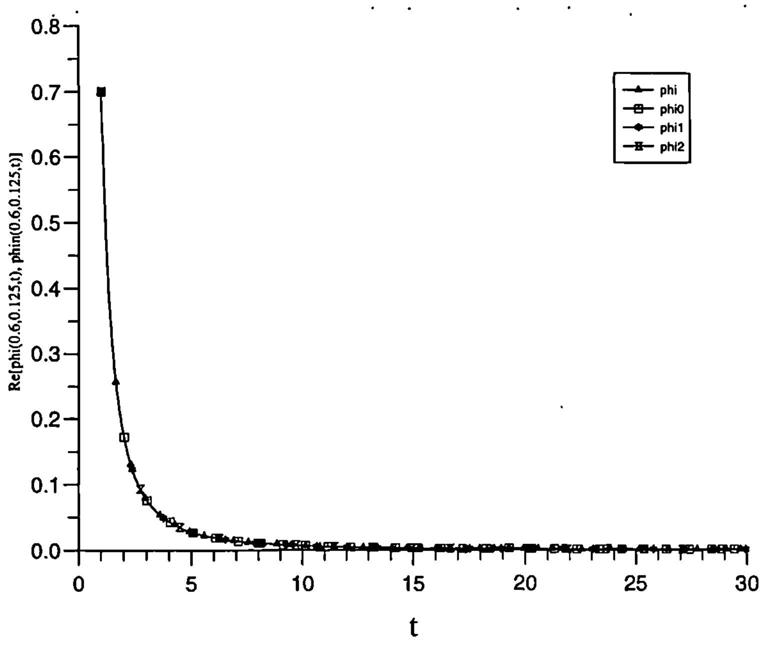
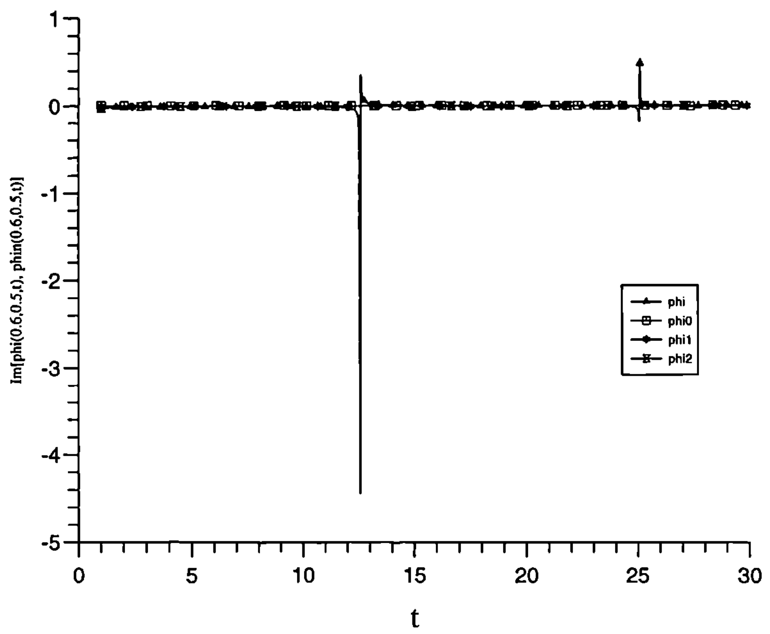


Figure 3.4.4



Graphs of the real or imaginary parts of $\phi(0.6, \varepsilon, t)$, $\phi_N(0.6, \varepsilon, t)$, $\varepsilon = 1/2^n$, $1 \le n \le 3$, $0 \le N \le 2$, and $1 \le t \le 30$. The curves are distinguished by the symbols, e.g., $\triangle \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$.

Figure 3.4.5

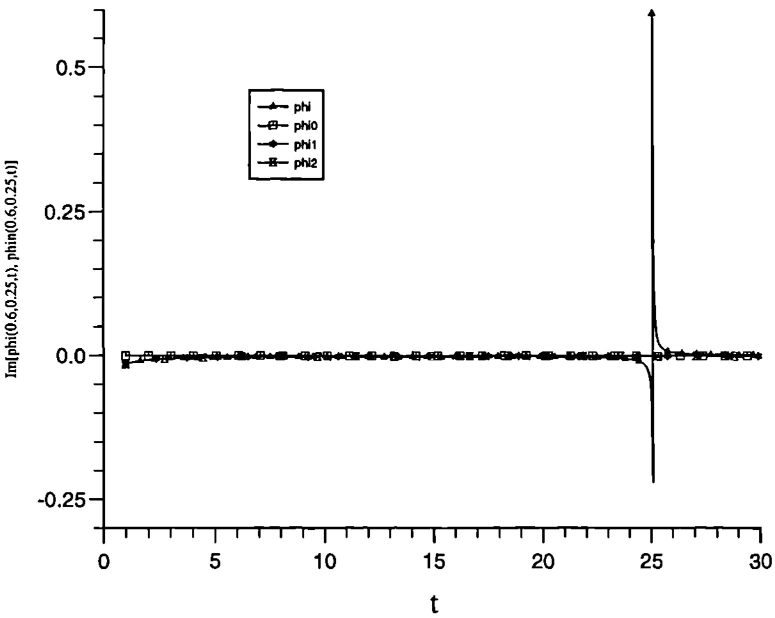
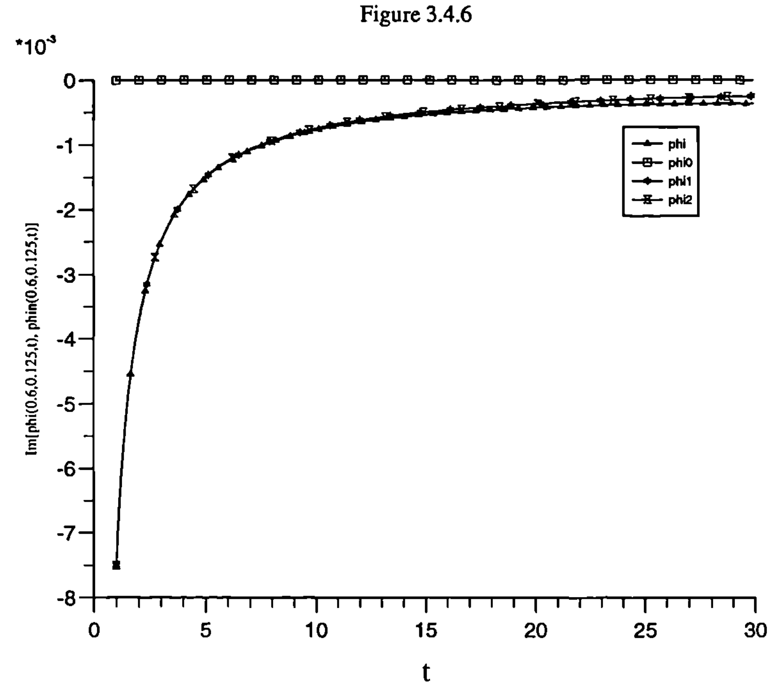


Figure 3.4.6



Graphs of the real or imaginary parts of $\phi(0.6, \varepsilon, t)$, $\phi_N(0.6, \varepsilon, t)$, $\varepsilon = 1/2^n$, $1 \le n \le 3$, $0 \le N \le 2$, and $1 \le t \le 30$. The curves are distinguished by the symbols, i.e., $\triangle \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$.

This is explained by the following identities for the errors $u^\varepsilon - u_N^\varepsilon, \partial(u^\varepsilon - u_N^\varepsilon)/\partial x$, obtained directly from the above expressions for $u^\varepsilon, u_N^\varepsilon$,

$$\begin{aligned} u^\varepsilon(x) - u_N^\varepsilon(x) &= \frac{\varepsilon^{N+1}}{2^{N+1}\pi^2}\left(\sin(2\pi x/\varepsilon)\cos(\pi x) - \frac{\varepsilon}{2}\cos(2\pi x/\varepsilon)\sin(\pi x)\right) S \\ \frac{\partial(u^\varepsilon - u_N^\varepsilon)}{\partial x}(x) &= \frac{\varepsilon^N}{2^N\pi}\left(1 - \frac{\varepsilon^2}{2^2}\right)\cos(2\pi x/\varepsilon)\cos(\pi x)\, S \end{aligned}$$

where $S = (1-\varepsilon/2)^{-1} = (1+\varepsilon^2/2^2+\varepsilon^4/2^4+\ldots)$. From these error equations the asymptotic error estimates (3.2.20), (3.4.24) now follow immediately with $C_1 = 2/\pi$, $C_2 = 3/2\pi^2$. The behaviour of the results tabulated in Tables 3.4.1 – 3.4.3 are also explained by these error identities.

3.5. Homogenization for Problems with Piecewise Smooth Data.

It has been shown above that boundary value problems, such as (3.1.1), with smooth coefficients lead to homogenizations, (3.2.1), which are nothing more than classical Taylor series expansions about an appropriate point in the $(\varepsilon,\, t)$–plane that converge in a generalized sense (compared to the classical concepts of pointwise or uniform convergence of formal power series expansions). By contrast we now consider problems of the type (3.1.1) but with non–smooth data; actually, piecewise smooth. We observe that the location of the singular points of $\phi(x, \bullet, \bullet)$, $x \in \mathcal{P}$ then depends on the coefficients and cannot, therefore, be easily determined for an abstract problem of this type. Thus, only the general characteristics are examined.

Let $a(x) = a_l(x)$, $x \in \mathcal{P}_l$, $l \in \mathbb{N}_m$ where $\overline{\mathcal{P}} = \cup_{l\in\mathbb{N}_m}\overline{\mathcal{P}}_l$, $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ if $i \neq j$ and $a_l \in C^1(\overline{\mathcal{P}}_l)$, $1 \le l \le m$ but $a \notin C^0(\mathcal{P})$. The weak formulation (3.1.9) is equivalent to (3.1.19)–(3.1.23) and the solution is given by $\phi(x,\varepsilon,t) = \phi_l(x,\varepsilon,t)$, $x \in \mathcal{P}_l$, $l \in \mathbb{N}_m$, $\varepsilon > 0$, $|t| > 0$ where

$$\phi_l(x,\varepsilon,t) = \frac{i\varepsilon}{t}e^{-it\varepsilon x}\int_{x_{l-1}}^{x}\frac{e^{it\varepsilon z}}{a(z)}\,dz + c_l(\varepsilon,\, t)\, e^{-it\varepsilon x}\int_{x_{l-1}}^{x}\frac{1}{a(z)}dz + d_l(\varepsilon,\, t)\, e^{-it\varepsilon x} \tag{3.5.1}$$

with the boundary and transition conditions (3.1.20)–(3.1.23) determining the arbitrary functions c_l, d_l, $l \in \mathbb{N}_m$. However, the resulting system of equations for these constants can be written

$$A(\varepsilon,t)\,\underline{\omega}(\varepsilon,t) = \underline{\tau}(\varepsilon,t) \tag{3.5.2}$$

where the column matrices $\underline{\tau}(\varepsilon,t)$, $\underline{\omega}(\varepsilon,t) \in \mathbb{C}^{2m}$ are as follows

$$\underline{\omega}(\varepsilon,t) = \Big[c_1(\varepsilon,t),\ d_1(\varepsilon,t),\ \ldots,\ c_{m-1}(\varepsilon,t),\ d_{m-1}(\varepsilon,t),\ c_m(\varepsilon,t),\ d_m(\varepsilon,t)\Big]^T \tag{3.5.3}$$

$$\underline{\tau}(\varepsilon,t) = \left[\frac{\varepsilon}{it}A_1(x_1,\varepsilon,t),\ 0,\ \ldots,\ \frac{\varepsilon}{it}A_{m-1}(x_{m-1},\varepsilon,t),\ 0,\ \frac{i\varepsilon}{t}A_m(1,\varepsilon,t),\ 0\right]^T \tag{3.5.4}$$

and the matrix $A(\varepsilon,t) \in \mathbb{C}^{2m,2m}$ has the coefficients specified in the identity below

$$A(\varepsilon,t) = \begin{bmatrix} B_1(x_1) & 1 & 0 & -1 & & & & & & & & \\ 1 & 0 & -1 & 0 & & & & & & & & \\ & & B_2(x_2) & 1 & 0 & -1 & & & & & & \\ & & 1 & 0 & -1 & 0 & \ddots & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & \ddots & B_{m-1}(x_{m-1}) & 1 & 0 & -1 \\ & & & & & & & 1 & 0 & -1 & 0 \\ 0 & e^{i\varepsilon t} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -B_m(1) & -1 \\ e^{i\varepsilon t} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 \end{bmatrix} \tag{3.5.5}$$

Rows $1,\dots,2m-2$ represent the interface conditions (3.1.20), (3.1.21) and rows $2m-1$, $2m$ represent the boundary conditions (3.1.22), (3.1.23). Furthermore, after examining (3.5.5), we observe some additional properties of the matrix function A below:

(1) If $(\varepsilon,t) \in \mathcal{H} \stackrel{\text{def}}{=} \cup_{n\in\mathcal{Z}\backslash\{0\}}\mathcal{H}_n$, $n \in \mathcal{Z}\backslash\{0\}$ then rows $2n$, $1 \le n \le m$ of $A(\varepsilon,t)$ are linearly dependent, e.g., $\sum_{n=1}^{m-1} r_{2n} = r_{2m}$ where r_n, $1 \le n \le 2m$ is row n of $A(\varepsilon,t)$ and, otherwise, for $(\varepsilon,t) \in \mathcal{O} \stackrel{\text{def}}{=} \mathbb{R}^2\backslash\mathcal{H}$, the rows of $A(\varepsilon,t)$ are linearly independent.

(2) The characteristic function, $p(A(\varepsilon,t)) = |A(\varepsilon,t) - \lambda(\varepsilon,t)\, I|$, has the quadratic form

$$p(A(\varepsilon,t)) = \gamma_1\, e^{2i\varepsilon t} + \gamma_2\, e^{i\varepsilon t} + \gamma_3 \tag{3.5.6}$$

where $\gamma_n \in \mathbb{R}$, $1 \le n \le 3$ are constants which are independent of ε, t. The characteristic equation, $p(A(\varepsilon,t)) = 0$, thus implies the identities

$$e^{i\varepsilon t} = -\frac{\gamma_2}{2\gamma_1} - \frac{1}{2\gamma_1}\sqrt{\gamma_2^2 - 4\gamma_1\gamma_3}\,,\; -\frac{\gamma_2}{2\gamma_1} + \frac{1}{2\gamma_1}\sqrt{\gamma_2^2 - 4\gamma_1\gamma_3} \tag{3.5.7}$$

$$\Rightarrow \quad \varepsilon t = -i\,\ln\left[-\frac{\gamma_2}{2\gamma_1} - \frac{1}{2\gamma_1}\sqrt{\gamma_2^2 - 4\gamma_1\gamma_3}\right],\; \ln\left[-\frac{\gamma_2}{2\gamma_1} + \frac{1}{2\gamma_1}\sqrt{\gamma_2^2 - 4\gamma_1\gamma_3}\right] \tag{3.5.8}$$

If $(\varepsilon,t) \notin \mathcal{H}$ then the components of the solution, $\underline{\omega}(\varepsilon,t)$, are given by the expressions

$$c_l(\varepsilon,t) = 0, \qquad d_l(\varepsilon,t) = \frac{i\varepsilon}{t}\left[\frac{A(1,\varepsilon,t)}{e^{i\varepsilon t}-1} + \sum_{s=1}^{l-1} A_s(x_s,\varepsilon,t)\right], \quad 1 \le l \le m \tag{3.5.9}$$

and, denoting by $\mathcal{S}(A)$ the set of all singular points defined according to the relation,

$$\mathcal{S}(A) \stackrel{\text{def}}{=} \left\{(\varepsilon,t) \in \mathbb{R}^2 \mid \|(\epsilon,\tau) - (\varepsilon,t)\|_2 \to 0 \Rightarrow \left\|\phi(\bullet,\epsilon,\tau); H^1(\mathcal{P})\right\| \to \infty\right\} \tag{3.5.10}$$

it is now evident from (3.5.9) that if singularities exist they arise, as in Section 3.2.1, along the hyperbolae $\mathcal{H}_n$, $n \in \mathcal{Z}\backslash\{0\}$, i.e., $\mathcal{S}(A) \subset \mathcal{H}$. Thus, $\phi(x,\bullet,\bullet)$, $x \in \mathcal{P}$ is holomorphic for $(\varepsilon,t) \in \mathcal{O} = \mathbb{R}^2\backslash\mathcal{H}$ and the analysis of the Taylor series representation performed in Section 3.2.1 is also valid here.

Conversely, if $(\varepsilon, t) \in \mathcal{H}$ then $A(\varepsilon, t)$ becomes singular with rank $2m-1$ and the coefficients $\underline{\omega}(\varepsilon, t)$ are underdetermined. However, it is clear from the definition of $A(\varepsilon, t)$ that $c_l(\varepsilon, t) = c(\varepsilon, t)$, $1 \leq l \leq m$ for unknown $c(\varepsilon, t)$ and the coefficients c, d_l, $2 \leq l \leq m$ can then be expressed in terms of d_1 as follows

$$c(\varepsilon, t) = \frac{1}{B(1)} \left[\frac{\varepsilon}{it} A(1, \varepsilon, t) - (m-1)\, d_1(\varepsilon, t) \right], \tag{3.5.11}$$

$$d_l(\varepsilon, t) = -\left[\frac{\varepsilon}{it} A(x_l, \varepsilon, t) - l d_1(\varepsilon, t) \right] - \frac{B(x_l)}{B(1)} \left[\frac{\varepsilon}{it} A(1, \varepsilon, t) - (m-1) d_1(\varepsilon, t) \right] \tag{3.5.12}$$

where A, B are defined in relation (3.2.24). Furthermore, the boundary condition $u^\varepsilon(0) = 0$ and the conjugate symmetry property (3.2.44) together imply the equations

$$\begin{aligned} u^\varepsilon(0) &= \sum_{n \in \mathbb{N} \setminus I(\varepsilon)} a_n \Big[\phi(0, \varepsilon, n\pi) - \phi(0, \varepsilon, -n\pi) \Big] + \sum_{n \in I(\varepsilon)} a_n \Big[d_1(\varepsilon, n\pi) - d_1(\varepsilon, -n\pi) \Big] \\ &= \sum_{k \in \mathbb{N} \setminus I(\varepsilon)} 2i\, a_n\, \Im[\phi(0, \varepsilon, n\pi)] + \sum_{n \in I(\varepsilon)} 2i\, a_n\, \Im[d_1(\varepsilon, n\pi)] \\ &= \sum_{n \in I(\varepsilon)} 2i\, a_n\, \Im[d_1(\varepsilon, n\pi)] \\ &= 0 \end{aligned} \tag{3.5.13}$$

However, because the function d_1 and the coefficients a_n, $n \in \mathcal{Z} \setminus \{0\}$ are independent from one another it follows that $d_1(\varepsilon, t) \in \mathbb{R}$ for $(\varepsilon, t) \in \mathcal{H}$. Thus, in the same fashion as Section 3.2.1, if one maintains the proviso that relation (3.5.13) is satisfied, then the choice of the function, $d_1(\varepsilon, t)$, is inconsequential insofar as it has no influence upon the solution u^ε.

The homogenization (3.2.1) is now applied to a number of sample problems with piecewise defined coefficients to determine the effects of low regularity on the behaviour of the asymptotic approximations obtained from this approach.

3.6. Sample problem: Piecewise smooth data, $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathbb{R})$.

Now let $f(x) = 1$, $x \in \Omega = (0, 1)$ and define $f_A(x)$, $x \in C = (-1, 1)$ and the coefficient a, on the canonical periodic cell, $\mathcal{P} = (0, 1)$, as follows

$$a(x) = \begin{cases} a_1 = 1, & 0 < x < 1/3 \\ a_2 = 10, & 1/3 \leq x < 2/3 \\ a_3 = 1, & 2/3 \leq x \leq 1 \end{cases}, \quad f_A(x) = \begin{cases} 1, & 0 < x \leq 1 \\ -1, & 1 < x \leq 2 \end{cases}, \quad a_n = \begin{cases} 2/n\pi i, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

where, then, f_C is the periodic extension of f_A to $\mathbb{R}$ defined by relation (3.1.5). In this instance $\alpha_1 = 0.1$, $\alpha_2 = 1$ and, clearly, $a \notin C^n(\mathcal{P})$, $n \geq 0$. However, a is a piecewise C^∞ function, see (3.1.18) with $a_l \in C^\infty(\overline{\mathcal{P}}_l)$, $\mathcal{P}_l = ((l-1)/3, l/3)$, $1 \leq l \leq 3$. With this data, the cell problem is then given by (3.1.19)-(3.1.23). The solution, ϕ, is, correspondingly, piecewise defined, i.e.,

$$\phi(x, \varepsilon, t) = \begin{cases} \phi_1(x, \varepsilon, t), & \text{if } \ 0 \leq x < 1/3 \\ \phi_2(x, \varepsilon, t), & \text{if } \ 1/3 \leq x < 2/3 \\ \phi_3(x, \varepsilon, t), & \text{if } \ 2/3 \leq x < 1 \end{cases} \tag{3.6.1}$$

where

$$\phi_1(x,\varepsilon,t) = \frac{1}{t^2} - \frac{9}{10t^2}\,\frac{e^{i\varepsilon t/3}}{1+e^{i\varepsilon t/3}+e^{2i\varepsilon t/3}}\,e^{-i\varepsilon t x} \tag{3.6.2}$$

$$\phi_2(x,\varepsilon,t) = \frac{1}{10t^2} + \frac{9}{10t^2}\,\frac{e^{2i\varepsilon t/3}+e^{i\varepsilon t}}{1+e^{i\varepsilon t/3}+e^{2i\varepsilon t/3}}\,e^{-i\varepsilon t x} \tag{3.6.3}$$

$$\phi_3(x,\varepsilon,t) = \frac{1}{t^2} - \frac{9}{10t^2}\,\frac{e^{4i\varepsilon t/3}}{1+e^{i\varepsilon t/3}+e^{2i\varepsilon t/3}}\,e^{-i\varepsilon t x} \tag{3.6.4}$$

Evidently, $\phi(x,\bullet,\bullet)$, $x \in \mathcal{P}$ is defined by relations (3.6.1)–(3.6.4) for all $(\varepsilon,t) \in \mathbb{R}^2\backslash\mathcal{H}$ where

$$\mathcal{S}(A) = \left\{(\varepsilon,t) \in \mathbb{R}^2 \mid 1 + e^{i\varepsilon t/3} + e^{i2\varepsilon t/3} = 0\right\} \tag{3.6.5}$$

However, the roots of the quadratic, $1 + e^{i\varepsilon t/3} + e^{i2\varepsilon t/3}$, are given by

$$e^{i\varepsilon t/3} = -1/2 + i\sqrt{3}/2,\ -1/2 - i\sqrt{3}/2 \tag{3.6.6}$$

$$\Rightarrow \qquad \varepsilon t = 2\pi + 6\pi n,\ 4\pi + 6\pi n,\ n \in \mathcal{Z} \tag{3.6.7}$$

It is now apparent that $\mathcal{S}(A) \subset \mathcal{H}$ where $\mathcal{H}$ is the family of hyperbolae $\mathcal{H}_n$, $n \in \mathcal{Z}\backslash\{0\}$ defined in Section 3.2.1. Thus, from direct knowledge of ϕ, we have determined that the singularities of $\phi(x,\bullet,\bullet)$, $x \in \mathcal{P}$ occur along hyperbolae, $\mathcal{H}$, in the (ε,t) plane, as indicated in Section 3.5. Evidently, $\phi(\bullet,\varepsilon,t) \in C^0(\mathcal{P})$ and $\phi(\bullet,\varepsilon,t) \notin C^n(\mathcal{P})$, $n \geq 1$ while $\phi_l(x,\bullet,\bullet) \in C^\infty(\mathcal{O})$, $x \in \mathcal{P}_l$, $1 \leq l \leq 3$ where $\mathcal{O} = \mathbb{R}^2\backslash\mathcal{H}$. One can therefore employ the classical Taylor series representation of $\phi(x,\bullet,\bullet)$ in the neighbourhood $(\varepsilon,t) \in B(0,2\sqrt{\pi},\ell_2)$, which are, to third order terms,

$$\phi_1(x,\varepsilon,t) = \frac{7}{10t^2} + \varepsilon\,\frac{3ix}{10t} + \varepsilon^2\,\frac{-2+27x^2}{180} + \varepsilon^3\,\frac{ixt}{180}(2-9x^2) + O(\varepsilon^4) \tag{3.6.8}$$

$$\begin{aligned}\phi_2(x,\varepsilon,t) = {} & \frac{7}{10t^2} + \varepsilon\frac{3i}{10t}\,(1-2x) + \varepsilon^2\,\frac{-11+54x-54x^2}{180} \\ & + \varepsilon^3\,\frac{it}{180}(-1+11x-27x^2+18x^3) + O(\varepsilon^4)\end{aligned} \tag{3.6.9}$$

$$\begin{aligned}\phi_3(x,\varepsilon,t) = {} & \frac{7}{10t^2} + \varepsilon\,\frac{3i}{10t}\,(-1+x) + \varepsilon^2\,\frac{25-54x+27x^2}{180} \\ & + \varepsilon^3\,\frac{it}{180}(7-25x+27x^2-9x^3) + O(\varepsilon^4)\end{aligned} \tag{3.6.10}$$

The proof of the equivalence between the homogenization, (3.2.1), and the Taylor series, (3.2.27), provided in Section 3.2.1 is clearly applicable here. Thus, the expansions (3.6.8) – (3.6.10) determine the homogenization (3.2.1) and the asymptotic approximations, ϕ_N, $N \geq 0$, defined in relation (3.2.13). Indeed we deduce the following identities from the asymptotic expansions (3.6.8)–(3.6.10)

$$A = \frac{10}{7},\quad g_0(t) = \frac{7}{10t^2},\quad \chi_1(x) = \begin{cases} 3x/7, & \text{if } \ 0 \leq x < 1/3 \\ 3/7 - 6x/7, & \text{if } \ 1/3 \leq x < 2/3 \\ -3/7 + 3x/7, & \text{if } \ 2/3 \leq x < 1 \end{cases} \tag{3.6.11}$$

where A is the homogenized coefficient occurring in the homogenized problem (3.2.15) and $\phi_1(x,t) = it\,g_0(t)\,\chi_1(x)$. Furthermore, from the asymptotic expansions (3.6.8) – (3.6.10) we deduce the following expressions for $\phi_2(x,t), \chi_3(x)$

$$\phi_2(x,t) = \begin{cases} \dfrac{-2+27x^2}{180}, & 0 < x < \frac{1}{3} \\ \dfrac{-11+54x-54x^2}{180}, & \frac{1}{3} < x < \frac{2}{3} \\ \dfrac{25-54x+27x^2}{180}, & \frac{2}{3} < x < 1 \end{cases}, \quad \chi_3(x) = \begin{cases} \dfrac{2x-9x^3}{180}, & 0 < x < \frac{1}{3} \\ \dfrac{-1+11x-27x^2+18x^3}{180}, & \frac{1}{3} < x < \frac{2}{3} \\ \dfrac{7-25x+27x^2-9x^3}{180}, & \frac{2}{3} < x < 1 \end{cases}$$

where $\phi_3(x,t) = it\,\chi_3(x)$ and, for this problem, therefore

$$g_k(t) = 0, \quad k = 1, 2, \ldots$$

The errors $\|u_\ell^\varepsilon - v; \mathcal{L}_2(\Omega)\|$, $|u_\ell^\varepsilon - v; H^1(\Omega)|$ have been computed, for $v = u_{N,\ell}^\varepsilon$, $\tilde{u}_{N,M,\ell}^\varepsilon$, $\ell = 1201$, in the same manner as for problem 3.4 and are reported in tables 3.6.1 – 3.6.4 below.

Table 3.6.1: $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\|u_\ell^\varepsilon - u_{0,\ell}; \mathcal{L}_2(\Omega)\|$	$\lvert u_\ell^\varepsilon - u_{0,\ell}; H^1(\Omega)\rvert$
0.5	3.24138702(−3)	3.97572749(−2)
0.25	1.48888933(−3)	4.15677910(−2)
0.125	7.27036348(−4)	4.20081448(−2)
0.0625	3.61309379(−4)	4.21174938(−2)
0.03125	1.80377530(−4)	4.21447906(−2)
0.015625	9.01540847(−5)	4.21516346(−2)
0.0078125	4.50727117(−5)	4.21533404(−2)
	$O(\varepsilon)$	$O(1)$

Table 3.6.2: $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\|u_\ell^\varepsilon - u_{1,\ell}^\varepsilon; \mathcal{L}_2(\Omega)\|$	$\lvert u_\ell^\varepsilon - u_{1,\ell}^\varepsilon; H^1(\Omega)\rvert$
0.5	1.22808159(−3)	9.99242444(−3)
0.25	3.07020378(−4)	4.99623845(−3)
0.125	7.67550388(−5)	2.49786206(−3)
0.0625	1.91889366(−5)	1.24907035(−3)
0.03125	4.79701252(−6)	6.24551693(−4)
0.015625	1.19894035(−6)	3.12262628(−4)
0.0078125	3.01178450(−7)	1.56140607(−4)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Although, in contrast to problem 3.4, the coefficient a is only piecewise smooth the figures 3.6.1–3.6.6 illustrate that the asymptotic functions, ϕ_N, $0 \leq N \leq 2$, provide accurate approximations of ϕ, the weak solution of (3.1.9). Indeed, we again observe that it is difficult to distinguish between the various curves which represent these approximations. This

Table 3.6.3: $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\|u_\ell^\varepsilon - u_{2,\ell}^\varepsilon; \mathcal{L}_2(\Omega)\|$	$\|u_\ell^\varepsilon - \tilde{u}_{2,1,\ell}^\varepsilon; H^1(\Omega)\|$
0.5	9.59140389(−4)	4.26226183(−3)
0.25	2.38943249(−4)	1.51247677(−3)
0.125	5.96432445(−5)	5.35989505(−4)
0.0625	1.49242031(−5)	1.89453716(−4)
0.03125	3.73256197(−6)	6.71881678(−5)
0.015625	9.33039470(−7)	2.37763798(−5)
0.0078125	2.35106500(−7)	8.44198637(−6)
	$O(\varepsilon^2)$	$O(\varepsilon^{3/2})$

Table 3.6.4: $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\|u_\ell^\varepsilon - \tilde{u}_{3,2,\ell}^\varepsilon; \mathcal{L}_2(\Omega)\|$	$\|u_\ell^\varepsilon - \tilde{u}_{3,1,\ell}^\varepsilon; H^1(\Omega)\|$
0.5	7.95161939(−4)	2.46108688(−3)
0.25	2.14111498(−4)	8.95024032(−4)
0.125	5.59894312(−5)	3.19064235(−4)
0.0625	1.43982448(−5)	1.12981244(−4)
0.03125	3.65790302(−6)	4.01071110(−5)
0.015625	9.22584800(−7)	1.41892658(−5)
0.0078125	2.33660560(−7)	5.14193646(−6)
	$O(\varepsilon^2)$	$O(\varepsilon^{3/2})$

supports, once more, the utility of the lower order approximations, ϕ_N, $0 \le N \le 2$. The large amplitudes, or *spikes*, apparent in $\phi(x, \varepsilon, \bullet)$ at the points $t = 2\pi n/\varepsilon$, $n \in \mathcal{Z}\backslash\{0\}$ are an obvious manifestation of the singularities, $\mathcal{H}$, observed above.

The computational results illustrated in tables 3.6.1 – 3.6.4, suggest, in contrast to problem 3.4, that the order of convergence of the approximations $u_{N,\ell}^\varepsilon$ never exceeds $O(\varepsilon^2)$ in the $\mathcal{L}_2(\Omega)$ norm topology and $O(\varepsilon)$ in the $H^1(\mathcal{C})$ norm topology. However, as demonstrated generally in Section 3.3, an important consequence of the low regularity $f_\mathcal{C} \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$ is that the higher order homogenization approximations, $u_{N,\ell}^\varepsilon$, $N \ge 3$, $\ell \in \mathbb{N}$, are unavailable, again contrasting with problem 3.4. This is evident from the homogenization (3.3.15) and series (3.2.14), for the ε^3 term in (3.2.14) has the asymptotic order $O(1)$ ($|n| \to \infty$) and $u_{N,\ell}^\varepsilon$, $N \ge 3$ therefore diverges as $\ell \to \infty$, i.e., $\|u_{N,\ell}^\varepsilon; \mathcal{L}_2(\Omega)\| \to \infty\,(\ell \to \infty)$. Thus, in Tables 3.6.3, 3.6.4 we examine instead the asymptotic approximations $\tilde{u}_{N,M,\ell}^\varepsilon$, $N \ge 1$, $1 \le M \le 2$, $\ell \in \mathbb{N}$ defined in Section 3.3, i.e.,

$$\tilde{u}_{N,M,\ell}^\varepsilon(x) = \sum_{n \in \mathcal{Z}_{\tau(\varepsilon)}\backslash\{0\}} a_n\, e^{n\pi x i}\phi_N(x/\varepsilon, \varepsilon, n\pi) + \sum_{n \in \mathcal{Z}_\ell \backslash \mathcal{Z}_{\tau(\varepsilon)}} a_n\, e^{n\pi x i}\phi_M(x/\varepsilon, \varepsilon, n\pi)$$

where $\tau(\varepsilon) = \{n \in \mathbb{N} \mid n < 2/\varepsilon\}$. The results suggest that, by employing these approximations, one can improve upon the accuracy, if not the order of convergence, of the $\mathcal{L}_2(\Omega)$

norm errors of the lower order approximations $u^{\varepsilon}_{N,\ell}$, $0 \le N \le 2$. Furthermore, the tables demonstrate that these approximations produce smaller $H^1(\Omega)$ semi–norm errors which also converge a half order of ε more rapidly as $\varepsilon \to 0$. The influence of low regularity is further examined in problem 3.7.

3.7. Sample problem: Mixed regularity data, $a \in C^\infty(\mathcal{P})$, $f_{\mathcal{C}} \in \mathcal{L}_2(\mathcal{C})\backslash H^1(\mathcal{C})$.

The previous problem demonstrated the consequences for convergence order and accuracy when both a and $f_{\mathcal{C}}$ have low regularity. The convergence rate quickly reached a finite upper limit in problem 3.6 while, by contrast, no such limit was observed in problem 3.4 and, comparing tables 3.6.1 – 3.6.4, 3.4.1 – 3.4.3, it is clear that the reduced regularity also degraded the accuracy of the approximations. We now attempt to isolate the different roles of a and $f_{\mathcal{C}}$ on the homogenization approach by considering the following related problem of mixed regularity where, now, $a \in C^\infty_{per}(\mathcal{P})$ and, once again, $f_{\mathcal{C}} \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$ are defined as follows

$$a(x) = \frac{1}{1+\frac{1}{2}\cos(2\pi x)}, \quad f_{\mathcal{A}}(x) = \begin{cases} 1, & \text{if } 0 < x \le 1 \\ -1, & \text{if } 1 < x \le 2 \end{cases}, \quad a_n = \begin{cases} 2/n\pi i, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \tag{3.7.1}$$

where $f_{\mathcal{C}}$ is then obtained via relation (3.1.5). The analytical expression for ϕ, the solution of the complex valued boundary value problem (3.1.9), is provided in problem 3.6. The errors $\|u^{\varepsilon}_{\ell} - v; \mathcal{L}_2(\Omega)\|$, $|u^{\varepsilon}_{\ell} - v; H^1(\Omega)|$ have been computed, for $v = u^{\varepsilon}_{N,\ell}, \tilde{u}^{\varepsilon}_{N,\ell}$, $\ell = 1201$, and are reported in the tables 3.7.1 – 3.7.2.

Table 3.7.1: $a \in C^\infty(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\|u^{\varepsilon}_{\ell} - u_{0,\ell}; \mathcal{L}_2(\Omega)\|$	$\lvert u^{\varepsilon}_{\ell} - u_{0,\ell}; H^1(\Omega)\rvert$
0.5	3.32870592(−3)	3.58210497(−2)
0.25	1.46891484(−3)	3.53030964(−2)
0.125	7.07923164(−4)	3.51720311(−2)
0.0625	3.50566358(−4)	3.51390935(−2)
0.03125	1.74856196(−4)	3.51308997(−2)
0.015625	8.73746467(−5)	3.51289137(−2)
0.0078125	4.36806389(−5)	3.51284249(−2)
	$O(\varepsilon)$	$O(1)$

Tables 3.7.1 – 3.7.2 demonstrate that, although the coefficient, a, is infinitely smooth, the homogenization exhibits the same characteristics as observed for problem 3.6 in which $a \in \mathcal{P}C^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$. Indeed, all of the characteristics noted for tables 3.6.1 – 3.6.2 concerning the asymptotic approximations $u^{\varepsilon}_{N,\ell}$, $\tilde{u}^{\varepsilon}_{N,M,\ell}$, $0 \le N \le 2$, $1 \le M \le 2$, $\ell \in \mathbb{N}$ are again apparent in this problem.

The restriction, $u^{\varepsilon}|_{\Omega}$, of the analytical solution, u^{ε}, can, evidently, be obtained directly by solving the boundary value problem (3.1.1). Performing this computation one obtains the

Figure 3 6 1

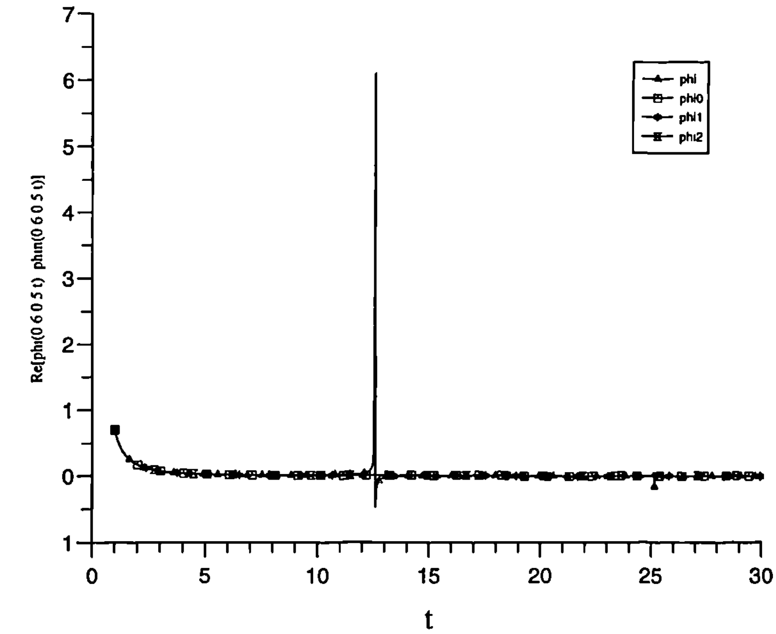
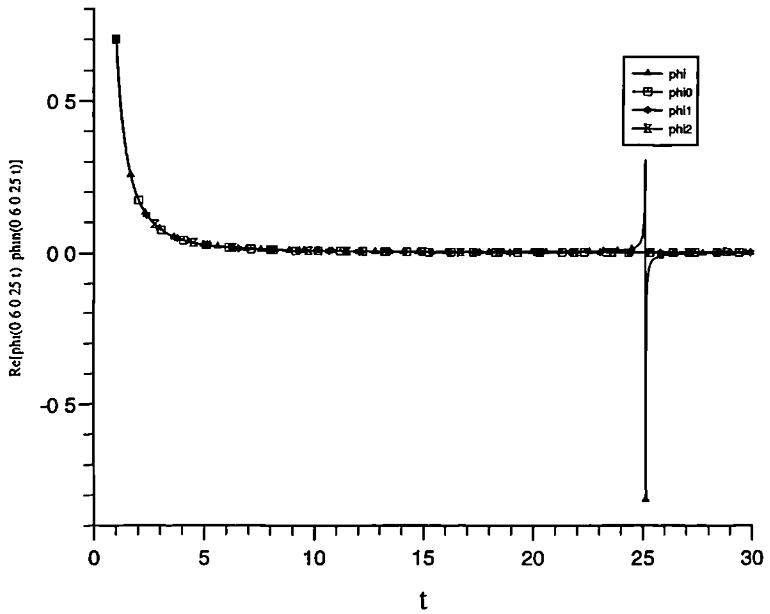


Figure 3 6 2



Graphs of the real or imaginary parts of $\phi(0\ 6,\varepsilon,t)$, $\phi_N(0\ 6,\varepsilon,t)$, $\varepsilon = 1/2^n$, $1 \le n \le 3$, $0 \le N \le 2$, and $1 \le t \le 30$ The curves are distinguished by the symbols, i e , $\triangle \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Figure 3 6 3

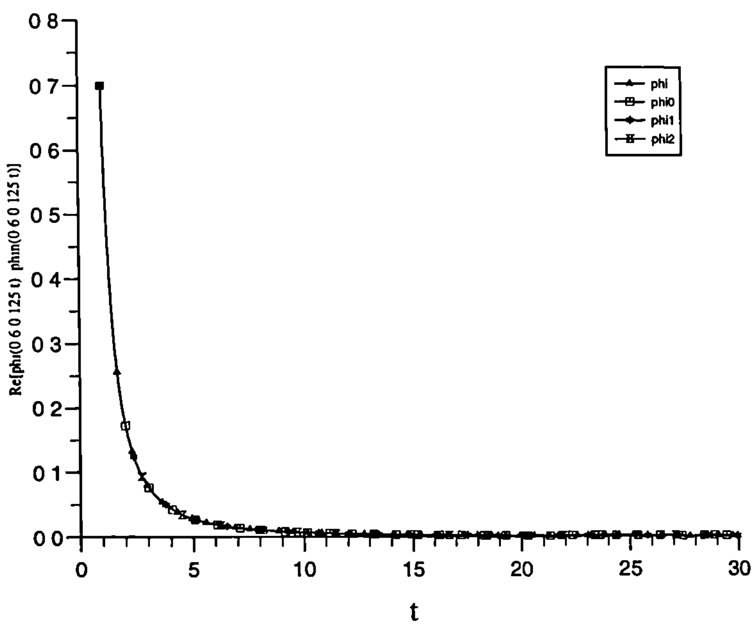
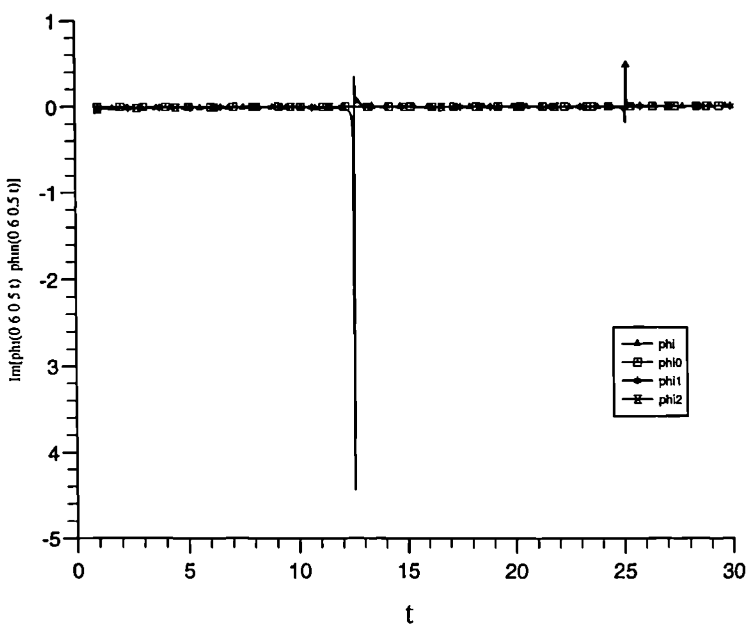


Figure 3 6 4



Graphs of the real or imaginary parts of $\phi(0\ 6, \varepsilon, t)$, $\phi_N(0\ 6, \varepsilon, t)$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3$, $0 \leq N \leq 2$, and $1 \leq t \leq 30$ The curves are distinguished by the symbols, i e , $\triangle \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Figure 3 6 5

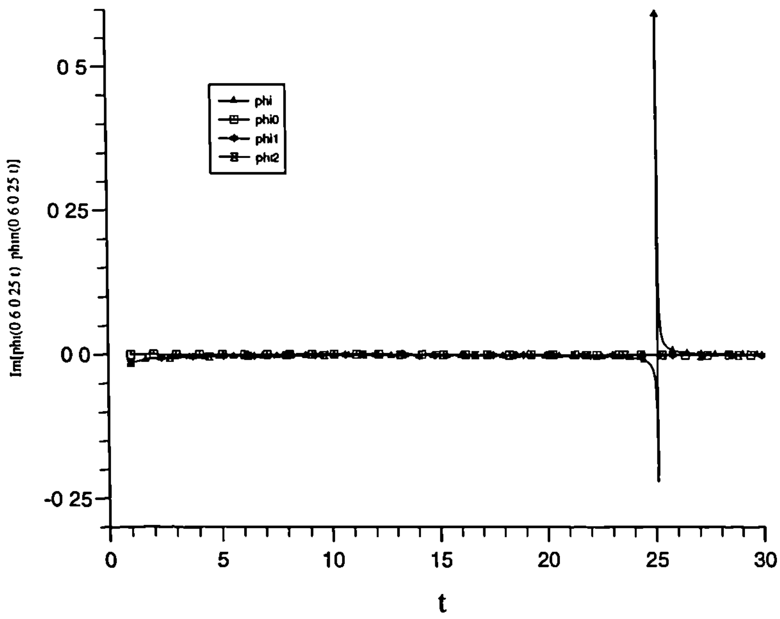
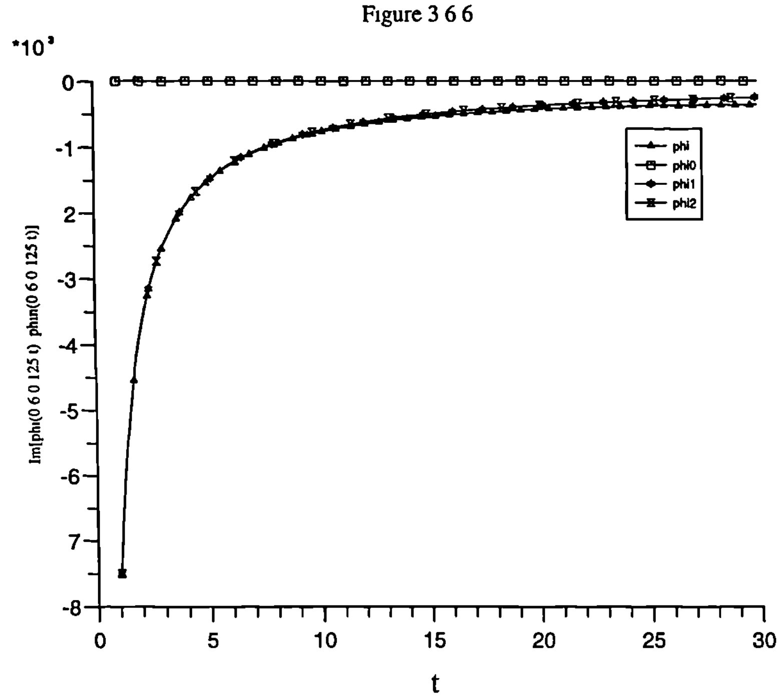


Figure 3 6 6



Graphs of the real or imaginary parts of $\phi(0\ 6, \varepsilon, t)$, $\phi_N(0\ 6, \varepsilon, t)$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3$, $0 \leq N \leq 2$, and $1 \leq t \leq 30$ The curves are distinguished by the symbols, i e , $\triangle \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Table 3 7 2 $a \in C^\infty(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\Vert u_\ell^\varepsilon - u_{1,\ell}^\varepsilon, \mathcal{L}_2(\Omega)\Vert$	$\vert u_\ell^\varepsilon - u_{1,\ell}^\varepsilon, H^1(\Omega)\vert$
0 5	1 33471402(−3)	9 68484915(−3)
0 25	3 33678477(−4)	4 84176303(−3)
0 125	8 34196183(−5)	2 42094460(−3)
0 0625	2 08548898(−5)	1 21065961(−3)
0 03125	5 21377082(−6)	6 05201079(−4)
0 015625	1 30343163(−6)	3 02621469(−4)
0 0078125	3 25858180(−7)	1 51314183(−4)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Table 3 7 3 $a \in C^\infty(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\Vert u_\ell^\varepsilon - u_{2,\ell}^\varepsilon, \mathcal{L}_2(\Omega)\Vert$	$\vert u_\ell^\varepsilon - \tilde{u}_{2,1,\ell}^\varepsilon, H^1(\Omega)\vert$
0 5	1 09169502(−3)	4 87789107(−3)
0 25	2 72684686(−4)	1 72902040(−3)
0 125	6 79958552(−5)	6 12103867(−4)
0 0625	1 70033284(−5)	2 16481353(−4)
0 03125	4 25200134(−6)	7 49622435(−5)
0 015625	1 06405310(−6)	2 72828034(−5)
0 0078125	2 66009860(−7)	9 73324486(−6)
	$O(\varepsilon^2)$	$O(\varepsilon^{3/2})$

Table 3 7 4 $a \in C^\infty(\mathcal{P})$, $f \in H^0(\mathcal{C})\backslash H^1(\mathcal{C})$

Cell Size, ε	$\Vert u_\ell^\varepsilon - \tilde{u}_{3,1,\ell}^\varepsilon, \mathcal{L}_2(\Omega)\Vert$	$\vert u_\ell^\varepsilon - \tilde{u}_{3,2,\ell}^\varepsilon, H^1(\Omega)\vert$
0 5	9 21893991(−4)	2 97189811(−3)
0 25	2 47140955(−4)	1 07162895(−3)
0 125	6 42423423(−5)	3 81218187(−4)
0 0625	1 64641798(−5)	1 36084073(−4)
0 03125	4 17605998(−6)	4 72273363(−5)
0 015625	1 05335661(−6)	1 70200860(−5)
0 0078125	2 64529370(−7)	6 15656966(−6)
	$O(\varepsilon^2)$	$O(\varepsilon^{3/2})$

following identity for $u^\varepsilon(x)$, $x \in \Omega$

$$u^\varepsilon(x) = \frac{1}{2}(x - x^2) + \varepsilon\,(1/2 - x)\,\frac{1}{4\pi}\,\sin(2\pi x/\varepsilon) + \varepsilon^2\left[-\frac{1}{8\pi^2}\cos(2\pi x/\varepsilon) + \frac{1}{8\pi^2}\right] \quad (3\ 7\ 2)$$

$$= u_0(x) + \varepsilon\,\frac{\partial u_0}{\partial x}(x)\,\chi_1(x/\varepsilon) + \varepsilon^2\left[\frac{\partial^2 u_0}{\partial x^2}(x)\,\chi_2(x/\varepsilon) + \frac{1}{8\pi^2}\right]$$

where it is assumed that $1/\varepsilon \in \mathbb{N}$ We now construct u^ε as the 2–periodic anti–symmetric extension of the solution $u^\varepsilon|_\Omega$ by computing, with the aid of Fourier series expansions, 2–periodic extensions of the functions $\alpha(x) = (x - x^2)/2$, $\beta(x) = (1/2 - x)$ of u^ε The respective antisymmetric and symmetric extensions of α and β are thus, for $x \in \mathbb{R}$,

$$\alpha(x) = \sum_{n\in\mathcal{Z}\setminus\{0\}} a_n \frac{1}{n^2\pi^2} e^{n\pi x i}, \qquad \beta(x) = \sum_{n\in\mathcal{Z}\setminus\{0\}} a_n \frac{i}{n\pi} e^{n\pi x i} \tag{3 7 3}$$

Substituting relations (3 7 3) into (3 7 2), the following 2–periodic antisymmetric extension is obtained for u^ε

$$\begin{aligned} u^\varepsilon(x) &= \sum_{n\in\mathcal{Z}\setminus\{0\}} a_n \left[\frac{1}{n^2\pi^2} + \varepsilon\,\frac{\sin(2\pi x/\varepsilon)}{4\pi}\,\frac{i}{n\pi} + \varepsilon^2\,\frac{(1-\cos(2\pi x/\varepsilon))}{8\pi^2}\right] e^{n\pi x i} \\ &= \sum_{n\in\mathcal{Z}\setminus\{0\}} a_n \left[\phi_0(n\pi) + \varepsilon\,\phi_1(x/\varepsilon, n\pi) + \varepsilon^2\left(\phi_2(x/\varepsilon, n\pi) + \frac{1}{8\pi^2}\right)\right] e^{n\pi x i} \end{aligned}$$

where the identity in the second line follows immediately from the expansion (3 4 11) However, from the homogenization (3 2 14) and the above Fourier series expression for u^ε, the following error estimates are now immediately apparent for the limit functions $u^\varepsilon_N \stackrel{\text{def}}{=} u^\varepsilon_{N,\infty}$, $0 \leq N \leq 2$

$$\|u^\varepsilon - u^\varepsilon_N, \mathcal{L}_2(\Omega)\| \leq C_1\,\varepsilon^{\min(N+1,2)}, \quad |u^\varepsilon - u^\varepsilon_N, H^1(\Omega)| \leq C_2\,\varepsilon^N,$$

where C_1, $C_2 > 0$ are constants independent of ε Furthermore, for this problem, if $N = 2$ then one can select $C_2 = 0$ Indeed, these error bounds are confirmed by the results illustrated in Tables 3 7 1 – 3 7 4 However, as observed in Section 3 3, the regularity property $f_C \in H^0(\mathcal{C})\setminus H^1(\mathcal{C})$ means that one cannot obtain, for $\ell \to \infty$, valid $H^1(\Omega)$ norm estimates of u^ε from the approximations $u^\varepsilon_{N,\ell}$, $N \geq 2$, $\ell \in \mathbb{N}$ or valid $\mathcal{L}_2(\Omega)$ norm estimates of u^ε from the approximations $u^\varepsilon_{N,\ell}$, $N \geq 3$, $\ell \in \mathbb{N}$ because of the nature of convergence of these functions as $\ell \to \infty$ Thus, we apply, as in problem 3 6, the functions $\tilde{u}^\varepsilon_{N\,M,\ell}$, $N \geq 2$, $1 \leq M \leq 2$, $\ell \in \mathbb{N}$ and the results provided in Tables 3 7 1 – 3 7 4 suggest the following error bounds, for $\ell \in \mathbb{N}$, $N \geq 2$,

$$\begin{aligned} \|u^\varepsilon_\ell - \tilde{u}^\varepsilon_{N\,M,\ell}, \mathcal{L}_2(\Omega)\| &\leq C_1\,\varepsilon^{\min(M+1\;2)}, \quad 1 \leq M \leq 2 \\ \|u^\varepsilon_\ell - \tilde{u}^\varepsilon_{N,M,\ell}, H^1(\Omega)\| &\leq C_2\,\varepsilon^{\min(N,3/2)}, \quad M = 1 \end{aligned}$$

In a private communication Professor Ivo Babuška has demonstrated that for a specific problem of the type being considered here the rate of convergence of $u^\varepsilon_{1,\ell}$ to u^ε_ℓ as $\varepsilon \to 0$ cannot exceed 3/2 Indeed, the results of Table 3 7 3 bear out this finding We observe that, although the level of regularity of a is an important factor in obtaining accurate asymptotic approximations derived from the homogenization approach, it does not affect the rate of convergence It is the regularity properties of f_C which exert the dominant influence on the convergence behaviour for $\varepsilon \to 0$ This property of the homogenization is examined further in problem 3 8

3 8 Sample problem Mixed regularity data, $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathcal{C})$

It has been determined from problems 3 6, 3 7 that the behaviour of the homogenization when $f_C \in H^0(\mathcal{P})\backslash H^1(\mathcal{P})$ and a is either piecewise or globally smooth is unchanged To emphasize the effect of the regularity of the function f_C on the homogenization, we consider, with respect to the regularity of the data a, f_C, the converse situation to the previous problem, 3 7, i e , define

$$a(x) = \begin{cases} a_1 = 1, & \text{if } 0 < x < 1/3 \\ a_2 = 10, & \text{if } 1/3 \le x < 2/3 \\ a_3 = 1, & \text{if } 2/3 \le x \le 1 \end{cases}, \qquad f_C(x) = \sin(\pi x) \tag{3 8 1}$$

The weak solution, ϕ, of the cell problem (3 1 19) – (3 1 23), which is also piecewise defined, is given in relations (3 6 1) – (3 6 4) and the weak solution, u^ε, of problem (3 1 1) is determined from relation (3 1 7) Once again, the errors, $\|u^\varepsilon - u^\varepsilon_N, \mathcal{L}_2(\Omega)\|$, $|u^\varepsilon - u^\varepsilon_N, H^1(\Omega)|$, have been computed and are reported in the tables 3 8 1 – 3 8 3

Table 3 8 1 $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon - u_0, \mathcal{L}_2(\Omega)\|$	$\|u^\varepsilon - u_0, H^1(\Omega)\|$
0 5	7 11253489(−3)	9 54929897(−2)
0 25	3 32217720(−3)	9 54929897(−2)
0 125	1 63344765(−3)	9 54929897(−2)
0 0625	8 13316124(−4)	9 54929897(−2)
0 03125	4 06233558(−4)	9 54929897(−2)
0 015625	2 03063761(−4)	9 54929897(−2)
0 0078125	1 01525255(−4)	9 54929897(−2)
	$O(\varepsilon)$	$O(1)$

Table 3 8 2 $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon - u^\varepsilon_1, \mathcal{L}_2(\Omega)\|$	$\|u^\varepsilon - u^\varepsilon_1, H^1(\Omega)\|$
0 5	1 73930827(−3)	2 04124196(−2)
0 25	4 05197388(−4)	1 02062098(−2)
0 125	9 95487155(−5)	5 10310490(−3)
0 0625	2 47792450(−5)	2 55155245(−3)
0 03125	6 18808814(−6)	1 27577623(−3)
0 015625	1 54660220(−6)	6 37888113(−4)
0 0078125	3 86624310(−7)	3 18944056(−4)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Thus, despite the low regularity of the coefficient a, the higher order approximations, u^ε_N, $N \ge 3$, are available once again and the lower order approximations, u^ε_N, $N = 0, 1, 2$, behave in an identical fashion to that observed for problem 3 4 which also possessed an

Table 3 8 3 $a \in \mathcal{PC}^\infty(\mathcal{P})\backslash C^0(\mathcal{P})$, $f \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon - u_2^\varepsilon, \mathcal{L}_2(\Omega)\|$	$\|u^\varepsilon - u_2^\varepsilon, H^1(\Omega)\|$
0 5	4 32587839(−4)	4 97495505(−3)
0 25	5 03594314(−5)	1 24373876(−3)
0 125	6 18519067(−6)	3 10934691(−4)
0 0625	7 69765770(−7)	7 77336726(−5)
0 03125	9 61153600(−8)	1 94334182(−5)
0 015625	1 20111300(−8)	4 85835454(−6)
0 0078125	1 50129000(−9)	1 21458863(−6)
	$O(\varepsilon^3)$	$O(\varepsilon^2)$

infinitely smooth inhomogeneous term f_C The problems 3 6–3 8 and their results are now analysed and explained Furthermore, a Theorem is proposed which both summarizes and generalizes the properties of the homogenization approach described here

3 9 Analysis and Conclusions

The homogenization (3 2 1) was observed, in problem 3 4, to provide asymptotic approximations u_N^ε, $N \geq 0$, defined by relation (3 2 14), of the solution, u^ε, of the boundary value problem (3 1 1), which become ever more accurate, as $N \to \infty$, in precise accordance with the Bakhvalov and Panasenko Theorem 3 2 2 This is exactly what one should expect for $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathcal{C})$ where also, therefore, $u^\varepsilon \in C^\infty(\mathcal{C})$ However, to determine both the roles and affects of the functions a, f_C on the homogenization we considered various problems with regularity characteristics lower than those displayed in problem 3 4

We assume that $f_C \in H^m(\mathcal{C})\backslash H^{m+1}(\mathcal{C})$ and observe from the regularity theory that $u_0 \in H^{m+2}(\mathcal{C})\backslash H^{m+3}(\mathcal{C})$ However, if we recall the *two-scale* expansion (3 3 15), i e ,

$$u_{N\,\ell}^\varepsilon(x) = u_{0\,\ell}(x) + \varepsilon\left[\chi_1(x/\varepsilon)\,\frac{\partial u_{0,\ell}}{\partial x}(x) + G_{1,\ell}(x)\right] + \varepsilon^2\left[\chi_2(x/\varepsilon)\,\frac{\partial^2 u_{0,\ell}}{\partial x^2}(x) + G_{2,\ell}(x)\right]$$
$$+ \quad + \varepsilon^N\left[\chi_N(x/\varepsilon)\,\frac{\partial^N u_{0,\ell}}{\partial x^N}(x) + G_{N,\ell}(x)\right] \qquad (3\,9\,1)$$

where

$$u_{0,\ell}(x) = \sum_{n\in Z_\ell\backslash\{0\}} a_n\, e^{n\pi x \imath}\phi_0(n\pi), \quad G_{k,\ell}(x) = \sum_{n\in Z_\ell\backslash\{0\}} a_n\, e^{n\pi x\imath} g_k(n\pi) \qquad (3\,9\,2)$$

then the property $u_0 \in H^{m+2}(\mathcal{C})\backslash H^{m+3}(\mathcal{C})$ suggests that the derivative $D^\alpha u_{0\,\ell}$, $\alpha \geq m+3$ and, therefore, the asymptotic approximation $u_{\alpha\,\ell}^\varepsilon$, $\alpha \geq m+3$, cannot converge as $\ell \to \infty$, in either $\mathcal{L}_2(\mathcal{C})$ or $H^1(\mathcal{C})$ norm topologies Indeed, as a consequence of the property $D^m f_C \in \mathcal{L}_2(\mathcal{C})$ it follows that $a_n(f_C) = o(|n|^{-m})$ $(|n| \to \infty)$, $|n^{m+k} a_n(f_C)| \to \infty$ $(|n| \to \infty)$, $k \geq 1$ and, therefore, because the modulus of the general term of $D^\alpha u_{0,\ell}$, $\alpha \geq m+3$ satisfies $|a_n(f_C)\,(n\pi\imath)^\alpha e^{n\pi x\imath}\phi_0(n\pi)| = A^{-1}\pi^\alpha |n^{\alpha-2} a_n(f_C)| \not\to 0$ $(|n| \to \infty)$ the termwise derivatives $D^\alpha u_{0,\ell}$, $\alpha \geq m+3$ all diverge as $\ell \to \infty$ as observed above Thus, for low regularity problems of this type we must consider alternative asymptotic approximations to $u^{\varepsilon}_{N,\ell}$, $\ell \in \mathbb{N}$ for $N \geq m+3$ It is for this reason that we introduced in Section 3 3 the approximations $\tilde{u}^{\varepsilon}_{N,M,\ell}$, $N \geq m+3$, $M \leq m+2$ which exploit the good approximation properties of ϕ_N within the region of analyticity of $\phi(x,\bullet,\bullet)$, $x \in \mathcal{P}$

Based on the analysis and computations performed in Sections 3 4, 3 6–3 8 we propose the following theorem for the general asymptotic behaviour of the homogenization approach founded on (3 2 1)

Conjecture 3 9 *Let $a \in \mathcal{PC}^{\infty}_{per}(\mathcal{P})$, $f_{\mathcal{C}} \in H^m(\mathcal{C})$ then the functions $u^{\varepsilon}_{N,\ell}$, $u^{\varepsilon}_N \stackrel{\text{def}}{=} u^{\varepsilon}_{N,\infty}$, and $\tilde{u}^{\varepsilon}_{N,M,\ell}$ have the following asymptotic approximation properties*

$$\|u^{\varepsilon} - u^{\varepsilon}_N, H^p(\mathcal{C})\| \leq C\,\varepsilon^{\min(N+1,m+2)-p}, \quad 0 \leq N \leq m+2 \tag{3 9 3}$$

$$\|u^{\varepsilon}_{\ell} - u^{\varepsilon}_{N,\ell}, H^p(\mathcal{C})\| \leq C\,\varepsilon^{\min(N+1,m+2)-p}, \quad 0 \leq N \leq m+2-p \tag{3 9 4}$$

$$\|u^{\varepsilon}_{\ell} - \tilde{u}^{\varepsilon}_{N,M,\ell}, H^p(\mathcal{C})\| \leq C\,\varepsilon^{\min(N+1,m+2)-p/2}, \quad N \geq m+2,\ M = m+2-p \tag{3 9 5}$$

where $0 \leq p \leq 1$, $\ell \in \mathbb{N}$, $C > 0$ is a constant independent of ε, and $u^{\varepsilon} \in H^{m+\lambda}(\mathcal{C}) \cap H^1_0(\mathcal{C})$, $1 < \lambda < 2$ is the weak solution of problem (3 1 1) ■

We have not included $H^1(\mathcal{C})$ error estimates for $u^{\varepsilon}_{m+2,\ell}$ in relation (3 9 4) because, as indicated above, $\|u^{\varepsilon}_{m+2,\ell}, H^1(\mathcal{C})\| \to \infty$ $(\ell \to \infty)$ and, consequently, this function cannot provide a valid $H^1(\mathcal{C})$ norm approximation of u^{ε} This occurs because the asymptotic approximation, $u^{\varepsilon}_{m+2,\ell}$, cannot be differentiated term by term – this was demonstrated in Section 3 7 However, in Sections 3 4, 3 6–3 8 it occurred that $g_k = 0$, $k \geq 1$ and, in such a circumstance, (3 9 1) then implies that, for $0 \leq N \leq m+2$, $x \in \mathcal{C}$,

$$u^{\varepsilon}_N(x) = u_0(x) + \varepsilon\,\chi_1(x/\varepsilon)\,\frac{\partial u_0}{\partial x}(x) + \varepsilon^2\,\chi_2(x/\varepsilon)\,\frac{\partial^2 u_0}{\partial x^2}(x) + \quad + \varepsilon^N\,\chi_N(x/\varepsilon)\,\frac{\partial^N u_0}{\partial x^N}(x) \tag{3 9 6}$$

It may then be preferable to seek the asymptotic approximations u^{ε}_N $(= u^{\varepsilon}_{N,\infty})$ in the form (3 9 6), cf BAKHVALOV & PANASENKO (1989), clearly, there are no series truncation errors and possibly no reduction in the convergence rates occasioned by termwise differentiation as observed in (3 9 5)

4 Homogenization of Two Dimensional Elliptic Boundary Value Problems

4 0 Introduction

As part of the route towards our stated goal we now move to problems with the next higher order of difficulty and follow the format of Chapter 3 Thus, we now consider elliptic boundary problems in $\mathbb{R}^2$ where the material properties of the medium, Ω, change periodically and irregularly on a scale, ε, due to the presence of composite materials The asymptotic approach developed in Chapter 3, i e , homogenization, is extended to include boundary value problems of this type However, we observe that, for $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the analytical expressions for u^ϵ and u_N^ϵ, $N \geq 0$ employed in the homogenization approach are generally unavailable In order to overcome this lack of analytical information we resort to using finite element techniques to construct accurate and robust discrete asymptotic approximations which are analogous to those employed in Chapter 3 In using finite element methods, we naturally wish to exploit known a priori estimates for the error Such estimates depend on the regularity of the solution, which, in turn, depends on the geometry of the domain, the geometry of the material interface and material properties With polygonal interfaces, singularities will occur at the vertices The approach adopted here is to take finite element meshes which coincide with these interfaces and to state the finite element error estimates in terms of parameters defining the dominant form of the singularity It is not our purpose here to embark on a detailed treatment of these singularities Guided by our experiments in the one dimensional setting in Chapter 3, we assess the behaviour of the combined homogenization/finite element approach for a variety of problems exhibiting various levels of regularity In this way we determine how the various regularity characteristics of the problem affect the homogenization approach

The difficulties caused by the presence, in the model problem, of rapidly changing coefficients of low regularity for the direct application of conventional finite element approaches were considered in the one dimensional case in the previous chapter, cf Section 3 0 It was observed that finite element techniques applied directly to the model problem could not resolve, within practical constraints, the variations of the coefficients necessary to construct accurate numerical approximations However, the observations in Theorem 3 0 1 of the asymptotic behaviour, as $\varepsilon \to 0$, of the coefficients and solutions of elliptic boundary value problems led to the approach called homogenization In Chapter 3 we observed that this approach introduces errors which decrease as $\varepsilon \to 0$, i e , as the variation of the coefficient, $V_{\mathcal{P}}(a)$, increases Indeed, for ε–periodic coefficients it was demonstrated that the asymptotic approximations, u_N^ε, $N \geq 0$, obtained from the homogenization approach, exhibit the following properties for $\Omega \subset\subset \mathbb{R}$

$$\|u^\varepsilon - u_N^\varepsilon, \mathcal{L}_2(\Omega)\| \to 0 \ (\varepsilon \to 0), \quad N \geq 0$$

$$\|u^\varepsilon - u_N^\varepsilon, H^1(\Omega)\| \to 0 \ (\varepsilon \to 0), \quad N \geq 1$$

where the rate of convergence, as $\varepsilon \to 0$, of the errors increase, irrespective of the regularity of the coefficient a, as $N \to \infty$ Thus, the approach based on homogenization, described in Chapter 3, is particularly well adapted for the treatment of the inherent difficulties caused by the rapid variation of low regularity coefficients

4 1 The Model Two Dimensional Problem

We employ the following elliptic boundary problem as the model two dimensional prototype to illustrate a combined approach based on both homogenization techniques and finite element discretizations Find the weak solution $u^\varepsilon \in H_0^1(\Omega)$ of the elliptic equation

$$-\sum_{k,l=1}^{2} \frac{\partial}{\partial x_k}\left[a_{kl}(\underline{x}/\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_l}(\underline{x})\right] = f(\underline{x}), \quad x \in \Omega \stackrel{\text{def}}{=} (0,1)^2 \tag{4 1 1}$$

where $f \in \mathcal{L}_2(\Omega)$ and $A = (a_{kl})_{k,l=1}^2 \in (\mathcal{L}_\infty(\mathcal{P}))^{2\times 2}$ is a symmetric 1–periodic matrix with elements satisfying the property, cf Figure 4 1,

$$\operatorname{Tr}(a_{kl})\Big|_{\Gamma_s} = \operatorname{Tr}(a_{kl})\Big|_{\Gamma_{s+2}}, \quad 1 \leq s \leq 2 \tag{4 1 2}$$

and, for almost all $\underline{x} \in \Omega$, $\varepsilon > 0$

$$0 < \alpha_1 \sum_{k=1}^{2} |\xi_k|^2 \leq \sum_{k,l=1}^{2} \xi_k\, a_{kl}(\underline{x}/\varepsilon)\, \xi_l \leq \alpha_2 \sum_{k=1}^{2} |\xi_k|^2 < \infty, \quad (\xi_1, \xi_2) \in \mathbb{R}^2 \tag{4 1 3}$$

where α_1, $\alpha_2 > 0$ are constants which are independent from ε The weak formulation of problem (4 1 1) can be obtained by multiplying relation (4 1 1) by $v \in H_0^1(\Omega)$ and integrating by parts to obtain the problem Find $u^\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega \sum_{k\,l=1}^{2} a_{kl}(\underline{x}/\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_k}(\underline{x})\,\frac{\partial v}{\partial x_l}(\underline{x})\,d\underline{x} = \int_\Omega f(\underline{x})\,v(\underline{x})\,d\underline{x}, \quad v \in H_0^1(\Omega) \tag{4 1 4}$$

Application of the Lax–Milgram lemma to the weak form (4 1 4) of (4 1 1) establishes the existence of a unique solution, $u^\varepsilon \in H_0^1(\Omega)$, which also exhibits the regularity property, cf MURAT & TARTAR (1994),

$$\|u^\varepsilon, H^1(\Omega)\| \leq C\,\|f, \mathcal{L}_2(\Omega)\| \tag{4 1 5}$$

where $C = C(f, \alpha_1) > 0$ is independent of ε If the data A are piecewise constant, i e ,

$$A(\underline{x}/\varepsilon) = A^{[r]}, \quad \underline{x} \in \Omega_r,\ A^{[r]} \in \mathbb{R}^{2,2}, \quad 1 \leq r \leq m_\varepsilon \tag{4 1 6}$$

where $\overline{\Omega} = \cup_{r=1}^{m_\varepsilon} \overline{\Omega}_r$ and Ω_r, $1 \leq r \leq m_\varepsilon$ are simply connected polygonal regions with $\Omega_r \cap \Omega_s = \emptyset$, $r \neq s$ then in a neighbourhood of the *vertices* of the interfaces $\Gamma_{rs} \overset{\text{def}}{=} \partial\Omega_r \cap \partial\Omega_s$, $1 \leq r, s \leq m_\varepsilon$ the solution, u^ε, of problem (4 1 1) will generally exhibit the characteristically singular behaviour commonly observed for problems with smooth coefficients formulated in nonconvex polygonal regions Indeed, following KELLOGG (1971) we define the Hilbert space

$$\mathcal{D}(\Omega, a) \overset{\text{def}}{=} \{v^\varepsilon \in H_0^1(\Omega) \mid \exists f \in \mathcal{L}_2(\Omega) \text{ s t } a(v^\varepsilon, w) = (f, w, \mathcal{L}_2(\Omega)),\ w \in H_0^1(\Omega)\} \tag{4 1 7}$$

$$(v, w, \mathcal{D}(\Omega, a)) \overset{\text{def}}{=} (Lv, Lw, \mathcal{L}_2(\Omega)), \quad v, w \in \mathcal{D}(\Omega, a) \tag{4 1 8}$$

where $a \in \mathcal{BL}(H_0^1(\Omega) \times H_0^1(\Omega), \mathbb{R})$ is the bilinear form associated with the weak formulation (4 1 4) and $L \in \mathcal{BL}(\mathcal{D}(\Omega, a), \mathcal{L}_2(\Omega))$ is the operator defined pointwise as $Lu^\varepsilon \overset{\text{def}}{=} f$, $f \in \mathcal{L}_2(\Omega)$ if, and only if, $u^\varepsilon \in H_0^1(\Omega)$ is the solution of the weak problem (4 1 4) It is shown in KELLOGG (1971) that u^ε can then be written in the form

$$u^\varepsilon = \sum_{j=1}^{M} \sigma_j v_j^\varepsilon + w^\varepsilon \overset{\text{def}}{=} v^\varepsilon + w^\varepsilon \tag{4 1 9}$$

where $\sigma_j \in \mathbb{R}$, w^ε, $v_j^\varepsilon \in \mathcal{D}(\Omega, a)$, $1 \leq j \leq M$ and

$$\|Lv^\varepsilon, \mathcal{L}_2(\Omega)\| + \|w^\varepsilon, H^1(\Omega)\| + \sum_{j=1}^{m_\varepsilon} \|w^\varepsilon, H^2(\Omega_j)\| \leq C\,\|Lu^\varepsilon, \mathcal{L}_2(\Omega)\| \tag{4 1 10}$$

The form of the singular functions $v_j^\varepsilon \in H^{1+\alpha_j}(\Omega), 0 < \alpha_j \leq 1$, $1 \leq j \leq M$ will depend precisely on the coefficients a_{kl}, $1 \leq k, l \leq 2$ and the geometry of the interfaces Γ_{rs}, $1 \leq r, s \leq m_\varepsilon$, cf BLUMENFELD (1985) The regularity properties of u^ε are clearly important because they determine how rapidly the errors introduced by finite element approximations diminish as $h \to 0$ Clearly, there are techniques of approximation which are particularly appropriate for problems of this type, e g , the class of a–posteriori adaptive methods and the non–conforming approach of BABUŠKA & OSBORN (1985) for which, in the norm $\|v\|^2 \overset{\text{def}}{=} \sum_{\tau \in \mathcal{T}_h(\Omega)} \|v, H^1(\tau)\|^2$, the optimal $O(h)$ error bound can be attained, however, we have found that, to assess our approach, it is sufficient to employ piecewise linear approximations constructed for triangulations, $\mathcal{T}_h(\Omega)$, $h > 0$, which have the property $\tau \cap \Gamma_{rs} \overset{\text{def}}{=} \emptyset$, $1 \leq r, s \leq m_\varepsilon$ for $\tau \in \mathcal{T}_h(\Omega)$, cf Section 2 2

We observe that problem (4 1 1) can be obtained as the restriction to Ω of the planar elliptic problem Find the weak solution $u^\varepsilon \in H^1_{loc}(\mathbb{R}^2) \stackrel{\text{def}}{=} \{v \ \mathbb{R}^2 \to \mathbb{C} \mid \text{For any open subset } \Omega \subset \subset \mathbb{R}^2,\ v \in H^1(\Omega)\}$ of the elliptic equation

$$-\sum_{k,l=1}^{2} \frac{\partial}{\partial x_k}\left[a_{kl}(\underline{x}/\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_l}(\underline{x})\right] = f_C(\underline{x}), \quad \underline{x} \in \mathbb{R}^2 \tag{4 1 11}$$

where the function f_C is defined as the periodic extension to $\mathbb{R}^2$ of the function f_A where f_A is defined as follows

$$f_A(\underline{x}) \stackrel{\text{def}}{=} \begin{cases} f(x_1,x_2), & \text{if } (x_1,x_2) \in \Omega \\ -f(-x_1,x_2), & \text{if } (-x_1,x_2) \in \Omega \\ f(-x_1,-x_2), & \text{if } (-x_1,-x_2) \in \Omega \\ -f(x_1,-x_2), & \text{if } (x_1,-x_2) \in \Omega \end{cases} \tag{4 1 12}$$

Thus, f_C is formally defined by the Fourier series expansion

$$f_C(\underline{x}) \stackrel{\text{def}}{=} \sum_{\underline{n}\in \mathcal{Z}^2\setminus\{0\}} a_{\underline{n}}\, e^{\underline{n}\,\underline{x}\,\pi i}, \qquad a_{\underline{n}} \stackrel{\text{def}}{=} \frac{1}{4}\int_C f_A(\underline{x})\, e^{\underline{n}\,\underline{x}\,\pi i}\, d\underline{x} \tag{4 1 13}$$

where $C \stackrel{\text{def}}{=} (-1,1)^2$ The partial differential equation (4 1 11) evidently implies (4 1 1) while the periodicity and antisymmetry of f_C imply the following properties of u^ε, for almost all $\underline{x} \in \Omega$,

$$u^\varepsilon(\underline{x} + 2\underline{n}) = u^\varepsilon(\underline{x}), \quad \underline{n} \in \mathcal{Z}^2 \tag{4 1 14}$$

$$u^\varepsilon((-1)^{m_1}x_1, (-1)^{m_2}x_2) = (-1)^{m_1+m_2}u^\varepsilon(x_1,x_2), \quad \underline{m} \in \mathbb{N}_0^2\setminus\{0\} \tag{4 1 15}$$

$$\int_{B(0,\rho,\infty)} u^\varepsilon(\underline{x})\, d\underline{x} = 0, \quad \rho > 0 \tag{4 1 16}$$

Furthermore, the regularity property $u^\varepsilon \in H^{1+\rho}(C)$ for some $\rho > 0$, the Sobolev embedding $H^{1+\rho}(C) \subset C^{0,\lambda}(\overline{C})$, $0 < \lambda < 1$, cf ADAMS (1975), and the antisymmetry of u^ε, cf (4 1 15), imply that $u^\varepsilon \in H^1_0(\Omega)$ Following BABUŠKA & MORGAN (1991i) we observe that for $f(\underline{x}) = e^{i\underline{t}\,\underline{x}}$ the mapping

$$\underline{x} \mapsto e^{i\underline{t}\,\underline{x}}\,\phi(\underline{x}/\varepsilon, \varepsilon, \underline{t}) \tag{4 1 17}$$

solves (4 1 11) where $\underline{x} \mapsto \phi(\underline{x}, \varepsilon, \underline{t})$ is a complex-valued, 1-periodic function that, in the weak sense, satisfies, for $\varepsilon > 0$, $\underline{t} \neq 0$, the partial differential equation

$$-\sum_{k\,l=1}^{2} \frac{\partial}{\partial x_k}\left[a_{kl}(\underline{x})\frac{\partial}{\partial x_l}\left(e^{i\varepsilon\underline{t}\,\underline{x}}\,\phi(\underline{x},\varepsilon,\underline{t})\right)\right] = \varepsilon^2 e^{i\varepsilon\underline{t}\,\underline{x}}, \quad \underline{x} \in \mathcal{P} = (0,1)^2 \tag{4 1 18}$$

and periodic boundary conditions on $\partial\mathcal{P}$, for $1 \le s \le 2$,

$$\operatorname{Tr}(\phi(\bullet,\varepsilon,\underline{t}))\Big|_{\Gamma_s} = \operatorname{Tr}(\phi(\bullet,\varepsilon,\underline{t}))\Big|_{\Gamma_{s+2}} \tag{4 1 19}$$

$$\operatorname{Tr}\left([A\nabla\phi(\bullet,\varepsilon,\underline{t})]\ \underline{n}\right)\Big|_{\Gamma_s} = \operatorname{Tr}\left([A\nabla\phi(\bullet,\varepsilon,\underline{t})]\ \underline{n}\right)\Big|_{\Gamma_{s+2}} \tag{4 1 20}$$

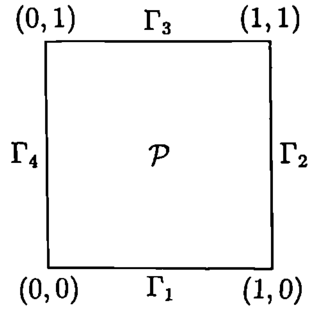


Figure 4 1 The periodic cell

where $\underline{n}(\underline{x})$ is the unit outward normal vector to the boundary, $\partial\mathcal{P}$, at the point $\underline{x}$, and Γ_s, $1 \le s \le 4$ are the boundary segments of the periodic cell, $\mathcal{P}$, illustrated in Figure 4 1

Thus, employing simple linear superposition, the solution, u^ε, can be written as follows, see BABUŠKA & MORGAN (1991ii) for the analysis,

$$u^\varepsilon(\underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^2 \setminus \{0\}} a_{\underline{n}}\, e^{\underline{n}\cdot\underline{x}\pi\imath}\, \phi(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) \tag{4 1 21}$$

Expression (4 1 21) now provides the opportunity to investigate the development of approximation techniques based on the cell problem (4 1 18)–(4 1 20) rather than the original boundary value problem (4 1 1) However, before considering techniques of approximation, the properties of the weak formulation of problem (4 1 18)–(4 1 20) and the respective weak solution, ϕ, will be studied

4 1 1 Properties of the Cell Problem

The weak formulation of the cell problem (4 1 18)–(4 1 20) is derived by multiplying equation (4 1 18) by the function $e^{-\imath\varepsilon\underline{t}\cdot\underline{x}}\overline{v(\underline{x})}$, $v \in H^1_{per}(\mathcal{P})$ and then integrating by parts to obtain the problem Find $\phi(\bullet, \varepsilon, \underline{t}) \in H^1_{per}(\mathcal{P})$ such that, for $v \in H^1_{per}(\mathcal{P})$,

$$\int_{\mathcal{P}} \sum_{k\,l=1}^{2} a_{kl}(x)\, \frac{\partial}{\partial x_k}\left(e^{\imath\varepsilon\underline{t}\cdot\underline{x}}\,\phi(\underline{x}, \varepsilon, \underline{t})\right) \frac{\partial}{\partial x_l}\left(e^{-\imath\varepsilon\underline{t}\cdot\underline{x}}\,\overline{v(\underline{x})}\right) d\underline{x} = \varepsilon^2 \int_{\mathcal{P}} \overline{v(\underline{x})}\, d\underline{x} \tag{4 1 22}$$

where it has been observed that the boundary term

$$\int_{\partial\mathcal{P}} \overline{v(\underline{x})} \left(\imath\varepsilon\,\phi(\underline{x}, \varepsilon, \underline{t})\,[A(\underline{x})\,\underline{t}] + A(\underline{x})\nabla_x\phi(\underline{x}, \varepsilon, \underline{t})\right) \cdot \underline{n}(\underline{x})\, d\underline{x} \tag{4 1 23}$$

vanishes as a consequence of the boundary trace properties of $A = (a_{kl})^2_{k,l=1}$, $\phi(\bullet, \varepsilon, \underline{t})$ specified in relations (4 1 2), (4 1 19), and (4 1 20) Observe that $\overline{v(\underline{x})} = \Re[v(\underline{x})] - \imath\,\Im[v(\underline{x})]$ is the complex conjugate of $v(\underline{x}) \in \mathbb{C}$ Clearly, for u, $v \in H^1_{per}(\mathcal{P})$, the sesquilinear form for this problem is defined as follows

$$\begin{aligned}
\Phi(\varepsilon, \underline{t})[u, v] &= \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x})\, \frac{\partial}{\partial x_k}\left(e^{\imath\varepsilon\underline{t}\cdot\underline{x}}\,u(\underline{x})\right) \frac{\partial}{\partial x_l}\left(e^{-\imath\varepsilon\underline{t}\cdot\underline{x}}\,\overline{v(\underline{x})}\right) d\underline{x} \\
&= \int_{\mathcal{P}} \sum_{k\,l=1}^{2} a_{kl}(\underline{x})\, \frac{\partial u}{\partial x_k}(\underline{x})\, \overline{\frac{\partial v}{\partial x_l}(\underline{x})}\, d\underline{x} + \imath\varepsilon \int_{\mathcal{P}} \sum_{k\,l=1}^{2} a_{kl}(\underline{x}) \left(t_k\, u(\underline{x})\, \overline{\frac{\partial v}{\partial x_l}(x)} - t_l\, \frac{\partial u}{\partial x_k}(\underline{x})\, \overline{v(\underline{x})}\right) d\underline{x} \\
&\quad + \varepsilon^2 \int_{\mathcal{P}} \sum_{k,l=1}^{2} t_k t_l\, a_{kl}(\underline{x})\, u(\underline{x})\, \overline{v(\underline{x})}\, d\underline{x} \\
&= \Phi_0[u, v] + \varepsilon\, \Phi_1(\underline{t})[u, v] + \varepsilon^2\, \Phi_2(\underline{t})[u, v]
\end{aligned}$$

The sesquilinear form is clearly Hermitian symmetric, i e , $\Phi(\varepsilon,\underline{t})[u,\, v] = \overline{\Phi(\varepsilon,\underline{t})[v,\, u]}$, u, $v \in H^1_{per}(\mathcal{P})$ Further, it follows from applications of relation (4 1 3) and the Cauchy-Schwarz inequality that the following relations are valid

$$|\Phi_0[u,v]| \leq \left| \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \, \frac{\partial u}{\partial x_k}(\underline{x}) \, \overline{\frac{\partial u}{\partial x_l}(\underline{x})} \, d\underline{x} \right|^{1/2} \left| \int_{\mathcal{P}} \sum_{k\,l=1}^{2} a_{kl}(\underline{x}) \, \frac{\partial v}{\partial x_k}(\underline{x}) \, \overline{\frac{\partial v}{\partial x_l}(\underline{x})} \, d\underline{x} \right|^{1/2}$$

$$\leq \alpha_2 \, \|u, H^1(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \qquad (4\,1\,24)$$

$$|\Phi_1(\underline{t})[u,v]| \leq \left| \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \, t_k u(\underline{x}) \, \overline{\frac{\partial v}{\partial x_l}(\underline{x})} \, d\underline{x} \right| + \left| \int_{\mathcal{P}} \sum_{k\,l=1}^{2} a_{kl}(\underline{x}) \, t_l \, \frac{\partial u}{\partial x_k}(\underline{x}) \, \overline{v(\underline{x})} \, d\underline{x} \right|$$

$$\leq \left| \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \, t_k \, t_l \, |u(\underline{x})|^2 \, d\underline{x} \right|^{1/2} \left| \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \, \frac{\partial v}{\partial x_k}(\underline{x}) \, \overline{\frac{\partial v}{\partial x_l}(\underline{x})} \, d\underline{x} \right|^{1/2} +$$

$$\left| \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \, \frac{\partial u}{\partial x_k}(\underline{x}) \, \overline{\frac{\partial u}{\partial x_l}(\underline{x})} \, d\underline{x} \right|^{1/2} \left| \int_{\mathcal{P}} \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \, t_k \, t_l \, |v(\underline{x})|^2 d\underline{x} \right|^{1/2}$$

$$\leq \alpha_2 \, \|\underline{t}\|_2 \left(\|u, \mathcal{L}_2(\mathcal{P})\| \, |v, H^1(\mathcal{P})| + |u, H^1(\mathcal{P})| \, \|v, \mathcal{L}_2(\mathcal{P})\| \right)$$

$$\leq 2\alpha_2 \, \|\underline{t}\|_2 \, \|u, H^1(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \qquad (4\,1\,25)$$

$$|\Phi_2(\underline{t})[u,v]| \leq \alpha_2 \|\underline{t}\|_2^2 \, |(u, v, \mathcal{L}_2(\mathcal{P}))| \qquad (4\,1\,26)$$

$$\leq \alpha_2 \|\underline{t}\|_2^2 \, \|u, \mathcal{L}_2(\mathcal{P})\| \, \|v, \mathcal{L}_2(\mathcal{P})\| \leq \alpha_2 \|\underline{t}\|_2^2 \, \|u, H^1(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \qquad (4\,1\,27)$$

$$\Rightarrow \quad |\Phi(\varepsilon,\underline{t})[u,\, v]| \leq |\Phi_0[u,v]| + \varepsilon \, |\Phi_1(\underline{t})[u,v]| + \varepsilon^2 \, |\Phi_2(\underline{t})[u,v]|$$

$$\leq C(\varepsilon,\, \underline{t}) \, \|u, H^1(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \qquad (4\,1\,28)$$

where $C(\varepsilon,\, \underline{t}) = \alpha_2(1 + 2\varepsilon \, \|\underline{t}\|_2 + \varepsilon^2 \|\underline{t}\|_2^2) > 0$ Thus, the mappings Φ_0, $\Phi_1(\underline{t})$, $\Phi_2(\underline{t})$ are sesquilinear and Φ_0 is also positive semi–definite over $H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P})$ In fact, from (4 1 24), $\Phi_0[1,v] = \Phi_0[u,1] = \Phi_0[1,1] = 0$ and, furthermore, from (4 1 25), $\Phi_1(\underline{t})[1,1] = 0$ To establish the $H^1_{per}(\mathcal{P})$–Ellipticity of $\Phi(\varepsilon,\underline{t})$ the following lemma is required

Lemma 4 2 2 *There exists a constant $C_1 > 0$ such that*

$$\frac{1}{C_1(1 + \|\underline{t}\|_2)} \, \|v, H^1(\mathcal{P})\| \leq \left\| v \, e^{i\varepsilon \, \underline{t} \, (\bullet)}, H^1(\mathcal{P}) \right\| \leq C_1(1 + \|\underline{t}\|_2) \, \|v, H^1(\mathcal{P})\| \qquad (4\,1\,29)$$

for all $v \in H^1(\mathcal{P})$, $\underline{t} \in \mathbb{R}^2$

Proof If $v \in H^1(\mathcal{P})$ then the inequality on the right follows from the following relations

$$\left\| v \, e^{i\varepsilon \, \underline{t} \, (\bullet)}, H^1(\mathcal{P}) \right\|^2 = \int_{\mathcal{P}} v(\underline{x}) \overline{v(\underline{x})} \, d\underline{x} + \int_{\mathcal{P}} \sum_{n=1}^{2} \frac{\partial}{\partial x_n} \left(v(\underline{x}) \, e^{i\varepsilon \, \underline{t} \, \underline{x}} \right) \frac{\partial}{\partial x_n} \left(\overline{v(\underline{x})} \, e^{-i\varepsilon \, \underline{t} \, \underline{x}} \right) d\underline{x}$$

$$= \|v, \mathcal{L}_2(\mathcal{P})\|^2 + |v, H^1(\mathcal{P})|^2 + \varepsilon^2 \|\underline{t}\|_2^2 \, \|v, \mathcal{L}_2(\mathcal{P})\|^2 + 2\varepsilon \, \Im \left[\int_{\mathcal{P}} \sum_{n=1}^{2} \overline{t_n \, v(\underline{x})} \, \frac{\partial v}{\partial x_n}(\underline{x}) \, d\underline{x} \right]$$

$$\leq \|v, \mathcal{L}_2(\mathcal{P})\|^2 + |v, H^1(\mathcal{P})|^2 + \varepsilon^2 \, \|\underline{t}\|_2^2 \, \|v, \mathcal{L}_2(\mathcal{P})\|^2 + 2\varepsilon \left| \int_{\mathcal{P}} \sum_{n=1}^{2} \overline{t_n \, v(\underline{x})} \, \frac{\partial v}{\partial x_n}(\underline{x}) \, d\underline{x} \right|$$

$$\leq \|v, \mathcal{L}_2(\mathcal{P})\|^2 + |v, H^1(\mathcal{P})|^2 + \varepsilon^2 \, \|\underline{t}\|_2^2 \, \|v, \mathcal{L}_2(\mathcal{P})\|^2 + 2\varepsilon \, \|\underline{t}\|_2 \, \|v, \mathcal{L}_2(\mathcal{P})\| \, |v, H^1(\mathcal{P})|$$

$$\leq \|v, \mathcal{L}_2(\mathcal{P})\|^2 + (1 + \varepsilon \|\underline{t}\|_2)^2 \|v, H^1(\mathcal{P})\|^2 \leq 2(1 + \varepsilon \|\underline{t}\|_2)^2 \|v, H^1(\mathcal{P})\|^2$$

The inequality on the left is similarly proved by applying the inequality on the right to the function $w(\underline{x}) = v(\underline{x})\, e^{\imath\varepsilon \underline{t}\,\underline{x}}$, i e ,

$$\|v, H^1(\mathcal{P})\| = \|w\, e^{-\imath\varepsilon \underline{t}\,(\bullet)}, H^1(\mathcal{P})\| \le C_1(1 + \|\underline{t}\|_2)\, \|w, H^1(\mathcal{P})\| \qquad \blacksquare$$

Lemma 4 2 *A constant $C_2 > 0$ exists such that*

$$\frac{1}{C_2}\, \|v\, e^{\imath\varepsilon \underline{t}\,(\bullet)}, H^1(\mathcal{P})\| \le |v\, e^{\imath\varepsilon \underline{t}\,(\bullet)}, H^1(\mathcal{P})| \le C_2\, \|v\, e^{\imath\varepsilon \underline{t}\,(\bullet)}, H^1(\mathcal{P})\|, \tag{4 1 30}$$

for $v \in H^1_{per}(\mathcal{P})$ when $\varepsilon\underline{t} \notin \mathcal{H}^2$ and for $v \in H^1_0(\mathcal{P})$ when $\varepsilon\underline{t} \in \mathcal{H}^2$

Proof The inequality on the right follows immediately for any $C_2 \ge 1$ Let $v \in C^\infty(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$ and define $w(\underline{x}) \overset{\text{def}}{=} v(\underline{x})\, e^{\imath\varepsilon\underline{t}\,\underline{x}}$, $\underline{x} \in \mathcal{P}$, $w(\underline{x}) \overset{\text{def}}{=} 0$, $\underline{x} \in \mathbb{R}^2\backslash\mathcal{P}$ then

$$|w(\underline{x})|^2 = \left| \int_{-\rho}^{x_1} w_{x_1}(\xi, x_2)\, d\xi \right|^2 \le (x_1 + \rho) \int_{-\rho}^{x_1} |w_{x_1}(\xi, x_2)|^2\, d\xi, \quad \rho > 0$$
$$\Rightarrow \quad |w(\underline{x})|^2 \le \int_0^1 \left| w_{x_1}(\xi, x_2) \right|^2 d\xi$$

Integrating this expression over $\mathcal{P}$ then yields the following inequality

$$\|w, \mathcal{L}_2(\mathcal{P})\| \le \|w_{x_1}, \mathcal{L}_2(\mathcal{P})\| \le |w, H^1(\mathcal{P})|$$

from which we deduce

$$\|v\, e^{\imath\varepsilon\underline{t}\,(\bullet)}, H^1(\mathcal{P})\|^2 = \|v e^{\imath\varepsilon\underline{t}\,(\bullet)}, \mathcal{L}_2(\mathcal{P})\|^2 + |v e^{\imath\varepsilon\underline{t}\,(\bullet)}, H^1(\mathcal{P})|^2 \le 2\, |v\, e^{\imath\varepsilon\underline{t}\,(\bullet)}, H^1(\mathcal{P})|^2$$

However, because $v \in C^\infty(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$ is arbitrary the norm equivalence claimed aboved is established for all $v \in C^\infty(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$ and $C_2 = 1/\sqrt{2}$ Furthermore, by completing the function space $C^\infty(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$ within $\mathcal{L}_2(\mathcal{P})$ using the $H^1(\mathcal{P})$ norm topology one obtains $H^1_{per}(\mathcal{P})$, i e , $C^\infty(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$ is densely embedded in $H^1_{per}(\mathcal{P})$ Thus, the norm equivalence follows also for the completion $H^1_{per}(\mathcal{P})$ of $C^\infty(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$, cf HACKBUSCH (1992) However, the norm equivalence represented by the above inequality fails when $\varepsilon\underline{t} \in \mathcal{H}^2$, this is apparent with $v(\underline{x}) = e^{-\imath\varepsilon\underline{t}\,\underline{x}}$, $\underline{x} \in \mathcal{P}$ for, then, $e^{\imath\varepsilon\underline{t}\,(\bullet)} \in H^1_{per}(\mathcal{P})$ But, replacing $C^\infty(\mathcal{P})$ with $C_0^\infty(\mathcal{P})$ in the above steps, the norm equivalence (4 1 30) then follows immediately ■

Thus, from Lemmas 4 1 and 4 2 the V–Ellipticity of $\Phi(\varepsilon, \underline{t})$ follows immediately from the inequalities below

$$\begin{aligned}
|\Phi(\varepsilon, \underline{t})[v, v]| &= \left| \int_{\mathcal{P}} \sum_{k\,l=1}^{2} a_{kl}(\underline{x})\, \frac{\partial}{\partial x_k}\left(e^{\imath\varepsilon\,\underline{t}\,\underline{x}}\, v(\underline{x})\right) \frac{\partial}{\partial x_l}\left(e^{-\imath\varepsilon\,\underline{t}\,\underline{x}}\, \overline{v(\underline{x})}\right) d\underline{x} \right| \\
&\ge \alpha_1 \int_{\mathcal{P}} \sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(e^{\imath\varepsilon\,\underline{t}\,\underline{x}}\, v(\underline{x})\right) \frac{\partial}{\partial x_k}\left(e^{-\imath\varepsilon\,\underline{t}\,\underline{x}}\, \overline{v(\underline{x})}\right) d\underline{x} \\
&\ge \alpha_1\, C_2^{-2}\, \|v\, e^{\imath\varepsilon\,\underline{t}\,(\bullet)}, H^1(\mathcal{P})\|^2 \\
&\ge C(\underline{t})\, \|v, H^1(\mathcal{P})\|^2
\end{aligned} \tag{4 1 31}$$

where $V = H^1_{per}(\mathcal{P})$ for $(\varepsilon, \underline{t}) \notin \mathcal{H}^2$, $V = H^1_0(\mathcal{P})$ for $(\varepsilon, \underline{t}) \in \mathcal{H}^2$, and $C(\underline{t}) = \alpha_1 C_2^{-2} C_1^{-2} (1 + \|\underline{t}\|_2)^{-2}$ is independent from ε Thus, treating ε, $\underline{t}$ as parameters, the Lax–Milgram lemma demonstrates that a unique solution $\phi(\bullet, \varepsilon, \underline{t}) \in H^1_{per}(\mathcal{P})$, $\varepsilon\underline{t} \notin \mathcal{H}^2$ exists for the weak problem (4 1 22) However, if $\varepsilon\underline{t} \in \mathcal{H}^2$ then the sesquilinear form $\Phi(\varepsilon, \underline{t})$ is not positive on $H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P})$, e g ,

$$\Phi(\varepsilon, \underline{t})[e^{-\imath\varepsilon\underline{t}\,(\bullet)}, e^{-\imath\varepsilon\underline{t}\,(\bullet)}] = 0, \quad e^{-\imath\varepsilon\underline{t}\,(\bullet)} \in H^1_{per,0}(\mathcal{P}) \subset H^1_{per}(\mathcal{P}) \tag{4 1 32}$$

and the weak formulation (4 1 22) does not then satisfy the $H^1_{per}(\mathcal{P})$–ellipticity condition of the Lax–Milgram lemma, however, the weak formulation Find $\phi(\bullet, \varepsilon, \underline{t}) \in H^1_0(\mathcal{P})$ such that

$$\Phi(\varepsilon, \underline{t})[\phi(\bullet, \varepsilon, \underline{t}), v] = \varepsilon^2 \int_{\mathcal{P}} \overline{v(\underline{x})}\, d\underline{x}, \quad v \in H^1_0(\mathcal{P}) \tag{4 1 33}$$

does satisfy the Lax–Milgram lemma Thus, from the direct sum decomposition $H^1_{per}(\mathcal{P}) = H^1_0(\mathcal{P}) \oplus \mathbb{C}$ and relation (4 1 32), we observe that any function defined according to the following relation is also a solution

$$\psi(\bullet, \varepsilon, \underline{t}) \stackrel{\text{def}}{=} \phi(\bullet, \varepsilon, \underline{t}) + e^{-\imath\varepsilon\underline{t}\,\underline{x}} \chi(\varepsilon, \underline{t}), \quad \varepsilon\underline{t} \in \mathcal{H}^2 \tag{4 1 34}$$

where χ is an arbitrary function satisfying $\chi(\varepsilon, \underline{t}) = \overline{\chi(\varepsilon, -\underline{t})}$, $\varepsilon > 0$, $\underline{t} \neq 0$ Furthermore, if a is symmetric about the lines $x_1 = 1/2$, $x_2 = 1/2$, i e ,

$$a(x_1, x_2) = a(1 - x_1, x_2) = a(x_1, 1 - x_2), \quad (x_1, x_2) \in \mathcal{P} \tag{4 1 35}$$

then, as demonstrated in Section 3 1 1, the following conjugate symmetry relations are satisfied

$$\phi(\underline{x}, \varepsilon, \underline{t}) = \overline{\phi((1 - x_1, x_2), \varepsilon, \underline{t})} = \overline{\phi((x_1, 1 - x_2), \varepsilon, \underline{t})}, \quad \underline{x} \in \mathcal{P},\ \varepsilon > 0,\ \underline{t} \neq 0 \tag{4 1 36}$$

$$\Rightarrow \quad \mathrm{Tr}\left[\Im(\phi(\bullet, \varepsilon, \underline{t}))\right] = 0 \tag{4 1 37}$$

We now define the index set $I(\varepsilon) \stackrel{\text{def}}{=} \{\underline{n} \in \mathcal{Z}^2\backslash\{0\} \times \mathbb{N} \mid (\varepsilon, \underline{n}\pi) \in \mathcal{H}^2\}$ and observe that the solution, u^ε, can be written

$$u^\varepsilon(\underline{x}) = \sum_{\underline{n} \in \mathcal{Z}^2\backslash\{0\}} a_{\underline{n}}\, e^{\underline{n}\,\underline{x}\pi\imath} \phi(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) + \sum_{\underline{n} \in I(\varepsilon)} 2\imath\, a_{\underline{n}} \Im(\chi(\varepsilon, \underline{n}\pi)) \tag{4 1 38}$$

Thus, applying the boundary condition $\mathrm{Tr}\,(u^\varepsilon) = 0$ and property (4 1 37) to equation (4 1 38) and noting the continuity of the trace operator, i e , $\mathrm{Tr} \in \mathcal{BL}(H^1(\mathcal{P}), H^{1/2}(\partial\mathcal{P}))$, we deduce the following identities

$$\begin{aligned} 0 &= \sum_{\underline{n} \in \mathcal{Z}\backslash\{0\} \times \mathbb{N}} 2\imath\, a_{\underline{n}}\, e^{n_1 x_1 \pi\imath}\, \mathrm{Tr}\left[\Im(\phi(\bullet/\varepsilon, \varepsilon, \underline{n}\pi))\right] + \sum_{\underline{n} \in I(\varepsilon)} 2\imath\, a_{\underline{n}}\, \Im(\chi(\varepsilon, \underline{n}\pi)) \\ &= \sum_{\underline{n} \in I(\varepsilon)} 2\imath\, a_{\underline{n}}\, \Im(\chi(\varepsilon, \underline{n}\pi)) \end{aligned}$$

However, the independence of the coefficients $a_{\underline{n}}, \underline{n} \in \mathcal{Z}^2\backslash\{0\}$ and the function χ suggests that, for $(\varepsilon, \underline{t}) \in \mathcal{H}^2$, $\chi(\varepsilon, \underline{t}) \in \mathbb{R}$ Indeed, with this proviso the choice of function χ is of no consequence to the construction of u^ε

In the same vein as the 1–dimensional treatment, we observe that, in the circumstance in which the elliptic boundary value problem (4 1 1) models a heterogeneous body comprised of different homogeneous materials, the coefficients are piecewise smooth, i e , $A \in [\mathcal{P}C^\infty(\mathcal{P})\backslash C^0(\mathcal{P})]^{2,2}$, cf (4 1 6) It is then evident from the weak formulation (4 1 22) of the cell problem that the following interface transition conditions for $1 \le r, s \le m_1$ are implied

$$\mathrm{Tr}\,[\phi_r(\bullet, \varepsilon, \underline{t})]\Big|_{\Gamma_{rs}} = \mathrm{Tr}\,[\phi_s(\bullet, \varepsilon, \underline{t})]\Big|_{\Gamma_{rs}} \tag{4 1 39}$$

$$\mathrm{Tr}\left[A\nabla\left(e^{i\varepsilon\,\underline{t}\,(\bullet)}\,\phi_r(\bullet, \varepsilon, \underline{t})\right)\right]\;\underline{n}\Big|_{\Gamma_{rs}} = \mathrm{Tr}\left[A\nabla(e^{i\varepsilon\,\underline{t}\,(\bullet)}\,\phi_s(\bullet, \varepsilon, \underline{t}))\right]\;\underline{n}\Big|_{\Gamma_{rs}} \tag{4 1 40}$$

where $\underline{n}(\underline{x})$ is a normal vector to the interface Γ_{rs}, $1 \le r, s \le m_1$ at the point $\underline{x} \in \Gamma_{rs}$, $\phi_l(\bullet, \varepsilon, \underline{t}) \stackrel{\text{def}}{=} \phi(\bullet, \varepsilon, \underline{t})|_{\mathcal{P}_l}$, $1 \le l \le m_1$ defines the restriction of the cell function, $\phi(\bullet, \varepsilon, \underline{t})$, to each homogeneous element, $\mathcal{P}_l$, of $\mathcal{P}$, and Tr is the linear operator which maps a function to its trace on the boundary of its domain of definition In the 1–dimensional setting analytical expressions for ϕ were employed to assess the asymptotic approach for a variety of different problems However, in a 2–dimensional setting the problem of computing analogous analytical expressions for ϕ, ϕ_l, $1 \le l \le m_1$ is often intractable Therefore we now consider the application of finite element techniques for the weak formulation (4 1 22) of problem (4 1 18)–(4 1 20) and, in this way, we compute approximations $\phi_h(\bullet, \varepsilon, \underline{t})$ of $\phi(\bullet, \varepsilon, \underline{t})$ for $\varepsilon > 0$, $\underline{t} \ne 0$ where $h > 0$ is the discretization parameter

4 1 2 Finite Element approximation of $\phi(\bullet, \varepsilon, \underline{t})$

The variables ε, $\underline{t}$ which appear in the formulation (4 1 22) are determined by the model (4 1 1), the period of the material, ε, is prescribed and $\underline{t}$ corresponds to a Harmonic component of the right hand side f_C Thus, these variables are subsequently interpreted as fixed parameters in (4 1 4) We begin by constructing the finite dimensional subspaces $S^h_{per}(\mathcal{P}) \subset H^1_{per}(\mathcal{P})$, $h > 0$ Let $S^h(\mathcal{P}) \subset H^1(\mathcal{P})$ be the finite dimensional space over the complex field, $\mathbb{C}$, of piecewise linear polynomials introduced in Chapter 2§1 and let $\mathcal{B}(S^h(\mathcal{P}))$ denote a basis for this function space The basis $\mathcal{B}(S^h(\mathcal{P}))$ can be partitioned into disjoint subsets $\mathcal{B}^h(\mathcal{P})$, $\mathcal{B}^h(\partial\mathcal{P}\backslash\mathcal{V})$, $\mathcal{B}^h(\mathcal{V})$, i e ,

$$\mathcal{B}(S^h(\mathcal{P})) = \mathcal{B}^h(\mathcal{P}) \cup \mathcal{B}^h(\partial\mathcal{P}\backslash\mathcal{V}) \cup \mathcal{B}^h(\mathcal{V}) \tag{4 1 41}$$

where $\mathcal{V} \stackrel{\text{def}}{=} \left\{v \in \overline{\mathcal{P}} \mid v \text{ is a vertex of } \partial\mathcal{P}\right\}$ and, for arbitrary $\mathcal{F} \subset \overline{\mathcal{P}}$, we define the subsets (bases), cf (2 1 4),

$$\mathcal{B}^h(\mathcal{F}) \stackrel{\text{def}}{=} \{\varphi \in \mathcal{B}(S^h(\mathcal{P})) \mid \varphi^{-1}(\{1\}) \subset \mathcal{F}\} \tag{4 1 42}$$

where it is assumed that $\|\varphi, C^0(\overline{\mathcal{P}})\| = 1$, $\varphi \in \mathcal{B}^h(S^h(\mathcal{P}))$ and $\varphi^{-1}(A) \stackrel{\text{def}}{=} \{x \in \overline{\mathcal{P}} \mid \phi(x) \in A\}$ is the inverse image of $A \subset \mathbb{R}$ Therefore, with $\mathcal{F}$ equal, respectively, to $\mathcal{P}$, $\partial\mathcal{P}\backslash\mathcal{V}$, $\mathcal{V}$ the bases $\mathcal{B}^h(\mathcal{P})$, $\mathcal{B}^h(\partial\mathcal{P}\backslash\mathcal{V})$, $\mathcal{B}^h(\mathcal{V})$ are obtained from (4 1 42) We now construct a basis, $\mathcal{B}(S^h_{per}(\mathcal{P}))$, of $S^h_{per}(\mathcal{P})$ according to the definition

$$\mathcal{B}(S^h_{per}(\mathcal{P})) \stackrel{\text{def}}{=} \mathcal{B}^h(\mathcal{P}) \cup \mathcal{B}^h_{per}(\partial\mathcal{P}\backslash\mathcal{V}) \cup \mathcal{B}^h_{per}(\mathcal{V}) \tag{4 1 43}$$

where $\mathcal{B}^h_{per}(\partial\mathcal{P}\backslash\mathcal{V}) = \cup_{s=1}^2 \mathcal{B}^h_s$ and the bases $\mathcal{B}^h_{per}(\mathcal{V})$, $\mathcal{B}^h_s$, $1 \le s \le 2$ are defined below, cf Figure 4 1,

$$\mathcal{B}^h_s \stackrel{\text{def}}{=} \left\{ \sum_{l=1}^{2} \varphi_l \mid \varphi_l \in \mathcal{B}^h(\Gamma_{s+2(l-1)}\backslash\mathcal{V}),\ 1 \le l \le 2,\ \Pi_s(\operatorname{supp}\varphi_1) = \Pi_s(\operatorname{supp}\varphi_2) \right\} \tag{4 1 44}$$

where Π_s $X_1 \times X_2 \to X_s$, $1 \le s \le 2$ is the projection operator and

$$\mathcal{B}^h_{per}(\mathcal{V}) \stackrel{\text{def}}{=} \left\{ \sum_{l=1}^{n} \varphi_l \mid \{\varphi_l\}_{l=1}^n = \mathcal{B}^h(\mathcal{V}) \right\} \tag{4 1 45}$$

It now follows immediately from Chapter 2 and the above relations that $S^h_{per}(\mathcal{P})$ is a conforming finite element space, i e , $S^h_{per}(\mathcal{P}) \subset H^1_{per}(\mathcal{P})$, $h > 0$ and, furthermore,

$$S^{h_1}_{per}(\mathcal{P}) \subset S^{h_2}_{per}(\mathcal{P}) \subset \quad \subset S^{h_r}_{per}(\mathcal{P}) \subset \quad \subset H^1_{per}(\mathcal{P}) \tag{4 1 46}$$

where $\mathcal{T}_{h_i}(\mathcal{P})$, $i \ge 2$ are successive refinements of the triangulation $\mathcal{T}_{h_1}(\mathcal{P})$ Thus, employing the Galerkin approach, we obtain the discretized problem Find $\phi_h(\bullet, \varepsilon, \underline{t}) \in S^h_{per}(\mathcal{P})$ such that

$$\Phi(\varepsilon, \underline{t})[\phi_h(\bullet, \varepsilon, \underline{t}), v_h] = \varepsilon^2 \int_{\mathcal{P}} \overline{v_h(\underline{x})}\, d\underline{x}, \quad v_h \in S^h_{per}(\mathcal{P}) \tag{4 1 47}$$

In Section 4 1 1 it was demonstrated that, for $(\varepsilon, \underline{t}) \notin \mathcal{H}^2$, the sesquilinear operator $\Phi(\varepsilon, \underline{t})$ $H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P}) \to \mathbb{C}$ is continuous and $H^1_{per}(\mathcal{P})$–elliptic However, because $S^h_{per}(\mathcal{P}) \subset H^1_{per}(\mathcal{P})$, these properties also hold when the domain is restricted to $S^h_{per}(\mathcal{P}) \times S^h_{per}(\mathcal{P})$ and, thus, the Lax–Milgram lemma can be applied to demonstrate the existence of a unique solution $\phi_h(\bullet, \varepsilon, \underline{t}) \in S^h_{per}(\mathcal{P})$ for the Galerkin problem (4 1 47) Similarly, if $(\varepsilon, \underline{t}) \in \mathcal{H}^2$ then we replace $S^h_{per}(\mathcal{P})$ by $S^h_0(\mathcal{P})$ in (4 1 47) and seek $\phi_h(\bullet, \varepsilon, \underline{t}) \in S^h_0(\mathcal{P})$

4 2 Homogenization Construction of the Asymptotic Expansion

We should like to begin here by commenting that Conjecture 3 1 1, asymptotic expansion (3 2 1), and Theorem 3 2 1 introduced in the one dimensional context in Chapter 3 generalize immediately to the 2–dimensional setting with only simple modifications and we shall, therefore, refer directly to these results as stated in Chapter 3 with the understanding that they are to be interpreted in the appropriate two dimensional context

The task of determining analytical expressions for the weak solution $\phi(\bullet, \varepsilon, \underline{t}) \in H^1_{per}(\mathcal{P})$, $\varepsilon > 0$, $\underline{t} \neq 0$ of problem (4 1 22) is usually intractable and, similarly, so is the problem of computing analytical expressions for the terms $\phi_n(\bullet,\underline{t}) \in H^1_{per}(\mathcal{P})$, $n \geq 0$ of the asymptotic expansion (cf Theorem 3 2 1),

$$\phi(\underline{x},\varepsilon,\underline{t}) = \sum_{n=0}^{\infty} \varepsilon^n \phi_n(\underline{x},\underline{t}), \quad \underline{x} \in \mathcal{P},\ (\varepsilon,\underline{t}) \in \widehat{G},\ \phi_n(\bullet,\underline{t}) \in H^1_{per}(\mathcal{P}) \tag{4 2 1}$$

Thus, we employ finite element techniques for the approximation of the terms $\phi_n(\bullet,\underline{t})$, $\underline{t} \neq 0$, $n \geq 0$ using, as a basis for approximation, the expressions (3 2 10) provided in Theorem 3 2 1 However, we observe that for problems of low regularity, i e , $f_C \in H^m(\mathcal{C})\backslash H^{m+1}(\mathcal{C})$, the parameter $\underline{t}$ is unbounded and, consequently, an approach based on the direct approximation of the functions $\chi_n(\bullet,\underline{t}) \in H^1_{per,0}(\mathcal{P})$, $n \geq 1$ (cf Theorem 3 2 1) would be impractical We demonstrate how this difficulty can be overcome by (i) Separating the variables $\underline{x}$, $\underline{t}$ for each function $\chi_n(\underline{x},\underline{t})$, $\underline{x} \in \mathcal{P}$, $\underline{t} \neq 0$, and then (ii) Approximating independently the separate $\underline{x}$, $\underline{t}$ components of χ_n, $n \geq 1$ The construction of approximating finite element subspaces $S^h_{per\,0}(\mathcal{P}) \subset H^1_{per\,0}(\mathcal{P})$, $h > 0$ is described together with their application to determine accurate and robust approximations $\chi_{n,h}(\bullet,\underline{t}) \in S^h_{per,0}(\mathcal{P})$ of $\chi_n(\bullet,\underline{t}) \in H^1_{per,0}(\mathcal{P})$ and the errors introduced by applying this finite element approach are analysed

4 2 1 Separating the variables in $\phi_n(\underline{x},\underline{t})$

The term $\phi_n(\underline{x},\underline{t})$ is, ultimately, employed in a series expansion of the form (4 2 1) in which the variable $\underline{t}$ corresponds to a specific Harmonic frequency of f_C, cf (4 1 13), and $\underline{x} \in \mathcal{P}$ However, we shall demonstrate that it is possible to deduce expressions for $\phi_n(\underline{x},\underline{t})$ in which the functional dependence on the variable $\underline{x}$ is separated from that of the variable $\underline{t}$, i e , ϕ_n can be written in the form

$$\phi_n(\underline{x},\underline{t}) = \sum_{k=0}^{n} \theta_k(\underline{x})\lambda_k(\underline{t}) \tag{4 2 2}$$

where $\theta_k \in H^1_{per,0}(\mathcal{P})$, $0 \leq k \leq n$ are obtained as the solution of a weak problem formulated in a Sobolev space setting and λ_k, $0 \leq k \leq n$ are rational functions whose coefficients are determined by the weak solutions $\theta_k \in H^1_{per\,0}(\mathcal{P})$, $0 \leq k \leq n$ The property (4 2 2) provides the opportunity to introduce finite element approximations $\theta_{k\,h}, \lambda_{k,h}$, $h > 0$ of, respectively, θ_k, λ_k where $\lambda_{k\,h}, \lambda_k$ differ only in the value of their coefficients and, in this way, we construct approximations $\phi_{n,h}$ of ϕ_n, i e ,

$$\phi_{n,h}(\underline{x},\underline{t}) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \theta_{k,h}(\underline{x})\lambda_{k,h}(\underline{t}) \tag{4 2 3}$$

The separated variable expression (4 2 2) is a direct corollary of the following theorem which demonstrates that the functions $\chi_n(\bullet,\underline{t}) \in H^1_{per\,0}(\mathcal{P})$, $\underline{t} \neq 0$, $n \geq 1$, introduced in Theorem 3 2 1, can be represented in the form (4 2 2)

Theorem 4 2 1 *The functions $\chi_n(\bullet,\underline{t}) \in H^1_{per\,0}(\mathcal{P})$, $\underline{t} \neq 0$, $n \geq 1$ defined in Theorem 3 2 1 can be written in the form, for $\alpha \in \mathbb{N}_0^2$,*

$$\chi_n(\underline{x},\underline{t}) = i^n \sum_{|\alpha|=n} \underline{t}^{\alpha} \chi_{\alpha}(\underline{x}), \quad \underline{x} \in \mathcal{P},\ \underline{t} \neq 0, \quad n \geq 1 \tag{4 2 4}$$

where $\chi_\alpha \in H^1_{per\,0}(\mathcal{P})$, $|\alpha| \geq 1$ is defined as the unique solution of the weak formulation

$$\Phi_0[\chi_\alpha, v] = \Theta^{(\alpha)}(v), \quad v \in H^1_{per,0}(\mathcal{P}) \tag{4 2 5}$$

where $\Theta^{(\alpha)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$, $|\alpha| \geq 1$ is defined in relations (4 2 8), (4 2 9) Furthermore, for $\underline{t} \neq 0$, $g_0(\underline{t}) = (\sum_{|\alpha|=2} \kappa_\alpha \underline{t}^\alpha)^{-1}$ and the functions g_n, $n \geq 1$ can be written

$$g_n(\underline{t}) = -g_0(\underline{t}) \sum_{j=0}^{n-1} i^{n-j}\, g_j(\underline{t}) \sum_{|\alpha|=n+2-j} \kappa_\alpha \underline{t}^\alpha, \quad \underline{t} \neq 0, \quad n \geq 1 \tag{4 2 6}$$

where the constants $\kappa_\alpha \in \mathbb{R}$, $|\alpha| \geq 2$ are given by

$$\kappa_\alpha \overset{\text{def}}{=} - \sum_{\substack{\alpha=\beta+\gamma \\ |\gamma|=1}} \Phi_1^{(\gamma)}[\chi_\beta, 1] + \sum_{\substack{\alpha=\beta+\gamma+\delta \\ |\gamma|\ |\delta|=1}} \Phi_2^{(\gamma,\delta)}[\chi_\beta, 1], \quad |\alpha| \geq 2 \tag{4 2 7}$$

and $\Phi_1^{(\gamma)}$, $\Phi_2^{(\beta\ \gamma)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$ for $|\beta|$, $|\gamma| = 1$

Proof We first define the mappings $\Theta^{(\alpha)}$, $\Phi_1^{(\gamma)}$, $\Phi_2^{(\gamma,\delta)}$ employed in relations (4 2 5) and (4 2 7) as follows, for $\alpha, \beta, \gamma, \delta \in \mathbb{N}_0^2$, $v \in H^1_{per,0}(\mathcal{P})$,

$$\text{if } |\delta| = 1, \quad \Theta^{(\delta)}(v) \overset{\text{def}}{=} -\Phi_1^{(\delta)}[1, v] \tag{4 2 8}$$

$$\text{if } |\delta| \geq 2, \quad \Theta^{(\delta)}(v) \overset{\text{def}}{=} - \sum_{\substack{\delta=\alpha+\beta \\ |\beta|=1}} \Phi_1^{(\beta)}[\chi_\alpha, v] + \sum_{\substack{\delta=\alpha+\beta+\gamma \\ |\beta|\ |\gamma|=1}} \Phi_2^{(\beta,\gamma)}[\chi_\alpha, v] \tag{4 2 9}$$

where $\chi_0 \overset{\text{def}}{=} 1$ and, for $|\alpha|$, $|\beta| = 1$, $u, v \in H^1_{per}(\mathcal{P})$,

$$\Phi_1^{(\alpha)}[u, v] \overset{\text{def}}{=} \sum_{|\beta|=1} \Phi_1^{(\alpha,\beta)}[u, v] \tag{4 2 10}$$

$$\Phi_1^{(\alpha\ \beta)}[u, v] \overset{\text{def}}{=} \int_{\mathcal{P}} a_{\alpha\beta}(\underline{x}) \Big(u(\underline{x})\, D^\beta v(\underline{x}) - D^\beta u(\underline{x}) v(\underline{x}) \Big)\, d\underline{x} \tag{4 2 11}$$

$$\Phi_2^{(\alpha\ \beta)}[u, v] \overset{\text{def}}{=} \int_{\mathcal{P}} a_{\alpha\beta}(\underline{x}) u(\underline{x}) v(\underline{x})\, d\underline{x} \tag{4 2 12}$$

where we have, evidently, employed the multi–index notation,

$$D^\alpha \overset{\text{def}}{=} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_1^{\alpha_2}}, \qquad a_{\alpha\beta} \overset{\text{def}}{=} a_{kl}, \quad |\alpha|, |\beta| = 1 \tag{4 2 13}$$

where $k \overset{\text{def}}{=} \alpha_1 + 2\alpha_2$ and $l \overset{\text{def}}{=} \beta_1 + 2\beta_2$ Clearly, that $\Phi_1^{(\alpha,\beta)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P}) \times H^1_{per\,0}(\mathcal{P})\,, \mathbb{R})$, for $|\alpha|, |\beta| = 1$, is apparent from the relations, for $u, v \in H^1_{per,0}(\mathcal{P})$,

$$\begin{aligned}
\left|\Phi_1^{(\alpha,\beta)}[u, v]\right| &\leq \left| \int_{\mathcal{P}} a_{\alpha\beta}(\underline{x}) u(\underline{x}) D^\beta v(\underline{x})\, d\underline{x} \right| + \left| \int_{\mathcal{P}} a_{\alpha\beta}(\underline{x}) v(\underline{x}) D^\beta u(\underline{x})\, d\underline{x} \right| \\
&\leq \left[\int_{\mathcal{P}} |a_{\alpha\beta}(\underline{x})|^2\, |u(\underline{x})|^2\, d\underline{x} \right]^{1/2} \left[\int_{\mathcal{P}} |D^\beta v(\underline{x})|^2\, d\underline{x} \right]^{1/2} + \\
&\quad \left[\int_{\mathcal{P}} |a_{\alpha\beta}(\underline{x})|^2\, |v(\underline{x})|^2\, d\underline{x} \right]^{1/2} \left[\int_{\mathcal{P}} |D^\beta u(\underline{x})|^2\, d\underline{x} \right]^{1/2} \\
&\leq \|a_{\alpha\beta}, \mathcal{L}_\infty(\mathcal{P})\| \Big(\|u, \mathcal{L}_2(\mathcal{P})\|\, |v, H^1(\mathcal{P})| + \|v, \mathcal{L}_2(\mathcal{P})\|\, |u, H^1(\mathcal{P})| \Big) \\
&\leq C\, \|a_{\alpha\beta}, \mathcal{L}_\infty(\mathcal{P})\|\, \|u, H^1(\mathcal{P})\|\, \|v, H^1(\mathcal{P})\|
\end{aligned} \tag{4 2 14}$$

where $C > 0$ is a constant independent of $u, v \in H^1_{per\,0}(\mathcal{P})$ It is, similarly, demonstrated in Lemma 4 2 2 that $\Phi_1^{(\alpha)}$, $\Phi_2^{(\alpha\,\beta)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P})\,,\,\mathbb{R})$ for $|\alpha|, |\beta| = 1$ and, thus, $\Theta^{(\alpha)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P})\,,\,\mathbb{R})$ for $|\alpha| \geq 1$

We now demonstrate that the functions χ_n, $n \geq 1$ defined in (4 2 5) satisfy relations (3 2 8) Let $n = 1$ in (4 2 4) and observe from (3 2 4), (4 2 5), and (4 2 8), that, for $v \in H^1_{per\,0}(\mathcal{P})$,

$$\begin{aligned}\Phi_0[\chi_1(\bullet,\underline{t}),v] &= \imath \sum_{|\alpha|=1} \underline{t}^\alpha \Phi_0[\chi_\alpha, v] \\ &= -\imath \sum_{|\alpha|=1} \underline{t}^\alpha \Phi_1^{(\alpha)}[1,v] \\ &= -\Phi_1(\underline{t})[1,v] \qquad (4\,2\,15)\end{aligned}$$

Thus, $\chi_1(\bullet,\underline{t}) \in H^1_{per,0}(\mathcal{P})$, as expressed in (4 2 4), uniquely satisfies (3 2 8) However, if $n \geq 2$ then, employing (4 2 4), (4 2 5), and (4 2 9), we deduce the following identities, for $v \in H^1_{per\,0}(\mathcal{P})$,

$$\begin{aligned}\Phi_0[\chi_n(\bullet,\underline{t}),v] &= \imath^n \sum_{|\alpha|=n} \underline{t}^\alpha\, \Phi_0[\chi_\alpha, v] \\ &= \imath^n \sum_{|\alpha|=n} \underline{t}^\alpha \left[-\sum_{\substack{\alpha=\beta+\gamma \\ |\gamma|=1}} \Phi_1^{(\gamma)}[\chi_\beta, v] + \sum_{\substack{\alpha=\beta+\gamma+\delta \\ |\gamma|\,|\delta|=1}} \Phi_2^{(\gamma,\delta)}[\chi_\beta, v] \right] \\ &= -\imath^n \sum_{|\beta|=n-1} \underline{t}^\beta \sum_{|\gamma|=1} \underline{t}^\gamma\, \Phi_1^{(\gamma)}[\chi_\beta, v] + \imath^n \sum_{|\beta|=n-2} \underline{t}^\beta \sum_{|\gamma|,|\delta|=1} \underline{t}^{\gamma+\delta}\, \Phi_2^{(\gamma,\delta)}[\chi_\beta, v] \\ &= -\imath^{n-1} \sum_{|\beta|=n-1} \underline{t}^\beta\, \Phi_1(\underline{t})[\chi_\beta, v] - \imath^{n-2} \sum_{|\beta|=n-2} \underline{t}^\beta\, \Phi_2(\underline{t})[\chi_\beta, v] \\ &= -\Phi_1(\underline{t})[\chi_{n-1}(\bullet,\underline{t}),v] - \Phi_2(\underline{t})[\chi_{n-2}(\bullet,\underline{t}),v] \qquad (4\,2\,16)\end{aligned}$$

This demonstrates the validity of the *separated variable* expression (4 2 4) We now substitute expression (4 2 4) for $\chi_1(\bullet,\underline{t}) \in H^1_{per,0}(\mathcal{P})$, $\underline{t} \neq 0$ into relation (3 2 9) to provide the following equations

$$\begin{aligned}(g_0(\underline{t}))^{-1} &= \imath \sum_{|\alpha|=1} \underline{t}^\alpha\, \Phi_1(\underline{t})[\chi_\alpha, 1] + \Phi_2(\underline{t})[1,1] \\ &= -\sum_{|\alpha|,|\beta|=1} \underline{t}^{\alpha+\beta}\, \Phi_1^{(\beta)}[\chi_\alpha, 1] + \sum_{|\alpha|,|\beta|=1} \underline{t}^{\alpha+\beta}\, \Phi_2^{(\alpha\,\beta)}[1,1] \\ &= \sum_{|\delta|=2} \underline{t}^\delta \left[-\sum_{\substack{\delta=\alpha+\beta \\ |\beta|=1}} \Phi_1^{(\beta)}[\chi_\alpha, 1] + \sum_{\substack{\delta=\alpha+\beta \\ |\beta|\,|\gamma|=1}} \Phi_2^{(\alpha,\beta)}[1,1] \right]\end{aligned}$$

and, employing definition (4 2 7), we obtain the expression $g_0(\underline{t}) = (\sum_{|\alpha|=2} \kappa_\alpha\, \underline{t}^\alpha)^{-1}$, $\underline{t} \neq 0$ Similarly, substituting expression (4 2 4) for $\chi_n(\bullet,\underline{t}) \in H^1_{per\,0}(\mathcal{P})$, $n \geq 1$, $\underline{t} \neq 0$ into relation (3 2 9) we deduce the following equations, for $\underline{t} \neq 0$,

$$g_n(\underline{t}) = -g_0(\underline{t}) \sum_{\jmath=0}^{n-1} g_\jmath(\underline{t}) \left[\imath^{n+1-\jmath} \sum_{|\alpha|=n+1-\jmath} \underline{t}^\alpha\, \Phi_1(\underline{t})[\chi_\alpha, 1] + \imath^{n-\jmath} \sum_{|\alpha|=n-\jmath} \underline{t}^\alpha\, \Phi_2(\underline{t})[\chi_\alpha, 1] \right]$$

$$= -g_0(\underline{t}) \sum_{j=0}^{n-1} g_j(\underline{t})\, i^{n-j} \Bigg[- \sum_{\substack{|\alpha|=n+1-j \\ |\beta|=1}} \underline{t}^{\alpha+\beta}\, \Phi_1^{(\beta)}[\chi_\alpha, 1] + \sum_{\substack{|\alpha|=n-j \\ |\beta|\,|\gamma|=1}} \underline{t}^{\alpha+\beta+\gamma}\, \Phi_2^{(\beta\,\gamma)}[\chi_\alpha, 1] \Bigg]$$

$$= -g_0(\underline{t}) \sum_{j=0}^{n-1} i^{n-j}\, g_j(\underline{t}) \sum_{|\delta|=n+2-j} \underline{t}^{\delta} \Bigg[- \sum_{\substack{\delta=\alpha+\beta \\ |\beta|=1}} \Phi_1^{(\beta)}[\chi_\alpha, 1] + \sum_{\substack{\delta=\alpha+\beta+\gamma \\ |\beta|\,|\gamma|=1}} \Phi_2^{(\beta\,\gamma)}[\chi_\alpha, 1] \Bigg] \quad (4\,2\,17)$$

Thus, comparing relation (4 2 15) with (4 2 5) and (4 2 6) and noting expression (4 2 7) for κ_α, $|\alpha| \geq 3$, the theorem is proved ■

From the Lax–Milgram Lemma and the knowledge gained from Theorem 4 2 1 it is clear that one can compute finite element approximations, $\chi_{\alpha,h} \in H^1_{per,0}(\mathcal{P})$, of the functions χ_α, $\alpha \in \mathbb{N}_0^2\backslash\{0\}$ which do not depend on the unbounded variable $\underline{t}$ Thus, we now consider techniques for the construction of finite element subspaces $S^h_{per\,0}(\mathcal{P}) \subset H^1_{per,0}(\mathcal{P})$ from which the approximations $\chi_{\alpha,h}$ will be selected

4 2 2 Construction of the finite element spaces $S^h_{per,0}(\mathcal{P}) \subset H^1_{per,0}(\mathcal{P})$.

Let $\mathcal{B}(S^h_{per}(\mathcal{P}))$ denote the basis for $S^h_{per}(\mathcal{P})$ introduced in Section 4 1 2 with elements φ_n, $1 \leq n \leq \mathcal{D}$ where $\mathcal{D} = \dim(S^h_{per}(\mathcal{P}))$, then, define the functions $\psi_n \in S^h_{per}(\mathcal{P})$, $1 \leq n \leq \mathcal{D}_0 \stackrel{\text{def}}{=} \mathcal{D}-1$, which span $S^h_{per,0}(\mathcal{P})$, according to the relation, for $1 \leq n \leq \mathcal{D}_0$,

$$\psi_n(\underline{x}) \stackrel{\text{def}}{=} \varphi_n(\underline{x}) - \frac{\|\varphi_n, \mathcal{L}_1(\mathcal{P})\|}{\|\varphi_{n+1}, \mathcal{L}_1(\mathcal{P})\|}\, \varphi_{n+1}(\underline{x}), \quad \underline{x} \in \mathcal{P} \quad (4\,2\,18)$$

$$\Rightarrow \quad \operatorname{supp}\psi_n = \operatorname{supp}\phi_n \cup \operatorname{supp}\phi_{n+1} \quad (4\,2\,19)$$

We claim that $\mathcal{B}(S^h_{per,0}(\mathcal{P})) \stackrel{\text{def}}{=} \{\psi_n\}_{n=1}^{\mathcal{D}_0}$ is then a basis for a finite element subspace $S^h_{per,0}(\mathcal{P}) \subset H^1_{per\,0}(\mathcal{P})$ Indeed, it is evident from the relation $S^h_{per,0}(\mathcal{P}) \subset S^h_{per}(\mathcal{P})$ that $\psi_n \in S^h_{per}(\mathcal{P})$ and, furthermore, $\int_{\mathcal{P}} \psi_n(\underline{x})\, d\underline{x} = 0$ because

$$\int_{\mathcal{P}} \psi_n(\underline{x})\, d\underline{x} = \int_{\mathcal{P}} \varphi_n(\underline{x})\, d\underline{x} - \frac{\|\varphi_n, \mathcal{L}_1(\mathcal{P})\|}{\|\varphi_{n+1}, \mathcal{L}_1(\mathcal{P})\|} \int_{\mathcal{P}} \varphi_{n+1}(\underline{x})\, d\underline{x}$$

$$= \|\varphi_n, \mathcal{L}_1(\mathcal{P})\| - \frac{\|\varphi_n, \mathcal{L}_1(\mathcal{P})\|}{\|\varphi_{n+1}, \mathcal{L}_1(\mathcal{P})\|} \|\varphi_{n+1}, \mathcal{L}_1(\mathcal{P})\|$$

$$= 0, \quad 1 \leq n \leq \mathcal{D}_0 \quad (4\,2\,20)$$

Now suppose there are constants α_n, $1 \leq n \leq \mathcal{D}_0$ such that

$$\alpha_1 \psi_n(\underline{x}) + \quad + \alpha_{\mathcal{D}_0} \psi_{\mathcal{D}_0}(\underline{x}), = 0, \quad \underline{x} \in \mathcal{P} \quad (4\,2\,21)$$

then this implies, for $\underline{x} \in \mathcal{P}$, the following identities

$$\sum_{n=1}^{\mathcal{D}_0} \alpha_n \psi_n(\underline{x}) = \sum_{n=1}^{\mathcal{D}_0} \alpha_n \left[\varphi_n(\underline{x}) - \frac{\|\varphi_n, \mathcal{L}_1(\mathcal{P})\|}{\|\varphi_{n+1}, \mathcal{L}_1(\mathcal{P})\|}\, \varphi_{n+1}(\underline{x}) \right]$$

$$= \sum_{n=1}^{\mathcal{D}} \beta_n \varphi_n(\underline{x}) = 0 \quad (4\,2\,22)$$

where

$$\beta_1 = \alpha_1, \quad \beta_n = \alpha_n - \alpha_{n-1} \frac{\|\varphi_{n-1}, \mathcal{L}_1(\mathcal{P})\|}{\|\varphi_n, \mathcal{L}_1(\mathcal{P})\|}, \; 1 < n < \mathcal{D}_0, \quad \beta_{\mathcal{D}} = -\alpha_{\mathcal{D}_0} \frac{\|\varphi_{\mathcal{D}_0}, \mathcal{L}_1(\mathcal{P})\|}{\|\varphi_{\mathcal{D}}, \mathcal{L}_1(\mathcal{P})\|} \quad (4\,2\,23)$$

Because $\{\varphi_n\}_{n=1}^{\mathcal{D}}$ is a basis for $S^h_{per}(\mathcal{P})$ it follows that $\beta_n = 0,\ 1 \le n \le \mathcal{D}$ and, therefore, relations (4 2 23) imply that $\alpha_n = 0,\ 1 \le n \le \mathcal{D}_0$ Thus, the set $\mathcal{B}(S^h_{per,0}(\mathcal{P})) = \{\psi_n\}_{n=1}^{\mathcal{D}_0}$ is a basis for the finite element subspace $S^h_{per,0}(\mathcal{P}) \subset H^1_{per,0}(\mathcal{P})$ Once again we observe that, because $S^h_{per,0}(\mathcal{P}) \subset H^1_{per,0}(\mathcal{P})$, the Lax–Milgram Lemma guarantees the existence of a unique solution $u_h \in S^h_{per,0}(\mathcal{P})$ of the abstract Galerkin problem Find $u_h \in S^h_{per,0}(\mathcal{P})$ such that

$$\Phi_0[u_h, v_h] = F(v_h), \quad v_h \in S^h_{per,0}(\mathcal{P}) \qquad (4\,2\,24)$$

where $F \in \mathcal{BL}(H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$

4.2.3. Analysis of the Finite Element Approximation Errors.

The discretization errors which arise as a consequence of the application of Galerkin finite element techniques to problems (4 2 5), e g , Find $\chi_\alpha \in H^1_{per,0}(\mathcal{P})$ such that

$$\Phi_0[\chi_\alpha, v] = \Theta^{(\alpha)}(v), \quad v \in H^1_{per,0}(\mathcal{P}), \quad |\alpha| \ge 1, \qquad (4\,2\,25)$$

are analysed below where we provide error bounds for the approximation errors $\chi_\alpha - \chi_{\alpha,h}$, $\alpha \in \mathbb{N}_0^2\backslash\{0\}$ in both $\mathcal{L}_2(\mathcal{P})$ and $H^1(\mathcal{P})$ norm topologies

We observe, cf (4 2 9), that the functionals $\Theta^{(\alpha)}$, $|\alpha| \ge 2$ are unknown elements of the Banach space $\mathcal{BL}(H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$ because they depend directly on the unknown weak solutions $\chi_\beta \in H^1_{per,0}(\mathcal{P})$, $|\beta| < |\alpha|$, $h > 0$ Clearly, however, one cannot base computational approaches on purely abstract problems of this type and we therefore employ finite element approximations $\chi_{\beta,h} \in S^h_{per,0}(\mathcal{P})$, $|\beta| < |\alpha|$, $h > 0$ to construct approximating functionals $\Theta^{(\alpha)}_h \in \mathcal{BL}(H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$ of $\Theta^{(\alpha)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$ which we define according to the relation

$$\Theta^{(\alpha)}_h(v) \stackrel{\text{def}}{=} -\sum_{\substack{\alpha=\beta+\gamma \\ |\gamma|=1}} \Phi_1^{(\gamma)}[\chi_{\beta,h}, v] + \sum_{\substack{\alpha=\beta+\gamma+\delta \\ |\gamma|\ |\delta|=1}} \Phi_2^{(\gamma,\delta)}[\chi_{\beta,h}, v], \quad v \in H^1_{per,0}(\mathcal{P}), \quad |\alpha| \ge 2 \quad (4\,2\,26)$$

and $\Theta^{(\alpha)}_h \stackrel{\text{def}}{=} \Theta^{(\alpha)}$, $h > 0$, $|\alpha| = 1$ Thus, we define the Galerkin problems as Find $\chi_{\alpha,h} \in S^h_{per,0}(\mathcal{P})$ such that

$$\Phi_0[\chi_{\alpha\,h}, v_h] = \Theta^{(\alpha)}_h(v_h), \quad v_h \in S^h_{per,0}(\mathcal{P}) \qquad (4\,2\,27)$$

where $\alpha \in \mathbb{N}_0^2\backslash\{0\}$

We now demonstrate in the Lemma 4 2 2 that the various mappings in (4 2 26) from which $\Theta^{(\alpha)}_h$, $\alpha \in \mathbb{N}_0^2\backslash\{0\}$ is composed are continuous The corollary of this Lemma is, of course, the conclusion that $\Theta^{(\alpha)}_h$, $\alpha \in \mathbb{N}_0^2\backslash\{0\}$ is a functional, i e , an element of $\mathcal{BL}(H^1_{per,0}(\mathcal{P})\,, \mathbb{R})$

Lemma 4 2 2 *The mappings* $\Phi_1^{(\alpha)}$, $\Phi_2^{(\alpha,\beta)}$ $H^1_{per\,0}(\mathcal{P}) \times H^1_{per\,0}(\mathcal{P}) \to \mathbb{R}$ *defined in relations (4 2 10) and (4 2 12) are continuous, i e , for* $u, v \in H^1_{per}(\mathcal{P})$, $|\alpha|, |\beta| = 1$,

$$|\Phi_1^{(\alpha)}[u,v]| \leq C_1 \, \|u, H^1(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \tag{4 2 28}$$

$$|\Phi_2^{(\alpha,\beta)}[u,v]| \leq C_2 \, \|u, H^1(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \tag{4 2 29}$$

where C_1, $C_2 > 0$ *are constants independent of* u, v

Proof It has been established in the proof of Theorem 4 2 1 that, for $|\alpha|, |\beta| = 1$, $\Phi_1^{(\alpha,\beta)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P}), \mathbb{R})$ and because $\Phi_1^{(\alpha)} = \sum_{|\beta|=1} \Phi_1^{(\alpha\,\beta)}$, $|\alpha| = 1$ it follows that $\Phi_1^{(\alpha)} \in \mathcal{BL}(H^1_{per\,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P}), \mathbb{R})$ Furthermore, from relation (4 2 14), it is clear that an upper bound for the $\mathcal{BL}(H^1_{per\,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P}), \mathbb{R})$ norm of $\Phi_1^{(\alpha)}$ is the following

$$\|\Phi_1^{(\alpha)}\| \leq 4 \max_{|\beta|=1} \|a_{\alpha\beta}, \mathcal{L}_\infty(\mathcal{P})\|, \quad |\alpha| = 1 \tag{4 2 30}$$

Similarly, from the Cauchy–Schwarz inequality, it is evident that

$$\begin{aligned}\left|\Phi_2^{(\alpha,\beta)}[u,v]\right| &\leq \left[\int_{\mathcal{P}} |a_{\alpha\beta}(\underline{x})|^2 \, |u(\underline{x})|^2 \, d\underline{x}\right]^{1/2} \left[\int_{\mathcal{P}} |v(\underline{x})|^2 \, d\underline{x}\right]^{1/2} \\ &\leq \|a_{\alpha\beta}, \mathcal{L}_\infty(\mathcal{P})\| \, \|v, H^1(\mathcal{P})\| \, \|u, H^1(\mathcal{P})\| \end{aligned} \tag{4 2 31}$$

Thus, for $|\alpha|, |\beta| = 1$, it follows that $\Phi_2^{(\alpha,\beta)} \in \mathcal{BL}(H^1_{per,0}(\mathcal{P}) \times H^1_{per,0}(\mathcal{P}), \mathbb{R})$ and $\|\Phi_2^{(\alpha,\beta)}\| \leq \|a_{\alpha\beta}, \mathcal{L}_\infty(\mathcal{P})\|$ ■

The rate at which the piecewise linear approximations $\chi_{\alpha,h} \in S^h_{per,0}(\mathcal{P})$ converge, as the finite element diameter $h \to 0$, to the analytical solution $\chi_\alpha \in H^1_{per,0}(\mathcal{P})$ for $\alpha \in \mathbb{N}_0^2\backslash\{0\}$ in the $H^p(\mathcal{P})$, $0 \leq p \leq 1$ norm topologies is detailed in the following Theorem

Theorem 4 2 3 *For* $\alpha \in \mathbb{N}_0^2\backslash\{0\}$ *let* $\chi_{\alpha\,h} \in S^h_{per,0}(\mathcal{P})$ *be the Galerkin solution of (4 2 5), i e , it satisfies (4 2 27) then, for* $0 \leq p \leq 1$,

$$\|\chi_\alpha - \chi_{\alpha,h}, H^p(\mathcal{P})\| \leq \; C_\alpha \, h^{(s-1)(2-p)}, \quad h > 0 \tag{4 2 32}$$

where $s \stackrel{\text{def}}{=} \max\{\, r \mid \chi_\beta \in H^r(\mathcal{P}) \cap H^1_{per\,0}(\mathcal{P}), \; |\beta| = 1 \,\}$ *and* $C_\alpha > 0$ *is a constant independent of* $h > 0$

Proof Let $\chi_\alpha \in H^1_{per\,0}(\mathcal{P})$, $\chi_{\alpha,h} \in S^h_{per,0}(\mathcal{P}) \subset H^1_{per,0}(\mathcal{P})$ be, respectively, the solutions of problem (4 2 5), Galerkin problem (4 2 27), then, for $v_h \in S^h_{per,0}(\mathcal{P})$ and $|\alpha| \geq 2$, we observe that

$$\begin{aligned}\Phi_0[\chi_\alpha - \chi_{\alpha,h}, v_h] &= \Phi_0[\chi_\alpha, v_h] - \Phi_0[\chi_{\alpha\,h}, v_h] \\ &= \Theta^{(\alpha)}(v_h) - \Theta_h^{(\alpha)}(v_h) \\ &= -\sum_{\substack{\alpha=\beta+\gamma \\ |\gamma|=1}} \Phi_1^{(\gamma)}[\chi_\beta - \chi_{\beta,h}, v_h] + \sum_{\substack{\alpha=\beta+\gamma+\delta \\ |\gamma|\,|\delta|=1}} \Phi_2^{(\gamma\,\delta)}[\chi_\beta - \chi_{\beta\,h}, v_h] \end{aligned} \tag{4 2 33}$$

The continuity of the mappings $\Phi_1^{(\gamma)}$, $\Phi_2^{(\gamma\,\delta)}$, demonstrated in Lemma 4 2 2 for $\gamma, \delta \in \mathbb{N}_0^2 \backslash \{0\}$ imply that there exist positive constants $K_{1\,\alpha}$, $K_{2\,\alpha}$, and K_α which are independent of the solutions χ_α, $\chi_{\alpha\,h}$, such that, for $v_h \in S^h_{per\,0}(\mathcal{P})$, $|\Phi_0[\chi_\alpha - \chi_{\alpha\,h}, v_h]| \leq$

$$\leq \left[K_{1,\alpha} \sum_{|\beta|=|\alpha|-1} \|\chi_\beta - \chi_{\beta,h}, H^1(\mathcal{P})\| + K_{2,\alpha} \sum_{|\beta|=|\alpha|-2} \|\chi_\beta - \chi_{\beta\,h}, H^1(\mathcal{P})\| \right] \|v_h, H^1(\mathcal{P})\|$$

$$\leq K_\alpha \sum_{|\beta|=|\alpha|-2}^{|\beta|=|\alpha|-1} \|\chi_\beta - \chi_{\beta,h}, H^1(\mathcal{P})\| \;\; \|v_h, H^1(\mathcal{P})\| \tag{4 2 34}$$

However, setting $v_h = \chi_\alpha - \chi_{\alpha\,h}$ in this relation and using the $H^1_{per,0}(\mathcal{P})$-Ellipticity of the sesquilinear form Φ_0 we deduce the following inequality

$$\|\chi_\alpha - \chi_{\alpha,h}, H^1(\mathcal{P})\| \leq \frac{K_\alpha}{C_E} \sum_{|\beta|=|\alpha|-2}^{|\beta|=|\alpha|-1} \|\chi_\beta - \chi_{\beta\,h}, H^1(\mathcal{P})\| \tag{4 2 35}$$

where $C_E > 0$ is the ellipticity constant of Φ_0 It is then evident that, if

$$\|\chi_\beta - \chi_{\beta,h}, H^1(\mathcal{P})\| \leq C_\beta\, h^\gamma, \quad |\beta| < |\alpha| \tag{4 2 36}$$

then there is a constant $C_\alpha > 0$, independent of χ_α and h, such that

$$\|\chi_\alpha - \chi_{\alpha,h}, H^1(\mathcal{P})\| \leq C_\alpha\, h^\gamma, \tag{4 2 37}$$

However, from Céa's Theorem, cf Section 2 2 2, we have, for $|\beta| = 1$,

$$\|\chi_\beta - \chi_{\beta,h}, H^1(\mathcal{P})\| \leq \frac{\|\Phi_0\|}{C_E} \inf\{ \|\chi_\alpha - v_h, H^1(\mathcal{P})\| \;\; v_h \in S^h_{per,0}(\mathcal{P}) \} \tag{4 2 38}$$

where $C_E > 0$ denotes the ellipticity constant of the bounded sesquilinear operator $\Phi_0 \in \mathcal{BL}(H^1_{per}(\mathcal{P}) \times H^1_{per}(\mathcal{P})\,, \mathbb{R})$ However, from the approximation property, cf HACKBUSCH (1992),

$$\inf\{ \|v - v_h, H^1(\mathcal{P})\| \;\; v_h \in S^h_{per,0}(\mathcal{P}) \} \leq C(\theta)\, h^{s-1} \|v, H^s(\mathcal{P})\|, \quad 1 \leq s \leq 2 \tag{4 2 39}$$

where v is an arbitrary element of $H^s(\mathcal{P}) \cap H^1_{per,0}(\mathcal{P})$ and θ is the minimum interior angle of any triangle in the set $\mathcal{T}_h(\mathcal{P})$ of finite elements, we thus have,

$$\|\chi_\beta - \chi_{\beta,h}, H^1(\mathcal{P})\| \leq \widetilde{C}(\theta)\, h^{s-1} \|\chi_\beta, H^s(\mathcal{P})\|, \quad h > 0, \quad |\beta| = 1 \tag{4 2 40}$$

Thus, if we define $s \stackrel{\text{def}}{=} \max\{ r \mid \chi_\beta \in H^r(\mathcal{P}) \cap H^1_{per\,0}(\mathcal{P}),\ |\beta| = 1 \}$ then, in (4 2 36), $\gamma = s - 1$ and the approximation property (4 2 40) and the error bound (4 2 35) imply the error bounds

$$\|\chi_\alpha - \chi_{\alpha,h}, H^1(\mathcal{P})\| \leq \frac{K_\alpha}{C_E} \widetilde{C}(\theta)\, h^{s-1} \sum_{|\beta|=1} \|\chi_\beta, H^s(\mathcal{P})\|, \quad |\alpha| = 2 \tag{4 2 41}$$

Clearly, inequality (4 2 32) now follows directly from (4 2 41) for $p = 1$ and $|\alpha| \leq 2$ and the remaining estimate for $p = 0$ is obtained with the application of the Aubin–Nitsche Theorem, cf Section 2 2 2, which provides the following alternative error estimate to (4 2 40)

$$\|\chi_\beta - \chi_{\beta,h}, \mathcal{L}_2(\mathcal{P})\| \leq C_1(\theta)\, h^{2(s-1)}\, \|\chi_\beta, H^s(\mathcal{P})\|, \quad h > 0, \quad |\beta| = 1 \qquad (4\,2\,42)$$

The error bound (4 2 35) and the error bound (4 2 32), now established for $|\alpha| \leq 2$, and the Aubin–Nitsche Theorem together imply the error bounds (4 2 32) for the higher order approximations $\chi_{\alpha,h} \in S^h_{per,0}(\mathcal{P})$, $|\alpha| \geq 3$, $h > 0$ ■

We observe, for the specific case of piecewise constant coefficients, cf (4 1 6), that with $|\beta| = 1$, $\chi_\beta \in H^{1+\sigma}(\mathcal{P})$, for some $\sigma > 0$ and Theorem 4 2 3 provides the error bounds, for $0 \leq p \leq 1$,

$$\|\chi_\alpha - \chi_{\alpha,h}, H^p(\mathcal{P})\| \leq C_\alpha\, h^{(2-p)\sigma}, \quad h > 0, \quad |\alpha| \geq 1 \qquad (4\,2\,43)$$

However, if the finite element triangulations $\mathcal{T}_h(\mathcal{P})$, $h > 0$ are constructed such that no finite element, $\tau \in \mathcal{T}_h(\mathcal{P})$, can overlap an interface boundary, Γ_{rs}, $1 \leq r, s \leq m$, cf Section 4 1, then the triangle inequality and the regularity property $\chi_\beta \in H^2(\mathcal{P}_h)$, $|\beta| = 1$ where $\mathcal{P}_h$ is any convex union of triangles $\tau \in \mathcal{T}_h(\mathcal{P})$, $h > 0$ satisfying $\text{dist}(\mathcal{P}_h, \mathcal{V}) > \rho > 0$ for ρ sufficiently large and where $\mathcal{V} \stackrel{\text{def}}{=} \{v \in \Gamma \mid v \text{ is a vertex}\}$ suggest the error estimate, for $0 \leq p \leq 1$,

$$\|\chi_\alpha - \chi_{\alpha\, h}, H^p(\mathcal{P}_h)\| \leq C(\theta)\, h^{2-p} \|\chi_\alpha, H^2(\mathcal{P}_h)\| + \|\chi_{\alpha,h} - \Pi_h \chi_\alpha, H^p(\mathcal{P}_h)\|, \quad |\alpha| \geq 1 \quad (4\,2\,44)$$

where $\Pi_h \;\; H^2(\mathcal{P}) \to S^h(\mathcal{P})$ is the interpolation operator and θ is the smallest interior angle of any $\tau \subset \mathcal{P}_h$, $h > 0$ The first term in (4 2 44) reflects the optimal approximation errors possible in each element, τ, as a consequence of the type of triangulation $\mathcal{T}_h(\mathcal{P})$ while the second term represents the pollution effect of the singularities on the region $\mathcal{P}_h \subset \mathcal{P}$ and will, consequently, have a lower asymptotic order with respect to h, cf NITSCHE & SCHATZ (1974) Thus, for $|\alpha| \geq 1$, we expect the approximations $\chi_{\alpha\, h} \in S^h_{per,0}(\mathcal{P})$ to converge to $\chi_\alpha \in H^1_{per,0}(\mathcal{P})$, as $h \to 0$, more rapidly than is indicated by the global error bound (4 2 43) for an arbitrary triangulation $\mathcal{T}_h(\mathcal{P})$ Indeed, we exploit the approximation properties (4 2 44) in the computational examples provided in Sections 4 4 1–4 4 4 for which the coefficients $a_{\alpha\beta}$, $|\alpha|, |\beta| = 1$ are piecewise constant

The constants κ_α, $|\alpha| \geq 2$ defined in relation (4 2 7) are unknown because they are defined in terms of the weak solutions $\chi_\beta \in H^1_{per,0}(\mathcal{P})$, $|\beta| < |\alpha|$ Thus, we define the approximations $\kappa_{\alpha\, h}$, $h > 0$ as follows

$$\kappa_{\alpha,h} \stackrel{\text{def}}{=} - \sum_{\substack{\alpha=\beta+\gamma \\ |\gamma|=1}} \Phi_1^{(\gamma)}[\chi_{\beta\, h}, 1] + \sum_{\substack{\alpha=\beta+\gamma+\delta \\ |\gamma|\, |\delta|=1}} \Phi_2^{(\gamma,\delta)}[\chi_{\beta\, h}, 1], \quad |\alpha| \geq 2 \qquad (4\,2\,45)$$

where $\chi_{\beta,h} \in S^h_{per\,0}(\mathcal{P})$, $|\beta| < |\alpha|$, $h > 0$ are the finite element approximations introduced in problem (4 2 27) The rate at which the error $\kappa_\alpha - \kappa_{\alpha\, h}$ decays is considered in the following Corollary to Theorem 4 2 3

Corollary 4 2 4 *There exist constants $C_\alpha > 0$, $|\alpha| \geq 2$, independent of $h > 0$, such that*

$$|\kappa_\alpha - \kappa_{\alpha\, h}| \leq C_\alpha h^{s-1}, \quad h > 0 \tag{4 2 46}$$

where $s \stackrel{\text{def}}{=} \max\{\, r \mid \chi_\beta \in H^r(\mathcal{P}) \cap H^1_{per,0}(\mathcal{P}),\ |\beta| = 1\,\}$ and κ_α, $\kappa_{\alpha\, h}$ are defined in relations (4 2 7) and (4 2 45) respectively

Proof The error bound (4 2 46) follows immediately from relations (4 2 34), (4 2 39) and (4 2 40) provided in the proof of Theorem 4 2 3 ∎

We observe, however, that if the coefficients $a_{\alpha\beta}$, $|\alpha|, |\beta| = 1$ are piecewise constant we obtain $|\kappa_\alpha - \kappa_{\alpha\, h}| = O(h^\sigma)$, $0 < \sigma \leq 1$, however, by constructing $\mathcal{T}_h(\mathcal{P})$, $h > 0$ as above we find that there are components of the error which are bounded by terms of the order $O(h)$ as $h \to 0$

4 3 Estimation of the Finite Element/Homogenization Error.

It has already been noted that, generally, there are no algorithms available which can be employed to provide explicit analytical expressions for the weak solutions, ϕ, u^ϵ, of problems (4 1 4) and (4 1 22) However, to assess our approach we require, at least, approximations, ϕ_h, $u^\epsilon_{\ell\, h}$, $\ell \in \mathbb{N}$, $h > 0$, with which the asymptotics

$$u^\epsilon_{N,\ell,h}(\underline{x}) \stackrel{\text{def}}{=} \sum_{\underline{n} \in \mathcal{Z}^2_\ell \setminus \{0\}} a_{\underline{n}} e^{\underline{n}\,\underline{x}\,\pi i} \phi_{N,h}(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi), \quad \underline{x} \in \mathbb{R}^2, \quad \ell \in \mathbb{N}, N \geq 0 \tag{4 3 1}$$

can be meaningfully compared, i e , such that the error $u^\epsilon_{\ell\, h} - u^\epsilon_{N,\ell\, h}$ closely parallels the actual error $u^\epsilon_\ell - u^\epsilon_{N,\ell\, h}$, $\ell \in \mathbb{N}$ for $h > 0$ sufficiently small Clearly, this requires accurate approximations ϕ_h, $u^\epsilon_{\ell\, h}$ of ϕ, u^ϵ_ℓ and, thus, we employ finite element techniques to construct approximations $\phi_h(\bullet, \varepsilon, \underline{t})$, $u^\epsilon_{\ell\, h}$, $\underline{t} \neq 0$, ε, $h > 0$ where

$$u^\epsilon_{\ell\, h}(\underline{x}) \stackrel{\text{def}}{=} \sum_{\underline{n} \in \mathcal{Z}^2_\ell \setminus \{0\}} a_{\underline{n}} e^{\underline{n}\,\underline{x}\,\pi i} \phi_h(\underline{x}/\varepsilon, \varepsilon, \pi\, \underline{n}), \quad \underline{x} \in \mathbb{R}^2 \tag{4 3 2}$$

The errors which these approximations introduce are analysed, and, finally, they are employed to investigate the errors $u^\epsilon - u^\epsilon_{N\, h}$, $N \geq 0$, $h > 0$

4 3 1 Finite Element Approximations $\phi_h(\bullet, \varepsilon, \underline{t})$, $h > 0$.

Let $S^h_{per}(\mathcal{P})$ be the function space of periodic, piecewise linear functions over the field $\mathbb{C}$, defined in Section 4 1 2, and define $S^h_0(\mathcal{P})$ as the subspace of functions of $S^h_{per}(\mathcal{P})$ with zero trace on the boundary, $\partial\mathcal{P}$ We now define the approximation $\phi_h(\bullet, \varepsilon, \underline{t})$ as the solution of the Galerkin problem Find $\phi_h(\bullet, \varepsilon, \underline{t}) \in \mathcal{V}_h$ such that

$$\Phi(\underline{t})[\phi_h(\bullet, \varepsilon, \underline{t}), v_h] = \varepsilon^2 \int_{\mathcal{P}} \overline{v_h(\underline{x})}\, d\underline{x}, \quad v_h \in \mathcal{V}_h \tag{4 3 3}$$

where $\mathcal{V}_h \stackrel{\text{def}}{=} S^h_{per}(\mathcal{P})$ if $(\varepsilon, \underline{t}) \notin \mathcal{H}^2$ and $\mathcal{V}_h \stackrel{\text{def}}{=} S^h_0(\mathcal{P})$ if $(\varepsilon, \underline{t}) \in \mathcal{H}^2$, cf 4 1 1 The errors introduced by this approximation are considered in the following Theorem

Theorem 4 3 1 *Let $\phi_h(\bullet,\varepsilon,\underline{t}) \in \mathcal{V}_h$, $h > 0$ be the solution of the Galerkin problem (4 3 3) then, for $0 \le p \le 1$,*

$$\|\phi(\bullet,\varepsilon,\underline{t}) - \phi_h(\bullet,\varepsilon,\underline{t}), H^p(\mathcal{P})\| \le C(\varepsilon,\underline{t})\, h^{(s-1)(2-p)}\, \|\phi(\bullet,\varepsilon,\underline{t}), H^s(\mathcal{P})\|, \quad h > 0 \qquad (4\,3\,4)$$

where $s \stackrel{\text{def}}{=} \max\{r \mid \phi(\bullet,\varepsilon,\underline{t}) \in H^r(\mathcal{P}) \cap H^1_{per}(\mathcal{P})\}$ and $C(\varepsilon,\underline{t}) = O(\|\underline{t}\|_2^4)\ (\|\underline{t}\|_2 \to \infty)$

Proof From Cea's Theorem, relations (4 1 28), (4 1 31), and the approximation property, cf HACKBUSCH (1992),

$$\inf\{\,\|v - v_h, H^1(\mathcal{P})\| \quad v_h \in S^h_{per}(\mathcal{P})\,\} \le C(\theta)\, h^{s-1}\, \|v, H^s(\mathcal{P})\|, \quad 1 \le s \le 2 \qquad (4\,3\,5)$$

where v is an arbitrary element of $H^s(\mathcal{P}) \cap H^1_{per}(\mathcal{P})$ and θ is the minimum interior angle of any triangle in the set $\mathcal{T}_h(\mathcal{P})$ of finite elements, we thus have,

$$\|\phi(\bullet,\varepsilon,\underline{t}) - \phi_h(\bullet,\varepsilon,\underline{t}), H^1(\mathcal{P})\| \le C(\varepsilon,\underline{t})\, h^{s-1}\, \|\phi(\bullet,\varepsilon,\underline{t}), H^s(\mathcal{P})\|, \quad h > 0 \qquad (4\,3\,6)$$

where the positive function $C(\varepsilon,\,\underline{t}) = C_1^2 C_2^2(\alpha_2/\alpha_1)(1+\varepsilon\|\underline{t}\|_2)^2(1+2\varepsilon\,\|\underline{t}\|_2+\varepsilon^2\|\underline{t}\|_2^2)$ and $s \stackrel{\text{def}}{=} \max\{r \mid \phi(\bullet,\varepsilon,\underline{t}) \in H^r(\mathcal{P}) \cap H^1_{per}(\mathcal{P})\}$ Thus, for $p = 1$, property (4 3 4) follows immediately from (4 3 6) and, for $p = 0$, we apply the Aubin–Nitsche Theorem to obtain (4 3 4) ■

The *local* finite element approximation $\phi_h(\bullet,\varepsilon,\underline{t}) \in S^h_{per}(\mathcal{P})$, $\varepsilon, h > 0$, $\underline{t} \ne 0$ shall be employed in the computational examples in Sections 4 4 1 – 4 4 4 to construct the *global* approximations $u^\varepsilon_{\ell\,h}$, $\ell \in \mathbb{N}$ defined in relation (4 3 2) The errors introduced by such an approximation over Ω are considered in Section 4 3 2

4 3 2 Analysis of the Global, Ω, Approximation Errors

The errors introduced by the approximations $\chi_{\alpha,h} \in S^h_{per,0}(\mathcal{P},\mathbb{R})$, $|\alpha| \ge 1$, $\phi_h(\bullet,\varepsilon,\underline{t}) \in S^h_{per}(\mathcal{P})$ for $\varepsilon > 0$, $\underline{t} \ne 0$, and $h > 0$ were analysed in Sections 4 2 3, 4 3 1 to determine the effects of approximation within the reference cell $\mathcal{P}$ However, to assess the homogenization approach we require some indication of the errors introduced over Ω by the *global* approximations $u^\varepsilon_{\ell,h}$, $u^\varepsilon_{N,\ell,h}$, cf (4 3 1), (4 3 2), which are constructed from these *local* approximations We perform an analysis to determine error bounds for $u^\varepsilon - u^\varepsilon_{N,\ell\,h}$ in the $H^p(\Omega)$, $0 \le p \le 1$ norm topologies

We begin by bounding the truncation error $u^\varepsilon - u^\varepsilon_\ell$ and the approximation error $u^\varepsilon_\ell - u^\varepsilon_{\ell,h}$ for $\ell \in \mathbb{N}$, $h > 0$ in Lemmas 4 3 2 and 4 3 3 below

Lemma 4 3 2 *Define $f_\ell \in \mathcal{L}_2(\Omega)$, $\ell \in \mathbb{N}$ by the following relation*

$$f_\ell(\underline{x}) \stackrel{\text{def}}{=} \sum_{\underline{n} \in \mathbb{Z}^2_\ell \setminus \{0\}} a_{\underline{n}}\, e^{i\underline{n}\,\underline{x}\pi}, \quad \underline{x} \in \mathbb{R}^2 \qquad (4\,3\,7)$$

and define $u^\varepsilon_\ell \in H^1_0(\Omega)$ to be the unique function which has the property

$$\int_\Omega \sum_{k,l=1}^{2} a_{kl}(\underline{x}/\varepsilon)\, \frac{\partial u^\varepsilon_\ell}{\partial x_k}(\underline{x})\, \frac{\partial v}{\partial x_l}(\underline{x})\, d\underline{x} = \int_\Omega f_\ell(\underline{x})\, v(\underline{x})\, d\underline{x}, \quad v \in H^1_0(\Omega) \qquad (4\,3\,8)$$

then, for $0 \le p \le 1$, *and* $\ell \in \mathbf{N}$,

$$\|u^\varepsilon - u_\ell^\varepsilon, H^p(\Omega)\| \le C_1 \|f - f_\ell, \mathcal{L}_2(\Omega)\| \le C_2 \|\mathcal{A}_\ell, \ell_2(\mathcal{Z}^2)\| \qquad (4\ 3\ 9)$$

where $\mathcal{A}_\ell = (\mathcal{A}_{\underline{n}}^\ell)$, $\underline{n} \in \mathcal{Z}^2$ *is the* $\ell_2(\mathcal{Z}^2)$ *sequence*

$$\mathcal{A}_{\underline{n}}^\ell = \begin{cases} 0, & \text{if } \underline{n} \in \mathcal{Z}_\ell^2 \\ a_{\underline{n}}, & \text{otherwise} \end{cases} \qquad (4\ 3\ 10)$$

and C_1, $C_2 > 0$ *are constants which are independent of* f, f_ℓ, *the weak solutions* u^ε, u_ℓ^ε, *and* a

Proof It is clear from (4 1 4) and (4 3 8) that the function $u^\varepsilon - u_\ell^\varepsilon \in H_0^1(\Omega)$ has the property

$$\int_\Omega \sum_{k,l=1}^{2} a_{kl}(\underline{x}) \frac{\partial(u^\varepsilon - u_\ell^\varepsilon)}{\partial x_k}(\underline{x}) \frac{\partial v}{\partial x_l}(\underline{x})\, d\underline{x} = \int_\Omega (f - f_\ell)(\underline{x})\, v(\underline{x})\, d\underline{x}, \quad v \in H_0^1(\Omega) \qquad (4\ 3\ 11)$$

Thus, employing the Cauchy–Schwarz inequality, the $H_0^1(\Omega)$–ellipticity of the bilinear form in relation (4 3 8), and Parseval's relation we deduce that relation (4 3 11) implies (4 3 9) ■

Lemma 4 3 3 *For finite, bounded* $\ell \in \mathbf{N}$ *the approximation errors* $u_\ell^\varepsilon - u_{\ell,h}^\varepsilon$ *are bounded above as follows*

$$\|u_\ell^\varepsilon - u_{\ell\, h}^\varepsilon, H^p(\Omega)\| \le C(\ell)\, h^{(2-p)(s-1)}, \quad h > 0, \quad 0 \le p \le 1 \qquad (4\ 3\ 12)$$

where $C(\ell) \to \infty$ $(\ell \to \infty)$ *is independent of* ε, $h > 0$

Proof The error $u_\ell^\varepsilon - u_{\ell,h}^\varepsilon$, $\ell \in \mathbf{N}$, $h > 0$ in the norm topologies $H^p(\Omega)$, $0 \le p \le 1$ can be written

$$\|u_\ell^\varepsilon - u_{\ell,h}^\varepsilon, H^p(\Omega)\| = \left\| \sum_{\underline{n} \in \mathcal{Z}_\ell^2 \setminus \{0\}} a_{\underline{n}} e^{\underline{n}\,(\bullet)\pi i} \Big(\phi(\bullet/\varepsilon, \varepsilon, \underline{n}\pi) - \phi_h(\bullet/\varepsilon, \varepsilon, \underline{n}\pi)\Big), H^p(\Omega) \right\| \qquad (4\ 3\ 13)$$

However, for finite $\ell \in \mathbf{N}$, the Holder inequality implies the relation, for $|\alpha| \le 1$,

$$\left| \sum_{\underline{n} \in \mathcal{Z}_\ell^2 \setminus \{0\}} a_{\underline{n}} D^\alpha \Big(e^{\underline{n}\,\underline{x}\pi i} \left(\phi(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) - \phi_h(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi)\right)\Big) \right| \le$$

$$\|f_\ell, \mathcal{L}_2(\mathcal{C})\| \left[\sum_{\underline{n} \in \mathcal{Z}_\ell^2 \setminus \{0\}} \left| D^\alpha \Big(e^{\underline{n}\,\underline{x}\pi i} \left(\phi(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) - \phi_h(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi)\right)\Big) \right|^2 \right]^{1/2} \qquad (4\ 3\ 14)$$

and, substituting this relation in (4 3 13), we obtain the upper bound, for $0 \le p \le 1$,

$$\|u_\ell^\varepsilon - u_{\ell\, h}^\varepsilon, H^p(\Omega)\|^2 \le C\, \|f_\ell, \mathcal{L}_2(\mathcal{C})\|^2 \sum_{\underline{n} \in \mathcal{Z}_\ell^2 \setminus \{0\}} \|e^{\underline{n}\,(\bullet)\pi i} (\phi(\bullet/\varepsilon, \varepsilon, \underline{n}\pi) - \phi_h(\bullet/\varepsilon, \varepsilon, \underline{n}\pi)), H^p(\mathcal{P})\|^2 \qquad (4\ 3\ 15)$$

where we have observed that $\mathcal{P} \equiv \Omega$ and, from Parseval's relation,

$$\|f_\ell, \mathcal{L}_2(\mathcal{C})\|^2 = \sum_{\underline{n} \in \mathcal{Z}_\ell^2 \setminus \{0\}} |a_{\underline{n}}|^2, \quad \ell \in \mathbb{N} \tag{4 3 16}$$

Furthermore, Lemma 4 2, the weak formulation (4 1 22), and the Cauchy–Schwarz inequality imply the relations

$$\begin{aligned}
&\|e^{\underline{t}\,(\bullet)\imath}\,((\phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t})), H^1(\mathcal{P})\|^2 \\
&\quad \le C_2\, |e^{\underline{t}\,(\bullet)\imath}\,((\phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t})), H^1(\mathcal{P})|^2 \\
&\quad \le C_2\, \alpha_1^{-1} |\Phi(\varepsilon,\underline{t})[\phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t}), \phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t})]| \\
&\quad = C_2\, \alpha_1^{-1} |\Phi(\varepsilon,\underline{t})[\phi(\bullet/\varepsilon,\varepsilon,\underline{t}), \phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t})]| \\
&\quad = C_2\, \alpha_1^{-1} \left| \int_{\mathcal{P}} \overline{\phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t})}\, d\underline{x} \right| \\
&\quad \le C_2\, \alpha_1^{-1} \|\phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t}), \mathcal{L}_2(\mathcal{P})\|
\end{aligned} \tag{4 3 17}$$

where we have observed that, for $v \in H^1_{per}(\mathcal{P})$, $\mathcal{P}_{\imath\jmath} \stackrel{\text{def}}{=} (\imath-1,\imath) \times (\jmath-1,\jmath)$, $1 \le \imath,\jmath \le 1/\varepsilon$,

$$\int_{\mathcal{P}} v(\underline{x})\, d\underline{x} = \int_{\mathcal{P}_{\imath\jmath}} v(\underline{x})\, d\underline{x}$$

Now, if $p = 1$ we employ inequality (4 3 17) in relation (4 3 15) and otherwise, if $p = 0$, we use identity

$$\|e^{\underline{t}\,(\bullet)\imath}\,(\phi(\bullet/\varepsilon,\varepsilon,\underline{t}) - \phi_h(\bullet/\varepsilon,\varepsilon,\underline{t})), \mathcal{L}_2(\mathcal{P})\| = \|\phi(\bullet,\varepsilon,\underline{t}) - \phi_h(\bullet,\varepsilon,\underline{t}), \mathcal{L}_2(\mathcal{P})\| \tag{4 3 18}$$

and, thus, from Theorem 4 3 1 we deduce the error estimate, for $0 \le p \le 1$,

$$\|u_\ell^\varepsilon - u_{\ell,h}^\varepsilon, H^p(\Omega)\| \le C\, h^{(2-p)(s-1)}\, \|f_\ell, \mathcal{L}_2(\mathcal{P})\| \left[\sum_{\underline{n} \in \mathcal{Z}_\ell^2 \setminus \{0\}} C_2(\underline{n})\, \|\phi(\bullet,\varepsilon,\underline{n}\pi), H^s(\mathcal{P})\| \right]^{1/2} \tag{4 3 19}$$

where $C_2(\underline{n}) \to \infty$ as $\|\underline{n}\|_2 \to \infty$ The functions $D^\alpha\phi(\underline{x},\bullet,\underline{n}\pi)$, $|\alpha| \le 1$ are Holomorphic for $|\varepsilon - s_r(\underline{m},\underline{n})| > \delta$ where $\delta > 0$ is fixed and $s_r(\underline{m},\underline{n}) = 2m_r/n_r$, $1 \le r \le 2$, $\underline{m},\underline{n} \in \mathcal{Z}^2 \setminus \{0\}$, cf Theorem 3 1 1 Thus, within this bounded domain the functions $D^\alpha\phi(\underline{x},\bullet,\underline{n}\pi)$, $|\alpha| \le 1$ can be bounded independently of ε and because $\|\phi(\bullet,\varepsilon,\underline{n}\pi), H^s(\mathcal{P})\|$ is defined in terms of these functions, e g , for $s = 1 + \sigma$,

$$\|\phi(\bullet,\varepsilon,\underline{n}\pi), H^s(\mathcal{P})\|^2 = \sum_{|\alpha| \le 1} \left[\|D^\alpha\phi(\bullet,\varepsilon,\underline{n}\pi), \mathcal{L}_2(\mathcal{P})\|^2 + \|D^\alpha\phi(\bullet,\varepsilon,\underline{n}\pi), H^\sigma(\mathcal{P})\|^2 \right]$$

it can also be bounded independently of ε The error bound (4 3 12) now follows directly from (4 3 19) ∎

We observe that the asymptotic property $C(\ell) \to \infty$ $(\ell \to \infty)$ precludes the use of Lemma 4 3 3 to deduce the asymptotic properties of the error $u^\varepsilon - u_h^\varepsilon \stackrel{\text{def}}{=} \lim_{\ell\to\infty}(u_\ell^\varepsilon - u_{\ell,h}^\varepsilon)$ (with the limit taken in the $H^1(\Omega)$ sense) Indeed, the asymptotic character of the function $C(\varepsilon, \underline{t})$, deduced in Theorem 4 3 1 using Cea's Theorem, suggests that we can do no better The triangle inequality and Lemma's 4 3 2, 4 3 3 are now applied to analyse the error $u^\varepsilon - u^\varepsilon_{N\,\ell\,h}$ into separate components as follows, for $\ell \leq \Lambda$, $0 \leq p \leq 1$,

$$\begin{aligned}\|u^\varepsilon - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\| &\leq \|u^\varepsilon - u^\varepsilon_\ell, H^p(\Omega)\| + \|u^\varepsilon_\ell - u^\varepsilon_{\ell\,h}, H^p(\Omega)\| + |\, u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\| \\ &\leq C_1 \, \|f - f_\ell, \mathcal{L}_2(\Omega)\| + C(\ell)\, h^{(2-p)(s-1)} + \|u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\| \quad (4\,3\,20)\end{aligned}$$

where Λ is a fixed positive integer Thus, by employing finite element triangulations $\mathcal{T}_h(\mathcal{P})$ with $h > 0$ sufficiently small and ℓ large, i e, such that the errors $\|u^\varepsilon_\ell - u^\varepsilon_{\ell\,h}, H^p(\Omega)\|$ and $\|f - f_\ell, \mathcal{L}_2(\Omega)\|$ are an order of magnitude smaller than $\|u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell,h}, H^p(\Omega)\|$, the behaviour of $\|u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\|$ provides an accurate guide to the character of the error $\|u^\varepsilon - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\|$ in the norm topologies $H^p(\Omega)$, $0 \leq p \leq 1$ Indeed, this analysis motivates the computations undertaken in Sections 4 4 1–4 4 4 which assess the errors $\|u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\|$, $0 \leq p \leq 1$ for a variety of problems possessing different regularity characteristics However, the task of constructing accurate approximations $\phi_h(\bullet, \varepsilon, \underline{t}) \in S^h_{per}(\mathcal{P})$, $u^\varepsilon_{\ell\,h}$ of $\phi(\bullet, \varepsilon, \underline{t}) \in H^1_{per}(\mathcal{P})$, $u^\varepsilon_\ell \in H^1_{per}(\mathcal{P})$ becomes impractical for very large ℓ and $\varepsilon \approx 0$ Indeed, to construct $u^\varepsilon_{\ell\,h}$ it is necessary to solve the Galerkin problem (4 3 3) for each $\underline{t} = \underline{n}\pi$, $|n_1|$, $|n_2| \leq \ell$ and, on any computer architecture, to assess the global errors $u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell,h}$ requires, as $\varepsilon \to 0$, an unboundedly increasing proportion of cpu time Thus, we attempt to obtain a reliable and accurate assessment of our approach by employing $\varepsilon = 1/r$, $1 \leq r \leq R$ with ℓ, R sufficiently large so that the principal approximation properties of $u^\varepsilon_{N\,\ell\,h}$ become apparent while remaining within the constraints imposed on time and space by the resources of a computer architecture

4 4 Computational Examples.

Following the one dimensional setting of Chapter 3 we now find it necessary to make some comments regarding the effect of problem regularity on the convergence properties of the asymptotic approximations $u^\varepsilon_{N\,\ell\,h}$ as $\ell \to \infty$ The functions $u^\varepsilon_{N\,\ell\,h}$, $N \geq 0$, $\ell \in \mathbb{N}$, $h > 0$ where

$$u^\varepsilon_{N\,\ell\,h}(\underline{x}) \stackrel{\text{def}}{=} \sum_{\underline{n} \in Z^2_\ell \setminus \{0\}} a_{\underline{n}}\, e^{\underline{n}\,\underline{x}\pi i}\, \phi_{N\,h}(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi), \quad \underline{x} \in \mathbb{R}^2, \quad \varepsilon > 0 \qquad (4\,4\,1)$$

are evidently constructed from the discrete approximations $\phi_{N,h}$, $N \geq 0$, $h > 0$ which are defined as follows

$$\phi_{N\,h}(\underline{x}, \varepsilon, \underline{t}) \stackrel{\text{def}}{=} \sum_{n=0}^{N} \varepsilon^n\, \phi_{n\,h}(\underline{x}, \underline{t}), \quad \underline{x} \in \mathcal{P},\ \underline{t} \neq 0 \qquad (4\,4\,2)$$

where $g_{0\,h}(\underline{t}) = (\sum_{|\alpha|=2} \kappa_{\alpha\,h}\, \underline{t}^\alpha)^{-1}$, $\underline{t} \neq 0$ and, for $\underline{x} \in \mathcal{P}$, $\underline{t} \neq 0$, $n \geq 1$,

$$\phi_{n\,h}(\underline{x}, \underline{t}) \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} g_{j\,h}(\underline{t})\, \chi_{n-j\,h}(\underline{x}, \underline{t}) + g_{n\,h}(\underline{t}) \qquad (4\,4\,3)$$

where

$$\chi_{n\,h}(\underline{x},\underline{t}) \stackrel{\text{def}}{=} \imath^n \sum_{|\alpha|=n} \underline{t}^\alpha \chi_{\alpha\,h}(\underline{x}), \quad g_{n,h}(\underline{t}) \stackrel{\text{def}}{=} -g_{0\,h}(\underline{t}) \sum_{\jmath=0}^{n-1} \imath^{n-\jmath}\, g_{\jmath,h}(\underline{t}) \sum_{|\alpha|=n+2-\jmath} \kappa_{\alpha,h}\, \underline{t}^\alpha \qquad (4\,4\,4)$$

and $\kappa_{\alpha\,h}$, $|\alpha| \geq 2$, $h > 0$ is defined in relation (4 2 45) However, from (4 1 25), (4 1 27), Lemma 10, and Theorem 9 of BABUŠKA & MORGAN (1991ii) it follows that there exist constants $\eta, \theta > 0$, which are independent of $\underline{t} \in \mathbb{R}^2\backslash\{0\}$, such that

$$g_{0\,h}(\underline{t}) \leq 1/(\gamma_1\, \|\underline{t}\|_2^2), \quad \|\phi_{k\,h}(\bullet,\underline{t}), H^1(\mathcal{P})\| \leq \eta\, g_{0,h}(\underline{t})\, \theta^k\, \|\underline{t}\|_2^k, \quad k \geq 0,\ \underline{t} \neq 0 \quad (4\,4\,5)$$

$$\Rightarrow \quad \|\phi_{k,h}(\bullet,\underline{t}), H^1(\mathcal{P})\| = O(\|\underline{t}\|_2^{k-2})\ (\|\underline{t}\|_2 \to \infty) \qquad (4\,4\,6)$$

Furthermore, if $f_C \in BV(C)$ then there exist functions $\varphi_\imath$, $\psi_\imath$, $1 \leq \imath \leq 2$ which are non–decreasing and non–negative and are such that $f_C = \varphi_1 - \psi_1 - \varphi_2 + \psi_2$ The second mean value theorem for integrals then shows that

$$\int_{\mathcal{P}} [\varphi_r(\underline{x}), \psi_r(\underline{x})]\, e^{-\underline{n}\,\underline{x}\pi\imath}\, d\underline{x} = O(|n_1 n_2|^{-1})\ (\|\underline{n}\|_2 \to \infty), \quad 1 \leq r \leq 2 \qquad (4\,4\,7)$$

$$\Rightarrow \quad a_{\underline{n}} = O(|n_1 n_2|^{-1})\ (\|\underline{n}\|_2 \to \infty) \qquad (4\,4\,8)$$

The convergence properties, as $\ell \to \infty$, of the approximations $u^\varepsilon_{N,\ell,h}$, $N \geq 0$, $\varepsilon, h > 0$ in the $H^p(\Omega)$, $0 \leq p \leq 1$ sense are now apparent from relations (4 4 5), (4 4 8) and

$$\|u^\varepsilon_{N,\ell,h}, H^p(\Omega)\| \leq \sum_{\underline{n}\in Z_\ell^2\backslash\{0\}} |a_{\underline{n}}|\, \|e^{\underline{n}\,\underline{x}\pi\imath}\, \phi_{N,h}(\bullet/\varepsilon, \varepsilon, \underline{n}\pi), H^p(\Omega)\| \qquad (4\,4\,9)$$

$$\leq \sum_{\underline{n}\in Z_\ell^2\backslash\{0\}} |a_{\underline{n}}|\, (1 + \|\underline{n}\|_2)^p\, \|\phi_{N,h}(\bullet/\varepsilon, \varepsilon, \underline{n}\pi), H^p(\Omega)\| \qquad (4\,4\,10)$$

$$\leq \sum_{\underline{n}\in Z_\ell^2\backslash\{0\}} |a_{\underline{n}}|\, (1 + \|\underline{n}\|_2)^p\, \varepsilon^{-p}\, \|\phi_{N,h}(\bullet, \varepsilon, \underline{n}\pi), H^p(\mathcal{P})\| \qquad (4\,4\,11)$$

for, by the comparison test, $u^\varepsilon_{N,\ell\,h} \to u^\varepsilon_{N,h}$ absolutely w r t $\|\bullet, H^p(\Omega)\|$, $0 \leq N + p \leq 1$ as $\ell \to \infty$, i e ,

$$|a_{\underline{n}}|\, (1 + \|\underline{n}\|_2)^p\, \varepsilon^{-p}\, \|\phi_{N\,h}(\bullet, \varepsilon, \underline{n}\pi), H^p(\mathcal{P})\| \leq K_1\, |a_{\underline{n}}|\, (1 + \|\underline{n}\|_2)^p\, \|\underline{n}\|_2^{N-2}$$
$$\leq K_2\, |a_{\underline{n}}|\, (n_1^2 + n_2^2)^{(N+p-2)/2} = K_2\, |a_{\underline{n}}|\, \|\underline{n}\|_2^{N+p-2}$$
$$\leq K_2\, 2^{(N+p-2)/2}\, |n_1 n_2|^{(N+p)/2-2} \leq K_3\, |n_1 n_2|^{-3/2} \qquad (4\,4\,12)$$

and, for $N+p \geq 3$, $\|u^\varepsilon_{N,\ell\,h}, H^p(\Omega)\| \to \infty$ $(\ell \to \infty)$ Furthermore, if $N+p = 2$ then (4 4 5) implies the asymptotic relation $\|\phi_{N,h}(\bullet, \underline{n}\pi), H^p(\mathcal{P})\| = O(1)$ $(\|\underline{n}\|_2 \to \infty)$ and therefore we need only establish the $H^p(\Omega)$ convergence of the term $\zeta^\varepsilon_{N\,\ell\,h}(\underline{x}) = \sum_{\underline{n}\in Z_\ell^2\backslash\{0\}} a_{\underline{n}}\, e^{\underline{n}\,\underline{x}\pi\imath}\, \phi_{N,h}(\underline{x}/\varepsilon, \underline{n}\pi)$ as $\ell \to \infty$ However, $\phi_{N\,h}(\bullet, \underline{t}) \in H^1_{per}(\mathcal{P})$, $\underline{t} \neq 0$ and therefore we can expand this function as a Fourier series, e g ,

$$\phi_{N\,h}(\underline{x}, \underline{t}) = \sum_{\underline{m}\in Z^2} \alpha^{N\,h}_{\underline{m}}(\underline{t})\, e^{2\pi\underline{m}\,\underline{x}\imath}, \quad \alpha^{N\,h}_{\underline{m}}(\underline{t}) = \frac{1}{4}\int_{\mathcal{P}} \phi_{N\,h}(\underline{x}, \underline{t})\, e^{-2\pi\underline{m}\,\underline{x}\imath}\, d\underline{x} \qquad (4\,4\,13)$$

and therefore

$$\zeta^{\varepsilon}_{N,\ell,h}(\underline{x}) = \sum_{\underline{n}\in Z^2_\ell\setminus\{0\}} \sum_{\underline{m}\in Z^2} a_{\underline{n}}\, \alpha^{N,h}_{\underline{m}}(\underline{n}\pi)\, e^{(\underline{n}+2\underline{m}/\varepsilon)\cdot \underline{x}\pi\imath} \tag{4.4.14}$$

We assume that $\phi_{N,h}(\bullet,\underline{t}) \in BV(\mathcal{P})$ and thus, from (4.4.5), $\alpha^{N,h}_{\underline{m}}(\underline{t}) = O(\|\underline{t}\|_2^{N-2}/|m_1 m_2|)$ as $\|\underline{m}\|_2 \to \infty$ and $\|\underline{t}\|_2 \to \infty$. The orthogonality of the exponential functions $e^{(\underline{n}+2\underline{m}/\varepsilon)\cdot(\bullet)\imath}$, $\underline{n} \in Z^2_\ell\setminus\{0\}$, $\underline{m} \in Z^2$ with respect to the $\mathcal{L}_2(\mathcal{C})$ inner product then suggests that

$$\begin{aligned}\|\zeta^{\varepsilon}_{2,\ell,h}, \mathcal{L}_2(\mathcal{C})\|^2 &= \sum_{\underline{n}\in Z^2_\ell\setminus\{0\}} \sum_{\underline{m}\in Z^2} |a_{\underline{n}}|^2\, |\alpha^{2,h}_{\underline{m}}(\underline{n}\pi)|^2 \\ &\le C_1 \sum_{\underline{n}\in Z^2_\ell\setminus\{0\}} |a_{\underline{n}}|^2 \sum_{\underline{m}\in Z^2} |m_1 m_2|^{-2} \le C_2\, \|f_{\mathcal{C}}, \mathcal{L}_2(\mathcal{C})\|^2\end{aligned} \tag{4.4.15}$$

Thus, the function $\zeta^{\varepsilon}_{2,\ell,h}$ converges in $\mathcal{L}_2(\mathcal{C})$ as $\ell \to \infty$ and, consequently, so does $u^{\varepsilon}_{2,\ell,h}$. The property of absolute convergence, as $\ell \to \infty$, of the approximations $u^{\varepsilon}_{1,\ell,h}$, $\ell \in \mathbb{N}$, $h > 0$, with respect to the $\mathcal{L}_2(\mathcal{C})$ norm, observed above, means that it is valid to differentiate the function $\zeta_{1,h} \overset{\text{def}}{=} \lim_{\ell\to\infty} \zeta^{\varepsilon}_{1,\ell,h}$ (with convergence in the $\mathcal{L}_2(\mathcal{C})$ sense) termwise, i.e., for $h > 0$,

$$D^\alpha \zeta_{1,h}(\underline{x}) = \sum_{\underline{n}\in Z^2\setminus\{0\}} a_{\underline{n}}\, e^{\underline{n}\cdot\underline{x}\pi\imath} \Big[\underline{n}^\alpha \pi\imath\, \phi_{1,h}(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) + \varepsilon^{-1} D^\alpha \phi_{1,h}(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi)\Big], \quad |\alpha| = 1 \tag{4.4.16}$$

The convergence of $u^{\varepsilon}_{1,\ell,h}$ in $H^1(\mathcal{C})$ as $\ell \to \infty$ now follows, as above, from the asymptotic relation (4.4.5), the series expansion (4.4.13), the $\mathcal{L}_2(\mathcal{C})$ orthogonality of the exponential functions $e^{(\underline{n}+2\underline{m}/\varepsilon)\cdot(\bullet)\imath}$, $\underline{n} \in Z^2_\ell\setminus\{0\}$, $\underline{m} \in Z^2$, and Bessel's inequality. We now follow the approach taken in the one dimensional setting and propose the $H^p(\Omega)$ convergent approximations $u^{\varepsilon}_{N,M,\ell,h}$, $N + p \ge 3$, $M, \ell \in \mathbb{N}$, $h > 0$ defined as follows

$$\tilde{u}^{\varepsilon}_{N,M,\ell,h}(x) \overset{\text{def}}{=} \sum_{\underline{n}\in Z^2_{\tau(\varepsilon)}\setminus\{0\}} a_{\underline{n}}\, e^{\underline{n}\cdot\underline{x}\pi\imath}\, \phi_{N,h}(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) + \sum_{\underline{n}\in Z^2_\ell\setminus Z^2_{\tau(\varepsilon)}} a_{\underline{n}}\, e^{\underline{n}\cdot\underline{x}\pi\imath}\, \phi_{M,h}(\underline{x}/\varepsilon, \varepsilon, \underline{n}\pi) \tag{4.4.17}$$

where $\tau(\varepsilon) = \max\{n \in \mathbb{N} \mid n < 2/\varepsilon\}$. Below, we apply our approach to the $\mathbb{R}^2$ counterparts of the boundary value problems investigated in Chapter 3 and assess their behaviour using the computational techniques described above. With this approach we expect to demonstrate that the features of the asymptotic approximations observed in the one–dimensional context readily generalize to the $\mathbb{R}^2$ setting.

4.4.1. Sample problem. Smooth Data, $a \in C^\infty(\mathcal{P})$, $f_{\mathcal{C}} \in C^\infty(\mathcal{C})$.

We define the coefficients $a_{kl} \overset{\text{def}}{=} \delta_{kl}\, a$, $1 \le k,l \le 2$, f, employed in the elliptic boundary value problem (4.1.1), below

$$a(\underline{x}) \overset{\text{def}}{=} \left[1 + \frac{1}{4}\sum_{n=1}^{2} \cos(2\pi x_n)\right]^{-1}, \quad f(\underline{x}) \overset{\text{def}}{=} \prod_{n=1}^{2} \sin(\pi x_n) \tag{4.4.18}$$

It is evident that $a, f \in C^\infty(\mathbb{R}^2)$ and f is antisymmetric and 2–periodic, i e , for $\underline{x} \in \mathbb{R}^2$,

$$f(\underline{x}+2\underline{n}) = f(\underline{x}), \quad \underline{n} \in \mathcal{Z}^2 \tag{4 4 19}$$

$$f((-1)^{m_1}x_1, (-1)^{m_2}x_2) = (-1)^{m_1+m_2} f(x_1, x_2), \quad \underline{m} \in \mathbb{N}_0^2 \backslash \{0\} \tag{4 4 20}$$

and, therefore, $f = f_C$ where f_C is given by the Fourier series expansion (4 1 13) and $a_{\underline{n}} \stackrel{\text{def}}{=} a_{n_1}a_{n_2}$, $\underline{n} \in \mathcal{Z}^2\backslash\{0\}$ where

$$a_n \stackrel{\text{def}}{=} \begin{cases} 1/2n\imath, & \text{if } n = \pm 1 \\ 0, & \text{if } n \neq \pm 1 \end{cases}, \quad n \in \mathcal{Z}\backslash\{0\} \tag{4 4 21}$$

Furthermore, a is a 1–periodic function which satisfies the periodic boundary condition (4 1 2), the ellipticity inequality (4 1 3) with $\alpha_1 = 2/3$, $\alpha_2 = 2$, and $u^\varepsilon \in C^2(\mathbb{R}^2)$, $\phi(\bullet, \varepsilon, \underline{t}) \in H^1_{per}(\mathcal{P}) \cap C^2(\mathcal{P}) \cap C^1(\overline{\mathcal{P}})$ are the *classical solutions* of problems (4 1 11), (4 1 22) respectively

We employ a uniform finite element triangulation, $\mathcal{U}_h(\mathcal{P})$, of $\mathcal{P}$ with $h = 1/16$, i e , each finite element $\tau \in \mathcal{U}_h(\mathcal{P})$ is obtained by translating and/or rotating the right angled triangle $T_h = \{(\xi,\eta) \mid \xi, \eta \geq 0,\ \xi+\eta \leq h\}$ Note that in Theorems 4 2 3, 4 3 1 and Corollary 4 2 4 the parameter $s = 2$ The errors $\|u^\varepsilon_h - u^\varepsilon_{N\,h}, H^p(\mathcal{P})\|$, $0 \leq p \leq 1$, $0 \leq N \leq 3$ have been computed and are presented in tables 4 4 1 1–4 4 1 3 where $\varepsilon = 2^{-r}$, $1 \leq r \leq 4$ and because, therefore, $2^{-r}\underline{1}\pi \neq 2\pi\underline{m}$, $r \geq 1$, $\underline{m} \in \mathcal{Z}^2\backslash\{0\}$ it follows that $\varepsilon\underline{n}\pi \notin \mathcal{H}^2$ where $n_i = \pm 1$, $1 \leq i \leq 2$ Each integral over $\tau \in \mathcal{U}_h(\mathcal{P})$ is approximated by a 7 point quadrature rule, cf AKIN (1982), and the algebraic equations which arise are solved by a Cholesky factorization technique We point out that there is no subscript $\ell \in \mathbb{N}$ in tables 4 4 1 1–4 4 1 3 because there is no truncation error committed in the computations, i e , the Fourier series is summed in its entirety

Table 4 4 1 1 $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_h - u_{0,h}, L_2(\Omega)\|$	$\|u^\varepsilon_h - u_{0,h}, H^1(\Omega)\|$
0 5	1 39403508(−3)	1 92809615(−2)
0 25	7 74303030(−4)	2 00011017(−2)
0 125	3 96255426(−4)	2 02073130(−2)
0 0625	1 99238516(−4)	2 02602928(−2)
	$O(\varepsilon)$	$O(1)$

Table 4 4 1 2 $a \in C^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_h - u^\varepsilon_{1,h}, L_2(\Omega)\|$	$\|u^\varepsilon_h - u_{1\,h}, H^1(\Omega)\|$
0 5	2 84813088(−4)	3 30893241(−3)
0 25	4 57597122(−5)	1 17921226(−3)
0 125	9 80590435(−6)	5 18420187(−4)
0 0625	2 34887912(−6)	2 49722298(−4)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Table 4 4 1 3 $a \in C^\infty(\mathcal{P})$, $f_{\mathcal{C}} \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u_h^\varepsilon - u_{2,h}^\varepsilon, L_2(\Omega)\|$	$\|u_h^\varepsilon - u_{2,h}, H^1(\Omega)\|$
0 5	2 44001085(−4)	2 74778793(−3)
0 25	2 68637426(−5)	6 66596767(−4)
0 125	3 22450617(−6)	1 65123796(−4)
0 0625	3 98734188(−7)	4 11817089(−5)
	$O(\varepsilon^3)$	$O(\varepsilon^2)$

The graphs of the real and imaginary components of $\phi_h(\underline{1}/2,\bullet,\bullet)$, $\phi_{N,h}(\underline{1}/2,\bullet,\bullet)$, $0 \le N \le 2$, $h > 0$ illustrated in Figures 4 4 1 1−4 4 1 6 clearly demonstrate the utility of the asymptotic approximations $\phi_{N,h}$, $0 \le N \le 2$, $h > 0$ of ϕ_h, indeed, as $t \to \infty$, it becomes difficult to distinguish between the various approximations The principal features evident in these graphs, i e , the monotone convergence of the approximations, $\phi_{N,h}^\varepsilon$, $0 \le N \le 2$, $h > 0$, to the asymptote $y = 0$ and the extrema of ϕ_h, $h > 0$ – which correspond to the singularities of ϕ – were also observed for the analogous analytical functions ϕ, ϕ_N, $0 \le N \le 2$ in the one dimensional setting of Chapter 3 Furthermore, we find it interesting that the graphs reveal that the functions $\phi_{N\,h}(\underline{x},\varepsilon,\bullet)$, $\underline{x} \in \mathcal{P}$, $0 \le N \le 2$ provide accurate approximations of $\phi_h(\underline{x},\varepsilon,\bullet)$, $\underline{x} \in \mathcal{P}$ *outside* the region, $\widehat{G}$, where the expansion (4 2 1) is analytically justified

Clearly, for f defined by relation (4 4 18) the Fourier series (4 1 13) has finitely many terms and, therefore, questions of convergence of the sums (4 1 21), (4 4 1) never arise, thus, one can construct asymptotic approximations $u_{N,h}^\varepsilon$, $h > 0$ of any order $N \in \mathbb{N}$ Indeed, the computational results presented in Tables 4 4 1 1−4 4 1 3 suggest the following property for $h > 0$ sufficiently small

$$\|u_h^\varepsilon - u_{N,h}^\varepsilon, H^p(\Omega)\| \le C(h)\,\varepsilon^{N+1-p}, \quad N \ge 0, \quad 0 \le p \le 1 \qquad (4\,4\,22)$$

where $C(h) > 0$ is a constant which is independent of $\varepsilon > 0$

4 4 2 Sample problem Piecewise smooth Data, $a \in C^\infty(\mathcal{P})$, $f_{\mathcal{C}} \in \mathcal{PC}^\infty(\mathcal{C})$

Let $a_{kl} \in C^\infty(\mathbb{R}^2)$, $1 \le k,l \le 2$ be defined as in Section 4 4 1 and define $f(\underline{x}) \stackrel{\text{def}}{=} 1$, $\underline{x} \in \Omega$ then $f_{\mathcal{A}} \in \mathcal{PC}^\infty(\mathcal{C})$ is a step function which extends f antisymmetrically to $\mathcal{C}$ and is given by relation (4 1 12) Similarly, the 2–periodic extension of $f_{\mathcal{A}}$ to $f_{\mathcal{C}} \in \mathcal{PC}^\infty(\mathbb{R}^2)$ is defined by the Fourier series expansion (4 1 13) where the coefficients are $a_{\underline{n}} \stackrel{\text{def}}{=} a_{n_1}a_{n_2}$, $\underline{n} \in \mathcal{Z}^2\backslash\{0\}$ and

$$a_n \stackrel{\text{def}}{=} \frac{1}{n\pi\imath}\Big[1 - (-1)^n\Big], \quad n \in \mathcal{Z}\backslash\{0\} \qquad (4\,4\,23)$$

The weak solutions $u^\varepsilon \in H_0^1(\Omega)$, $\phi(\bullet,\varepsilon,\underline{t}) \in H_{per}^1(\mathcal{P})$ are, as in Section 4 4 1, *classical solutions* of (4 1 4), (4 1 22) respectively, i e , $u^\varepsilon \in C^2(\Omega)\cap C^0(\overline{\Omega})$, $\phi(\bullet,\varepsilon,\underline{t}) \in H_{per}^1(\mathcal{P})\cap C^2(\mathcal{P})\cap C^1(\overline{\mathcal{P}})$, however, in contrast to Section 4 4 1, u^ε is **not** a *classical solution* of problem (4 1 11), i e , $u^\varepsilon \notin C^2(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$ but $u^\varepsilon \in H^2(\mathcal{C}) \cap H_0^1(\mathcal{C})$, cf Theorem 9 1 22 of HACKBUSCH (1992), and, because $f_{\mathcal{C}} \in H^{1/2-\rho}(\mathcal{C})$, $\rho > 0$, $u^\varepsilon \in H^{5/2-\rho}(\mathcal{B})$ for any open ball $\mathcal{B} \subset\subset \mathcal{C}$

Figure 4 4 1 1

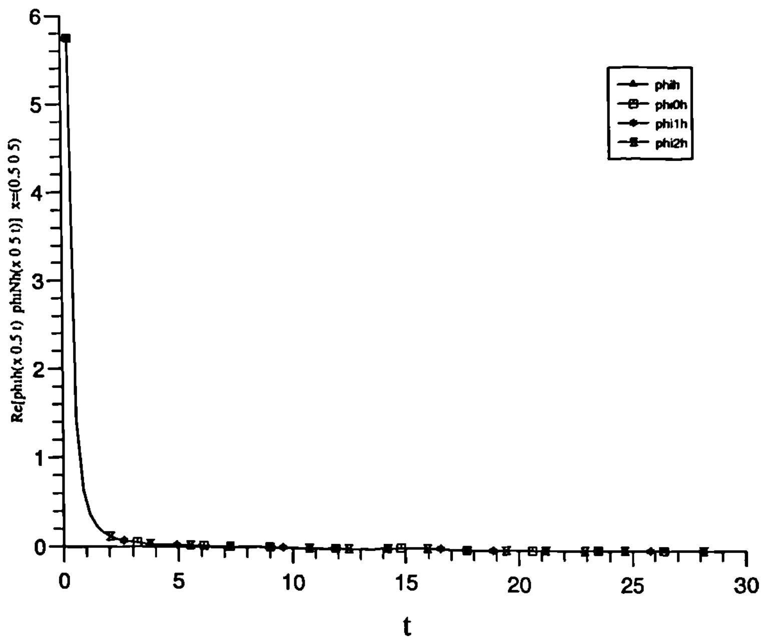
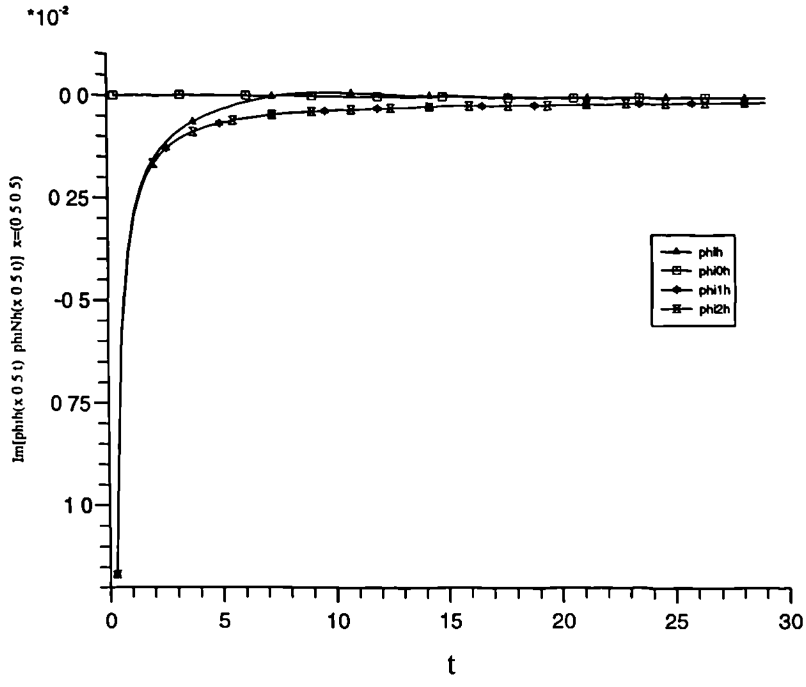


Figure 4 4 1 2



Graphs of the real or imaginary parts of $\phi_h(\underline{x}, \varepsilon, t)$, $\phi_{N\,h}(\underline{x}, \varepsilon, t)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3$, $1 \leq t \leq 30$, $0 \leq N \leq 2$, $h = 1/16$ The curves are distinguished by the symbols, e g , $\Delta \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Figure 4 4 1 3

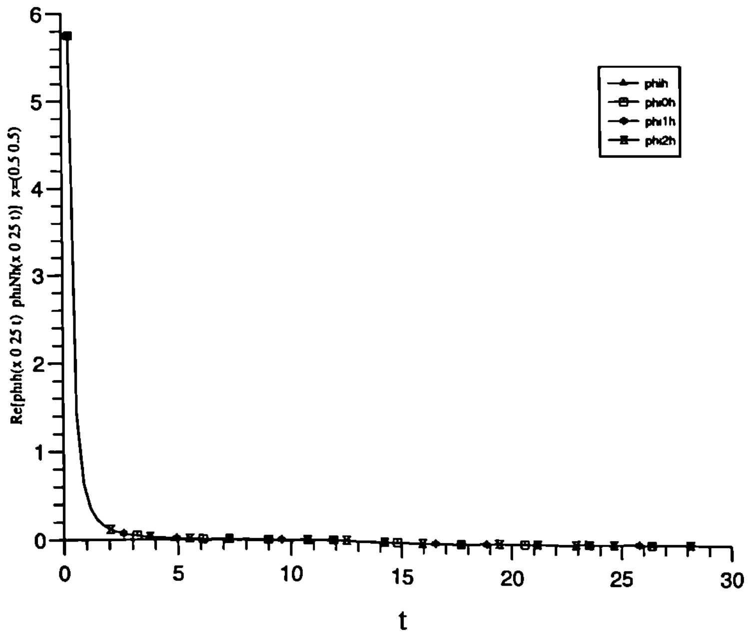
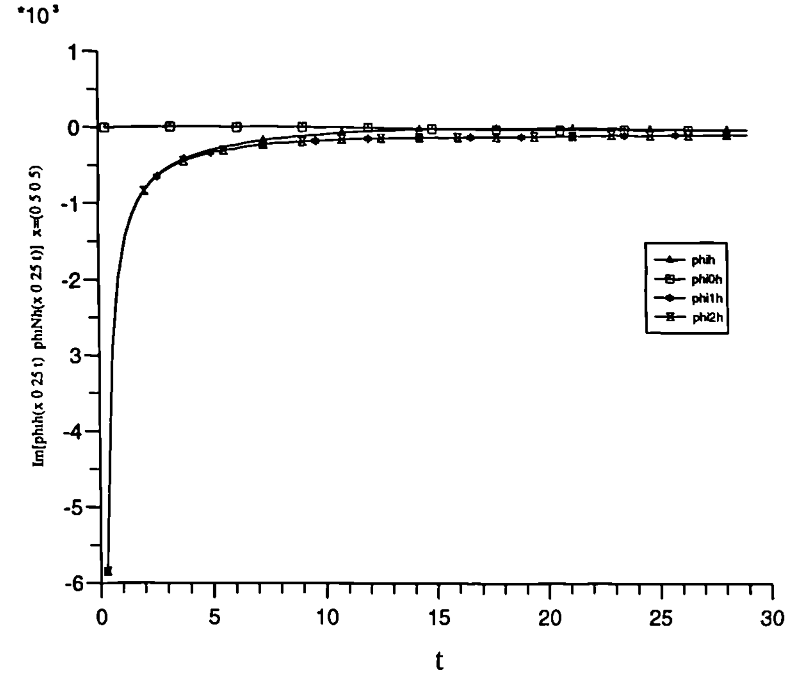


Figure 4 4 1 4



Graphs of the real or imaginary parts of $\phi_h(\underline{x}, \varepsilon, t)$, $\phi_{N\,h}(\underline{x}, \varepsilon, t)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3, 1 \leq t \leq 30$, $0 \leq N \leq 2$, $h = 1/16$ The curves are distinguished by the symbols, e g , $\Delta \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Figure 4 4 1 5

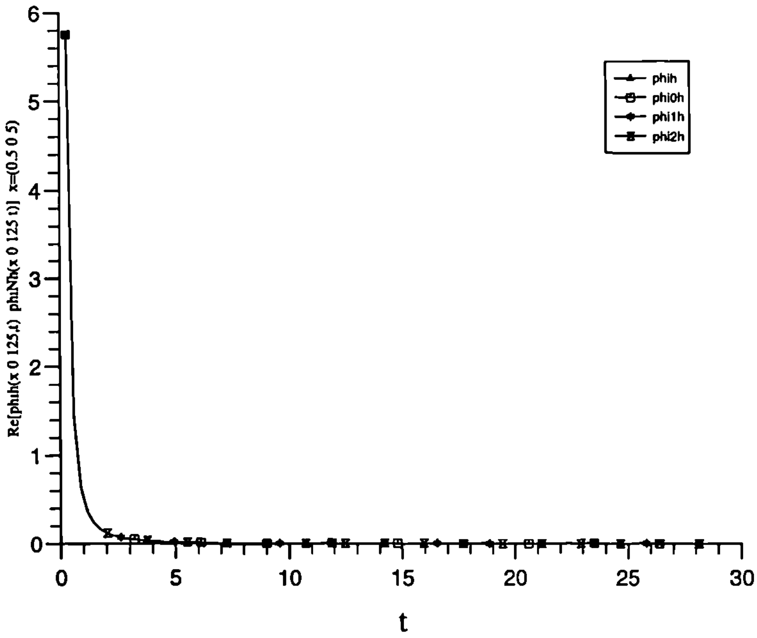
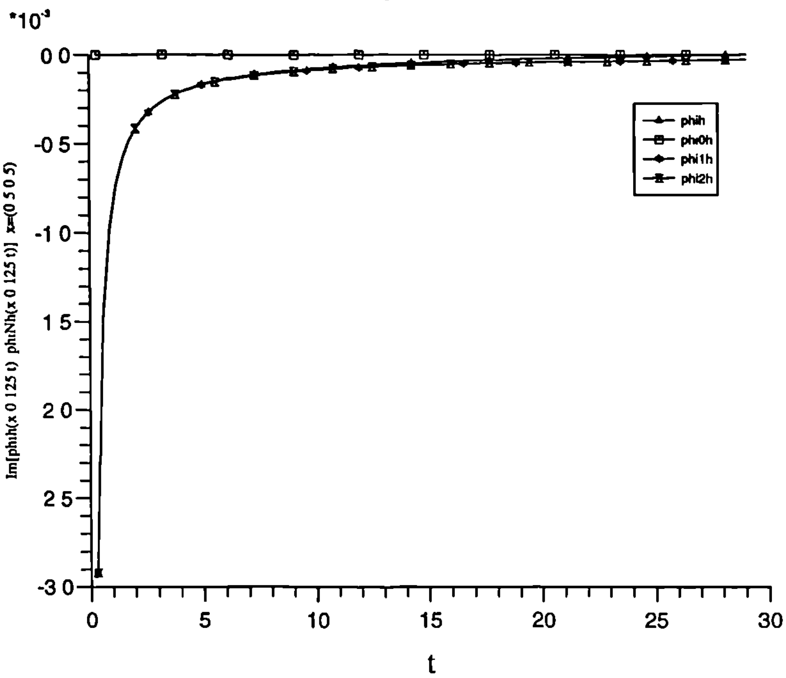


Figure 4 4 1 6



Graphs of the real or imaginary parts of $\phi_h(\underline{x}, \varepsilon, t)$, $\phi_{N\,h}(\underline{x}, \varepsilon, t)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 1/2^n$, $1 \le n \le 3, 1 \le t \le 30$, $0 \le N \le 2$, $h = 1/16$ The curves are distinguished by the symbols, e g , $\Delta \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

The errors $\|u^{\varepsilon}_{\ell,h} - u^{\varepsilon}_{N\,\ell\,h}, H^p(\mathcal{P})\|$, $0 \le p \le 1$, $0 \le N \le 3$ have been computed and are presented in tables 4 4 2 1–4 4 2 3 where $\ell = 71$, $\varepsilon = 2^{-r}$, $1 \le r \le 4$ and, thus, $(\varepsilon, t_q) \notin \mathcal{H}$, $1 \le q \le 2$ because $2^{-r}(2\underline{n} + \underline{1})\pi \neq 2\pi\underline{m}$, $r \ge 1$, $\underline{n}$, $\underline{m} \in \mathcal{Z}^2\backslash\{0\}$ The finite element triangulation $\mathcal{U}_h(\mathcal{P})$, $h = 1/16$ is employed once again to obtain the computational results reported in the tables

Table 4 4 2 1 $a \in C^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathcal{C})$

Cell Size, ε	$\|u^{\varepsilon}_{\ell,h} - u_{0\,\ell\,h}, L_2(\Omega)\|$	$\|u^{\varepsilon}_{\ell,h} - u_{0,\ell\,h}, H^1(\Omega)\|$
0 5	2 55209846(−3)	3 30043356(−2)
0 25	1 33536187(−3)	3 36109462(−2)
0 125	6 65648382(−4)	3 37290018(−2)
0 0625	3 32510592(−4)	3 37623695(−2)
	$O(\varepsilon)$	$O(1)$

Table 4 4 2 2 $a \in C^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathcal{C})$

Cell Size, ε	$\|u^{\varepsilon}_{\ell,h} - u^{\varepsilon}_{1,\ell,h}, L_2(\Omega)\|$	$\|u^{\varepsilon}_{\ell,h} - u_{1,\ell,h}, H^1(\Omega)\|$
0 5	7 19260110(−4)	6 98363635(−3)
0 25	2 62528987(−4)	4 34283106(−3)
0 125	6 51723448(−5)	2 43346296(−3)
0 0625	1 54817208(−5)	1 29349317(−3)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Table 4 4 2 3 $a \in C^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathcal{C})$

Cell Size, ε	$\|u^{\varepsilon}_{\ell,h} - u^{\varepsilon}_{2,\ell,h}, L_2(\Omega)\|$	$\|u^{\varepsilon}_{\ell,h} - u_{2\,1,\ell,h}, H^1(\Omega)\|$
0 5	9 29159899(−4)	2 02676373(−2)
0 25	2 57005360(−4)	5 35188282(−3)
0 125	5 10135998(−5)	1 48569648(−3)
0 0625	8 96991395(−6)	4 20775584(−4)
	$O(\varepsilon^2)$	$O(\varepsilon^{1+\alpha})$

4 4 3 Sample problem Piecewise smooth Data, $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathcal{C})$

Define f as in Section 4 4 2 and the 1-periodic coefficients $a_{kl} \stackrel{\text{def}}{=} \delta_{kl}\, a$, $1 \le k, l \le 2$ where, for $\underline{x} \in \mathcal{P}$, a is the step function

$$a(\underline{x}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \underline{x} \in \mathcal{P}\backslash(1/4, 3/4)^2 \\ 10, & \text{if } \underline{x} \in (1/4, 3/4)^2 \end{cases} \tag{4 4 24}$$

and, therefore, there exists a partition of Ω

$$\overline{\Omega} = \cup_{r=1}^{m_\varepsilon} \overline{\Omega}^{\varepsilon}_r, \quad \Omega^{\varepsilon}_i \cap \Omega^{\varepsilon}_j,\ i \neq j \tag{4 4 25}$$

such that $a(\underline{x}/\varepsilon) = a^{[r]} \in \mathbb{R}$, $\underline{x} \in \Omega_r^\varepsilon$, $1 \le r \le m_\varepsilon$ It is evident from definition (4 4 18) that $a \in \mathcal{PC}^\infty(\mathbb{R}^2)$ satisfies the boundary condition (4 1 2) and the ellipticity inequality (4 1 3) with $\alpha_1 = 1$, $\alpha_2 = 2$ Furthermore, Theorem 9 1 26 of HACKBUSCH (1992) shows that, for any open ball $\mathcal{B} \subset\subset \Omega_r^\varepsilon$, $1 \le r \le m_\varepsilon$, there is the interior regularity $u^\varepsilon \in H^k(\mathcal{B})$, $k \in \mathbb{N}$ (cf HACKBUSCH (1992)), however, the continuous embedding $H^{j+2}(\mathcal{B}) \to C^{j,\lambda}(\mathcal{B})$, $j \in \mathbb{N}_0$, $0 < \lambda < 1$ (cf ADAMS (1975)), and the weak formulation (4 1 4) then imply that the weak solution $u^\varepsilon \in H_0^1(\Omega)$ is also a *classical* solution in the region $\Omega\backslash\Gamma$ where $\Gamma \stackrel{\text{def}}{=} \cup_{r,s=1}^{m_\varepsilon}(\partial\Omega_r^\varepsilon \cap \partial\Omega_s^\varepsilon)$ and, on Γ, satisfies the weak continuity condition

$$\sum_{r=1}^{m_\varepsilon} \int_{\partial\Omega_r^\varepsilon} a^{[r]}\, \nabla u^\varepsilon(\underline{x}) \;\; \underline{n}^{[r]}(\underline{x})\, v(\underline{x})\, d\underline{x} = 0, \quad v \in C_0^\infty(\Omega) \tag{4 4 26}$$

where $\underline{n}^{[r]}(\underline{x}) \in \mathbb{R}^2$ is the unit outward normal vector to the boundary $\partial\Omega_r^\varepsilon$ at the point $\underline{x} \in \partial\Omega_r^\varepsilon$ If, however, $u^\varepsilon \in W_\infty^1(\Omega)$ then, for $\sigma < 1/2$, it is clear that (cf (1 2 15))

$$\|u^\varepsilon, H^{1+\sigma}(\Omega)\|^2 \le \|u^\varepsilon, H^1(\Omega)\|^2 + \iint_{\Omega\times\Omega} \frac{1}{\|\underline{x}-\underline{z}\|_2^{2+2\sigma}}\, d\underline{x}\, d\underline{z} < \infty \tag{4 4 27}$$

i e , $u^\varepsilon \in H_0^1(\Omega) \cap H^{3/2-\rho}(\Omega)$, $\rho > 0$ Indeed, it is the interior interface vertices $((2n+1)p/4, (2m+1)q/4)$, $0 \le m,n \le 1$, $p,q \in \mathbb{N}_0$ which cause the singular components of the solution to arise and, therefore, the reduced regularity of u^ε (compared to Section 4 4 1)

The errors $\|u_{\ell,h}^\varepsilon - u_{N,\ell,h}^\varepsilon, H^p(\mathcal{P})\|$, $0 \le p \le 1$, $0 \le N \le 3$ have been computed and are presented in tables 4 4 3 1–4 4 3 3 where $\ell = 71$, $\varepsilon = 2^{-r}$, $1 \le r \le 4$, and $(\varepsilon, t_q) \notin \mathcal{H}$, $1 \le q \le 2$ because $2^{-r}(2\underline{n}+\underline{1})\pi \ne 2\pi\underline{m}$, $r \ge 1$, $\underline{n}, \underline{m} \in \mathcal{Z}^2\backslash\{0\}$ The finite element triangulation $\mathcal{U}_h(\mathcal{P})$, $h = 1/16$ is employed to obtain the computational results reported in the tables where, clearly, the finite elements $\tau \in \mathcal{U}_h(\mathcal{P})$ do not cross the interface boundaries, i e , $\tau \cap \partial\mathcal{P}_r = \emptyset$, $1 \le r \le m_1$ where $\mathcal{P}_r \stackrel{\text{def}}{=} \partial\Omega_r^1$, see (4 4 25) We recall the analysis of Section 4 4 and observe that the termwise derivative of the approximation $u_{2,\ell,h}^\varepsilon$ diverges as $\ell \to \infty$ and we therefore employ the approximation $u_{2,1,\ell,h}^\varepsilon$ instead

Table 4 4 3 1 $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_{\mathcal{C}} \in \mathcal{PC}^\infty(\mathcal{C})$

Cell Size, ε	$\|u_{\ell,h}^\varepsilon - u_{0\,\ell,h}, L_2(\Omega)\|$	$\lvert u_{\ell,h}^\varepsilon - u_{0,\ell\,h}, H^1(\Omega)\rvert$
0 5	5 13260128(−3)	7 22495894(−2)
0 25	2 59876887(−3)	7 53652399(−2)
0 125	1 29971219(−3)	7 65957443(−2)
0 0625	6 50236166(−4)	7 70283492(−2)
	$O(\varepsilon)$	$O(1)$

The graphs of the approximations $\phi_h(\underline{x}, \varepsilon, \bullet)$, $\phi_{N\,h}(\underline{x}, \varepsilon, \bullet)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 2^{-n}$, $1 \le n \le 3$, $0 \le N \le 2$, $h = 1/16$ presented in Figures 4 4 3 1–4 4 3 6 reveal the now familiar features observed during the preceding computations It is also apparent from the graphs that the

Table 4 4 3 2 $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_{\ell,h} - u^\varepsilon_{1,\ell,h}, L_2(\Omega)\|$	$\|u^\varepsilon_{\ell,h} - u_{1\,\ell\,h}, H^1(\Omega)\|$
0 5	1 22649269(−3)	2 04409797(−2)
0 25	4 43631691(−4)	1 23931444(−2)
0 125	1 21902193(−4)	6 63802373(−3)
0 0625	3 12648593(−5)	3 44159199(−3)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Table 4 4 3 3 $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_C \in \mathcal{PC}^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_{\ell,h} - u^\varepsilon_{2,\ell,h}, L_2(\Omega)\|$	$\|u^\varepsilon_{\ell,h} - u_{2\,1\,\ell\,h}, H^1(\Omega)\|$
0 5	1 64063523(−3)	3 41231714(−2)
0 25	3 98818364(−4)	8 94704807(−3)
0 125	7 57837897(−5)	2 45068320(−3)
0 0625	1 30882661(−5)	6 85638290(−4)
	$O(\varepsilon^2)$	$O(\varepsilon^{1+\alpha})$

discontinuities, cf (4 4 24), do not significantly reduce the quality or utility of the asymptotic approximations $\phi_{N\,h}$ of ϕ_h

The computational results obtained in Tables 4 4 3 1–4 4 3 3 suggest the following error bounds, for $0 \le N \le 2$, $h = 1/16$, $\ell = 71$,

$$\begin{aligned} \|u^\varepsilon_{\ell\,h} - u^\varepsilon_{N\,\ell\,h}, H^p(\Omega)\| &\le C_1(h)\,\varepsilon^{N+1-p}, \quad 0 \le N + p \le 2 \\ \|u^\varepsilon_{\ell\,h} - u^\varepsilon_{2,1\,\ell,h}, H^1(\Omega)\| &\le C_2(h)\,\varepsilon^{1+\alpha} \end{aligned} \tag{4 4 28}$$

where $C_1(h)$, $C_2(h) > 0$ are constants independent of ε and $0 < \alpha < 1$ Thus, the computed errors converge in a similar manner to the analogous approximations computed analytically in the one dimensional examples of Chapter 3 This suggests – while, clearly, not proving – that, with our choice of h, ℓ, the error

$$\|u^\varepsilon_\ell - u^\varepsilon_{\ell\,h}, H^p(\Omega)\| \le C(\ell)\,h^{(2-p)(s-1)} \tag{4 4 29}$$

is sufficiently small that one can obtain meaningful results by investigating the errors $\|u^\varepsilon_{\ell,h} - u^\varepsilon_{N,\ell\,h}, H^p(\Omega)\|$ and $\|u^\varepsilon_{\ell,h} - u^\varepsilon_{N\,M\,\ell\,h}, H^p(\Omega)\|$ as in Tables 4 4 3 1 – 4 4 3 3

4 4 4 Sample problem Piecewise smooth Data, $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_C \in C^\infty(\mathcal{C})$.

Define the coefficients $a_{kl} \stackrel{\text{def}}{=} \delta_{kl}\, a$, $1 \le k, l \le 2$ and f as follows

$$a(\underline{x}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \underline{x} \in \mathcal{P}\backslash(1/4, 3/4)^2 \\ 10, & \text{if } \underline{x} \in (1/4, 3/4)^2 \end{cases}, \qquad f(\underline{x}) \stackrel{\text{def}}{=} \prod_{n=1}^{2} \sin(\pi x_n) \tag{4 4 30}$$

The properties of the functions a, f, f_C have been studied in problems 4 4 1 – 4 4 3, furthermore, the weak solution $u^\varepsilon \in H^1_0(\Omega)$ exhibits the same regularity properties as observed in

Figure 4 4 3 1

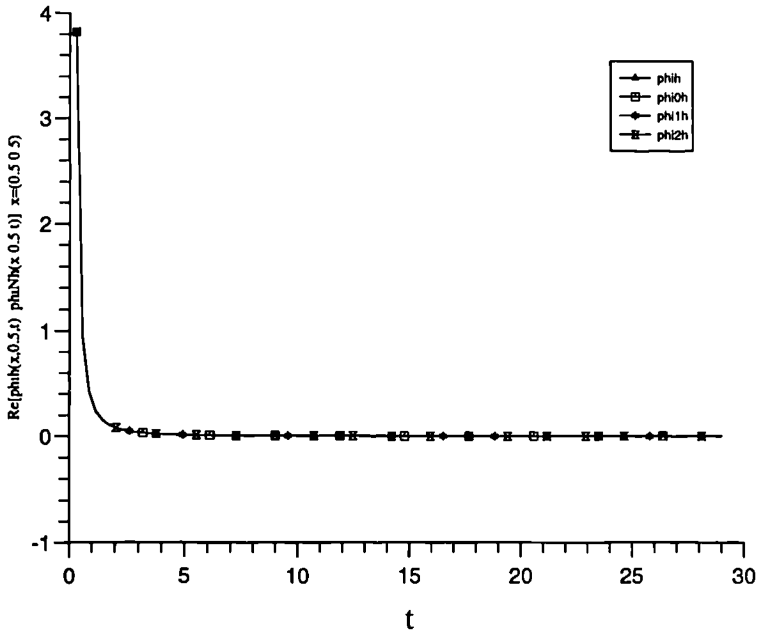
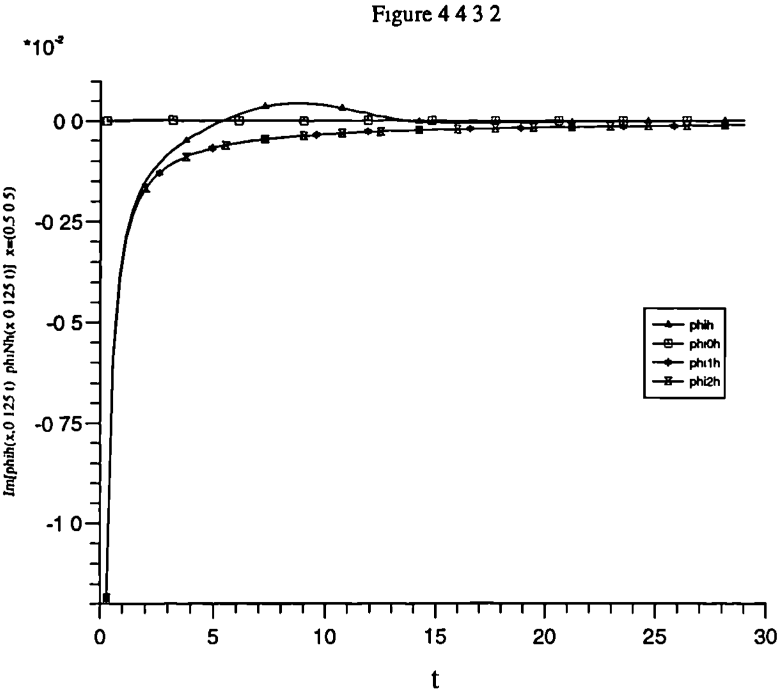


Figure 4 4 3 2



Graphs of the real or imaginary parts of $\phi_h(\underline{x}, \varepsilon, t)$, $\phi_{N,h}(\underline{x}, \varepsilon, t)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3$, $1 \leq t \leq 30$, $0 \leq N \leq 2$, $h = 1/16$ The curves are distinguished by the symbols, e g , $\Delta \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Figure 4 4 3 3

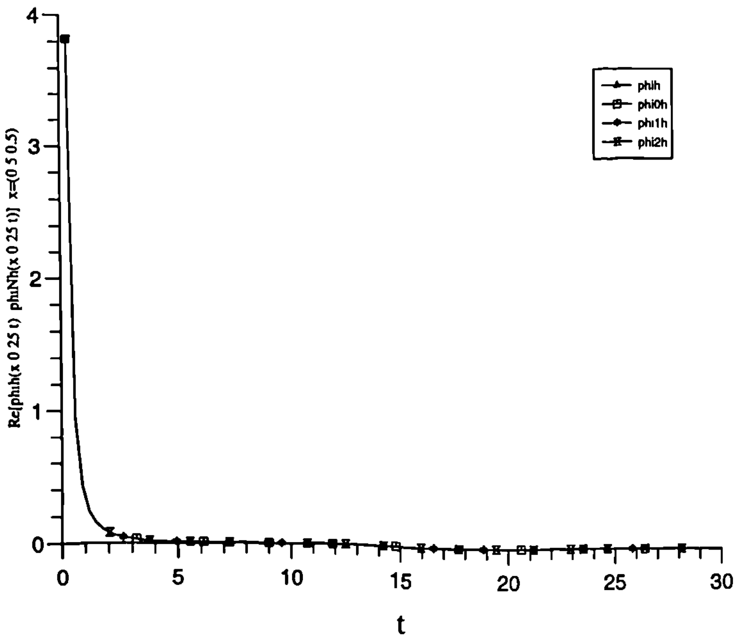
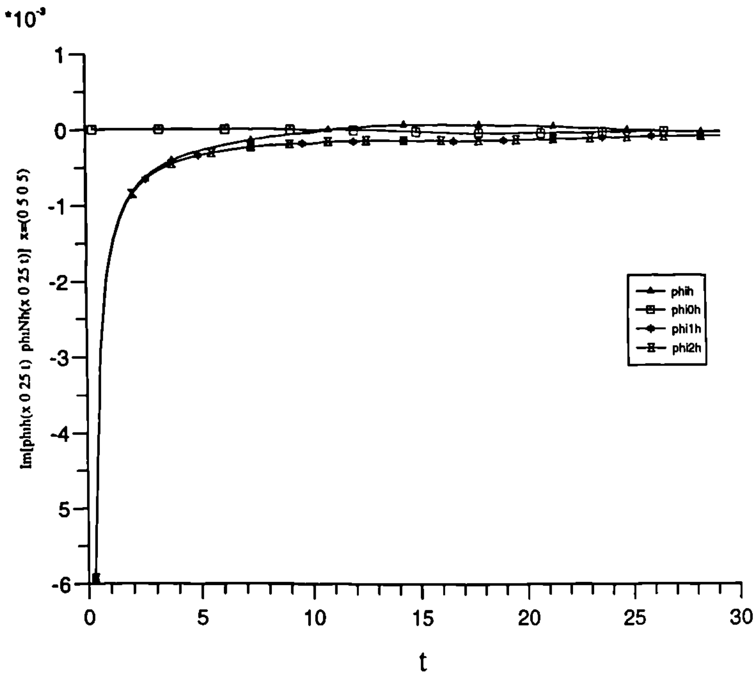


Figure 4 4 3 4



Graphs of the real or imaginary parts of $\phi_h(\underline{x}, \varepsilon, t)$, $\phi_{N\,h}(\underline{x}, \varepsilon, t)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3$, $1 \leq t \leq 30$, $0 \leq N \leq 2$, $h = 1/16$ The curves are distinguished by the symbols, e g , $\Delta \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

Figure 4 4 3 5

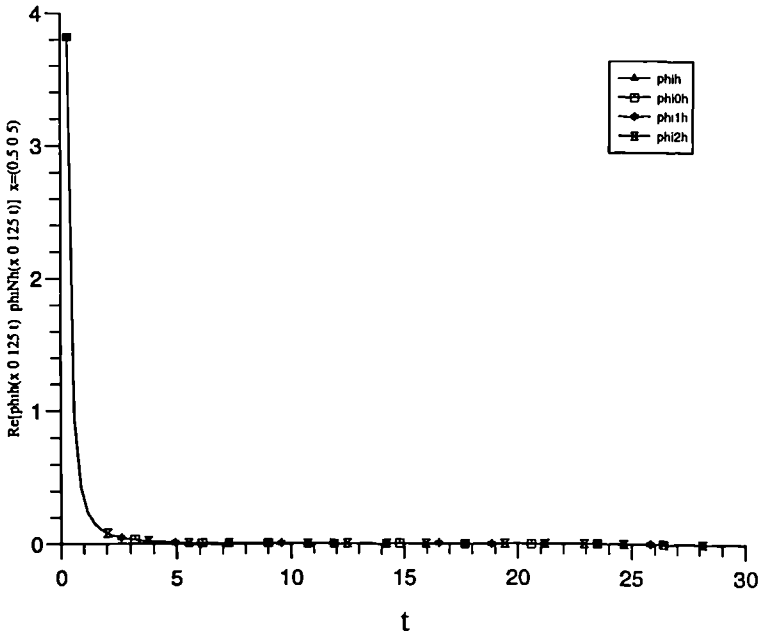
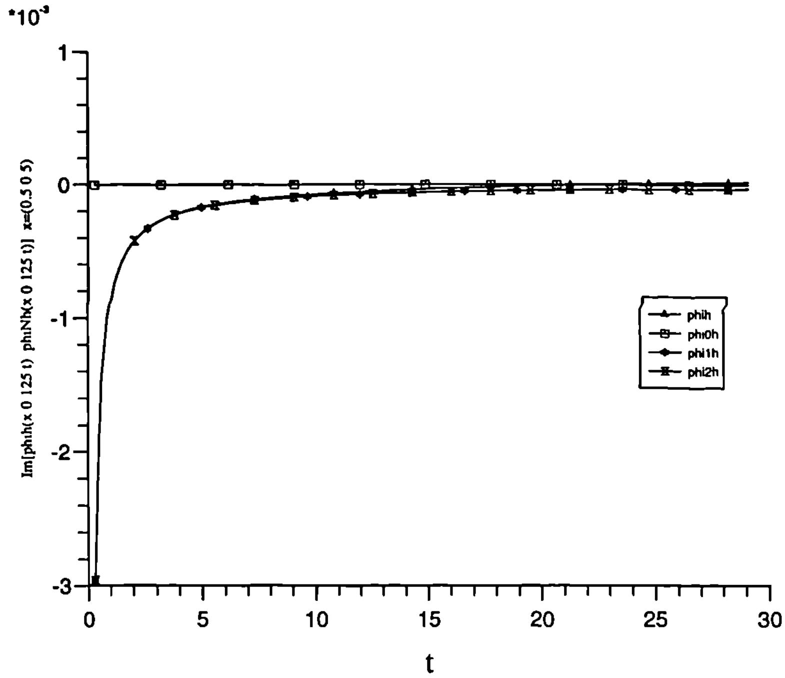


Figure 4 4 3 6



Graphs of the real or imaginary parts of $\phi_h(\underline{x}, \varepsilon, t)$, $\phi_{N,h}(\underline{x}, \varepsilon, t)$, $\underline{x} = \underline{1}/2$, $\varepsilon = 1/2^n$, $1 \leq n \leq 3, 1 \leq t \leq 30$, $0 \leq N \leq 2$, $h = 1/16$ The curves are distinguished by the symbols, e g , $\Delta \Rightarrow \phi$, $\square \Rightarrow \phi_0$, $\star \Rightarrow \phi_1$, $\bowtie \Rightarrow \phi_2$

problem 4 4 3, i e , u^ε has singularities at the interior interface vertex points, $u^\varepsilon \in C^2(\Omega\backslash\Gamma)$, $u^\varepsilon \in H^k(\mathcal{B})$, $k \in \mathbb{N}$ for any open ball $\mathcal{B} \subset\subset \Omega^\varepsilon_r$, $1 \le r \le m_\varepsilon$, and if $u^\varepsilon \in W^1_\infty(\Omega)$ then $u^\varepsilon \in H^{3/2-\rho}(\Omega)$, $\rho > 0$

The errors $\|u^\varepsilon_h - u^\varepsilon_{N,h}, H^p(\mathcal{P})\|$, $0 \le p \le 1$, $0 \le N \le 3$ have been computed and are presented in tables 4 4 4 1 – 4 4 4 3 where $\varepsilon = 2^{-r}$, $1 \le r \le 4$ and because, therefore, $2^{-r}\underline{1}\pi \neq 2\pi\underline{m}$, $r \ge 1$, $\underline{m} \in \mathcal{Z}^2\backslash\{0\}$ it follows that $\varepsilon\underline{n}\pi \notin \mathcal{H}^2$ where $n_i = \pm 1$, $1 \le i \le 2$ The finite element triangulation $\mathcal{U}_h(\mathcal{P})$, $h = 1/16$ is employed to obtain the computational results reported in the tables where, clearly, the finite elements $\tau \in \mathcal{U}_h(\mathcal{P})$ do not cross the interface boundaries, i e , $\tau \cap \partial\mathcal{P}_r = \emptyset$, $1 \le r \le m_1$ where $\mathcal{P}_r \stackrel{\text{def}}{=} \partial\Omega^1_r$, see (4 4 25)

Table 4 4 4 1 $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_\mathcal{C} \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_h - u_{0,h}, L_2(\Omega)\|$	$\|u^\varepsilon_h - u_{0,h}, H^1(\Omega)\|$
0 5	3 04183197(−3)	4 61609913(−2)
0 25	1 55530030(−3)	4 63470369(−2)
0 125	7 79908828(−4)	4 63884111(−2)
0 0625	3 90161435(−4)	4 63983690(−2)
	$O(\varepsilon)$	$O(1)$

Table 4 4 4 2 $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_\mathcal{C} \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_h - u^\varepsilon_{1,h}, L_2(\Omega)\|$	$\|u^\varepsilon_h - u_{1,h}, H^1(\Omega)\|$
0 5	7 98611323(−4)	1 28522393(−2)
0 25	1 94706196(−4)	6 29434012(−3)
0 125	4 86812019(−5)	3 13265547(−3)
0 0625	1 21750108(−5)	1 56458260(−3)
	$O(\varepsilon^2)$	$O(\varepsilon)$

Table 4 4 4 3 $a \in \mathcal{PC}^\infty(\mathcal{P})$, $f_\mathcal{C} \in C^\infty(\mathcal{C})$

Cell Size, ε	$\|u^\varepsilon_h - u^\varepsilon_{2,h}, L_2(\Omega)\|$	$\|u^\varepsilon_h - u_{2,h}, H^1(\Omega)\|$
0 5	3 57718390(−4)	4 52235561(−3)
0 25	3 58931520(−5)	1 04462517(−3)
0 125	4 18362046(−6)	2 55424836(−4)
0 0625	5 13263414(−7)	6 34934365(−5)
	$O(\varepsilon^3)$	$O(\varepsilon^2)$

4 5 Conclusions

Our aim in Section 4 4 was to demonstrate that the asymptotic approach introduced in Chapter 3 could be generalized to the two dimensional setting and combined with finite element techniques of approximation to produce functions $u^\varepsilon_{N\,\ell\,h}$, $N \ge 0$, $\ell \in \mathbb{N}$, $h > 0$ which approximate the weak solutions, u^ε, of scalar elliptic problems (4 1 1) such that the errors decrease, as $\varepsilon \to 0$, in the $H^p(\Omega)$, $0 \le p \le 1$ norm topologies

The computational results obtained in Section 4 4 and the analysis of Section 4 3 – which led to the error estimate (4 3 20) – evidently generalize the computational/analytical results of Chapter 3, i e , for $f_C \in H^m(\mathcal{C})\backslash H^{m+1}(\mathcal{C})$, $u^\varepsilon \in H^{1+\sigma}(\Omega)$, $\sigma > 0$ and $0 \le p \le 1$ we have

$$\|u^\varepsilon - u^\varepsilon_{N,\ell,h}, H^p(\Omega)\| \le C_1 \|f - f_\ell, \mathcal{L}_2(\Omega)\| + C(\ell)\, h^{(2-p)\sigma} + C_2\, \varepsilon^{\min(N+1,m+2)-p} \qquad (4\ 5\ 1)$$

where $0 \le N \le m+2-p$ and $\ell \le \Lambda$, Λ a fixed positive integer The analysis of Section 4 3 2 suggested that $C(\ell) \to \infty$ as $\ell \to \infty$ and, indeed, whether it is possible to replace $C(\ell)$ by a constant which can be bounded independently of $\ell \in \mathbb{N}$ is an open question However, because the asymptotic approximations $u^\varepsilon_{N,\ell,h}$, $h > 0$ converge as $\ell \to \infty$ for functions $f_C \in BV(\mathcal{C})$, cf Section 4 4, we expect such a constant to exist The computational results obtained in our assessment of the approximation $\tilde{u}^\varepsilon_{2,1,\ell,h}$ were, as commented in Section 4 4, inconclusive However, based on the definition of $\tilde{u}^\varepsilon_{N,M,\ell,h}$ (cf (4 4 17)) and the computational results obtained we suggest that there exists an α, $0 < \alpha \le 1$ such that

$$\|u^\varepsilon - \tilde{u}^\varepsilon_{N,M,\ell,h}, H^p(\Omega)\| \le C_1 \|f - f_\ell, \mathcal{L}_2(\Omega)\| + C(\ell)\, h^{(2-p)\sigma} + C_3\, \varepsilon^{\min(N+1,m+2)-\alpha p} \qquad (4\ 5\ 2)$$

where $N \ge m+2$, $M = m+2-p$ and $C_3 > 0$ is a constant independent of ε

5 DOMAIN DECOMPOSITION FOR TWO DIMENSIONAL LINEARLY ELASTIC MODELS OF HETEROGENEOUS MATERIALS

5 0. Introduction

In chapters 3 and 4 we have been able to use homogenization techniques which employ asymptotic expansions to treat problems with rough coefficients of large variation because the problems considered had periodic and asymptotic structures However, these characteristics are not always present and, even if they are, asymptotic parameters such as ε, which are not within the control of the numerical analyst, may simply be too large to obtain accurate approximations Thus, if there is no periodic structure and/or ε is large it becomes necessary to consider alternative methods and, here, as a general approach we use the technique of non–overlapping domain decomposition with preconditioning algorithms to obtain approximate solutions of linear elastic models of heterogeneous materials This will lead to algorithms which can be efficiently implemented on parallel machines with MIMD type architectures In particular, we extend the domain decomposition with preconditioning approach first introduced for scalar elliptic boundary value problems in MANDEL (1993) to two–dimensional elastic problems over Lipschitz domains Ω and demonstrate, both theoretically and computationally, that the convergence properties established there remain valid

Boundary value problems which are formulated to describe physical problems over regions Ω with complex geometry can be difficult to solve in the classical sense of the continuously differentiable C^n type spaces However, if Ω can be viewed as the union of a number, in this case two, smooth, geometrically elementary, overlapping subdomains $\Omega_i \in C^{2,\lambda}$, $0 < \lambda < 1, 1 \le i \le 2$, i e ,

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 \neq \emptyset \tag{5 0 1}$$

and analogous boundary value problems formulated over each subdomain Ω_i, $1 \le i \le 2$ can be solved analytically, then, for suitable boundary conditions and decompositions (5 0 1), cf KANTOROVICH & KRYLOV (1964), Schwarz's alternating method SCHWARZ (1890) demonstrates that the Harmonic function u, $u|_{\partial\Omega} = g$ can be synthesized from the pointwise limits of the solutions of the boundary value problems Find $u_i^{(n)} \in C^{2,\lambda}(\overline{\Omega}_i)$, $1 \le i \le 2$ such that, for $n \ge 1$,

$$\nabla^2 u_i^{(n)}(\underline{x}) = 0, \quad \underline{x} \in \Omega_i \tag{5 0 2}$$

$$u_i^{(n)}(\underline{x}) = g(\underline{x}), \quad \underline{x} \in \partial\Omega \cap \partial\Omega_i \tag{5 0 3}$$

$$u_i^{(n)}(\underline{x}) = u_{3-i}^{(n-2+i)}(\underline{x}), \quad \underline{x} \in \partial\Omega_i \cap \Omega_{3-i} \tag{5 0 4}$$

where $u_1^{(0)} \stackrel{\text{def}}{=} \varphi$ on $\partial\Omega_1 \cap \Omega_2$ for arbitrary $\varphi \in C^{2\,\lambda}(\overline{\Omega})$ such that the Dirichlet boundary values in (5 0 3), (5 0 4) define Holder continuous functions on $\partial\Omega_i$ with exponents $\nu_i \in (0,1)$, $1 \le i \le 2$ Thus, $u|_{\Omega_i} = \lim_{n\to\infty} u_i^{(n)}$, $1 \le i \le 2$ and, if $\varphi = u|_{\partial\Omega_1 \cap \Omega_2}$ then the iteration (5 0 2)–(5 0 4) converges in one step, i e , $u|_{\Omega_i} = u_i^{(1)}$, $1 \le i \le 2$ Schwarz's decomposition concept found renewed interest with the advent of modern parallel computer architectures where the approach based on the recurrence equations (5 0 2)–(5 0 4) became known as the *multiplicative* Schwarz method However, the need to obtain an algorithm which is better suited for a parallel machine architecture led to the innovation of the *additive* Schwarz method in which the coupling conditions (5 0 4) are modified as follows

$$u_i^{(n)}(\underline{x}) = u_{3-i}^{(n-1)}(\underline{x}), \quad \underline{x} \in \partial\Omega_i \cap \Omega_{3-i}, \quad 1 \le i \le 2$$

where $u_{3-i}^{(0)} \stackrel{\text{def}}{=} \varphi$ on $\partial\Omega_i \cap \Omega_{3-i}$, $1 \le i \le 2$ This modification removed the need to strictly alternate the order of iteration between adjacent subdomains and therefore freed the processing nodes from having to synchronize their computations at each iterative step Further generalizations of the Schwarz approach have led to decompositions which allow more than two subdomains with each subdomain having lower regularity than $C^{2,\lambda}$, $0 < \lambda < 1$, cf LETALLEC (1994) However, by constructing non–overlapping domain decompositions of Ω, i e , subsets $\Omega_i \subset \Omega$, $1 \le i \le k$ such that

$$\overline{\Omega} = \cup_{i=1}^{k} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \Leftrightarrow i \ne j \tag{5 0 5}$$

a new class of domain decomposition techniques arose in which the global problem was reformulated as a system of local problems, each pertaining to a specific subdomain, Ω_i, $1 \le i \le k$, and an interfacing problem on Γ where

$$\Gamma \stackrel{\text{def}}{=} \cup_{i=1}^{k} \Gamma_i, \qquad \Gamma_i \stackrel{\text{def}}{=} \overline{\partial\Omega_i \backslash \partial\Omega} \tag{5 0 6}$$

Thus, as one may infer from Schwarz's approach, one first solves the interface problem on Γ for a trace function, $\underline{u}_\Gamma$, and then, using $\underline{u}_\Gamma$, solves the problems on Ω_i, $1 \le i \le k$ Non-overlapping domain decomposition algorithms generally interface local problems by employing either Lagrange multipliers to enforce *weak* continuity between the local solutions, $\underline{u}_{\Omega_i}$, $1 \leq i \leq k$, i e ,

$$(\mathrm{Tr}\,(\underline{u}_{\Omega_i} - \underline{u}_{\Omega_j}), \underline{v}, (H^{1/2}(\partial\Omega_i \cap \partial\Omega_j))^2) = 0, \quad \underline{v} \in (H^{1/2}(\partial\Omega_i \cap \partial\Omega_j))^2 \quad 1 \leq i,j \leq k \quad (5\,0\,7)$$

leading to an interface problem of the form Find $\underline{\lambda}_\Gamma \in \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ such that

$$S^{-1}\underline{\lambda}_\Gamma = \underline{t}, \quad \underline{t} \in (H^{1/2}(\Gamma))^2 \quad (5\,0\,8)$$

where $S\ (H^{1/2}(\Gamma))^2 \to \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ is the global Steklov–Poincaré operator, or enforce *strong* continuity conditions

$$\mathrm{Tr}\,(\underline{u}_{\Omega_i}) = \mathrm{Tr}\,(\underline{u}_{\Omega_j}) \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \quad 1 \leq i,j \leq k \quad (5\,0\,9)$$

using Steklov–Poincare operators to reformulate the boundary value problem and obtain the interface problem Find $\underline{u} \in (H^{1/2}(\Gamma))^2$ such that

$$\langle S\underline{u}_\Gamma, \underline{v}\rangle = \langle L, \underline{v}\rangle, \quad \underline{v} \in (H^{1/2}(\Gamma))^2 \quad (5\,0\,10)$$

where $L \in \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ The Lagrange multiplier approach leads to a saddle point problem in which the auxiliary unknown $\underline{\lambda}_\Gamma \in \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ can be interpreted as the normal stress $\sigma(\underline{u}) \circ \underline{n}$ on the interface Γ, cf FARHAT (1991), BREZZI (1974) However, with this interfacing approach, subdomains Ω_i, Ω_j, $1 \leq i,j \leq k$ are coupled if, and only if, $\sigma(\partial\Omega_i \cap \partial\Omega_j) > 0$ This contrasts with the *strong* interfacing approach of (5 0 10) in which subdomains Ω_i, Ω_j are coupled if $\partial\Omega_i \cap \partial\Omega_j \neq \emptyset$ Thus, the weak interfacing approach leads to subproblems with a greater level of independence than the strong interfacing approach and therefore requires fewer costly interprocessor communications on a MIMD machine to interface the system, however, it does introduce the additional unknown $\underline{\lambda}_\Gamma \in \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ and we therefore employ approach (5 0 10)

In particular, we will employ non–overlapping domain decompositions to construct problem (5 0 10) for linearly elastic models of heterogeneous materials We recall that the weak formulation of the elastic model of material deformation has the form Find $\underline{u} \in (H_0^1(\Omega, \partial\Omega_D))^2$ such that, for $\underline{v} \in (H_0^1(\Omega, \partial\Omega_D))^2$,

$$\int_\Omega \sum_{i,j,k,l=1}^{2} a_{ijkl}(\underline{x}) \frac{\partial u_i}{\partial x_j}(\underline{x}) \frac{\partial v_k}{\partial x_l}(\underline{x})\, d\underline{x} = \int_\Omega f(\underline{x})\ \underline{v}(\underline{x})\, d\underline{x} + \int_{\partial\Omega_T} \underline{t}(\underline{x})\ \underline{v}(\underline{x})\, d\underline{x} \quad (5\,0\,11)$$

where $f \in (\mathcal{L}_2(\Omega))^2$ is the body force acting over Ω, $\underline{t} \in (\mathcal{L}_2(\partial\Omega_T))^2$ is the surface traction acting across the open subset $\partial\Omega_T$ of the boundary $\partial\Omega$, and a_{ijkl}, $1 \leq i,j,k,l \leq 2$ are material coefficients given in terms of the Lame functions, cf (1 3 11),

$$\lambda(\underline{x}) \stackrel{\text{def}}{=} \frac{\nu\, E(\underline{x})}{1-\nu^2}, \quad \mu(\underline{x}) \stackrel{\text{def}}{=} \frac{E(\underline{x})}{2(1+\nu)}, \quad \underline{x} \in \Omega \quad (5\,0\,12)$$

where $\nu \in \mathbb{R}$ is Poisson's ratio and $E \in \mathcal{L}_\infty(\Omega)$ is Young's Modulus of elasticity for the material Ω We then construct a preconditioner M_h, $h > 0$ and treat problem (5 0 10) with a preconditioned conjugate gradient algorithm, cf AXELSSON (1994) We analyse the spectrum $\sigma(M_h^{-1}S_h)$ of the preconditioned interface operator $M_h^{-1}S_h$, $h > 0$ and obtain an upper bound for the condition number $\kappa(M_h^{-1}S_h) \stackrel{\text{def}}{=} \|M_h^{-1}S_h\|_2 \ \|(M_h^{-1}S_h)^{-1}\|_2$ We confirm the validity of the condition number bound by applying our approach to a number of problems and compare the computational results with the condition number bound obtained in our analysis

5 1 Elements of the Theory of Domain Decomposition

It has been observed that the domain decomposition concept was originally conceived to answer a purely theoretical question concerning the existence of Harmonic functions over regions, Ω, with complex geometries However, domain decomposition concepts have also been prevalent among engineers where subdomains Ω_i, $1 \leq i \leq k$ correspond to distinct, elemental substructures of a system and, in this context, the Steklov–Poincaré problem (5 0 10) models the physics of the interfaces between adjacent substructures Indeed, a common engineering approach was to discretize (5 0 10) to obtain the Schur complement system

$$S_h \underline{u}_{\Gamma,h} = \underline{L}_h, \quad h > 0 \tag{5 1 1}$$

where $h > 0$ is the discretization parameter, Γ the union of the physical interfaces, S_h is the matrix representing the discretized Steklov–Poincaré operator, and then solve the resulting equations using a direct solution technique However, for systems with many substructures the Schur complement system (5 1 1) can have many parameters and the computational cost of constructing and then solving the resulting equations can be impractical The advent of practical iterative conjugate gradient algorithms allowed one to solve systems, such as (5 1 1), without explicitly constructing S_h and, thus, provided the opportunity to employ substructuring concepts where previously they were impractical and, furthermore, to consider the possibility of devising solution techniques based on decompositions of Ω where the subdomains Ω_i, $1 \leq i \leq k$ have no physical significance, cf BJORSTAD & HVIDSTEN (1987), BJORSTAD & WIDLUND (1986) The Steklov–Poincare operator, S, is a continuous linear operator which, when discretized using finite element techniques yields, however, a Schur complement matrix, S_h, with condition number $\kappa(S_h) \stackrel{\text{def}}{=} \|S_h\|_2 \ \|S_h^{-1}\|_2 = O(1/H^2 + 1/(Hh))\,(h, H \to 0)$ where $H \stackrel{\text{def}}{=} \max_{1\leq i\leq k} \operatorname{diam}(\Omega_i)$ Consequently, $\kappa(S_h)$ grows rapidly as $h, H \to 0$ and the application of simple conjugate gradient algorithms usually suffer from poor convergence properties, as one should anticipate from the error estimate, cf AXELSSON (1994),

$$\|\underline{u}_{\Gamma h}^{(n)} - \underline{u}_{\Gamma,h}\|_{S_h} \leq 2\left[\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right]^n \|\underline{u}_{\Gamma h}^{(0)} - \underline{u}_{\Gamma h}\|_{S_h}, \quad n \geq 1 \tag{5 1 2}$$

Thus, we investigate how one can construct a symmetric positive definite preconditioner P_h^{-1}, $h > 0$ which can be efficiently implemented and is such that the preconditioned system

$$P_h^{-1} S_h \underline{u}_{\Gamma,h} = P_h^{-1} \underline{L}_h \tag{5 1 3}$$

has a condition number $\kappa(P_h^{-1} S_h)$ which grows slowly compared to $\kappa(S_h)$ as $h, H \to 0$ so that the conjugate gradient algorithm, applied to the symmetric form of system (5 1 3), produces iterates $\underline{u}_{\Gamma,h}^{(n)}$, $n \geq 1$ which converge rapidly to $\underline{u}_{\Gamma,h}$ as $n \to \infty$

5 1 1. The Interface Problem

Let Ω be partitioned into k non-overlapping subdomains Ω_i, $1 \leq i \leq k$ satisfying

$$\overline{\Omega} = \cup_{i=1}^{k} \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset \Leftrightarrow i \neq j \tag{5 1 1 1}$$

and define each subdomain interface, Γ_i, $1 \leq i \leq k$, and the global interface, Γ, as follows

$$\Gamma_i \stackrel{\text{def}}{=} \overline{\partial\Omega_i \backslash \partial\Omega}, \qquad \Gamma \stackrel{\text{def}}{=} \cup_{i=1}^{k} \Gamma_i \tag{5 1 1 2}$$

Then, corresponding to each local interface Γ_i, $1 \leq i \leq k$ and the global interface Γ, we let $\partial\Omega_{i,D} \stackrel{\text{def}}{=} \partial\Omega_i \cap \partial\Omega_D$ and define the respective trace spaces $H_{00}^{1/2}(\Gamma_i)$, $H^{1/2}(\Gamma)$ as follows

$$H_{00}^{1/2}(\Gamma_i) \stackrel{\text{def}}{=} \left\{ \underline{v} \mid \mathcal{D}(\underline{v}) = \Gamma_i \text{ and } \exists \underline{w} \in H_0^1(\Omega_i, \partial\Omega_{i,D}) \text{ such that } \operatorname{Tr}(\underline{w})\Big|_{\Gamma_i} = \underline{v} \right\} \tag{5 1 1 3}$$

$$H^{1/2}(\Gamma) \stackrel{\text{def}}{=} \left\{ \underline{v} \mid \mathcal{D}(\underline{v}) = \Gamma,\ \underline{v}\Big|_{\Gamma_i} \in H_{00}^{1/2}(\Gamma_i),\ 1 \leq i \leq k \right\} \tag{5 1 1 4}$$

and we define $a_i \in \mathcal{BL}(H^1(\Omega_i) \times H^1(\Omega_i), \mathbb{R})$, $F_i \in \mathcal{BL}(H^1(\Omega_i), \mathbb{R})$, $1 \leq i \leq k$ to be the respective restrictions to Ω_i of the bilinear form $a \in \mathcal{BL}(H^1(\Omega) \times H^1(\Omega), \mathbb{R})$ and the functional $F \in \mathcal{BL}(H^1(\Omega), \mathbb{R})$, cf (1 3 16), i e , for $\underline{u}, \underline{v} \in H^1(\Omega_i)$

$$a_i(\underline{u}, \underline{v}) \stackrel{\text{def}}{=} \int_{\Omega_i} \sum_{k\,l\,m,n=1}^{2} a_{klmn}(\underline{x}) \frac{\partial u_k}{\partial x_l}(\underline{x}) \frac{\partial v_m}{\partial x_n}(\underline{x})\, d\underline{x},\ F_i(\underline{v}) \stackrel{\text{def}}{=} \int_{\Omega_i} f(\underline{x})\ \underline{v}(\underline{x})\, d\underline{x} + \int_{\partial\Omega_{i\,T}} \underline{t}(\underline{x})\ \underline{v}(\underline{x})\, d\sigma(\underline{x}) \tag{5 1 1 5}$$

where $\partial\Omega_{i\,T} \stackrel{\text{def}}{=} \partial\Omega_i \cap \partial\Omega_T$, $1 \leq i \leq k$ and $\partial\Omega_T \subset \partial\Omega$ is the subset of the boundary where surface traction forces apply Furthermore, it will be required to define extension operators E_i $(H_{00}^{1/2}(\Gamma_i))^2 \to (H^1(\Omega_i))^2$ which are right inverses of the trace operators $\operatorname{Tr} \in \mathcal{BL}((H^1(\Omega_i))^2, (H^{1/2}(\partial\Omega_i))^2)$ on Γ_i, $1 \leq i \leq k$ and, for this purpose, we identify E_i, $1 \leq i \leq k$ with the Harmonic extension operators defined as follows Let $\underline{u} \in (H_{00}^{1/2}(\Gamma_i))^2$ and define $E_i\underline{u} \in (H^1(\Omega_i))^2$ to be the function which has the properties $\operatorname{Tr}(E_i\underline{u})|_{\Gamma_i} = \underline{u}$, $\operatorname{Tr}(E_i\underline{u})|_{\partial\Omega\ D} = 0$ and

$$a_i(E_i\underline{u}, \underline{v}) = 0, \quad \underline{v} \in (H_0^1(\Omega_i, \Upsilon))^2 \tag{5 1 1 6}$$

where $\Upsilon_i \stackrel{\text{def}}{=} \Gamma_i \cup \partial\Omega_{i,D}$ Clearly, the properties of the bilinear form a_i and the Lax Milgram Lemma, cf Section 1 1 1, guarantee the existence of a unique Harmonic extension $E_i\underline{u} \in$

$(H^1(\Omega_i))^2$ for any $\underline{u} \in (H_{00}^{1/2}(\Gamma_i))^2$, $1 \leq i \leq k$. The continuity of the linear operators E_i, $1 \leq i \leq k$ follow from the inequality, for $\underline{u} \in (H_{00}^{1/2}(\Gamma_i))^2$, cf. DEROECK & LETALLEC (1991),

$$\|E_i\underline{u}, (H^1(\Omega_i))^2\| \leq C_1 \|\underline{u}_0, (H^1(\Omega_i))^2\| \leq C_2 \|\mathrm{Tr}\,(\underline{u}_0), (H^{1/2}(\partial\Omega_i))^2\| = C_2 \|\underline{u}, (H^{1/2}(\Gamma_i))^2\| \quad (5.1.1.7)$$

where $C_1, C_2 > 0$ are constants independent of $\underline{u} \in (H_{00}^{1/2}(\Gamma_i))^2$ and $\underline{u}_0 \in (H^1(\Omega_i))^2$ is the Harmonic extension of $\underline{u}$ satisfying $\mathrm{Tr}\,(\underline{u}_0)|_{\Gamma_i} = \underline{u}$, $\mathrm{Tr}\,(\underline{u}_0)|_{\partial\Omega_i\setminus\Gamma_i} = 0$. The global Harmonic extension operator $E : (H^{1/2}(\Gamma))^2 \to (H^1(\Omega))^2$ is then defined according to the relation

$$E\underline{u}\Big|_{\Omega_i} \stackrel{\text{def}}{=} E_i R_{\Gamma_i}\underline{u}, \quad \underline{u} \in (H^{1/2}(\Gamma))^2, \quad 1 \leq i \leq k \quad (5.1.1.8)$$

where $R_{\Gamma_i} : (H^{1/2}(\Gamma))^2 \to (H_{00}^{1/2}(\Gamma_i))^2$ is the restriction operator defined by $R_{\Gamma_i}\underline{u}_\Gamma \stackrel{\text{def}}{=} \underline{u}_\Gamma|_{\Gamma_i}$. However, in accordance with the decomposition (5.1.1.1) of the domain Ω, the Sobolev space $(H_0^1(\Omega, \partial\Omega_D))^2$ can be decomposed into the local spaces $E((H^{1/2}(\Gamma))^2)$, $(H_0^1(\Omega_i, \Upsilon_i))^2$, $1 \leq i \leq k$, i.e.,

$$(H_0^1(\Omega, \partial\Omega_D))^2 = E((H^{1/2}(\Gamma))^2) \oplus (H_0^1(\Omega_1, \Upsilon_1))^2 \oplus \dots \oplus (H_0^1(\Omega_k, \Upsilon_k))^2 \quad (5.1.1.9)$$

where $E((H^{1/2}(\Gamma))^2) = \{E\underline{u} \mid \underline{u} \in (H^{1/2}(\Gamma))^2\}$ and elements in $(H_0^1(\Omega_i, \Upsilon_1))^2$, $1 \leq i \leq k$ are extended by zero to Ω. It then follows that the global problem: Find $\underline{u} \in (H^1(\Omega))^2$ such that $\mathrm{Tr}\,(\underline{u})|_{\partial\Omega_D} = \underline{u}_D$ and

$$\sum_{i=1}^{k} a_i(\underline{u}, \underline{v}) = \sum_{i=1}^{k} F_i(\underline{v}), \quad \underline{v} \in (H_0^1(\Omega, \partial\Omega_D))^2 \quad (5.1.1.10)$$

can be replaced by the equivalent formulation: Find $\underline{u}_\Gamma \in (H^{1/2}(\Gamma))^2$, $\underline{u}_{\Omega_i} \in (H^1(\Omega_i))^2$, $1 \leq i \leq k$ such that $\mathrm{Tr}\,(\underline{u}_{\Omega_i})|_{\Gamma_i} = \underline{u}_\Gamma|_{\Gamma_i}$, $\mathrm{Tr}\,(\underline{u}_{\Omega_i})|_{\partial\Omega_{i,D}} = \underline{u}_D|_{\partial\Omega_{i,D}}$ and

$$a_i(\underline{u}_{\Omega_i}, \underline{v}) = F_i(\underline{v}), \quad \underline{v} \in (H_0^1(\Omega_i, \Upsilon_i))^2 \quad (5.1.1.11)$$

$$\sum_{i=1}^{k} a_i(\underline{u}_{\Omega_i}, E\underline{v}) = \sum_{i=1}^{k} F_i(E\underline{v}), \quad \underline{v} \in (H^{1/2}(\Gamma))^2 \quad (5.1.1.12)$$

The problems (5.1.1.10) and (5.1.1.11), (5.1.1.12) are then equivalent in the sense that

$$\underline{u}\Big|_{\Omega_i} = \underline{u}_{\Omega_i}, \qquad \mathrm{Tr}\,(\underline{u}|_{\Omega_i})\Big|_{\Gamma_i} = \underline{u}_\Gamma\Big|_{\Gamma_i}, \quad 1 \leq i \leq k \quad (5.1.1.13)$$

Thus, problems (5.1.1.11) and (5.1.1.12) form a coupled system in which (5.1.1.11) models the problem locally, i.e., within each subdomain Ω_i, $1 \leq i \leq k$ and (5.1.1.12) models the interfacing problem on Γ between the subdomains. It is this problem which we study in Section 5.1.3, discretize using finite element techniques, and finally solve using preconditioned conjugate gradient methods. However, we first observe, from the hypothesis of linear elasticity, the relation

$$\sigma_{mn}(\underline{u}_{\Omega_i}(\underline{x})) = \sum_{p,q=1}^{2} a_{mnpq}(\underline{x}) \frac{\partial u_{\Omega_i,p}}{\partial x_q}(\underline{x}), \quad 1 \leq m,n \leq 2, \quad \underline{x} \in \Omega_i, \quad 1 \leq i \leq k \quad (5.1.1.14)$$

where, if we assume that $\nabla \sigma(\underline{u}_{\Omega_i}) \in (\mathcal{L}_2(\Omega_i))^2$ then, employing Green's theorem, we deduce the following identities on Γ_i, $1 \leq i \leq k$, for $\underline{v} \in (H^{1/2}(\Gamma))^2$,

$$\begin{aligned} a_i(\underline{u}_{\Omega_i}, E\underline{v}) - F_i(E\underline{v}) &= \int_{\Omega_i} \sigma(\underline{u}_{\Omega_i}(\underline{x})) \; \nabla E\underline{v}(\underline{x})\, d\underline{x} - \int_{\Omega_i} \underline{f}(\underline{x}) \; E\underline{v}(\underline{x})\, d\underline{x} \\ &\quad - \int_{\partial\Omega_{i,T}} \underline{t}(\underline{x}) \; \underline{w}(\underline{x})\, d\sigma(\underline{x}) \\ &= -\int_{\Omega_i} \left[\nabla \sigma(\underline{u}_{\Omega_i}(\underline{x})) + \underline{f}(\underline{x})\right] \; E\underline{v}(\underline{x})\, d\underline{x} + \int_{\partial\Omega_i} [\sigma(\underline{u}_{\Omega_i}(\underline{x})) \circ \underline{n}_i(\underline{x})] \; \underline{w}(\underline{x})\, d\sigma(\underline{x}) \\ &\quad - \int_{\partial\Omega_{i,T}} \underline{t}(\underline{x}) \; \underline{w}(\underline{x})\, d\sigma(\underline{x}) \\ &= \int_{\Gamma_i} [\sigma(\underline{u}_{\Omega_i}(\underline{x})) \circ \underline{n}_i(\underline{x})] \; \underline{v}(\underline{x})\, d\sigma(\underline{x}) \end{aligned} \qquad (5\ 1\ 1\ 15)$$

where $\underline{w} \stackrel{\text{def}}{=} \mathrm{Tr}\,(E\underline{v})$, $\underline{n}_i(\underline{x})$ is the unit outward normal vector to $\partial\Omega_i$ at $\underline{x}$, and, for $\underline{x} \in \Omega_i$, $1 \leq i \leq k$, $\underline{v} \in (H^1(\Omega))^2$, $1 \leq p, q \leq 2$,

$$\nabla\underline{v}(\underline{x}) \stackrel{\text{def}}{=} \left[\frac{\partial v_p}{\partial x_q}(\underline{x})\right] \in \mathbb{R}^{2,2}, \; \sigma(\underline{u}_{\Omega_i}(\underline{x})) \quad \nabla\underline{v}(\underline{x}) \stackrel{\text{def}}{=} \sum_{p,q=1}^{2} \sigma_{pq}(\underline{u}_{\Omega_i}(\underline{x})) \frac{\partial v_p}{\partial x_q}(\underline{x}) \in \mathbb{R}$$

$$\nabla \sigma(\underline{u}_{\Omega_i}(\underline{x})) \stackrel{\text{def}}{=} \left[\sum_{q=1}^{2} \frac{\partial \sigma_{pq}}{\partial x_q}(\underline{u}_{\Omega_i}(\underline{x}))\right] \in \mathbb{R}^2, \; \sigma(\underline{u}_{\Omega_i}(\underline{x})) \circ \underline{n}(\underline{x}) \stackrel{\text{def}}{=} \left[\sum_{q=1}^{2} \sigma_{pq}(\underline{u}_{\Omega_i}(\underline{x}))\, n_q(\underline{x})\right] \in \mathbb{R}^2$$

However, the interface problem (5 1 1 12) then implies the following property

$$\sum_{i=1}^{k} \left[a_i(\underline{u}_{\Omega_i}, E\underline{v}) - F_i(E\underline{v})\right] = \sum_{i=1}^{k} \int_{\Gamma_i} (\sigma(\underline{u}_{\Omega_i}(\underline{x})) \circ \underline{n}_i(\underline{x})) \; \underline{v}(\underline{x})\, d\sigma(\underline{x}) = 0, \quad \underline{v} \in (H^{1/2}(\Gamma))^2 \qquad (5\ 1\ 1\ 16)$$

Thus, the problem of determining a global solution $\underline{u} \in (H^1(\Omega, \partial\Omega_D))^2$ of (5 1 1 10) is equivalent to the problem of finding a function defined on the interface Γ, e g , $\underline{u}_\Gamma \in (H^{1/2}(\Gamma))^2$, such that the local solutions $\underline{u}_{\Omega_i} \in (H^1(\Omega_i))^2$ of problems (5 1 1 11) have normal stress tensors, $\sigma(\underline{u}_{\Omega_i}) \circ \underline{n}_i$, which are continuous across the interface Γ, i e , they satisfy (5 1 1 16)

5 1 2 Steklov–Poincaré Operators and the Interface Problem

In this section we reformulate the interface problem (5 1 1 12), which is central to domain decomposition methods, to obtain an equivalent problem posed solely on the interface Γ in terms of a family of linear operators called Steklov–Poincaré operators Then, using finite element techniques to obtain approximating discretized Steklov–Poincaré operators we demonstrate how one obtains the Schur complement system (5 1 1) and, furthermore, we demonstrate how this system can be solved using conjugate gradient techniques without explicitly constructing the discretized operators

Let $a_i(\bullet,\bullet)$, $a(\bullet,\bullet)$, E_i, E, $1 \leq i \leq k$ be, respectively, the local and global bilinear forms and Harmonic extension operators defined above, the local Steklov–Poincare operator S_i $(H_{00}^{1/2}(\Gamma_i))^2 \to \mathcal{BL}((H_{00}^{1/2}(\Gamma_i))^2, \mathbb{R})$ is then defined according to the relation

$$\langle S_i\underline{u}, \underline{v}\rangle \stackrel{\text{def}}{=} a_i(E_i\underline{u}, E_i\underline{v}), \quad \underline{u}, \underline{v} \in (H_{00}^{1/2}(\Gamma_i))^2 \qquad (5\ 1\ 1\ 17)$$

and the corresponding global Steklov–Poincare operator $S \; (H^{1/2}(\Gamma))^2 \to \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ is defined as follows

$$\langle S\underline{u}, \underline{v}\rangle \stackrel{\text{def}}{=} \sum_{i=1}^{k} \langle S_i R_{\Gamma_i}\underline{u}, R_{\Gamma_i}\underline{v}\rangle = \sum_{i=1}^{k} a_i(E_i R_{\Gamma_i}\underline{u}, E_i R_{\Gamma_i}\underline{v}), \quad \underline{u}, \underline{v} \in (H^{1/2}(\Gamma))^2 \qquad (5\,1\,1\,18)$$

If $\underline{u}_\Gamma \in (H^{1/2}(\Gamma))^2$ denotes the $E((H^{1/2}(\Gamma))^2)$ component of the solution of problem (5 1 1 10) then we observe that the solutions $\underline{u}_{\Omega_i} \in (H^1(\Omega_i))^2$, $1 \le i \le k$ of problems (5 1 1 11) can be expressed as the sum $\underline{u}_{\Omega_i} = E_i R_\Gamma \, \underline{u}_\Gamma + \underline{w}_{\Omega_i}$ where $\underline{w}_{\Omega_i} \in (H^1(\Omega_i))^2$ is uniquely defined as the function with the following properties $\mathrm{Tr}\,(\underline{w}_{\Omega_i})|_{\partial\Omega_{i,D}} = \underline{u}_D|_{\partial\Omega_{i,D}}$, $\mathrm{Tr}\,(\underline{w}_{\Omega_i})|_{\Gamma_i} = 0$ and

$$a_i(\underline{w}_i, \underline{v}) = F_i(\underline{v}), \quad \underline{v} \in (H_0^1(\Omega_i, \Upsilon_i))^2 \qquad (5\,1\,1\,19)$$

However, given this decomposition of $\underline{u}_{\Omega_i}$ the interface problem (5 1 1 12) can be *rewritten* in terms of the operators S, S_i, $1 \le i \le k$ as follows, for $\underline{v} \in (H^{1/2}(\Gamma))^2$,

$$\sum_{i=1}^{k} a_i(\underline{u}_{\Omega_i}, E_i R_{\Gamma_i}\underline{v}) = \sum_{i=1}^{k} a_i(E_i R_{\Gamma_i}\underline{u}_\Gamma + \underline{w}_{\Omega_i}, E_i R_{\Gamma_i}\underline{v})$$

$$= \sum_{i=1}^{k} \langle S_i R_{\Gamma_i}\underline{u}_\Gamma, R_{\Gamma_i}\underline{v}\rangle + \sum_{i=1}^{k} a_i(\underline{w}_{\Omega_i}, E_i R_{\Gamma_i}\underline{v}) = \sum_{i=1}^{k} F_i(E_i R_{\Gamma_i}\underline{v}) \qquad (5\,1\,1\,20)$$

Thus, we define $L_i \in \mathcal{BL}((H_{00}^{1/2}(\Gamma_i))^2, \mathbb{R})$, $1 \le i \le k$ according to the relation

$$\langle L_i, \underline{v}\rangle \stackrel{\text{def}}{=} F_i(E_i\underline{v}) - a_i(\underline{w}_{\Omega_i}, E_i\underline{v}), \quad \underline{v} \in (H_{00}^{1/2}(\Gamma_i))^2 \qquad (5\,1\,1\,21)$$

and (5 1 1 20) becomes

$$\sum_{i=1}^{k} \langle S_i R_\Gamma \, \underline{u}_\Gamma, R_{\Gamma_i}\underline{v}\rangle = \sum_{i=1}^{k} \langle L_i, E_i R_{\Gamma_i}\underline{v}\rangle \qquad (5\,1\,1\,22)$$

Finally, we employ the transpose operators $R_{\Gamma_i}^T \; \mathcal{BL}((H_{00}^{1/2}(\Gamma_i))^2, \mathbb{R}) \to \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$, $E_i^T \; \mathcal{BL}((H^1(\Omega_i))^2, \mathbb{R}) \to \mathcal{BL}((H_{00}^{1/2}(\Gamma_i))^2, \mathbb{R})$, $1 \le i \le k$ and define the global interface problem in terms of the Steklov–Poincaré operators as follows Find $\underline{u}_\Gamma \in (H^{1/2}(\Gamma))^2$ such that

$$\Big\langle \sum_{i=1}^{k} R_\Gamma^T \, S_i R_{\Gamma_i}\underline{u}_\Gamma, \underline{v}\Big\rangle = \Big\langle \sum_{i=1}^{k} R_{\Gamma_i}^T E_i^T L_i, \underline{v}\Big\rangle, \quad \underline{v} \in (H^{1/2}(\Gamma))^2 \qquad (5\,1\,1\,23)$$

However, if we define $S \; (H^{1/2}(\Gamma))^2 \to \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$, $L \in \mathcal{BL}((H^{1/2}(\Gamma))^2, \mathbb{R})$ as follows

$$S \stackrel{\text{def}}{=} \sum_{i=1}^{k} R_{\Gamma_i}^T S_i R_{\Gamma_i}, \qquad L \stackrel{\text{def}}{=} \sum_{i=1}^{k} R_\Gamma^T \, E_i^T L_i \qquad (5\,1\,1\,24)$$

then the interface problem is Find $\underline{u}_\Gamma \in (H^{1/2}(\Gamma))^2$ such that

$$\langle S\underline{u}_\Gamma, \underline{v}\rangle = \langle L, \underline{v}\rangle, \quad \underline{v} \in (H^{1/2}(\Gamma))^2 \qquad (5\,1\,1\,25)$$

In Section 5 1 3 we demonstrate how the interface problem (5 1 1 25) can be discretized to obtain a linear system of symmetric, positive definite algebraic equations

5 1 3 The discretized Interface Problem Schur Complement Systems

Let $\mathcal{T}_h(\Omega)$, $h > 0$ be a triangulation of Ω, cf Section 2 1, where $h > 0$ is the mesh diameter and assume that each subdomain Ω_i, $1 \le i \le k$ is the union of some subset of elements of $\mathcal{T}_h(\Omega)$, i e , there exist triangulations $\mathcal{T}_h(\Omega_i) \subset \mathcal{T}_h(\Omega)$, $1 \le i \le k$ We now assume, without loss of generality, that the Dirichlet and traction boundary conditions are homogeneous and replace the infinite dimensional Sobolev spaces

$$(H_0^1(\Omega_i, \partial\Omega_{i\,D}))^2, \quad (H_0^1(\Omega, \partial\Omega_D))^2, \quad (H_{00}^{1/2}(\Gamma_i))^2, \quad (H^{1/2}(\Gamma))^2$$

with the respective approximating finite dimensional subspaces

$$(S_0^h(\Omega_i, \partial\Omega_{i,D}))^2, \quad (S_0^h(\Omega, \partial\Omega_D))^2, \quad (S^h(\Gamma_i))^2, \quad (S^h(\Gamma))^2$$

of piecewise linear polynomials where, for $1 \le i \le k$,

$$S^h(\Gamma_i) \stackrel{\text{def}}{=} \left\{ \underline{v} \mid \mathcal{D}(\underline{v}) = \Gamma_i \text{ and } \exists \underline{w} \in S^h(\Omega_i) \text{ such that } \underline{w}\Big|_{\Gamma_i} = \underline{v} \right\} \tag{5 1 1 26}$$

$$S^h(\Gamma) \stackrel{\text{def}}{=} \left\{ \underline{v} \mid \mathcal{D}(\underline{v}) = \Gamma \text{ and } \exists \underline{w} \in S^h(\Omega) \text{ such that } \underline{w}\Big|_{\Gamma_i} = \underline{v}\Big|_{\Gamma_i} \right\} \tag{5 1 1 27}$$

and $S_0^h(\Omega, \partial\Omega_D)$, $S_0^h(\Omega_i, \partial\Omega_{i,D})$, $1 \le i \le k$ are constructed as in section 2 1 The continuous operators R_{Γ_i}, E_i are thus replaced by their discrete counterparts $R_{\Gamma_i,h}, E_{i,h}$ and, similarly, the continuous Steklov–Poincare operators S, S_i are replaced by their discrete analogues $S_h, S_{i,h}$, $1 \le i \le k$ Given a basis $\mathcal{B}((S_0^h(\Omega_i, \partial\Omega_{i\,D}))^2)$ of $(S_0^h(\Omega_i, \partial\Omega_{i,D}))^2$ define $\mathcal{B}_h(\Gamma_i) \subset \mathcal{B}((S_0^h(\Omega_i, \partial\Omega_{i,D}))^2)$ to be the subset which contains those basis functions associated with a node $\underline{v} \in \Gamma_i$ of $\mathcal{T}_h(\Omega_i)$ and define $\mathcal{B}_h(\Omega_i) \stackrel{\text{def}}{=} \mathcal{B}((S_0^h(\Omega_i, \partial\Omega_{i,D}))^2) \backslash \mathcal{B}_h(\Gamma_i)$ then

$$\mathcal{B}(S_0^h(\Omega_i, \partial\Omega_{i,D})) = \mathcal{B}_h(\Omega_i) \cup \mathcal{B}_h(\Gamma) \tag{5 1 1 28}$$

and $N_i = N_{\Omega_i} + N_{\Gamma_i}$ where $2N_i \stackrel{\text{def}}{=} |\mathcal{B}((S_0^h(\Omega_i, \partial\Omega_{i,D}))^2)|$, $2N_{\Omega_i} \stackrel{\text{def}}{=} |\mathcal{B}_h(\Omega_i)|$, $2N_{\Gamma_i} \stackrel{\text{def}}{=} |\mathcal{B}_h(\Gamma_i)|$ Observing that a linear operator B $(S_0^h(\Omega_i, \partial\Omega_{i,D}))^2 \to \mathcal{BL}((S_0^h(\Omega_i, \partial\Omega_{i\,D}))^2, \mathbb{R})$ can be represented by a matrix $M \in \mathbb{R}^{2N_i, 2N_i}$ in the sense that, for $F \in \mathcal{BL}((S_0^h(\Omega_i, \partial\Omega_{i\,D}))^2, \mathbb{R})$,

$$\langle B\underline{u}, \underline{v}\rangle = \langle F, \underline{v}\rangle, \quad \underline{u}, \underline{v} \in (S_0^h(\Omega_i, \partial\Omega_{i,D}))^2 \quad \Longleftrightarrow \quad \underline{U}^T M \underline{V} = \underline{F}^T \underline{V}, \tag{5 1 1 29}$$

where, for $\mathcal{B}((S_0^h(\Omega_i, \partial\Omega_{i,D}))^2) = \{\underline{e}_r \phi_s^{(i)}\}_{r,s=1}^{2,N_i}$, functions $\underline{u}, \underline{v} \in (S_0^h(\Omega_i, \partial\Omega_{i,D}))^2$ can be written

$$\underline{u}(\underline{x}) = \sum_{r=1}^{N_i} \underline{u}_r \phi_r^{(i)}(\underline{x}), \qquad \underline{v}(\underline{x}) = \sum_{r=1}^{N_i} \underline{v}_r \phi_r^{(i)}(\underline{x}), \qquad \underline{x} \in \Omega_i \tag{5 1 1 30}$$

and the *block* matrix entries of $M \in \mathbb{R}^{2N_i, 2N_i}$, $\underline{F} \in \mathbb{R}^{2N_i}$ are given by the relations

$$M_{rs} \stackrel{\text{def}}{=} \begin{bmatrix} \langle B\underline{e}_1\phi_r^{(i)}, \underline{e}_1\phi_s^{(i)}\rangle & \langle B\underline{e}_1\phi_r^{(i)}, \underline{e}_2\phi_s^{(i)}\rangle \\ \langle B\underline{e}_2\phi_r^{(i)}, \underline{e}_1\phi_s^{(i)}\rangle & \langle B\underline{e}_2\phi_r^{(i)}, \underline{e}_2\phi_s^{(i)}\rangle \end{bmatrix}, \quad F_s \stackrel{\text{def}}{=} \begin{bmatrix} \langle F, \underline{e}_1\phi_s^{(i)}\rangle \\ \langle F, \underline{e}_2\phi_s^{(i)}\rangle \end{bmatrix}, \quad 1 \le r, s \le N_i \tag{5 1 1 31}$$

Thus, the linear operators $A_{i,h}$, $E_{i,h}$, $1 \le i \le k$ are represented by the matrices

$$A_{i,h} \stackrel{\text{def}}{=} \begin{bmatrix} A_{\Omega_i} & A_{\Omega_i,\Gamma_i} \\ A^T_{\Omega_i,\Gamma_i} & A_{\Gamma_i} \end{bmatrix} \in \mathbb{R}^{2N_i,2N_i}, \qquad E_{i,h} \stackrel{\text{def}}{=} \begin{bmatrix} -A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i} \\ I \end{bmatrix} \in \mathbb{R}^{2N_i,2N_{\Gamma_i}} \qquad (5\,1\,1\,32)$$

Let $\underline{x}^{(i)}_p$, $1 \le p \le N_{\Gamma_i}$ be the $\mathcal{T}_h(\Omega_i)$ nodes on Γ_i then the restriction operator $R_{\Gamma_i,h}$ is represented by the matrix $R_{\Gamma_i,h} \in \mathbb{R}^{2N_{\Gamma_i},2N}$ whose 2×2 *block* entries are defined as follows

$$(R_{\Gamma_i,h})_{p,q} \stackrel{\text{def}}{=} \begin{cases} I, & \text{if } G_i(p) = q \\ 0, & \text{if } G_i(p) \ne q \end{cases}, \quad 1 \le p \le N_{\Gamma_i}, \quad 1 \le q \le N \qquad (5\,1\,1\,33)$$

where $I \in \mathbb{R}^{2,2}$ is the identity matrix and G_i $\{1, \ldots, N_i\} \to \{1, \ldots, N\}$ maps the local *block parameter indices*, $\{1, \ldots, N_i\}$, of subdomain Ω_i to their global values, $\{1, \ldots, N\}$ Furthermore, it is apparent from relation (5 1 1 17) that $S_{i,h} = E^T_{i,h} A_{i,h} E_{i,h}$ and therefore the discrete local Steklov–Poincaré operator $S_{i,h}$, $1 \le i \le k$ can be represented by the matrix

$$\begin{aligned} S_{i,h} &= \begin{bmatrix} -A^T_{\Omega_i,\Gamma_i} A^{-T}_{\Omega_i} & I \end{bmatrix} \begin{bmatrix} A_{\Omega_i} & A_{\Omega_i,\Gamma_i} \\ A^T_{\Omega_i,\Gamma_i} & A_{\Gamma_i} \end{bmatrix} \begin{bmatrix} -A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i} \\ I \end{bmatrix} \\ &= A_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i} \in \mathbb{R}^{2N_{\Gamma_i},2N_{\Gamma_i}} \end{aligned} \qquad (5\,1\,1\,34)$$

and the Global Steklov–Poincaré operator, S_h, is represented by the matrix

$$S_h = \sum_{i=1}^{k} R^T_{\Gamma_i,h} S_{i,h} R_{\Gamma_i,h} = \sum_{i=1}^{k} R^T_{\Gamma_i,h} (A_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i}) R_{\Gamma_i,h} \in \mathbb{R}^{2N,2N} \qquad (5\,1\,1\,35)$$

Similarly, after discretization, the expressions $E^T_i L_i$, $1 \le i \le k$ are approximated by the analogous expressions $E^T_{i,h} L_{i,h}$, $1 \le i \le k$ which are represented by the following matrix–vector identities

$$\begin{aligned} E^T_{i,h} \underline{L}_{i,h} &= \begin{bmatrix} -A^T_{\Omega_i,\Gamma_i} A^{-T}_{\Omega_i} & I \end{bmatrix} \begin{bmatrix} \underline{F}_{\Omega_i} \\ \underline{F}_{\Gamma_i} \end{bmatrix} - \begin{bmatrix} -A^T_{\Omega_i,\Gamma_i} A^{-T}_{\Omega_i} & I \end{bmatrix} \begin{bmatrix} A_{\Omega_i} & A_{\Omega_i,\Gamma_i} \\ A^T_{\Omega_i,\Gamma_i} & A_{\Gamma_i} \end{bmatrix} \begin{bmatrix} \underline{w}_{\Omega_i} \\ 0 \end{bmatrix} \\ &= \underline{F}_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega_i} \underline{F}_{\Omega_i} \in \mathbb{R}^{2N_{\Gamma_i}} \end{aligned} \qquad (5\,1\,1\,36)$$

where $\underline{F}_i = [\underline{F}_{\Omega_i}, \underline{F}_{\Gamma_i}] \in \mathbb{R}^{2N}$ represents the functional $F_i \in \mathcal{BL}((S^h_0(\Omega_i, \partial\Omega_{i,D}))^2, \mathbb{R})$, cf (5 1 1 5) Thus, the right hand side of the discretized interface problem, illustrated in (continuous) operator form in relation (5 1 1 22), has the matrix form

$$\underline{L}_h = \sum_{i=1}^{k} R^T_{\Gamma_i,h} (\underline{F}_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega_i} \underline{F}_{\Omega_i}) \in \mathbb{R}^{2N} \qquad (5\,1\,1\,37)$$

Therefore, by discretizing the linear Steklov–Poincare operators and the associated restriction and extension operators, one obtains the following discrete Schur complement system

$$\begin{aligned} \sum_{i=1}^{k} R^T_{\Gamma_i,h} (A_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i}) R_{\Gamma_i,h} \underline{u}_{\Gamma,h} &= \sum_{i=1}^{k} R^T_{\Gamma_i,h} (\underline{F}_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega_i} \underline{F}_{\Omega_i}) && (5\,1\,1\,38) \\ \iff \qquad S_h \underline{u}_{\Gamma,h} &= \underline{L}_h && (5\,1\,1\,39) \end{aligned}$$

The symmetry of $S_h \in \mathbb{R}^{2N,2N}$ follows immediately from (5 1 1 34), (5 1 1 35) and, from the definition of the bilinear forms $a_i(\bullet,\bullet)$, $1 \le i \le k$, it is clear that

$$\langle S_h\underline{u}, \underline{u}\rangle = \sum_{i=1}^{k} a_i(E_{i,h}R_{\Gamma_i h}\underline{u}, E_{i h}R_{\Gamma_i h}\underline{u}) \ge 0, \quad \underline{u} \in (S^h(\Gamma))^2 \tag{5 1 1 40}$$

and, thus,

$$\langle S_h\underline{u}, \underline{u}\rangle = 0 \quad \Longleftrightarrow \quad a_i(E_{i h}R_{\Gamma_i h}\underline{u}, E_{i,h}R_{\Gamma_i,h}\underline{u}) = 0, \quad 1 \le i \le k \tag{5 1 1 41}$$

However, (5 1 1 41) holds only if $E_{i,h}R_{\Gamma_i,h}\underline{u}$ is a rigid body motion such that $\sigma(E_{i h}R_{\Gamma_i h}\underline{u})\circ\underline{n}_i$ has zero trace on the boundary, $\partial\Omega_i$, i e , $E_{i,h}R_{\Gamma_i,h}\underline{u} = \underline{a} + R(r,\theta)\,\underline{x}$, $\underline{a} \in \mathbb{R}^2$, $r \in \mathbb{R}$ where

$$R(r,\theta) \stackrel{\text{def}}{=} \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}, \quad \theta = (2n+1)\pi/2,\ n \in \mathcal{Z} \tag{5 1 1 42}$$

However, assuming that, for some $p \in \mathbb{N}_k$, subdomain Ω_p satisfies $\sigma(\partial\Omega_p \cap \partial\Omega_D) > 0$ then there exist constants C_1, $C_2 > 0$ such that

$$C_1\,\|\underline{u}, (H^1(\Omega_p))^2\|^2 \le a_p(\underline{u},\underline{u}) \le C_2\,\|\underline{u}, (H^1(\Omega_p))^2\|^2, \quad \underline{u} \in (S_0^h(\Omega_p, \partial\Omega_{p,D}))^2 \tag{5 1 1 43}$$

Relations (5 1 1 41) and (5 1 1 43) then imply that $E_{p h}R_{\Gamma_p,h}\underline{u} = 0$ and therefore $R_{\Gamma_p,h}\underline{u} = 0$ The zero trace $\underline{u}|_{\Gamma_p} = 0$ propagates to each subdomain to give $R_{\Gamma_i,h}\underline{u} = 0$, $1 \le i \le k$, i e , $\underline{u} = 0$ and the positive definiteness of S_h follows immediately

We now aim to develop preconditioners which allow one to solve the interface problem (5 1 1 39) efficiently using the conjugate gradient approach However, we first observe that the conjugate gradient approach, applied to (5 1 1 39), requires one to evaluate, at each iteration, the matrix–vector product $S_h\underline{d}$ for a given $\underline{d} \in \mathbb{R}^{2N}$ This can be achieved without explicitly constructing $S_h \in \mathbb{R}^{2N,2N}$ as follows Given $\underline{d} \in \mathbb{R}^{2N}$ define $\underline{d}_{\Gamma_i} \stackrel{\text{def}}{=} R_{\Gamma_i,h}\underline{d} \in \mathbb{R}^{2N_{\Gamma_i}}$, $1 \le i \le k$ and construct the Harmonic extension, $E_{i h}\underline{d}_{\Gamma_i} \in \mathbb{R}^{2N_i}$, by first solving the systems

$$A_{\Omega_i}\underline{x}_{\Omega} = -A_{\Omega_i\Gamma}\,\underline{d}_{\Gamma_i}, \quad 1 \le i \le k \tag{5 1 1 44}$$

and then observing that $E_{i,h}\underline{d}_{\Gamma_i} = [-A_{\Omega_i}^{-1}A_{\Omega_i,\Gamma_i}\underline{d}_{\Gamma_i}, \underline{d}_{\Gamma_i}] = [\underline{x}_{\Omega_i}, \underline{d}_{\Gamma_i}]$ The product $S_{i,h}\underline{d}_{\Gamma_i}$ is *then obtained from the relation*

$$\begin{bmatrix} 0 \\ S_{i h} \end{bmatrix}\underline{d}_{\Gamma_i} = \begin{bmatrix} A_{\Omega} & A_{\Omega_i,\Gamma_i} \\ A_{\Omega\,\Gamma}^T & A_{\Gamma} \end{bmatrix}\begin{bmatrix} -A_{\Omega_i}^{-1}A_{\Omega_i,\Gamma_i} \\ I \end{bmatrix}\underline{d}_{\Gamma_i} = \begin{bmatrix} A_{\Omega_i} & A_{\Omega_i,\Gamma_i} \\ A_{\Omega_i,\Gamma}^T & A_{\Gamma_i} \end{bmatrix}\begin{bmatrix} \underline{x}_i \\ \underline{d}_{\Gamma_i} \end{bmatrix} \tag{5 1 1 45}$$

Thus, by summing over each subdomain we obtain $S_h\underline{d} = \sum_{i=1}^{k} R_{\Gamma\,,h}^T S_{i h}\underline{d}_{\Gamma}$ The linear system of algebraic equations (5 1 1 44) is obtained from the definition of the discrete Harmonic operator $E_{i h}$ $(S^h(\Gamma_i))^2 \to (S_0^h(\Omega_i, \partial\Omega_i\backslash\Gamma_i))^2$, cf (5 1 1 6), and the Lax–Milgram lemma therefore guarantees the existence of a unique solution $\underline{x}_i \in \mathbb{R}^{2N_{\Omega_i}}$, $1 \le i \le k$ of system

$\mathcal{ALG}$ 1 Conjugate Gradient Algorithm $S_h \underline{u}_{\Gamma,h} = \underline{L}_h$

Determine an initial approximation $\underline{u}_{\Gamma,h}^{(0)}$,

$n \leftarrow 0$,

$\underline{e}_i^{(n)} \leftarrow -A_{\Omega_i}^{-1} A_{\Omega_i,\Gamma_i} R_{\Gamma_i,h} \underline{u}_{\Gamma,h}^{(n)}, \quad 1 \le i \le k$,

$\underline{z}^{(n)} \leftarrow \sum_{i=1}^{k} R_{\Gamma_i,h}^T (A_{\Gamma_i} R_{\Gamma_i,h} \underline{u}_{\Gamma,h}^{(n)} + A_{\Omega_i,\Gamma_i}^T \underline{e}_i^{(n)}) = S_h \underline{u}_{\Gamma,h}^{(n)}$,

$\underline{r}^{(n)} \leftarrow \underline{L}_h - S_h \underline{u}_{\Gamma,h}^{(n)} = \underline{L}_h - \underline{z}^{(n)}$,

$\underline{d}^{(n)} \leftarrow \underline{r}^{(n)}$,

While $n < n_{\max}$ and $\kappa(S_h)\,|(\underline{r}^{(n)}, \underline{r}^{(n)})|/|(\underline{L}_h, \underline{L}_h)| < \tau^2$

{

$\underline{e}_i^{(n)} \leftarrow -A_{\Omega_i}^{-1} A_{\Omega_i,\Gamma_i} R_{\Gamma_i,h} \underline{d}^{(n)}, \quad 1 \le i \le k$,

$\underline{z}^{(n)} \leftarrow \sum_{i=1}^{k} R_{\Gamma_i,h}^T (A_{\Gamma_i} R_{\Gamma_i,h} \underline{d}^{(n)} + A_{\Omega_i,\Gamma_i}^T \underline{e}_i^{(n)}) = S_h \underline{d}^{(n)}$,

$\alpha^{(n)} \leftarrow (\underline{r}^{(n)}, \underline{r}^{(n)})/(\underline{d}^{(n)}, \underline{z}^{(n)})$,

$\underline{u}_{\Gamma,h}^{(n+1)} \leftarrow \underline{u}_{\Gamma,h}^{(n)} + \alpha^{(n)} \underline{d}^{(n)}$,

$\underline{r}^{(n+1)} \leftarrow \underline{L}_h - S_h \underline{u}_{\Gamma,h}^{(n)} = \underline{r}^{(n)} - \alpha^{(n)} \underline{z}^{(n)}$,

$\beta^{(n+1)} \leftarrow (\underline{r}^{(n+1)}, \underline{r}^{(n+1)})/(\underline{r}^{(n)}, \underline{r}^{(n)})$,

$\underline{d}^{(n+1)} \leftarrow \underline{r}^{(n+1)} + \beta^{(n+1)} \underline{d}^{(n+1)}$,

$n \leftarrow n + 1$

}

(5 1 1 44) The conjugate gradient algorithm, as applied to the discretized interface system (5 1 1 39), is given in $\mathcal{ALG}$ 1

The rate at which the conjugate gradient iterations $\underline{u}_{\Gamma,h}^{(n)}$ converge to $\underline{u}_{\Gamma,h}$ as $n \to \infty$ will depend on the eigenvalue distribution of the Schur complement matrix S_h Indeed, the error bound (5 1 2) suggests that the condition number $\kappa(S_h)$ is the critical factor in such an approach However, for quasi–uniform triangulations $\mathcal{T}_h(\Omega)$, $h > 0$ of Ω, Ω a polygonal domain, it is known that, cf LETALLEC (1994),

$$\kappa(S_h) \le C H^{-2} \left[1 + \max\{H_i h_i^{-1} \mid 1 \le i \le k\}\right] \qquad (5\,1\,1\,46)$$

where $C > 0$ is a constant independent of h_i, H_i, h, H and

$$h_i \stackrel{\text{def}}{=} \max\{\text{diam}(\tau) \mid \tau \in \mathcal{T}_h(\Omega_i)\}, \qquad H_i \stackrel{\text{def}}{=} \text{diam}(\Omega_i), \qquad 1 \le i \le k \qquad (5\,1\,1\,47)$$

$$h \stackrel{\text{def}}{=} \max\{h_i \mid 1 \le i \le k\}, \qquad H \stackrel{\text{def}}{=} \max\{H_i \mid 1 \le i \le k\} \qquad (5\,1\,1\,48)$$

Thus, it is apparent from (5 1 1 46) that the condition number $\kappa(S_h)$ is of the order $O(H^{-2}(1+Hh^{-1}))$ as $h, H \to 0$ Therefore, the convergence factor $C(S_h)$ has the property

$$C(S_h) \stackrel{\text{def}}{=} \frac{\sqrt{\kappa(S_h)} - 1}{\sqrt{\kappa(S_h)} + 1} \nearrow 1 \quad (H, h \to 0) \qquad (5\,1\,1\,49)$$

and the error bound (5 1 2) reveals that the *rate of decay* of the error $\|\underline{u}_{\Gamma,h} - \underline{u}^{(n)}_{\Gamma h}\|_{S_h}$ decreases both rapidly and monotonically for an increasing number of subdomains, k, and decreasing mesh diameter, h Thus, we shall investigate ways to construct preconditioners $P_h^{-1} \in \mathbb{R}^{2N\,2N}$ such that (1) $\kappa(P_h^{-1}S_h) \ll \kappa(S_h)$, $H, h > 0$, (2) $\kappa(P_h^{-1}S_h)$ grows slowly as $H, h \to 0$ compared to $\kappa(S_h)$ and employ the preconditioned conjugate gradient algorithm The preconditioned conjugate gradient algorithm requires one to solve, at each iteration, a system of the form $P_h \underline{z} = \underline{r}$ for $\underline{z}, \underline{r} \in \mathbb{R}^{2N}$ and it is necessary, therefore, that this system is more easily solved than is $S_h \underline{z} = \underline{r}$ In the following sections preconditioning strategies are investigated which, in addition to the above properties, can be implemented by performing computations which are local to each subdomain, Ω_i, $1 \le i \le k$, and are therefore inherently parallel

5 2 The Neumann–Neumann Preconditioner

It has been demonstrated how finite element techniques can be applied to discretize the Steklov–Poincaré operators S_i, $1 \le i \le k$ thereby allowing one to approximate the interface problem (5 1 1 25) by the algebraic system of linear equations $S_h \underline{u}_{\Gamma,h} = \underline{L}_h$ where

$$S_h = \sum_{i=1}^{k} R^T_{\Gamma_i,h} S_{i,h} R_{\Gamma_i,h}, \qquad \underline{L}_h = \sum_{i=1}^{k} R^T_{\Gamma_i,h} \underline{L}_{i,h} \tag{5 2 1}$$

$$S_{i\,h} = A_{\Gamma_i} - A^T_{\Omega_i,\Gamma_i} A^{-1}_{\Omega} A_{\Omega_i,\Gamma_i}, \qquad \underline{L}_{i,h} = \underline{F}_{\Gamma_i} - A^T_{\Omega_i\,\Gamma_i} A^{-1}_{\Omega_i} \underline{F}_{\Omega_i} \tag{5 2 2}$$

It is apparent from Section 5 1 that in order to solve the discretized interface problem efficiently with the conjugate gradient approach it is necessary to employ a preconditioner Thus, we now introduce the preconditioner, $N_h^{-1} \in \mathbb{R}^{2N,2N}$, proposed by, among others, BOURGAT, GLOWINSKI, LETALLEC, & VIDRASCU (1989) and obtained by constructing weighted sums of the inverses, $S^{-1}_{i\,h}$, $1 \le i \le k$, of the Schur complement matrices $S_{i\,h}$, $1 \le i \le k$ We describe how the preconditioner is implemented, note its desirable features and assess the preconditioning properties of N_h^{-1} by examining an upper bound of the condition number $\kappa(N_h^{-1}S_h)$ provided in LETALLEC (1994)

If the decomposition (5 1 1 1) is constructed such that the vertices of the boundary, $\partial\Omega_i$, of each subdomain Ω_i, $1 \le i \le k$ belong to $\partial\Omega$ and the boundary conditions are such that the Steklov–Poincaré operators $S_{i,h}$ $(S^h(\Gamma_i))^2 \to \mathcal{BL}((S^h(\Gamma_i))^2, \mathbb{R})$, $1 \le i \le k$ are invertible then the preconditioner

$$P_h^{-1} \overset{\text{def}}{=} \sum_{i=1}^{k} (\alpha_i R^T_{\Gamma_i,h}) S^{-1}_{i,h} (\alpha_i R_{\Gamma_i,h}) \in \mathbb{R}^{2N,2N}, \qquad \sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i \ge 0,\ 1 \le i \le k \tag{5 2 3}$$

has the following property, cf LETALLEC (1994),

$$\kappa(P_h^{-1}S_h) \le C, \quad h > 0 \tag{5 2 4}$$

where $C > 0$ is a constant independent of $h > 0$ Indeed, if $k = 2$, (5 1 1 1) is a uniform decomposition of Ω and the triangulations $\mathcal{T}_h(\Omega_i)$, $1 \le i \le k$ are similar then, for appropriate boundary conditions and coefficients $a_{mnpq} \in \mathcal{L}_\infty(\Omega)$, $1 \le m,n,p,q \le 2$, $R^T_{\Gamma_1,h} S_{1,h} R_{\Gamma_1,h} = R^T_{\Gamma_2,h} S_{2,h} R_{\Gamma_2,h}$, $S_h = 2\, R^T_{\Gamma_i,h} S_{i,h} R_{\Gamma_i,h}$, $1 \le i \le 2$,

$$S_h^{-1} = \frac{1}{2} R^T_{\Gamma_i,h} S^{-1}_{i,h} R_{\Gamma_i,h} = \sum_{i=1}^{2} \frac{1}{4} R^T_{\Gamma_i,h} S^{-1}_{i,h} R_{\Gamma_i,h}, \quad i = 1,2 \tag{5 2 5}$$

Thus, with $\alpha_i \stackrel{\text{def}}{=} 1/2$, $i = 1,2$ in (5 2 3) we obtain $C = 1$ in (5 2 4) In general, however, $C > 1$, although the independence of the constant $C > 0$ from $h > 0$ suggests that the convergence factor $C(P_h^{-1} S_h)$ will not change significantly as $h \to 0$, cf (5 1 1 49) The task of determining the function $S^{-1}_{i,h} L_i \in (S^h(\Gamma_i))^2$ for $L_i \in \mathcal{BL}((S^h(\Gamma_i))^2, \mathbb{R})$, $S^{-1}_{i,h}\ \mathcal{BL}((S^h(\Gamma_i))^2, \mathbb{R}) \to (S^h(\Gamma_i))^2$ is equivalent to that of computing the product $S^{-1}_{i,h} \underline{L}_i \in \mathbb{R}^{2N_{\Gamma_i}}$, cf (5 1 1 29) Thus, from the definition of the Steklov–Poincare operators (5 1 1 17) we determine $S^{-1}_{i,h} L_i \in (S^h(\Gamma_i))^2$ as follows Find $\underline{z}_i \in (S^h_0(\Omega_i, \partial\Omega_{i,D}))^2$ such that

$$a_i(\underline{z}_i, \underline{v}) = \langle L_i, \underline{v}|_{\Gamma_i} \rangle, \quad \underline{v} \in (S^h_0(\Omega_i, \partial\Omega_{i,D}))^2 \tag{5 2 6}$$

then $S^{-1}_{i,h} L_i = \underline{z}_i|_{\Gamma_i}$ The equivalent system of algebraic equations obtained from this problem are then

$$\begin{bmatrix} A_{\Omega_i} & A_{\Omega_i,\Gamma_i} \\ A^T_{\Omega_i,\Gamma_i} & A_{\Gamma_i} \end{bmatrix} \begin{bmatrix} \underline{z}_{\Omega_i} \\ \underline{z}_{\Gamma_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{L}_{\Gamma_i} \end{bmatrix} \tag{5 2 7}$$

and $S^{-1}_{i,h} \underline{L}_i = \underline{z}_{\Gamma_i}$ The independence of the subproblems (5 2 6) allows one to implement the preconditioner, P_h^{-1}, using parallel computations and the conditioning property (5 2 4) ensures that the number of iterations required to achieve convergence will not rapidly increase if one employs more refined triangulations $\mathcal{T}_h(\Omega)$ or domain decompositions These properties are clearly desirable and motivate the generalization of the preconditioner P_h^{-1} to include general boundary conditions and decompositions (5 1 1 1) which, in particular, include interior crosspoints, i e , points $\underline{x}_c \in \text{int}(\Gamma)$ that are common to more than two distinct subdomains However, more general boundary conditions and decompositions allow the possibility that there exists a $p \in \mathbb{N}_k$ such that $\sigma(\partial\Omega_{p,D}) = 0$ and therefore a solution $\underline{z}_p \in (S^h(\Omega_p))^2$ of problem (5 2 6) exists and is unique, except for elements of $\mathcal{N}(A_{p,h})$, if, and only if, $L_p \in \mathcal{R}(S_{p,h})$

Thus, we define $\mathcal{S} \stackrel{\text{def}}{=} \{i \in \mathbb{N}_k \mid \sigma(\partial\Omega_{i,D}) = 0\}$ and for $i \in \mathbb{N}_k \backslash \mathcal{S}$ let $b_i \stackrel{\text{def}}{=} a_i$, cf (5 1 1 5), and for $i \in \mathcal{S}$ let $b_i \in \mathcal{BL}((S^h(\Omega_i))^2 \times (S^h(\Omega_i))^2, \mathbb{R})$ be some positive, symmetric bilinear form, i e , for $\underline{u}, \underline{v} \in (S^h(\Omega_i))^2$,

$$b_i(\underline{u}, \underline{v}) = b_i(\underline{v}, \underline{u}), \tag{5 2 8}$$

$$b_i(\underline{v}, \underline{v}) \ge 0, \quad b_i(\underline{v}, \underline{v}) = 0 \Leftrightarrow \underline{v} = 0 \tag{5 2 9}$$

which is equivalent with a_i on $(S^h(\Omega_i))^2 \backslash \mathcal{N}(A_{i,h})$, i e , there exists a constant $C_1 > 0$ which is independent of H, h such that

$$C_1\, b_i(\underline{v}, \underline{v}) \le a_i(\underline{v}, \underline{v}) \le b_i(\underline{v}, \underline{v}), \quad \underline{v} \in (S^h(\Omega_i))^2 \backslash \mathcal{N}(A_{i,h}) \tag{5 2 10}$$

and which, furthermore, satisfies the global equivalence property

$$C \sum_{i=1}^{k} b_i(\underline{v},\underline{v}) \le a(\underline{v},\underline{v}) \le \sum_{i=1}^{k} b_i(\underline{v},\underline{v}), \quad \underline{v} \in (S_0^h(\Omega,\partial\Omega_D))^2 \tag{5 2 11}$$

where $C > 0$ Let $\tilde{S}_{i\,h}$ $(S^h(\Gamma_i))^2 \to \mathcal{BL}((S^h(\Gamma_i))^2, \mathbb{R})$, $i \in \mathcal{S}$ be the discrete Steklov–Poincare operators associated with the bilinear forms $b_i \in \mathcal{BL}((S^h(\Omega_i))^2 \times (S^h(\Omega_i))^2, \mathbb{R})$, $i \in \mathcal{S}$, cf (5 1 1 17) then, following DEROECK & LETALLEC (1991), we define the preconditioner $N_h^{-1} \in \mathbb{R}^{2N,2N}$ as follows

$$N_h^{-1} \stackrel{\text{def}}{=} \sum_{i=1}^{k} R_{\Gamma_i,h}^T W_{i,h}^T B_{i,h}^{-1} W_{i,h} R_{\Gamma_i,h} \tag{5 2 12}$$

where, for $\sigma(\partial\Omega_{i,D}) > 0$, we define $B_{i,h}^{-1} \stackrel{\text{def}}{=} S_{i\,h}^{-1}$ and, for $\sigma(\partial\Omega_{i,D}) = 0$, we define $B_{i,h}^{-1} \stackrel{\text{def}}{=} \tilde{S}_{i,h}^{-1}$ The symmetric matrix $W_{i,h} \in \mathbb{R}^{2N_{\Gamma_i},2N_{\Gamma_i}}$ represents the weighting operator $W_{i,h}$ $(S^h(\Gamma_i))^2 \to (S^h(\Gamma_i))^2$ defined, for $w(i,r) \ge 0$, $1 \le r \le N_\Gamma$, $1 \le i \le k$, according to the relation

$$\underline{u} = \sum_{r=1}^{N_{\Gamma_i}} \underline{u}_r \psi_r^{(i)} \in (S^h(\Gamma_i))^2 \longmapsto W_{i,h}\underline{u} = \sum_{r=1}^{N_{\Gamma_i}} w(i,r)\, \underline{u}_r \psi_r^{(i)} \in (S^h(\Gamma_i))^2 \tag{5 2 13}$$

where $S^h(\Gamma_i) = \text{span}\{\psi_r^{(i)}\}_{r=1}^{N_{\Gamma_i}}$ and the weights $w(i,r)$, $1 \le r \le N_{\Gamma_i}$, $1 \le i \le k$ are chosen such that $W_{i,h}$, $1 \le i \le k$ form a partition of unity on Γ, i e , for $\underline{u} \in (S^h(\Gamma))^2$,

$$\sum_{i=1}^{k} (W_{i,h}\, \underline{u}|_{\Gamma_i})(\underline{x}) = \underline{u}(\underline{x}), \quad \underline{x} \in \Gamma \tag{5 2 14}$$

The operators $W_{i,h}$ $(S^h(\Gamma_i))^2 \to (S^h(\Gamma_i))^2$, $1 \le i \le k$ generalize the constant weights introduced in (5 2 3) because they allow one to weight each $(S^h(\Gamma_i))^2$ component of a function $\underline{u}_i \in (S^h(\Gamma_i))^2$, $1 \le i \le k$ differently and, in this way, one can define these operators such that $\kappa(N_h^{-1} S_h)$ is independent of the magnitude of any discontinuous changes in the coefficients a_{mnpq}, $1 \le m,n,p,q \le 2$ when they are piecewise continuous, cf Section 5 4 The partition of unity property (5 2 14) must, however, be satisfied, cf LETALLEC, DEROECK, & VIDRASCU (1991) Thus, for $L_i \in \mathcal{BL}((S^h(\Gamma_i))^2, \mathbb{R})$, $B_{i,h}^{-1} L_i = \underline{z}_i|_{\Gamma_i}$ where $\underline{z}_i \in (S^h(\Omega_i))^2$ is the solution of the Neumann problem Find $\underline{z}_i \in (S_0^h(\Omega_i, \partial\Omega_{i\,D}))^2$ such that

$$b_i(\underline{z}_i, \underline{v}) = \langle L_i, \underline{v}|_{\Gamma_i} \rangle, \quad \underline{v} \in (S_0^h(\Omega_i, \partial\Omega_{i\,D}))^2 \tag{5 2 15}$$

This problem can be represented in matrix form as follows

$$\begin{bmatrix} B_{\Omega_i} & B_{\Omega_i,\Gamma_i} \\ B_{\Omega_i,\Gamma_i}^T & B_{\Gamma_i} \end{bmatrix} \begin{bmatrix} \underline{z}_{\Omega_i} \\ \underline{z}_{\Gamma_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{L}_i \end{bmatrix} \tag{5 2 16}$$

with $B_{i\,h}^{-1} \underline{L}_i = \underline{z}_{\Gamma_i}$ In section 5 4 we shall employ, for $i \in \mathcal{S}$, the positive, symmetric bilinear forms $b_i \in \mathcal{BL}((H^1(\Omega_i))^2 \times (H^1(\Omega_i))^2, \mathbb{R})$ defined according to the relation

$$b_i(\underline{u},\underline{v}) \stackrel{\text{def}}{=} a_i(\underline{u},\underline{v}) + (\underline{u},\underline{v},(\mathcal{L}_2(\Omega_i))^2), \quad \underline{u},\underline{v} \in (H^1(\Omega_i))^2 \tag{5 2 17}$$

where $(\underline{u},\underline{v},(\mathcal{L}_2(\Omega_i))^2) \stackrel{\text{def}}{=} \int_\Omega \underline{u}(\underline{x})\ \ \underline{v}(\underline{x})\, d\underline{x}$ is the $(\mathcal{L}_2(\Omega_i))^2$ inner product The continuity of the mappings b_i, $i \in \mathcal{S}$ follow immediately from the Cauchy-Schwarz inequality and the property $a_i \in \mathcal{BL}((H^1(\Omega_i))^2 \times (H^1(\Omega_i))^2, \mathbb{R})$ while the $(H^1(\Omega_i))^2$–ellipticity is proved in the following Lemma

Lemma 5 2 *There exists a positive constant $\rho > 0$ such that*

$$b_i(\underline{v},\underline{v}) = a_i(\underline{v},\underline{v}) + (\underline{v},\underline{v},(\mathcal{L}_2(\Omega_i))^2) \geq \rho\,\|\underline{v},(H^1(\Omega_i))^2\|^2, \quad \underline{v} \in (H^1(\Omega_i))^2 \tag{5 2 18}$$

where $i \in \mathcal{S}$

Proof We first observe, cf BRENNER & RIDGWAY SCOTT (1994), that $(H^1(\Omega_i))^2$, $i \in \mathcal{S}$ can be written as a direct sum of closed subspaces as follows

$$(H^1(\Omega_i))^2 = \widehat{H}^1(\Omega_i) \oplus \mathcal{N}(A_i), \quad i \in \mathcal{S} \tag{5 2 19}$$

where, for $i \in \mathcal{S}$,

$$\widehat{H}^1(\Omega_i) = \left\{\underline{v} \in (H^1(\Omega_i))^2 \mid \int_{\Omega_i} \underline{v}(\underline{x})\,d\underline{x} = 0,\ \int_{\Omega_i} \mathrm{rot}\underline{v}(\underline{x})\,d\underline{x} = 0\right\} \tag{5 2 20}$$

$$\mathcal{N}(A_i) = \left\{\underline{v} \in (H^1(\Omega_i))^2 \mid \underline{v} = \underline{a} + R(r,\theta)\,\underline{x},\ \underline{a} \in \mathbb{R}^2, r \in \mathbb{R},\ \theta = (2n+1)\pi/2,\ n \in \mathcal{Z}\right\} \tag{5 2 21}$$

However, the continuity of the projection operators $P_1\ (H^1(\Omega_i))^2 \to \widehat{H}^1(\Omega_i), P_2\ (H^1(\Omega_i))^2 \to \mathcal{N}(A_i)$ suggests, cf BROWN & PAGE (1970), the existence of a constant $C > 0$ satisfying

$$C\,(\|P_1\underline{v},(H^1(\Omega_i))^2\| + \|P_2\underline{v},(H^1(\Omega_i))^2\|) \leq \|\underline{v},(H^1(\Omega_i))^2\|, \quad \underline{v} \in (H^1(\Omega_i))^2 \tag{5 2 22}$$

We now prove the result by Reductio ad Absurdum Assume that a constant $\rho > 0$ satisfying (5 2 18) does not exist, then, for $\rho = 1/n$, $n \in \mathbb{N}$ there must exist a $\underline{v}_n \in (H^1(\Omega_i))^2$ with the property

$$\|\underline{v}_n,(H^1(\Omega_i))^2\| = 1, \qquad b_i(\underline{v}_n,\underline{v}_n) < 1/n \tag{5 2 23}$$

It now follows from the definition of b_i, cf (5 2 17), relation (5 2 23) and the second Korn inequality, cf BRENNER & RIDGWAY SCOTT (1994), that there exists a $C_1 > 0$ such that, for $n \in \mathbb{N}$,

$$C_1\,\|P_1\underline{v}_n,(H^1(\Omega_i))^2\| \leq a_i(P_1\underline{v}_n,P_1\underline{v}_n) = a_i(\underline{v}_n,\underline{v}_n) \leq b_i(\underline{v}_n,\underline{v}_n) < 1/n \tag{5 2 24}$$

$$\Rightarrow \quad \|P_1\underline{v}_n,(H^1(\Omega_i))^2\| \to 0\ (n \to \infty) \tag{5 2 25}$$

However, it is apparent from (5 2 22) that $\{P_2\underline{v}_n\}_{n\geq 1}$ is a bounded sequence in the finite dimensional space $\mathcal{N}(A_i)$ $(\dim(\mathcal{N}(A_i)) = 3)$ and, thus, there exists a convergent subsequence $\{P_2\underline{v}_{n_j}\}_{j\geq 1}$ with limit $\underline{v} \in \mathcal{N}(A_i)$ Relations (5 2 23) then imply the contradictory conclusions $\|\underline{v},(H^1(\Omega_i))^2\| = 1$ and $\|\underline{v},(\mathcal{L}_2(\Omega_i))^2\| = 0$ ∎

The local and global spectral equivalence properties (5 2 10), (5 2 11) now follow immediately However, for uniform decompositions (5 1 1 1) and triangulations $\mathcal{T}_h(\Omega_i) \subset \mathcal{T}_h(\Omega)$, $1 \leq i \leq k$ it is demonstrated in LETALLEC & DEROECK (1991) that the preconditioner $N_h^{-1} \in \mathbb{R}^{2N,2N}$ has the property

$$\kappa(N_h^{-1}S_h) \leq C\,H^{-2}\Big[1 + \log(H/h)\Big]^2 \tag{5 2 26}$$

$\mathcal{ALG}$ 2 Conjugate Gradient Algorithm $N_h^{-1} S_h \underline{u}_{\Gamma,h} = N_h^{-1} \underline{L}_h$

Determine an initial approximation $\underline{u}_{\Gamma,h}^{(0)}$,

$n \leftarrow 0$,

$\underline{e}_i^{(n)} \leftarrow -A_{\Omega_i}^{-1} A_{\Omega_i,\Gamma_i} R_{\Gamma_i,h} \underline{u}_{\Gamma,h}^{(n)}, \quad 1 \leq i \leq k,$

$\underline{z}^{(n)} \leftarrow \sum_{i=1}^{k} R_{\Gamma_i,h}^T (A_{\Gamma_i} R_{\Gamma_i,h} \underline{u}_{\Gamma,h}^{(n)} + A_{\Omega_i,\Gamma_i}^T \underline{e}_i^{(n)}) = S_h \underline{u}_{\Gamma,h}^{(n)},$

$\underline{r}^{(n)} \leftarrow \underline{L}_h - \underline{z}^{(n)} = \underline{L}_h - S_h \underline{u}_{\Gamma,h}^{(n)},$

$\underline{r}_i^{(n)} \leftarrow R_{\Gamma_i,h} \underline{r}^{(n)}, \quad 1 \leq i \leq k,$

$\underline{w}_i^{(n)} \leftarrow W_{i,h} \underline{r}_i^{(n)}, \quad 1 \leq i \leq k,$

$\underline{\tilde{e}}_i^{(n)} \leftarrow -B_{\Omega_i}^{-1} B_{\Omega_i,\Gamma_i} \underline{w}_i^{(n)}, \quad 1 \leq i \leq k,$

$\underline{d}^{(n)} \leftarrow \sum_{i=1}^{k} R_{\Gamma_i,h}^T W_{i,h}^T (B_{\Gamma_i} \underline{w}_i^{(n)} + B_{\Omega_i,\Gamma_i}^T \underline{\tilde{e}}_i^{(n)}) = N_h^{-1} \underline{r}^{(n)},$

While $n < n_{\max}$ and $\kappa(N_h^{-1} S_h) \, |(N_h^{-1} \underline{r}^{(n)}, \underline{r}^{(n)})| / |(N_h^{-1} \underline{L}_h, \underline{L}_h)| < \tau^2$

{

$\underline{e}_i^{(n)} \leftarrow -A_{\Omega_i}^{-1} A_{\Omega_i,\Gamma_i} R_{\Gamma_i,h} \underline{d}^{(n)}, \quad 1 \leq i \leq k,$

$\underline{z}^{(n)} \leftarrow \sum_{i=1}^{k} R_{\Gamma_i,h}^T (A_{\Gamma_i} R_{\Gamma_i,h} \underline{u}_{\Gamma,h}^{(n)} + A_{\Omega_i,\Gamma_i}^T \underline{e}_i^{(n)}) = S_h \underline{d}^{(n)},$

$\alpha^{(n)} \leftarrow (\underline{r}^{(n)}, \underline{v}^{(n)}) / (\underline{d}^{(n)}, \underline{z}^{(n)}), \quad \underline{v}^{(0)} = \underline{d}^{(0)},$

$\underline{u}_{\Gamma,h}^{(n+1)} \leftarrow \underline{u}_{\Gamma,h}^{(n)} + \alpha^{(n)} \underline{d}^{(n)},$

$\underline{r}^{(n+1)} \leftarrow \underline{L}_h - S_h \underline{u}_{\Gamma,h}^{(n)} = \underline{r}^{(n)} - \alpha^{(n)} \underline{z}^{(n)},$

$\underline{r}_i^{(n+1)} \leftarrow R_{\Gamma_i,h} \underline{r}^{(n+1)}, \quad 1 \leq i \leq k,$

$\underline{w}_i^{(n+1)} \leftarrow W_{i,h} \underline{r}_i^{(n+1)}, \quad 1 \leq i \leq k,$

$\underline{\tilde{e}}_i^{(n+1)} \leftarrow -B_{\Omega_i}^{-1} B_{\Omega_i,\Gamma_i} \underline{w}_i^{(n+1)}, \quad 1 \leq i \leq k,$

$\underline{v}^{(n+1)} \leftarrow \sum_{i=1}^{k} R_{\Gamma_i,h}^T W_{i,h}^T (B_{\Gamma_i} \underline{w}_i^{(n)} + B_{\Omega_i,\Gamma_i}^T \underline{\tilde{e}}_i^{(n+1)}) = N_h^{-1} \underline{r}^{(n+1)},$

$\beta^{(n+1)} \leftarrow (\underline{r}^{(n+1)}, \underline{v}^{(n+1)}) / (\underline{r}^{(n)}, \underline{v}^{(n)}),$

$\underline{d}^{(n+1)} \leftarrow \underline{v}^{(n+1)} + \beta^{(n+1)} \underline{d}^{(n)},$

$n \leftarrow n + 1$

}

where $C > 0$ is a constant independent of H, h Thus, for fixed subdomain diameter H, $\kappa(N_h^{-1} S_h) = O(\log h^{-1})\,(h \to 0)$ and, observing that $\kappa(S_h) = O(h^{-1})\,(h \to 0)$, the conjugate gradient algorithm $\mathcal{ALG}$ 2 satisfies $\kappa(N_h^{-1} S_h) \ll \kappa(S_h)$, $|\kappa(S_h) - \kappa(N_h^{-1} S_h)| \to \infty$ as $h \to 0$, H fixed However, for $H/h \leq \rho$ (ρ independent of H, h), $\kappa(N_h^{-1} S_h) = O(H^{-2})\,(H \to 0)$ and $C(N_h^{-1} S_h)$ increases rapidly to 1 as $H \to 0$ thereby slowing the rate of convergence of $\mathcal{ALG}$ 2 until this approach becomes impractical Thus, the preconditioner N_h^{-1} provides improved asymptotic conditioning with respect to h but the practicality of this approach is restricted by the rapid growth of $C(N_h^{-1} S_h)$ as $H \to 0$ The conjugate gradient algorithm, as it applies to the interface system (5 1 1 39) with the preconditioner $N_h^{-1} \in \mathbb{R}^{2N,2N}$ defined in terms of the bilinear forms a_i, $i \in \mathbb{N}_k \backslash \mathcal{S}$ and b_i, $i \in \mathcal{S}$, cf (5 2 17), is given in $\mathcal{ALG}$ 2

5 3 The Coarse problem and the Balancing Preconditioner

The introduction of the positive bilinear forms b_i, $i \in \mathcal{S}$ allowed us to construct the preconditioner $N_h^{-1} \in \mathbb{R}^{2N,2N}$ when $P_h^{-1} \in \mathbb{R}^{2N\,2N}$, cf (5 2 3), was undefined and then to apply algorithm $\mathcal{ALG}$ 2 to linearly elastic problems with general boundary conditions using decompositions with interior crosspoints However, the resulting preconditioner, N_h^{-1}, is not uniquely defined because it depends on the choice of the b_i, $i \in \mathcal{S}$ and, as already observed in section 5 2, the $O(H^{-2})$ behaviour of the condition number $\kappa(N_h^{-1}S_h)$ causes algorithm $\mathcal{ALG}$ 2 to become impractical as $H \to 0$ We therefore demonstrate how to construct a preconditioner $M_h^{-1} \in \mathbb{R}^{2N,2N}$, for planar linear elastic problems, which employs a global problem of low dimension compared to (5 1 1 39) (the *coarse problem*) following a similar approach first proposed in MANDEL (1993) for scalar elliptic boundary value problems This approach is essentially a modification of the Neumann–Neumann preconditioning approach, cf $\mathcal{ALG}$ 2, and is devised such that the ambiguity of choice of the b_i, $i \in \mathcal{S}$ and the limiting $O(H^{-2})$ behaviour of $\kappa(N_h^{-1}S_h)$ are removed, i e , such that $\kappa(M_h^{-1}S_h) = O(1)$ $(H \to 0)$ where $H/h \leq \rho$ with $\rho > 0$ independent of H, h Indeed, the preconditioner will follow directly from the requirements that problems (5 2 6) are solvable and that M_h^{-1} does not depend on the choice of the solution of (5 2 6)

Thus, we begin by assuming that $\sigma(\partial\Omega_{i,D}) = 0$, $i \in \mathcal{S}$ and that problem (5 2 6) is solvable, i e , for $L \in \mathcal{BL}((S^h(\Gamma_i))^2, \mathbb{R})$, $\langle L, W_{i,h}\underline{v}\rangle = 0$, $\underline{v} \in \mathcal{N}(S_{i,h}) = \{\underline{v}|_{\Gamma_i} \mid \underline{v} \in \mathcal{N}(A_{i,h})\}$, $i \in \mathcal{S}$, then there exists a $\underline{z}_i \in (S^h(\Omega_i))^2$ such that

$$a_i(\underline{z}_i, \underline{v}) = \langle L, W_{i,h}\underline{v}|_{\Gamma_i}\rangle, \quad \underline{v} \in (S^h(\Omega_i))^2 \tag{5 3 1}$$

However, because $\mathcal{N}(A_{i\,h}) \neq \emptyset$, $i \in \mathcal{S}$, the solution $\underline{z}_i \in (S^h(\Omega_i))^2$ is not unique, i e , $\underline{z}_i + \underline{v}_i$ is also a solution of (5 3 1) for any $\underline{v}_i \in \mathcal{N}(A_{i,h})$ Therefore, we now describe how one can determine a unique solution of (5 3 1) in $\widehat{H}^1(\Omega_i)$ For problems of planar linear elasticity we observe that $\nu_i \stackrel{\text{def}}{=} \dim(\mathcal{N}(A_{i,h})) \in \{0,1,3\}$ and, for $1 \leq i \leq k$, $\mathcal{N}(A_{i\,h})$ includes all the rigid body motions of the linear operator $A_{i,h}$ If $\nu_i = 1$ then the only rigid body motions of $A_{i\,h}$ are rotations, i e , $\underline{a} = 0$ in (5 1 1 42), and we define $b_i \in \mathcal{BL}((H^1(\Omega_i))^2 \times (H^1(\Omega_i))^2, \mathbb{R})$ as follows

$$b_i(\underline{u}, \underline{v}) \stackrel{\text{def}}{=} a_i(\underline{u}, \underline{v}) + \int_{\Omega_i} \mathrm{rot}\underline{u}(\underline{x})\, d\underline{x} \int_{\Omega_i} \mathrm{rot}\underline{v}(\underline{x})\, d\underline{x}, \quad \underline{u}, \underline{v} \in (H^1(\Omega_i))^2 \tag{5 3 2}$$

where $\mathrm{rot}\underline{w} \stackrel{\text{def}}{=} \partial w_1/\partial x_2 - \partial w_2/\partial x_1$, $\underline{w} \in (H^1(\Omega_i))^2$ However, if $\nu_i = 3$ then $\mathcal{N}(A_{i\,h})$ contains all possible rigid body motions, cf (5 1 1 42), and we define the bilinear operator $b_i \in \mathcal{BL}((H^1(\Omega_i))^2 \times (H^1(\Omega_i))^2, \mathbb{R})$ as follows for $\underline{u}, \underline{v} \in (H^1(\Omega_i))^2$

$$b_i(\underline{u}, \underline{v}) \stackrel{\text{def}}{=} a_i(\underline{u}, \underline{v}) + \int_{\Omega_i} \underline{u}(\underline{x})\, d\underline{x} \int_{\Omega} \underline{v}(\underline{x})\, d\underline{x} + \int_{\Omega_i} \mathrm{rot}\underline{u}(\underline{x})\, d\underline{x} \int_{\Omega_i} \mathrm{rot}\underline{v}(\underline{x})\, d\underline{x}, \tag{5 3 3}$$

If $\nu_i = 1$ for some $i \in \mathcal{S}$ then $\partial\Omega_i \cap \partial\Omega_{i\,D}$ is a boundary point and, therefore, $\sigma(\partial\Omega_i \cap \partial\Omega_{i,D}) = 0$, $H_0^1(\Omega_i, \partial\Omega_{i,D}) = H^1(\Omega_i)$, and, at the continuous level, we therefore consider only the case $\nu_i = 3$, $i \in \mathcal{S}$ It is apparent from definition (5 3 3) that the bilinear forms b_i, $1 \le i \le k$ are symmetric and that, for $\underline{u} \in (H^1(\Omega_i))^2$,

$$b_i(\underline{u}, \underline{u}) = 0 \quad \Leftrightarrow \quad a_i(\underline{u}, \underline{u}) = 0, \int_\Omega \underline{u}(\underline{x})\, d\underline{x} = 0, \int_{\Omega_i} \mathrm{rot}\underline{u}(\underline{x})\, d\underline{x} = 0, \tag{5 3 4}$$

Thus, from the decomposition $(H^1(\Omega_i))^2 = \widehat{H}^1(\Omega_i) \oplus \mathcal{N}(A_i)$, $1 \le i \le k$ we can write, for any $\underline{u} \in (H^1(\Omega_i))^2$, $\underline{u} = \hat{\underline{u}} + \underline{a} + R(r,\theta)\,\underline{x}$ where $\hat{\underline{u}} \in \widehat{H}^1(\Omega_i)$, $\underline{a} \in \mathbb{R}^2$, $r \in \mathbb{R}$ and the positivity of the bilinear forms b_i, $i \in \mathcal{S}$ then follow from the observations that (1) $a_i(\underline{u}, \underline{u}) = a_i(\hat{\underline{u}}, \hat{\underline{u}}) = 0 \;\Leftrightarrow\; \hat{\underline{u}} = 0$, (2) $\int_{\Omega_i} \mathrm{rot}\underline{u}(\underline{x})\, d\underline{x} = \int_{\Omega_i} \mathrm{rot}[R(r,\theta)\,\underline{x}]\, d\underline{x} = 2r\,\mu(\Omega_i) = 0 \;\Leftrightarrow\; r = 0$, (3) $\int_{\Omega_i} \underline{u}(\underline{x})\, d\underline{x} = \underline{a}\,\mu(\Omega_i) = 0 \;\Leftrightarrow\; \underline{a} = 0$ Furthermore, we define the norm $|||\underline{u}, (H^1(\Omega_i))^2||| \stackrel{\text{def}}{=} \max(\|\hat{\underline{u}}, (H^1(\Omega_i))^2\|, \|\tilde{\underline{u}}, (H^1(\Omega_i))^2\|)$, $\underline{u} \in (H^1(\Omega_i))^2$ where $\underline{u} = \hat{\underline{u}} + \tilde{\underline{u}}$, $\hat{\underline{u}} \in \widehat{H}^1(\Omega_i)$, $\tilde{\underline{u}} \in \mathcal{N}(A_i)$ and deduce the $|||\bullet, (H^1(\Omega_i))^2|||$ continuity of the bilinear forms b_i, $i \in \mathcal{S}$ from the Cauchy–Schwarz inequality as follows, for $\underline{u}, \underline{v} \in (H^1(\Omega_i))^2$,

$$\begin{aligned}
|b_i(\underline{u}, \underline{v})| &\le |a_i(\hat{\underline{u}}, \hat{\underline{v}})| + \left| \int_{\Omega_i} \tilde{\underline{u}}(\underline{x})\, d\underline{x} \;\; \int_{\Omega_i} \tilde{\underline{v}}(\underline{x})\, d\underline{x} \right| + \left| \int_{\Omega_i} \mathrm{rot}\tilde{\underline{u}}(\underline{x})\, d\underline{x} \right| \; \left| \int_{\Omega_i} \mathrm{rot}\tilde{\underline{v}}(\underline{x})\, d\underline{x} \right| \\
&\le C\, \|\hat{\underline{u}}, (H^1(\Omega_i))^2\|\, \|\hat{\underline{v}}, (H^1(\Omega_i))^2\| + 2\mu(\Omega_i)\, \|\tilde{\underline{u}}, (\mathcal{L}_2(\Omega_i))^2\|\, \|\tilde{\underline{v}}, (\mathcal{L}_2(\Omega_i))^2\| \\
&\quad + \prod_{\underline{w} = \tilde{\underline{u}}\ \tilde{\underline{v}}} \left(\left| \int_{\Omega_i} \mathcal{D}^{(0,1)} w_1(\underline{x})\, d\underline{x} \right| + \left| \int_{\Omega_i} \mathcal{D}^{(1,0)} w_2(\underline{x})\, d\underline{x} \right| \right) \\
&\le C_1\, |||\underline{u}, (H^1(\Omega_i))^2|||\; |||\underline{v}, (H^1(\Omega_i))^2||| \\
&\quad + \mu(\Omega_i) \prod_{\underline{w} = \tilde{\underline{u}}\ \tilde{\underline{v}}} \left(\left[\int_{\Omega} |\mathcal{D}^{(0,1)} w_1(\underline{x})|^2\, d\underline{x} \right]^{1/2} + \left[\int_{\Omega_i} |\mathcal{D}^{(1,0)} w_2(\underline{x})|^2\, d\underline{x} \right]^{1/2} \right) \\
&\le C_2\, |||\underline{u}, (H^1(\Omega_i))^2|||\; |||\underline{v}, (H^1(\Omega_i))^2|||
\end{aligned} \tag{5 3 5}$$

where $\underline{u} = \hat{\underline{u}} + \tilde{\underline{u}}$, $\underline{v} = \hat{\underline{v}} + \tilde{\underline{v}}$, $\hat{\underline{u}}, \hat{\underline{v}} \in \widehat{H}^1(\Omega_i)$, $\tilde{\underline{u}}, \tilde{\underline{v}} \in \mathcal{N}(A_i)$ and $C_2 > 0$ depends on Ω_i alone The $(H^1(\Omega_i))^2$–ellipticity of the bilinear forms b_i, $i \in \mathcal{S}$ with respect to the $|||\bullet, (H^1(\Omega_i))^2|||$ norm follows immediately from Korn's second inequality, cf BRENNER & RIDGWAY SCOTT (1994), and the observation that all norms are equivalent on finite dimensional spaces, i e , for $\underline{u} \in (H^1(\Omega_i))^2$,

$$\begin{aligned}
b_i(\underline{u}, \underline{u}) &= a_i(\hat{\underline{u}}, \hat{\underline{u}}) + \left[\int_\Omega \tilde{\underline{u}}(\underline{x})\, d\underline{x} \right]^2 + \left[\int_{\Omega_i} \mathrm{rot}\tilde{\underline{u}}(\underline{x})\, d\underline{x} \right]^2 \\
&\ge \rho\, \|\hat{\underline{u}}, (H^1(\Omega_i))^2\|^2 + \gamma\, \|\tilde{\underline{u}}, (H^1(\Omega_i))^2\|^2 \ge \min(\rho, \gamma)\, |||\underline{u}, (H^1(\Omega_i))^2|||^2
\end{aligned} \tag{5 3 6}$$

where $\rho > 0$ is the ellipticity constant arising from Korn's second inequality

$$a_i(\hat{\underline{u}}, \hat{\underline{u}}) \ge \rho\, \|\hat{\underline{u}}, (H^1(\Omega_i))^2\|^2, \quad \hat{\underline{u}} \in \widehat{H}^1(\Omega_i) \tag{5 3 7}$$

and $\gamma > 0$ is the constant arising from the norm equivalence relation

$$\gamma \,\|\underline{\tilde{u}}, (H^1(\Omega_i))^2\|^2 \le \left[\int_{\Omega_i} \underline{\tilde{u}}(\underline{x})\, d\underline{x}\right]^2 + \left[\int_{\Omega_i} \mathrm{rot}\underline{\tilde{u}}(\underline{x})\, d\underline{x}\right]^2 \le \delta \,\|\underline{\tilde{u}}, (H^1(\Omega_i))^2\|^2, \quad \underline{\tilde{u}} \in \mathcal{N}(A_i) \tag{5 3 8}$$

Thus, b_i, $i \in \mathcal{S}$ satisfies the conditions of the Lax–Milgram lemma and defining $\underline{z}_i \in (S^h(\Omega_i))^2$ to be the unique solution of the problem Find $\underline{z}_i \in (S^h(\Omega_i))^2$ such that

$$b_i(\underline{z}_i, \underline{v}) = \langle L, W_{i,h}\underline{v}|_\Gamma \rangle, \quad \underline{v} \in (S^h(\Omega_i))^2 \tag{5 3 9}$$

it follows from (5 3 3) and (5 3 8) that (1) $\underline{v} \stackrel{\text{def}}{=} \underline{e}_r \in \mathcal{N}(A_{i\,h})$, $1 \le r \le 2 \Rightarrow \int_{\Omega_i} \underline{z}_i(\underline{x})\, d\underline{x} = 0$, and (2) $\underline{v} \stackrel{\text{def}}{=} R(1, \pi/2)\,\underline{x} \in \mathcal{N}(A_{i,h}) \Rightarrow \int_{\Omega_i} \mathrm{rot}\underline{z}_i(\underline{x})\, d\underline{x} = 0$ and, thus, $\underline{z}_i \in \widehat{H}^1(\Omega_i)$ is also a solution of problem (5 3 1)

Let $B_{i,h} \in \mathbb{R}^{2N_{\Gamma_i}, 2N_{\Gamma_i}}$, $1 \le i \le k$ be the matrices representing the Steklov–Poincare operators $\tilde{S}_{i,h}$, $1 \le i \le k$ associated with the bilinear forms b_i, $1 \le i \le k$ ($b_i = a_i$, $i \in \mathbb{N}_k \backslash \mathcal{S}$) in the sense of (5 1 1 29) and, with $\mathcal{N}_h \stackrel{\text{def}}{=} \prod_{i=1}^k \mathcal{N}(S_{i,h}) \subset \prod_{i=1}^k \mathbb{R}^{2N_{\Gamma_i}}$, $\nu \stackrel{\text{def}}{=} \dim(\mathcal{N}_h) = \sum_{i=1}^k \nu_i$, define, for $\underline{L} \in \mathbb{R}^{2N}$,

$$P_h^{-1}(\underline{z})\underline{L} \stackrel{\text{def}}{=} \sum_{i=1}^k R^T_{\Gamma_i,h} W^T_{i,h} (B^{-1}_{i,h} W_{i\,h} R_{\Gamma_i,h} \underline{L} + \underline{z}_i), \quad \underline{z} \stackrel{\text{def}}{=} \prod_{i=1}^k \underline{z}_i \in \mathcal{N}_h \tag{5 3 10}$$

where we have assumed that $W_{i,h} R_{\Gamma_i,h}\underline{L} \in \mathcal{R}(S_{i,h})$, $1 \le i \le k$ The preconditioner $M_h^{-1} \in \mathbb{R}^{2N\,2N}$ is then obtained by selecting $\underline{z} = \underline{z}^c$ in (5 3 4), i e , $M_h^{-1} \stackrel{\text{def}}{=} P_h^{-1}(\underline{z}^c)$, where $\underline{z}^c \in \mathcal{N}_h$ is defined to be the unique solution of the *coarse* variational problem Find $\underline{z}^c \in \mathcal{N}_h$ such that

$$J(\underline{z}^c) = \min\{J(\underline{z}) \mid \underline{z} \in \mathcal{N}_h\} \tag{5 3 11}$$

where, for $\underline{z} \in \mathcal{N}_h$,

$$J(\underline{z}) \stackrel{\text{def}}{=} ((P_h^{-1}(\underline{z}) - S_h^{-1})\underline{L}, (P_h^{-1}(\underline{z}) - S_h^{-1})\underline{L})_{S_h} \tag{5 3 12}$$

Thus, M_h^{-1} is obtained by modifying the local solutions of problems (5 2 6) with rigid body motions, i e , elements of $\mathcal{N}(S_{i,h})$, $1 \le i \le k$ such that $M_h^{-1} - S_h^{-1}$ is a minimum with respect to the energy norm in (5 3 12) Indeed, for $\Lambda(\underline{z}) \stackrel{\text{def}}{=} \sum_{i=1}^k R^T_{\Gamma_i\,h} W^T_{i,h} \underline{z}_i$, $\underline{z} \in \mathcal{N}_h$, it follows that

$$\begin{aligned} J(\underline{z}) &= ((N_h^{-1} + \Lambda(\underline{z}) - S_h^{-1})\underline{L}, (N_h^{-1} + \Lambda(\underline{z}) - S_h^{-1})\underline{L})_{S_h} \\ &= (\Lambda(\underline{z})\underline{L}, \Lambda(\underline{z})\underline{L})_{S_h} + 2((N_h^{-1} - S_h^{-1})\underline{L}, \Lambda(\underline{z})\underline{L})_{S_h} + ((N_h^{-1} - S_h^{-1})\underline{L}, (N_h^{-1} - S_h^{-1})\underline{L})_{S_h} \end{aligned} \tag{5 3 13}$$

and, therefore, J is a minimum at $\underline{z}^c \in \mathcal{N}_h$ if, and only if,

$$J^{(1)}[\underline{z}^c, \underline{z}] \stackrel{\text{def}}{=} \frac{\partial J}{\partial \tau}(\underline{z}^c + \tau \underline{z})\Big|_{\tau=0} = 0, \quad \underline{z} \in \mathcal{N}_h \tag{5 3 14}$$

i e , $\underline{z}^c \in \mathcal{N}_h$ is the unique solution of the problem Find $\underline{z}^c \in \mathcal{N}_h$ such that

$$\sum_{i=1}^k (R^T_{\Gamma_i,h} W^T_{i,h} \underline{z}^c_i, R^T_{\Gamma_j,h} W^T_{j,h} \underline{z}_j)_{S_h} = -((N_h^{-1} - S_h^{-1})\underline{L}, R^T_{\Gamma_j\,h} W^T_{j,h} \underline{z}_j)_{S_h}, \quad \prod_{j=1}^k \underline{z}_j \in \mathcal{N}_h \tag{5 3 15}$$

where $N_h^{-1} \in \mathbb{R}^{2N,2N}$ is the preconditioner defined in Section 5 2 for the bilinear forms defined in (5 3 2), (5 3 3) It is necessary, however, to compute at each step of the conjugate gradient algorithm, the product $M_h^{-1}\underline{r}^{(k)}$, $k \in \mathbb{N}$ where $\underline{r}^{(k)} \stackrel{\text{def}}{=} \underline{L} - S_h\underline{u}_{\Gamma,h}^{(k)}$ However, according to the definition of the preconditioner $M_h^{-1} \in \mathbb{R}^{2N\,2N}$, the product $M_h^{-1}\underline{r}^{(k)}$ is only defined for $W_{i\,h}R_{\Gamma_i,h}\underline{r}^{(k)} \in \mathcal{R}(S_{i\,h})$, i e , if $(W_{i,h}R_{\Gamma_i,h}\underline{r}^{(k)}, \underline{v}) = 0$, $\underline{v} \in \mathcal{N}(S_{i,h})$ Thus, we define the initial approximation, $\underline{u}_{\Gamma,h}^{(0)}$, as follows

$$\underline{u}_{\Gamma,h}^{(0)} \stackrel{\text{def}}{=} \Lambda(\underline{\theta}^c) = \sum_{i=1}^{k} R_{\Gamma_i\,h}^T W_{i,h}^T \underline{\theta}_i^c \tag{5 3 16}$$

where $\underline{\theta}^c \in \mathcal{N}_h$ is defined to be the solution of the problem Find $\underline{\theta}^c \in \mathcal{N}_h$ such that

$$(\underline{L} - S_h\underline{u}_{\Gamma,h}^{(0)}, R_{\Gamma_i,h}^T W_{i,h}^T \underline{v}) = 0, \quad \underline{v} \in \mathcal{N}_h \tag{5 3 17}$$

The property $W_{i,h}R_{\Gamma_i,h}\underline{L} \in \mathcal{R}(S_{i,h}) \;\Leftrightarrow\; (W_{i,h}R_{\Gamma_i,h}\underline{L}, \underline{v}) = 0$, $\underline{v} \in \mathcal{N}(S_{i,h})$, $1 \le i \le k$ then implies that the right hand side of the coarse problem (5 3 15) can be rewritten as follows

$$-((N_h^{-1} - S_h^{-1})\underline{L}, R_{\Gamma_j,h}^T W_{j\,h}^T \underline{z}_j)_{S_h} = -(S_h N_h^{-1}\underline{L}, R_{\Gamma_j,h}^T W_{j,h}^T \underline{z}_j) \tag{5 3 18}$$

The choice (5 3 16) of $\underline{u}_{\Gamma,h}^{(0)}$ ensures that $W_{i,h}R_{\Gamma_i,h}\underline{r}^{(n)} \in \mathcal{R}(S_{i,h})$, $n \ge 1$, $1 \le i \le k$ where $\underline{r}^{(n)} = \underline{L} - S_h\underline{u}_{\Gamma,h}^{(n)}$ This is established inductively as follows If $W_{i\,h}R_{\Gamma_i,h}\underline{r}^{(m)}$, $W_{i,h}R_{\Gamma_i,h}S_h\underline{d}^{(m)} \in \mathcal{R}(S_{i\,h})$, $1 \le i \le k$ for $m \le n$ and some $n \in \mathbb{N}$ then, observing that $\underline{r}^{(n+1)} = \underline{r}^{(n)} - \alpha^{(n)}S_h\underline{d}^{(n)}$, it follows that

$$\begin{aligned} S_h\underline{d}^{(n)} &= S_h\underline{v}^{(n)} + \beta^{(n)}S_h\underline{d}^{(n-1)} \\ &= S_h M_h^{-1}\underline{r}^{(n)} + \beta^{(n)}S_h\underline{d}^{(n-1)} \\ &= S_h \sum_{i=1}^{k} R_{\Gamma_i,h}^T W_{i\,h}^T (B_{i,h}^{-1} W_{i,h} R_{\Gamma_i\,h}\underline{r}^{(n)} + \underline{z}_i^c) + \beta^{(n)}S_h\underline{d}^{(n-1)} \end{aligned} \tag{5 3 19}$$

However, because $\underline{z}^c \in \mathcal{N}_h$ is determined such that $W_{i\,h}R_{\Gamma_i\,h}S_h\underline{v}^{(n)} \in \mathcal{R}(S_{i,h})$, $1 \le i \le k$, it follows that $W_{i,h}R_{\Gamma_i\,h}S_h\underline{d}^{(n)} \in \mathcal{R}(S_{i,h})$, $1 \le i \le k$ and, thus, $W_{i,h}R_{\Gamma_i\,h}\underline{r}^{(n+1)} \in \mathcal{R}(S_{i,h})$, $1 \le i \le k$ The property then follows immediately from the observation that, by the choice of $\underline{u}_{\Gamma,h}^{(0)}$, $W_{i,h}R_{\Gamma_i,h}\underline{r}^{(0)} \in \mathcal{R}(S_{i,h})$, $1 \le i \le k$ and $\underline{d}^{(-1)} = 0$

We observe that, defining $N_{i,h} \stackrel{\text{def}}{=} [\underline{n}_1^{(i)}, \quad , \underline{n}_{\nu_i}^{(i)}] \in \mathbb{R}^{2N_{\Gamma_i},\nu_i}$ where $\mathcal{R}(N_{i\,h}) = \mathcal{N}(S_{i,h})$ and writing $\underline{z}_i^c = N_{i,h}\underline{\lambda}_i^c$, $\underline{\lambda}_i^c \in \mathbb{R}^{\nu_i}$, $1 \le i \le k$, the matrix, $B \in \mathbb{R}^{\nu,\nu}$, and vector, $\underline{X} \in \mathbb{R}^{\nu}$, of the coarse problem (5 3 15) can be determined in block form as follows, $1 \le i, j \le k$,

$$B_{i,j} = N_{j\,h}^T W_{j\,h} R_{\Gamma_j,h} S_h R_{\Gamma_i,h}^T W_{i\,h}^T N_{i\,h} \tag{5 3 20}$$

$$= \sum_{p=1}^{k} N_{j\,h}^T W_{j\,h} R_{\Gamma_j,h} R_{\Gamma_p\,h}^T S_{p\,h} R_{\Gamma_p,h} R_{\Gamma_i\,h}^T W_{i\,h}^T N_{i,h} \in \mathbb{R}^{\nu_j,\nu_i} \tag{5 3 21}$$

$$\underline{X}_j = -\sum_{i,p=1}^{k} N_{j\,h}^T W_{j,h} R_{\Gamma_j,h} R_{\Gamma_p,h}^T S_{p,h} R_{\Gamma_p\,h} R_{\Gamma_i\,h}^T W_{i,h}^T B_{i,h}^{-1} W_{i,h} R_{\Gamma_i\,h}\underline{L} \in \mathbb{R}^{\nu_j} \tag{5 3 22}$$

and, therefore,

$$(B_{i,j})_{rs} = B_{kl}, \quad (\underline{X}_i)_r = (\underline{X})_k, \quad k = \sum_{m=1}^{i-1} \nu_m + r,\ l = \sum_{n=1}^{j-1} \nu_n + s,\ r \in \mathbb{N}_{\nu_j},\ s \in \mathbb{N}_{\nu_i},\ i,j \in \mathbb{N}_k$$

To determine the respective matrix and vector $B \in \mathbb{R}^{\nu\,\nu}$, $\underline{X} \in \mathbb{R}^{\nu}$ it is necessary to compute, as described above, the products $S_{r\,h}\underline{v}$, $B_{s\,h}^{-1}\underline{w}$ for some $r, s \in \mathbb{N}_k$, $\underline{v}, \underline{w} \in \mathbb{R}^{2N_{\Gamma_i}}$ However, because $R_{\Gamma_i,h}R_{\Gamma_p,h}^T \neq 0 \Leftrightarrow \overline{\Omega}_i \cap \overline{\Omega}_p \neq \emptyset$, $B, \underline{X}$ can be computed efficiently and, furthermore, if $\sigma(\partial\Omega_{i,D}) > 0$ then the blocks corresponding to subdomain Ω_i can be neglected since $\mathcal{N}(S_{i\,h}) = \emptyset$ The modified algorithm is presented in $\mathcal{ALG}$ 3

5 3 1 Condition Number bound

The distribution, $\sigma(M_h^{-1}S_h)$, of the eigenvalues of the preconditioned Schur complement matrix $M_h^{-1}S_h$, $h > 0$ is fundamentally important in our approach because it determines how rapidly the iterations produced by the conjugate gradient algorithm converge, cf (5 1 2) Clearly, the spectrum $\sigma(M_h^{-1}S_h)$ is affected by, for example, the shape regularity of the elements of the mesh $\mathcal{T}_h(\Omega)$, the mesh diameter $h > 0$, the shape regularity of the subdomains Ω_i, $1 \leq i \leq k$ in the decomposition (5 1 1 2), the variation and regularity of the coefficients a_{ijkl}, $1 \leq i, j, k, l \leq 2$, and the magnitude of the discontinuities α_i, $1 \leq i \leq k$, cf definition 5 2 However, following the analysis performed by BREZINA & MANDEL (1993), BRAMBLE, PASCIAK, & SCHATZ (1986) for scalar elliptic boundary value problems, we demonstrate that, for systems of elliptic equations with irregular coefficients, one can obtain the bound $\kappa(M_h^{-1}S_h) \leq C[1 + \log(H/h)]^2$ where $C > 0$ is independent of h, H and the jumps α_i, $1 \leq i \leq k$ by appropriately constructing the weight matrices $W_{i,h}$, $1 \leq i \leq k$ We begin with some definitions

Definition 5 1 *Let $\mathcal{V}(\Gamma)$ be the set of vertices of $\partial\Omega_i$, $1 \leq i \leq k$ which lie on the interface Γ and let $\underline{v}_1 \to \underline{v}_2$ be the straight line connecting vertex $\underline{v}_1 \in \mathcal{V}(\Gamma)$ to vertex $\underline{v}_2 \in \mathcal{V}(\Gamma)$ Then we define*

$$\mathcal{G}(\Gamma) \stackrel{\text{def}}{=} \{\gamma \subset \Gamma \mid \gamma \in \mathcal{V}(\Gamma) \text{ or } \gamma \cap \mathcal{V}(\Gamma) = \emptyset,\ \gamma = \operatorname{int}(\underline{v}_1 \to \underline{v}_2) \text{ for some } \underline{v}_1, \underline{v}_2 \in \mathcal{V}(\Gamma)\} \tag{5 3 23}$$

and, for $\gamma \subset \Gamma$, we define the boolean matrix $I_\gamma \in \mathbb{R}^{2N,2N}$ in terms of its 2×2 block entries $(I_\gamma)_{r\,s} \in \mathbb{R}^{2,2}$, $1 \leq r, s \leq N$ as follows

$$(I_\gamma)_{r,s} \stackrel{\text{def}}{=} \begin{cases} \Lambda_{r,s}, & \text{if the } \mathcal{T}_h(\Omega) \text{ node } \underline{x}_r \in \gamma \\ 0, & \text{if the } \mathcal{T}_h(\Omega) \text{ node } \underline{x}_r \notin \gamma \end{cases}, \quad 1 \leq r, s \leq N \tag{5 3 24}$$

where $\Lambda_{r\,s} \stackrel{\text{def}}{=} \delta_{rs}I \in \mathbb{R}^{2,2}$ and a point $\underline{x} \in \Gamma$ is defined to be a node of the finite element triangulation $\mathcal{T}_h(\Omega)$ if it is a vertex of some element $\tau \in \mathcal{T}_h(\Omega)$ Finally, we define the boolean matrix $I_\gamma^{ji} \stackrel{\text{def}}{=} R_{\Gamma_j,h}I_\gamma R_{\Gamma_i,h}^T \in \mathbb{R}^{2N_{\Gamma_j}\,2N_{\Gamma_i}}$ ∎

Thus, $\mathcal{G}(\Gamma)$ contains the vertices of the subdomain boundaries and the interiors of the straight lines in Γ which connect them and, for $\gamma \in \mathcal{G}(\Gamma)$, the matrices $I_\gamma \in \mathbb{R}^{2N,2N}$ map vectors

$\mathcal{ALG}$ 3 Conjugate Gradient Algorithm $M_h^{-1} S_h \underline{u}_{\Gamma,h} = M_h^{-1}\underline{L}_h$

$$H_{p,i} \leftarrow R_{\Gamma_p,h} R^T_{\Gamma_i,h} W^T_{i,h} N_{i,h}, \quad 1 \le i,p \le k,$$
$$E_{p,i} \leftarrow -A^{-1}_{\Omega_p} A_{\Omega_p,\Gamma_p} H_{p,i}, \quad 1 \le i,p \le k,$$
$$B_{ij} \leftarrow \textstyle\sum_{p=1}^{k} H^T_{p,j}(A_{\Gamma_p} H_{p,i} + A_{\Omega_p,\Gamma_p} E_{p,i}), \quad 1 \le i,j \le k,$$
$$\underline{X}_i \leftarrow N^T_{i,h} W_{i,h} R_{\Gamma_i,h} \underline{L}_h, \quad 1 \le i \le k,$$
$$n \leftarrow 0, \quad \underline{u}^{(n)}_{\Gamma,h} \leftarrow \Lambda(B^{-1}\underline{X}),$$

$$\underline{e}^{(n)}_i \leftarrow -A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i} R_{\Gamma_i,h} \underline{u}^{(n)}_{\Gamma,h}, \quad 1 \le i \le k,$$
$$\underline{z}^{(n)} \leftarrow \textstyle\sum_{i=1}^{k} R^T_{\Gamma_i,h}(A_{\Gamma_i} R_{\Gamma_i,h} \underline{u}^{(n)}_{\Gamma,h} + A^T_{\Omega_i,\Gamma_i} \underline{e}^{(n)}_i) = S_h \underline{u}^{(n)}_{\Gamma,h},$$

$$\underline{r}^{(n)} \leftarrow \underline{L}_h - \underline{z}^{(n)} = \underline{L}_h - S_h \underline{u}^{(n)}_{\Gamma,h},$$
$$\underline{r}^{(n)}_i \leftarrow R_{\Gamma_i,h} \underline{r}^{(n)}, \quad 1 \le i \le k,$$
$$\underline{w}^{(n)}_i \leftarrow W_{i,h} \underline{r}^{(n)}_i, \quad 1 \le i \le k,$$
$$\underline{\tilde{e}}^{(n)}_i \leftarrow -B^{-1}_{\Omega_i} B_{\Omega_i,\Gamma_i} \underline{w}^{(n)}_i, \quad 1 \le i \le k,$$
$$\underline{s}^{(n)} \leftarrow \textstyle\sum_{i=1}^{k} R^T_{\Gamma_i,h} W^T_{i,h}(B_{\Gamma_i} \underline{w}^{(n)}_i + B^T_{\Omega_i,\Gamma_i} \underline{\tilde{e}}^{(n)}_i) = N_h^{-1} \underline{r}^{(n)},$$
$$\underline{X}^{(n)}_i \leftarrow -(\underline{s}^{(n)}, R^T_{\Gamma_i,h} W^T_{i,h} N_{i,h})_{S_h}, \quad 1 \le i \le k,$$
$$\underline{z}^c \leftarrow B^{-1}\underline{X}^{(n)}, \quad \underline{d}^{(n)} \leftarrow \underline{s}^{(n)} + \Lambda(\underline{z}^c) = M_h^{-1}\underline{r}^{(n)},$$

While $n < n_{\max}$ and $\kappa(M_h^{-1}S_h)\,|(M_h^{-1}\underline{r}^{(n)},\underline{r}^{(n)})|/|(M_h^{-1}\underline{L}_h,\underline{L}_h)| < \tau^2$

{

$$\underline{e}^{(n)}_i \leftarrow -A^{-1}_{\Omega_i} A_{\Omega_i,\Gamma_i} R_{\Gamma_i,h} \underline{d}^{(n)}, \quad 1 \le i \le k,$$
$$\underline{z}^{(n)} \leftarrow \textstyle\sum_{i=1}^{k} R^T_{\Gamma_i,h}(A_{\Gamma_i} R_{\Gamma_i,h} \underline{u}^{(n)}_{\Gamma,h} + A^T_{\Omega_i,\Gamma_i} \underline{e}^{(n)}_i) = S_h \underline{d}^{(n)},$$

$$\alpha^{(n)} \leftarrow (\underline{r}^{(n)}, \underline{v}^{(n)})/(\underline{d}^{(n)}, \underline{z}^{(n)}), \quad \underline{v}^{(0)} = \underline{d}^{(0)},$$
$$\underline{u}^{(n+1)}_{\Gamma,h} \leftarrow \underline{u}^{(n)}_{\Gamma,h} + \alpha^{(n)} \underline{d}^{(n)},$$

$$\underline{r}^{(n+1)} \leftarrow \underline{L}_h - S_h \underline{u}^{(n)}_{\Gamma,h} = \underline{r}^{(n)} - \alpha^{(n)} \underline{z}^{(n)},$$
$$\underline{r}^{(n+1)}_i \leftarrow R_{\Gamma_i,h} \underline{r}^{(n+1)}, \quad 1 \le i \le k,$$
$$\underline{w}^{(n+1)}_i \leftarrow W_{i,h} \underline{r}^{(n+1)}_i, \quad 1 \le i \le k,$$
$$\underline{\tilde{e}}^{(n+1)}_i \leftarrow -B^{-1}_{\Omega_i} B_{\Omega_i,\Gamma_i} \underline{w}^{(n+1)}_i, \quad 1 \le i \le k,$$
$$\underline{s}^{(n+1)} \leftarrow \textstyle\sum_{i=1}^{k} R^T_{\Gamma_i,h} W^T_{i,h}(B_{\Gamma_i} \underline{w}^{(n)}_i + B^T_{\Omega_i,\Gamma_i} \underline{\tilde{e}}^{(n+1)}_i) = N_h^{-1} \underline{r}^{(n+1)},$$
$$\underline{X}^{(n+1)}_i \leftarrow -(\underline{s}^{(n+1)}, R^T_{\Gamma_i,h} W^T_{i,h} N_{i,h})_{S_h}, \quad 1 \le i \le k,$$
$$\underline{z}^c \leftarrow B^{-1}\underline{X}^{(n+1)}, \quad \underline{v}^{(n+1)} \leftarrow \underline{s}^{(n+1)} + \Lambda(\underline{z}^c) = M_h^{-1}\underline{r}^{(n+1)},$$

$$\beta^{(n+1)} \leftarrow (\underline{r}^{(n+1)}, \underline{v}^{(n+1)})/(\underline{r}^{(n)}, \underline{v}^{(n)}),$$
$$\underline{d}^{(n+1)} \leftarrow \underline{v}^{(n+1)} + \beta^{(n+1)} \underline{d}^{(n)},$$
$$n \leftarrow n + 1$$

}

$\underline{u} \in \mathbb{R}^{2N} \mapsto \underline{u}_\gamma \in \mathbb{R}^{2N}$ where $\underline{u}_\gamma$ differs from $\underline{u}$ only in that those entries which do not correspond to degrees of freedom of $\mathcal{T}_h(\Omega)$ on γ are zero Some elementary properties of the matrices $I_\gamma \in \mathbb{R}^{2N,2N}$, $\gamma \in \mathcal{G}(\Gamma)$ are provided in Lemma 5 1 below

Lemma 5 1 *Let $\gamma \in \mathcal{G}(\Gamma)$ then, for $1 \le i, j \le k$,*

$$\gamma \subset \partial\Omega_i \cap \partial\Omega_j \iff I_\gamma^{ji} \neq 0, \qquad \gamma \subset \partial\Omega_i \iff I_\gamma^{ii} \neq 0 \tag{5 3 25}$$

$$\sum_{\gamma\in\mathcal{G}(\Gamma)} I_\gamma = I_\Gamma \in \mathbb{R}^{2N,2N}, \quad \sum_{\gamma\in\mathcal{G}(\Gamma)} I_\gamma^{ji} = R_{\Gamma_j,h} R_{\Gamma_i,h}^T, \quad I_\gamma^{ji} = I_\gamma^{ji} I_\gamma^{ii} \tag{5 3 26}$$

and, furthermore, $\Gamma = \cup_{\gamma\in\mathcal{G}(\Gamma)}\gamma$

Proof Let $\gamma \subset \partial\Omega_i \cap \partial\Omega_j$ and, for $q \in \{1, \ldots, N\}$, let the $\mathcal{T}_h(\Omega)$ node $\underline{x}_q \in \mathcal{G}(\Gamma)$ be a vertex belonging to γ then

$$\begin{aligned}(I_\gamma^{ji})_{q_j,q_i} = (R_{\Gamma_j,h} I_\gamma R_{\Gamma_i,h}^T)_{q_j\, q_i} &= \sum_{m=1}^{N} (R_{\Gamma_j\, h})_{q_j,m} \sum_{p=1}^{N} (I_\gamma)_{m,p} (R_{\Gamma_i,h}^T)_{p,q_i} \\ &= \sum_{m=1}^{N} (R_{\Gamma_j,h})_{q_j,m} (I_\gamma)_{m,m} (R_{\Gamma_i\, h}^T)_{m,q_i}\end{aligned} \tag{5 3 27}$$

where $q_r = G_r^{-1}(q)$, $r = i, j$ However, because $(R_{\Gamma_j,h})_{q_j,q}(I_\gamma)_{q\, q}(R_{\Gamma_i\, h}^T)_{q\, q_i} = I \in \mathbb{R}^{2,2}$ it is clear that sum (5 3 27) is positive and, therefore, $I_\gamma^{ji} \neq 0$ The second relation in (5 3 25) follows similarly The final relation in (5 3 26) can be demonstrated as follows assume $\gamma \subset \partial\Omega_i \cap \partial\Omega_j$ then

$$\begin{aligned}I_\gamma^{ji} I_\gamma^{ii} &= R_{\Gamma_j,h} I_\gamma R_{\Gamma_i,h}^T R_{\Gamma_i,h} I_\gamma R_{\Gamma_i,h}^T \\ &= R_{\Gamma_j,h} I_\gamma I_{\Gamma_i} I_\gamma R_{\Gamma_i,h}^T \\ &= R_{\Gamma_j,h} I_\gamma R_{\Gamma_i,h}^T = I_\gamma^{ji}\end{aligned} \tag{5 3 28}$$

The first relation in (5 3 26) follows immediately from the definition (5 3 24) while the second is clear from the relations

$$\sum_{\gamma\in\mathcal{G}(\Gamma)} I_\gamma^{ji} = \sum_{\gamma\in\mathcal{G}(\Gamma)} R_{\Gamma_j,h} I_\gamma R_{\Gamma_i,h}^T = R_{\Gamma_j,h}\Big(\sum_{\gamma\in\mathcal{G}(\Gamma)} I_\gamma\Big) R_{\Gamma_i\, h}^T = R_{\Gamma_j,h} I_\Gamma R_{\Gamma_i,h}^T \tag{5 3 29}$$

and the observation that $R_{\Gamma_j,h} I_\Gamma = R_{\Gamma_j,h}$ ■

The weight matrices, $W_{i\, h}$, $1 \le i \le k$ employed in the definition of the preconditioner M_h in Section 5 2 can now be defined in terms of the block matrices I_γ^{ji}, $1 \le i, j \le k$ as follows

Definition 5 2 *For $\gamma \in \mathcal{G}(\Gamma)$, $i, j \in \{1, \ldots, k\}$ let $\mathcal{G}_{ji}(\Gamma) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{G}(\Gamma) \mid I_\gamma^{ji} \neq 0\}$, $a(i,\gamma) \stackrel{\text{def}}{=} \{j \mid I_\gamma^{ji} \neq 0\}$ and define the block matrices*

$$W_{i\, h} \stackrel{\text{def}}{=} \sum_{\gamma\in\mathcal{G}_{ii}(\Gamma)} w(i,\gamma,p)\, I_\gamma^{ii}, \quad 1 \le i \le k, \quad p \ge 1/2 \tag{5 3 30}$$

and the weights according to the relation

$$w(i,\gamma,p) \stackrel{\text{def}}{=} \frac{\alpha_i^p}{\sum_{j\in a(i\,\gamma)} \alpha_j^p}, \quad 1 \le i \le k, \quad \gamma \in \mathcal{G}(\Gamma), \quad p \ge 1/2 \tag{5 3 31}$$

where $a_{klmn} = \alpha\, b_{klmn}$, $\alpha(\underline{x}) = \alpha_i$, $\underline{x} \in \Omega_i$, $b_{klmn} \in \mathcal{L}_\infty(\Omega) \cap C^0(\overline{\Omega})$, $1 \le k, l, m, n \le 2$ ■

We observe that $\mathcal{G}_{ji}(\Gamma)$ contains all the geometrical elements of Γ which intersect $\partial\Omega_i \cap \partial\Omega_j$ and $a(i,\gamma)$ is a list of all subdomains whose boundaries intersect $\gamma \cap \partial\Omega_i$. The following Theorem, proved in BREZINA & MANDEL (1993), is fundamental for our analysis because it provides an inequality from which we subsequently obtain a bound on the condition number $\kappa(M_h^{-1} S_h)$.

Theorem 5.2 *Let* I_γ^{ji}, $S_{i,h}$, $N_{i,h}$, $W_{i,h}$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$ *be the matrices defined above then the* $W_{i,h} \in \mathbb{R}^{2N_{\Gamma_i}, 2N_{\Gamma_i}}$, $1 \le i \le k$ *have the partition of unity property (5.2.14) and, if there exists a number* $R > 0$ *such that, for* $\underline{u}_i \in \mathcal{N}(S_{i,h})^\perp \cap \mathcal{R}(S_{i,h} N_{i,h})^\perp$, $1 \le i \le k$ *and* $\gamma \in \mathcal{G}(\Gamma)$,

$$\alpha_j^{-1} \, \|I_\gamma^{ji} \underline{u}_i\|_{S_j,h}^2 \le \alpha_i^{-1} R \, \|\underline{u}_i\|_{S_i,h}^2 \tag{5.3.32}$$

then the preconditioner $M_h \in \mathbb{R}^{2N, 2N}$ *satisfies*

$$\kappa(M_h^{-1} S_h) \le K^2 L^2 R \tag{5.3.33}$$

where $K = \max_{1 \le i \le k} |\{j \mid R_{\Gamma_j,h} R_{\Gamma_i,h}^T \ne 0\}|$ *and* $L = \max_{1 \le i,j \le k} |\{\gamma \in \mathcal{G}(\Gamma) \mid I_\gamma^{ji} \ne 0\}|$ ■

We observe that the numbers K, L are parameters of the decomposition (5.1.1.2) of Ω into the subdomains Ω_i, $1 \le i \le k$, e.g., K is the maximum number of domains adjacent to any domain plus one and L is the maximum number of geometrical components, $\gamma \in \mathcal{G}(\Gamma)$, of any subdomain interface. A critical element of Theorem 5.2 is inequality (5.3.32) and the number $R > 0$, the analysis of (5.3.32) for problems of planar linear elasticity will lead to a logarithmic term in the upper bound (5.3.33).

In our analysis below we assume that the decomposition (5.1.1.2) of Ω has the following property. There exist bijective mappings $T_i\ \overline{\mathcal{S}} \to \overline{\Omega}_i$, $1 \le i \le k$, $\mathcal{S} \stackrel{\text{def}}{=} (0,1)^2$,

$$T_i \underline{s} \stackrel{\text{def}}{=} \underline{a}_0^i + \underline{a}_1^i H\, s_1 + \underline{a}_2^i H\, s_2, \quad \underline{s} \in \overline{\mathcal{S}} \tag{5.3.34}$$

where $\underline{a}_r^i \in \mathbb{R}^2$, $0 \le r \le 2$ are constants independent of $H > 0$. Thus, for $1 \le i \le k$, $\operatorname{diam}(\Omega_i) = O(H)$, $0 < \mu(\Omega_i) = |\mathcal{J}(T_i)| \le C\, H^2$ where $\mathcal{J}(T_i)$ is the Jacobian of the mapping T_i and $C > 0$ is a constant independent of H. Furthermore, following BRAMBLE, PASCIAK, & SCHATZ (1986), for $\underline{v} \in (H^{1/2}(\partial\Omega_i))^2$, $1 \le i \le k$, we define the scaled Sobolev norm, cf. Section 1.2,

$$\|\underline{v}, (H^{1/2}(\partial\Omega_i))^2\|_S^2 \stackrel{\text{def}}{=} \sum_{r\ i=1}^{2,m} \left[H^{-1} \|(\sigma_i v_r) \circ \alpha_i^{-1}, (\mathcal{L}_2(\mathbb{R}))^2\|^2 + |(\sigma_i v_r) \circ \alpha_i^{-1}, (H^{1/2}(\mathbb{R}))^2|^2 \right] \tag{5.3.35}$$

However, instead of $\|\bullet, (H^{1/2}(\partial\Omega_i))^2\|_S$ we shall employ the equivalent norm, cf. Section 1.2, $|||\bullet, (H^{1/2}(\partial\Omega_i))^2|||$ defined as follows for $\underline{v} \in (H^{1/2}(\partial\Omega_i))^2$, $1 \le i \le k$

$$|||\underline{v}, (H^{1/2}(\partial\Omega_i))^2|||^2 \stackrel{\text{def}}{=} \sum_{r=1}^{2} \left[H^{-1} \int_{\partial\Omega_i} |v_r(\underline{x})|^2 \, d\sigma(\underline{x}) + \iint_{\partial\Omega_i \times \partial\Omega_i} \frac{|v_r(\underline{x}) - v_r(\underline{z})|^2}{\|\underline{x} - \underline{z}\|_2^2} \, d\sigma(\underline{x}) \, d\sigma(\underline{z}) \right] \tag{5.3.36}$$

where σ is the surface measure defined in relation (1 2 31), cf WLOKA (1987) We now intend to demonstrate that there exists a real number $R > 0$ which satisfies relation (5 3 32) uniformly, i e , independently of $i, j \in \{1, \quad , k\}$, and, as a step towards this goal, we employ the results originally obtained by BRAMBLE, PASCIAK, SCHATZ (1986), DRYJA (1988), and BREZINA & MANDEL (1993) Furthermore, by establishing property (5 3 32) the required upper bound for $\kappa(M_h^{-1} S_h)$ will follow immediately from relation (5 3 33) of Theorem 5 2

The equivalence of the semi–norm $|\bullet, (H^{1/2}(\partial\Omega_i))^2|$ associated with the norm defined in (5 3 36) and the scaled energy norm $\alpha_i^{-1} \| \bullet \|_{S_{i,h}}$ defined for appropriate functions $\underline{v} \in (H^{1/2}(\partial\Omega_i))^2$, $1 \le i \le k$ is established by the following lemma, cf BREZINA & MANDEL (1993)

Lemma 5 3 *There exist constants C_r, $1 \le r \le 2$ which are independent of h, H and α_r, $1 \le r \le k$ such that*

$$C_1 \, |\underline{v}_h, (H^{1/2}(\partial\Omega_i))^2|^2 \le \alpha_i^{-1} \|M^{-1}\underline{v}_h\|_{S_{i,h}}^2 \le C_2 \, |\underline{v}_h, (H^{1/2}(\partial\Omega_i))^2|^2, \quad \underline{v}_h \in (S^h(\partial\Omega_i))^2 \tag{5 3 37}$$

where $|\bullet, (H^{1/2}(\partial\Omega_i))^2|$ is the Sobolev semi–norm (5 3 14), $M^{-1}\underline{v}_h \in \mathcal{N}(S_{i,h})^\perp \subset \mathbb{R}^{2N_{\Gamma_i} - \nu}$ is the vector of nodal values of $\underline{v}_h \in (S^h(\partial\Omega_i))^2$, and $S^h(\partial\Omega_i) \stackrel{\text{def}}{=} \{v_h|_{\partial\Omega_i} \mid v_h \in S^h(\Omega_i)\}$ ■

For problems of two–dimensional linear elasticity the polygonal boundaries, $\partial\Omega_i \subset \mathbb{R}^2$, $1 \le i \le k$ have measure $\sigma(\partial\Omega_i) = O(H) \, (H \to 0)$ and can, therefore, be parameterized in the form $\partial\Omega_i = \{T_{\partial\Omega_i}(s) \in \mathbb{R}^2 \mid 0 \le s \le H\}$ where $T_{\partial\Omega_i} \ (0, H) \to \partial\Omega_i$ is a bijective mapping However, because $\underline{v} \in (H^{1/2}(\partial\Omega_i))^2 \Leftrightarrow \underline{v} \circ T_{\partial\Omega_i} \in (H^{1/2}(0,H))^2$ one may equivalently consider elements either of $(H^{1/2}(\partial\Omega_i))^2$ or $(H^{1/2}(0,H))^2$

We shall employ Lemma's 5 4, 5 5, established by BREZINA & MANDEL (1993) from the work of DRYJA (1988), BRAMBLE, PASCIAK, & SCHATZ (1986), to obtain a bound on the semi–norm of the functions $I_\gamma \underline{v}_h$, $\underline{v}_h \in (S^h(\partial\Omega_i))^2$, $\gamma \in \mathcal{G}(\Gamma)$ where $I_\gamma \ S^h(\Gamma) \to S^h(\Gamma)$ denotes the linear operator represented by the matrix $I_\gamma \in \mathbb{R}^{2N, 2N}$ defined in (5 3 24), i e , for $\underline{u}_h \in S^h(\Gamma)$, $I_\gamma \underline{u}_h = \sum_{r=1}^{n_\gamma} (\underline{u}_h)_r \, \varphi_{n_r}|_\Gamma$ where $\{\varphi_i\}_{i=1}^N$ is the canonical basis for $S^h(\Omega)$, cf Section 2 2 1, and $\varphi_{n_r}^{-1}(\{1\}) \subset \gamma$, $1 \le r \le n_\gamma$ We point out that, in the lemma's below, $(S^h(0,H))^2$ (respectively $(S^h(\mathbb{R}))^2$) denotes the space of piecewise linear functions over the domain $(0,H)$ (respectively $\mathbb{R}$) corresponding to the *uniform partition* $0 < h < 2h < \quad < nh = H$, $n \in \mathbb{N}$ (respectively $\quad < 0 < h < \quad < nh = H < \quad$)

Lemma 5 4 *There exists a constant $C > 0$ such that*

$$|\delta_h(\underline{v}_h), (H^{1/2}(\mathbb{R}))^2|^2 \le C\Big[1 + \log(H/h)\Big] \, |||\underline{v}_h, (H^{1/2}(0,H))^2|||^2, \quad \underline{v}_h \in (S^h(0,H))^2 \tag{5 3 38}$$

where $\delta_h(\underline{v}_h) \in (S^h(\mathbb{R}))^2$, $\delta_h(\underline{v}_h)(0) \stackrel{\text{def}}{=} \underline{v}_h(0)$, $\delta_h(\underline{v}_h)(x) \stackrel{\text{def}}{=} 0$, $|x| \ge h$ and C is independent of $\underline{v}_h$ and the parameters h, H

Proof The results follows immediately from the definition of the norm $|||\bullet,(H^{1/2}((0,H)))^2|||$ (cf (5 3 36)), the semi-norm $|\bullet,(H^{1/2}(\mathbb{R}))^2|$ (cf Section 1 2), and Lemma 4 4 of BREZINA & MANDEL (1993) ∎

Lemma 5 5 *There exists a constant $C>0$ such that*

$$|\underline{w}_h,(H^{1/2}(\mathbb{R}))^2|^2 \le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_h,(H^{1/2}(0,H))^2|||^2, \quad \underline{v}_h\in(S^h(0,H))^2 \qquad (5\,3\,39)$$

where $\underline{w}_h\in(S^h(\mathbb{R}))^2$ is defined as follows

$$\underline{w}_h(x)\stackrel{\text{def}}{=}\begin{cases}\underline{v}_h(x), & \text{if } h\le x\le H-h\\ 0, & \text{if } x\le 0,\ x\ge H\end{cases} \qquad (5\,3\,40)$$

and C is independent of h, H, $\underline{v}_h$

Proof Use the norm definitions provided above and in Section 1 2 and apply Lemma 4 5 of BREZINA & MANDEL (1993) ∎

We now employ the above Lemma's to prove the following important result

Theorem 5 6 *There exists a constant $C>0$ such that, for any $\gamma\in\mathcal{G}(\Gamma)$, $\underline{v}_{i,h}\in(S^h(\partial\Omega_i))^2$, $1\le i\le k$,*

$$|I_\gamma\underline{v}_{i,h},(H^{1/2}(\partial\Omega_i))^2|^2 \le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{i,h},(H^{1/2}(\mathcal{C}))^2|||^2 \qquad (5\,3\,41)$$

where $\gamma\subset\overline{\mathcal{C}}$, $\mathcal{C}\in\mathcal{G}(\Gamma)$ and C is independent of $h,H,\underline{v}_{i,h}$

Proof Clearly, for $\gamma\in\mathcal{G}(\Gamma)$ there exists a bijective mapping $T_{\partial\Omega_i}$ $(0,\alpha H)\to\partial\Omega_i$, $\alpha\ge 1$ such that if γ is a vertex then $T_{\partial\Omega_i}(0)=\gamma$ else $T_{\partial\Omega_i}(0,H)=\gamma$ and, therefore, for $\underline{v}_{i,h}\in(S^h(\partial\Omega_i))^2$, $\underline{v}_{i,h}\circ T_{\partial\Omega_i}\in(S^h(0,\alpha H))^2$ and

$$\begin{aligned}|I_\gamma\underline{v}_{i,h},(H^{1/2}(\partial\Omega_i))^2|^2 &= \sum_{r=1}^{2}\iint_{\partial\Omega_i\times\partial\Omega_i}\frac{|I_\gamma v_{i,h,r}(\underline{x})-I_\gamma v_{i,h,r}(\underline{z})|^2}{\|\underline{x}-\underline{z}\|_2^2}\,d\sigma(\underline{x})\,d\sigma(\underline{z})\\ &\le C\sum_{r=1}^{2}\iint_{(0,\alpha H)\times(0,\alpha H)}\frac{|I_\gamma v_{i,h,r}(T_{\partial\Omega_i}(x))-I_\gamma v_{i,h,r}(T_{\partial\Omega_i}(z))|^2}{|x-z|^2}\,dx\,dz\\ &= C\,|I_\gamma\underline{v}_{i,h}\circ T_{\partial\Omega_i},(H^{1/2}(0,\alpha H))^2|^2 \end{aligned}\qquad (5\,3\,42)$$

where $C>0$ is independent of h,H If $\gamma\in\mathcal{G}(\Gamma)$ is a vertex then we observe that $I_\gamma\underline{v}_{i,h}\circ T_{\partial\Omega_i}$ coincides with the function $\delta_h(\underline{v}_{i,h})\in(S^h(\mathbb{R}))^2$ defined in Lemma 5 4 and we deduce the inequality,

$$\begin{aligned}|I_\gamma\underline{v}_{i,h}\circ T_{\partial\Omega_i},(H^{1/2}(0,\alpha H))^2|^2 &\le C\Big[1+\log(H/h)\Big]\,|||\underline{v}_{i,h}\circ T_{\partial\Omega_i},(H^{1/2}(0,H))^2|||^2\\ &\le C_1\Big[1+\log(H/h)\Big]\,|||\underline{v}_{i,h},(H^{1/2}(\mathcal{C}))^2|||^2\end{aligned}\qquad (5\,3\,43)$$

where $C = T_{\partial\Omega_i}(0,H) \in \mathcal{G}(\Gamma)$ If, however, $\gamma = \operatorname{int}(\underline{v}_1 \to \underline{v}_2) \in \mathcal{G}(\Gamma)$, $\underline{v}_1, \underline{v}_2 \in \mathcal{V}(\Gamma)$, cf (5 3 23), then we observe that $I_\gamma \underline{v}_{i,h} \circ T_{\partial\Omega_i}$ coincides with the function $\underline{w}_h \in (S^h(\mathbb{R}))^2$ defined in relation (5 3 40) of Lemma 5 5 and we deduce the inequality

$$|I_\gamma \underline{v}_{i,h} \circ T_{\partial\Omega_i}, (H^{1/2}(0,\alpha H))^2|^2 \le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{i,h} \circ T_{\partial\Omega_i}, (H^{1/2}(0,H))^2|||^2$$
$$\le C_1\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{i,h}, (H^{1/2}(C))^2|||^2 \qquad (5\ 3\ 44)$$

where $C = T_{\partial\Omega_i}(0,H) = \gamma$ and the constants $C, C_1 > 0$ are independent of h, H Inequality (5 3 41) now follows from (5 3 42), (5 3 43), and (5 3 44) ■

We now employ the above results to establish the bound for $\kappa(M_h^{-1}S_h)$ presented in Theorem 5 7 below

Theorem 5 7 *Let $\Omega \subset \mathbf{R}^2$ be a polygonal domain partitioned into subdomains Ω_i, $1 \le i \le k$ satisfying (5 1 1 1) and let $W_{i,h} \in \mathbf{R}^{2N_{\Gamma_i}, 2N_{\Gamma_i}}$, $1 \le i \le k$ be the weight matrices defined according to relation (5 3 30), (5 3 31) then there exists a constant $C > 0$ such that*

$$\kappa(M_h^{-1}S_h) \le C\Big[1+\log(H/h)\Big]^2, \quad h, H > 0 \qquad (5\ 3\ 45)$$

where, for a triangulation $\mathcal{T}_h(\Omega)$, S_h is the global Schur complement matrix (5 1 1 35), M_h is the preconditioner defined in Section 5 3, and C is independent of the parameters h, H where $\operatorname{diam}(\Omega_i) = O(H)$ $(H \to 0)$, $1 \le i \le k$

Proof Clearly, this result can be established by demonstrating the validity of inequality (5 3 32) for $R = C\,[1+\log(H/h)]^2$ However, it is apparent from Lemma 5 3 that (5 3 32) can be written equivalently as follows, for $\gamma \in \mathcal{G}(\Gamma)$, $\underline{v}_{i,h} \in (S^h(\partial\Omega_i))^2 \cap \mathcal{N}(S_{i,h})^\perp$,

$$|I_\gamma \underline{v}_{i,h}, (H^{1/2}(\partial\Omega_j))^2|^2 \le CR|\underline{v}_{i,h}, (H^{1/2}(\partial\Omega_i))^2|^2, \qquad (5\ 3\ 46)$$

Let $\gamma \subset \partial\Omega_i \cap \partial\Omega_j$ If $\gamma = \operatorname{int}(\underline{v}_1 \to \underline{v}_2)$, $\underline{v}_1, \underline{v}_2 \in \mathcal{V}(\Gamma)$ then it follows from Theorem 5 6 that

$$|I_\gamma \underline{v}_{i,h}, (H^{1/2}(\partial\Omega_j))^2|^2 \le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{i,h}, (H^{1/2}(\gamma))^2|||^2$$
$$\le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{i,h}, (H^{1/2}(\partial\Omega_i))^2|||^2 \qquad (5\ 3\ 47)$$

while, if γ is a vertex and $\underline{v}_{ij,h} \in (S^h(\partial\Omega_i \cup \partial\Omega_i))^2$ is any extension of $\underline{v}_i \in (S^h(\partial\Omega_i))^2$ to $\partial\Omega_j$ then Theorem 5 7 implies the inequality

$$|I_\gamma \underline{v}_{ij,h}, (H^{1/2}(\partial\Omega_j))^2|^2 \le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{ij\,h}, (H^{1/2}(C))^2|||^2 \qquad (5\ 3\ 48)$$

Indeed, with $\underline{v}_{ij\,h} \overset{\text{def}}{=} \underline{v}_{i,h}(\gamma)$ on $\partial\Omega_j$ we use Lemma 1 of DRYJA (1988) to obtain

$$|||\underline{v}_{ij\,h}, (H^{1/2}(C))^2|||^2 = H^{-1}\|\underline{v}_{ij\,h}, (\mathcal{L}_2(C))^2\|^2 \le C\,\|\underline{v}_{ij\,h}, (\mathcal{L}_\infty(C))^2\|^2$$
$$\le C\,\|\underline{v}_{i,h}, (\mathcal{L}_\infty(\partial\Omega_i))^2\|^2 \le C\Big[1+\log(H/h)\Big]^2 |||\underline{v}_{i,h}, (H^{1/2}(\partial\Omega_i))^2|||^2 \; (5\ 3\ 49)$$

where $C > 0$ is a constant which is independent from $i, j \in \mathbb{N}_k$, h, H. Thus, inequalities (5.3.48) and (5.3.49) show that (5.3.47) holds when $\gamma \in \mathcal{G}(\Gamma)$ is a vertex. However, we shall assume that the decomposition (5.1.1.1) is constructed such that there exists a constant $C > 0$ which is independent of $i \in \{1, \ldots, k\}$ with the property that if $\partial\Omega_i \cap \partial\Omega_D \neq \emptyset$ then $\sigma(\partial\Omega_i \cap \partial\Omega_D) \geq C\,\sigma(\partial\Omega_i)$. This enables one to uniformly apply the Poincare inequality

$$\|\underline{v}_{i,h}, (\mathcal{L}_2(\partial\Omega_i))^2\|^2 \leq CH|\underline{v}_{i,h}, (H^{1/2}(\partial\Omega_i))^2|^2 \tag{5.3.50}$$

to any subdomain $\Omega_i, 1 \leq i \leq k$. Thus, applying the Poincaré inequality (5.3.50) to relation (5.3.41) of Theorem 5.6 we replace the scaled norm, $|||\bullet, (H^{1/2}(\partial\Omega_i))^2|||$, with the semi–norm, $|\bullet, (H^{1/2}(\partial\Omega_i))^2|$, and obtain the relation

$$|I_\gamma \underline{v}_{i,h}, (H^{1/2}(\partial\Omega_j))^2|^2 \leq C\Big[1 + \log(H/h)\Big]^2 |\underline{v}_{i,h}, (H^{1/2}(\partial\Omega_i))^2|^2 \tag{5.3.51}$$

which is equivalent to inequality (5.3.32) and the theorem is thus proved. ∎

Finally, we observe that the constant C in (5.3.45) will depend on the parameters K, L defined in Theorem 5.2, the continuous coefficients b_{klmn}, $1 \leq k, l, m, n \leq 2$, cf. definition 5.2, and the admissible triangulation $\mathcal{T}_h(\Omega)$ of Ω.

5.4 Computational Examples

We apply our domain decomposition algorithm of Section 5.3 to a variety of problems with varying levels of material regularity, e.g., $a_{ijkl}, 1 \leq i, j, k, l \leq 2$ smooth or piecewise continuous with discontinuities of varying magnitude and, in particular, we consider linear elastic boundary value problems for which a_{ijkl}, $1 \leq i, j, k, l \leq 2$ is periodic, cf. Chapters 3, 4, or is randomly defined. The *effectiveness* of our domain decomposition approach is assessed by comparing the results obtained with algorithms $\mathcal{ALG}$ 1 (conjugate gradients with no preconditioner) and $\mathcal{ALG}$ 2 (conjugate gradients with the Neumann–Neumann preconditioner) for a variety of values of the problem and discretization parameters $\varepsilon, \alpha_i, h, H$, $1 \leq i \leq k$ where, in the computational examples below, we employ uniform domain decompositions (5.1.1.1), i.e., $H_i = H$, $1 \leq i \leq k$ and Ω_i, $1 \leq i \leq k$ can be obtained by translating and rotating the square

$$\Omega_H = \{(\xi, \eta) \mid 0 < \xi, \eta < H\}, \tag{5.4.1}$$

and uniform triangulations $\mathcal{T}_h(\Omega_i)$ of each subdomain Ω_i, $1 \leq i \leq k$, i.e., each $\tau \in \mathcal{T}_h(\Omega_i)$ is obtained by translating and rotating the right angled triangle

$$T_h = \{(\xi, \eta) \mid \xi, \eta > 0, \ \xi + \eta > h\}, \quad h > 0 \tag{5.4.2}$$

It is apparent from the error bound (5.1.2) that the condition number,

$$\kappa(P_h^{-1}S_h) = \|P_h^{-1}S_h\|_2 \, \|(P_h^{-1}S_h)^{-1}\|_2 = \lambda_{\max}(P_h^{-1}S_h)/\lambda_{\min}(P_h^{-1}S_h) \geq 1, \tag{5.4.3}$$

of the preconditioned matrix $P_h^{-1}S_h$ determines how rapidly the iterates $\underline{u}_{\Gamma,h}^{(n)}$, $n \geq 0$ converge to $\underline{u}_{\Gamma,h}$ as $n \to \infty$ However, we require some convergence criteria for our algorithm and, for this, we employ the following bound on the relative energy norm error, cf ASHBY & MANTEUFFEL (1990),

$$\frac{\|\underline{e}^{(n)}\|_{S_h}^2}{\|\underline{u}_{\Gamma,h}\|_{S_h}^2} \leq \kappa_{S_h}(P_h^{-1}S_h)\frac{|(P_h^{-1}S_h\underline{e}^{(n)},\underline{r}^{(n)})|}{|(P_h^{-1}S_h\underline{u}_{\Gamma,h},\underline{L}_h)|} = \kappa_{S_h}(P_h^{-1}S_h)\frac{|(P_h^{-1}\underline{r}^{(n)},\underline{r}^{(n)})|}{|(P_h^{-1}\underline{L}_h,\underline{L}_h)|} \quad (5\,4\,4)$$

where $\kappa_{S_h}(P_h^{-1}S_h) = \|P_h^{-1}S_h\|_{S_h}\|(P_h^{-1}S_h)^{-1}\|_{S_h}$ and $\underline{e}^{(n)} = \underline{u}_{\Gamma,h}-\underline{u}_{\Gamma,h}^{(n)}$, $\underline{r}^{(n)} = \underline{L}_h - S_h\underline{u}_{\Gamma,h}^{(n)}$, $n \geq 0$ However, we observe that

$$\begin{aligned}\|P_h^{-1}S_h\|_{S_h}^2 &= \sup_{\underline{x}\neq 0}\frac{\|P_h^{-1}S_h\underline{x}\|_{S_h}^2}{\|\underline{x}\|_{S_h}^2} = \sup_{\underline{x}\neq 0}\frac{(S_h^{1/2}P_h^{-1}S_h^{1/2}\underline{x},\underline{x})}{\|\underline{x}\|^2}\\ &= \lambda_{\max}(S^{1/2}P_h^{-1}S_h^{1/2}) = \lambda_{\max}(P_h^{-1}S_h)\end{aligned} \quad (5\,4\,5)$$

where $S_h = S_h^{1/2}S_h^{1/2}$, $S_h^{1/2} \in \mathbb{R}^{2N,2N}$, $h > 0$ and, similarly,

$$\|(P_h^{-1}S_h)^{-1}\|_{S_h}^2 = \lambda_{\max}((P_h^{-1}S_h)^{-1}) = 1/\lambda_{\min}(P_h^{-1}S_h) \quad (5\,4\,6)$$

Therefore, $\kappa_{S_h}(P_h^{-1}S_h) = \sqrt{\kappa(P_h^{-1}S_h)}$, $h > 0$ and we can ensure that $\|\underline{e}^{(n)}\|_{S_h}/\|\underline{u}_{\Gamma,h}\|_{S_h} \leq \tau$ by iterating, cf $\mathcal{ALG}$ 1, 2, 3, until

$$\kappa(P_h^{-1}S_h)\frac{|(P_h^{-1}\underline{r}^{(n)},\underline{r}^{(n)})|}{|(P_h^{-1}\underline{L}_h,\underline{L}_h)|} \leq \tau^2 \quad (5\,4\,7)$$

The parameters computed at each step of a conjugate gradient algorithm allow one to compute the leading tridiagonal submatrices $T_h^{(n)} \in \mathbb{R}^{n,n}$, $n \leq 2N$ of $T_h = T_h^{(2N)}$ where $T_h = Q_h^T P_h^{-1}S_hQ_h$ for some orthogonal matrix $Q_h \in \mathbb{R}^{2N,2N}$ The rapid convergence of the extreme eigenvalues of $T_h^{(n)}$, $n \geq 1$ to those of $P_h^{-1}S_h$, $h > 0$ with increasing n is established by the Kaniel–Paige convergence Theory, cf GOLUB & VAN LOAN (1989) We employ the rational QR algorithm with Newton Shift detailed in REINSCH & BAUER (1968) to compute approximations of the condition number $\kappa(P_h^{-1}S_h)$, $h > 0$ and use these in the convergence criteria (5 4 7) Algorithms $\mathcal{ALG}$ 1, 2, 3 have been implemented in C++ code and the results are presented in Sections 5 4 1–5 4 3 below

5 4 1 Plane stress sample problem Smooth Data

We define Poisson's ratio, ν, Young's modulus of elasticity, $E(\underline{x})$, $\underline{x} \in \Omega \stackrel{\text{def}}{=} (0,1)^2$, the material parameters $\lambda, \mu \in \mathbb{R}$, and the body force f according to the relations

$$\nu \stackrel{\text{def}}{=} 3/10, \quad E(\underline{x}) \stackrel{\text{def}}{=} 1, \quad \lambda(\underline{x}) \stackrel{\text{def}}{=} \frac{\nu E(\underline{x})}{1-\nu^2}, \quad \mu(\underline{x}) \stackrel{\text{def}}{=} \frac{E(\underline{x})}{2(1+\nu)}, \quad f(\underline{x}) \stackrel{\text{def}}{=} 0, \quad \underline{x} \in \Omega \quad (5\,4\,8)$$

and we determine the coefficients $a_{ijkl} \in C^\infty(\Omega)$, $1 \leq i,j,k,l \leq 2$ from relations (1 3 11) We employ the following boundary values of displacement, $\underline{u}$, and stress, σ,

$$\underline{u}(\underline{x}) \stackrel{\text{def}}{=} \begin{bmatrix} x_1 \\ \nu(1/2 - x_2)\end{bmatrix}, \quad \underline{x} \in \partial\Omega_D, \quad \sigma(\underline{x}) \stackrel{\text{def}}{=} \begin{bmatrix}\lambda(1-\nu)+2\mu & 0 \\ 0 & \lambda(1-\nu)-2\mu\nu\end{bmatrix}, \quad \underline{x} \in \partial\Omega_T \quad (5\,4\,9)$$

where $\partial\Omega_D \stackrel{\text{def}}{=} \{\underline{x} \mid x_1 = 0,\ 0 \le x_2 \le 1\} \cup \{\underline{x} \mid 0 \le x_1 \le 1,\ x_2 = 1\}$ and the surface tractions on $\partial\Omega_T$ are $\underline{t} = \sigma \circ \underline{n}$ The computational results obtained with (1) Uniform decompositions, (5 1 1 1), (2) Uniform triangulations, $\mathcal{T}_h(\Omega_i)$, $1 \le i \le k$, cf Section 5 4, (3) The weights, $w(i,\gamma,p)$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$, defined according to (5 3 31) where $\alpha_i = 1$, $1 \le i \le k$ and $p \stackrel{\text{def}}{=} 1$, (4) Convergence criteria (5 4 7) with the relative error parameter, $\tau = 10^{-\sqrt{18}}$, and (5) The number of iterations, n, limited by $n_{\max} = 80$ are provided in Table 5 4 1

Table 5 4 1 $a_{ijkl} \in C^\infty(\Omega)$, $1 \le i,j,k,l \le 2$

		$\mathcal{ALG}$ 1		$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
h	H	n	$\kappa(S_h)$	n	$\kappa(N_h^{-1}S_h)$	n	$\kappa(M_h^{-1}S_h)$
1/8	1/2	31	4 3072(+1)	14	8 5326(+0)	11	5 3153(+0)
1/16	1/2	49	8 3661(+1)	16	1 1108(+1)	12	7 2720(+0)
1/32	1/2	77	1 6507(+2)	17	1 4157(+1)	14	1 0121(+1)
1/16	1/4	NC	2 1497(+2)	52	1 0129(+2)	16	5 8160(+0)
1/32	1/4	NC	4 2650(+2)	64	1 2789(+2)	21	9 1396(+0)
1/64	1/4	NC	8 2186(+2)	73	1 5635(+2)	24	1 2505(+1)
1/32	1/8	NC	9 2200(+2)	NC	4 7860(+2)	23	5 9004(+0)
1/64	1/8	NC	1 6059(+3)	NC	6 0360(+2)	27	9 3509(+0)
1/128	1/8	NC	2 7264(+3)	NC	7 3753(+2)	31	1 2781(+1)

NC≡ No convergence after 80 iterations, $w(i,\gamma,1) = 1/|a(i,\gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

It is clearly apparent from Table 5 4 1 that, in contrast with algorithms $\mathcal{ALG}$ 1,2, the rate of convergence of algorithm $\mathcal{ALG}$ 3 does not slow significantly as $H, h \to 0$, indeed, the computational results confirm the logarithmic behaviour of $\kappa(M_h^{-1}S_h)$ established in Theorem 5 7 This is apparent when one compares Table 5 4 1 with the following table of values

H/h	4	8	16
$[1+\log(H/h)]^2$	5 6944008	9 4829602	14 23242

5 4 2 Plane stress sample problem Discontinuous Data

We now demonstrate that the convergence rates produced by the preconditioner M_h^{-1}, $h > 0$ ($w(i,\gamma,1)$, $1 \le i \le k, \gamma \in \mathcal{G}(\Gamma)$ defined by relation (5 3 31)) are independent of any coefficient discontinuities which are aligned with the subdomain boundaries and, to do this, we apply domain decomposition algorithms $\mathcal{ALG}$ 1,2,3 to a linear elastic analogue of the scalar, periodic boundary value problem investigated in Chapters 3, 4, i e , a problem of the form Find $\underline{u}^\varepsilon \in (H_0^1(\Omega, \partial\Omega_D))^2$ such that

$$\int_\Omega \sum_{i,j,k,l=1}^{2} a_{ijkl}(\underline{x}/\varepsilon)\,\frac{\partial u_i^\varepsilon}{\partial x_j}(\underline{x})\,\frac{\partial v_k}{\partial x_l}(\underline{x})\,d\underline{x} = \int_\Omega \underline{f}(\underline{x})\ \underline{v}(\underline{x})\,d\underline{x}, \quad \underline{v} \in (H_0^1(\Omega, \partial\Omega_D))^2 \qquad (5\ 4\ 10)$$

where the functions $a_{ijkl} \in \mathcal{L}_\infty(\mathcal{P})$, $1 \le i,j,k,l \le 2$ are 1–periodic and $\varepsilon > 0$ For $\sigma > 0$ we begin by defining the 1–periodic function $\mathcal{E}(\bullet, \sigma)$ on the cell, $\mathcal{P}$, as follows

$$\mathcal{E}(\underline{x}, \sigma) \stackrel{\text{def}}{=} \begin{cases} \sigma, & \text{if } \underline{x} \in [1/4, 3/4]^2 \\ 1, & \text{otherwise} \end{cases} \tag{5 4 11}$$

Young's modulus of elasticity is then defined according to the relation $E(\underline{x}) \stackrel{\text{def}}{=} \mathcal{E}(\underline{x}, \sigma)$, $\underline{x} \in \Omega$ and ν, λ, μ, f are given by relations (5 4 8) The boundary conditions employed are again given by relations (5 4 9), the triangulations, $\mathcal{T}_h(\Omega_i)$, $1 \le i \le k$, and domain decompositions (5 1 1 1) are uniform, cf Section 5 4 1, the iteration parameters have values $\tau = 10^{-\sqrt{18}}$, $n_{\max} = 80$, and the weights, $w(i, \gamma, p)$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$, are defined by relation (5 3 31) with $p \stackrel{\text{def}}{=} 1$ We construct the decomposition $\overline{\Omega} = \cup_{i=1}^{k} \overline{\Omega}_i$ such that $H = \varepsilon/4$, $a_{ijkl}(\bullet/\varepsilon)$, $1 \le i,j,k,l \le 2$ is constant in each subdomain Ω_i, $1 \le i \le k$ (with constant value σ or 1) and, cf (5 1 1 1) and (5 4 1),

$$\Omega_i = (p, q)\, H + \Omega_H, \quad H = \varepsilon/4, \quad 1 \le i \le k, \tag{5 4 12}$$

where $i = (\sqrt{k}+1)p + q$, $0 \le p, q \le \sqrt{k}$ The computational results obtained for this problem are provided in Tables 5 4 2a–f We demonstrate the effectiveness of the weights defined in relation (5 3 31) by repeating the computations with the alternative interface weights $w(i, \gamma, 1) \stackrel{\text{def}}{=} 1/|a(i, \gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$, the results are presented in Tables 5 4 2d–f

Table 5 4 2a $a_{ijkl} \in \mathcal{PC}^\infty(\Omega)$, $1 \le i,j,k,l \le 2$

			$\mathcal{ALG}$ 1		$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
ε	H	h	n	$\kappa(S_h)$	n	$\kappa(N_h^{-1}S_h)$	n	$\kappa(M_h^{-1}S_h)$
1	1/4	1/16	NC	1 4670(+3)	NC	7 6144(+2)	17	4 9512(+0)
1	1/4	1/32	NC	2 3360(+3)	NC	9 3890(+2)	20	7 8869(+0)
1	1/4	1/64	NC	4 0347(+3)	NC	1 1471(+3)	24	1 0971(+1)
1/2	1/8	1/32	NC	3 8933(+3)	NC	3 0049(+3)	21	5 3880(+0)
1/2	1/8	1/64	NC	5 2498(+3)	NC	3 6740(+3)	27	8 5498(+0)
1/2	1/8	1/128	NC	5 4075(+3)	NC	4 4593(+3)	32	1 1756(+1)
1/4	1/16	1/64	NC	4 9957(+3)	NC	1 1336(+4)	22	5 4137(+0)
1/4	1/16	1/128	NC	5 2769(+3)	NC	1 3800(+4)	30	8 5728(+0)
1/4	1/16	1/256	NC	5 1447(+3)	NC	1 6629(+4)	36	1 1779(+1)

$\sigma = 10$, $w(i, \gamma, 1) = \alpha_i / \sum_{j \in a(i,\gamma)} \alpha_j$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

The results presented in Tables 5 4 2a–c confirm the theoretical results obtained in Section 5 3 because they demonstrate that algorithm $\mathcal{ALG}$ 3 is not significantly affected by the presence of large discontinuities in a_{ijkl}, $1 \le i,j,k,l \le 2$ if the interface weights $w(i, \gamma, 1)$, $1 \le i \le k$, $\mathcal{G}(\Gamma)$ are defined according to relation (5 3 31) This is clearly not the case for algorithm $\mathcal{ALG}$ 1,

Table 5 4 2b $a_{ijkl} \in \mathcal{PC}^{\infty}(\Omega)$, $1 \le i, j, k, l \le 2$

			$\mathcal{ALG}$ 1		$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
ε	H	h	n	$\kappa(S_h)$	n	$\kappa(N_h^{-1}S_h)$	n	$\kappa(M_h^{-1}S_h)$
1	1/4	1/16	NC	3 9096(+2)	49	9 0650(+1)	23	1 5212(+1)
1	1/4	1/32	NC	8 3613(+2)	56	1 1324(+2)	23	1 2982(+1)
1	1/4	1/64	NC	1 6446(+3)	60	1 3823(+2)	25	1 3944(+1)
1/2	1/8	1/32	NC	1 4567(+3)	NC	5 0294(+2)	24	8 7873(+0)
1/2	1/8	1/64	NC	2 8022(+3)	NC	6 0488(+2)	27	1 0654(+1)
1/2	1/8	1/128	NC	4 5822(+3)	NC	7 1161(+2)	32	1 4176(+1)
1/4	1/16	1/64	NC	4 3856(+3)	NC	1 7969(+3)	26	7 1524(+0)
1/4	1/16	1/128	NC	5 2908(+3)	NC	2 2263(+3)	32	1 0664(+1)
1/4	1/16	1/256	NC	5 0519(+3)	NC	2 7042(+3)	37	1 4186(+1)

$\sigma = 1/18$, $w(i,\gamma,1) = \alpha_i / \sum_{j \in a(i,\gamma)} \alpha_j$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

Table 5 4 2c $a_{ijkl} \in \mathcal{PC}^{\infty}(\Omega)$, $1 \le i, j, k, l \le 2$

			$\mathcal{ALG}$ 1		$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
ε	H	h	n	$\kappa(S_h)$	n	$\kappa(N_h^{-1}S_h)$	n	$\kappa(M_h^{-1}S_h)$
1	1/4	1/16	NC	1 4305(+3)	53	1 7255(+2)	35	5 2338(+1)
1	1/4	1/32	NC	1 7579(+3)	61	2 0968(+2)	31	2 9882(+1)
1	1/4	1/64	NC	1 7290(+3)	73	2 5436(+2)	29	1 9844(+1)
1/2	1/8	1/32	NC	1 8337(+3)	NC	5 7397(+2)	31	2 0565(+1)
1/2	1/8	1/64	NC	2 9227(+3)	NC	7 1107(+2)	31	1 5048(+1)
1/2	1/8	1/128	NC	4 6207(+3)	NC	8 4675(+2)	32	1 4309(+1)
1/4	1/16	1/64	NC	4 5866(+3)	NC	1 8046(+3)	29	1 0251(+1)
1/4	1/16	1/128	NC	5 4589(+3)	NC	2 2262(+3)	32	1 0818(+1)
1/4	1/16	1/256	NC	5 3536(+3)	NC	2 7046(+3)	37	1 4315(+1)

$\sigma = 1/114$, $w(i,\gamma,1) = \alpha_i / \sum_{j \in a(i,\gamma)} \alpha_j$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

in fact, if one employs the alternative definition $w(i,\gamma,1) \stackrel{\text{def}}{=} 1/|a(i,\gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$, then, compared with the results reported in Tables 5 4 2a–c, the larger number of iterations, n, and condition numbers obtained in Tables 5 4 2d–f suggest that the behaviour of algorithm $\mathcal{ALG}$ 3 is no longer independent of the coefficient discontinuities which exist in the problem this confirms the importance of the choice of the interface weights $w(i,\gamma,p)$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$, $p \ge 1/2$

5 4 3 Plane stress sample problem Randomly Discontinuous Data

To demonstrate the effectiveness of the preconditioner M_h^{-1}, $h > 0$ for problems with dis-

Table 5 4 2d $a_{ijkl} \in \mathcal{PC}^{\infty}(\Omega)$, $1 \leq i, j, k, l \leq 2$

			$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
ε	H	h	n	$\kappa(N_h^{-1} S_h)$	n	$\kappa(M_h^{-1} S_h)$
1	1/4	1/16	NC	6 9794(+2)	32	2 3326(+1)
1	1/4	1/32	NC	8 7378(+2)	39	3 5578(+1)
1	1/4	1/64	NC	1 0406(+3)	45	5 0303(+1)
1/2	1/8	1/32	NC	2 6969(+3)	37	2 3270(+1)
1/2	1/8	1/64	NC	3 3622(+3)	48	3 5985(+1)
1/2	1/8	1/128	NC	4 0522(+3)	56	5 0861(+1)
1/4	1/16	1/64	NC	1 0393(+4)	42	2 3287(+1)
1/4	1/16	1/128	NC	1 2906(+4)	52	3 5987(+1)
1/4	1/16	1/256	NC	1 5398(+4)	60	5 0865(+1)

$\sigma = 10$, $w(i, \gamma, 1) = 1/|a(i, \gamma)|$, $1 \leq i \leq k$, $\gamma \in \mathcal{G}(\Gamma)$

Table 5 4 2e $a_{ijkl} \in \mathcal{PC}^{\infty}(\Omega)$, $1 \leq i, j, k, l \leq 2$

			$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
ε	H	h	n	$\kappa(N_h^{-1} S_h)$	n	$\kappa(M_h^{-1} S_h)$
1	1/4	1/16	63	6 8076(+1)	34	1 4305(+1)
1	1/4	1/32	73	8 9823(+1)	35	1 5061(+1)
1	1/4	1/64	NC	1 1624(+2)	45	2 5084(+1)
1/2	1/8	1/32	NC	4 4321(+2)	33	1 0332(+1)
1/2	1/8	1/64	NC	5 4593(+2)	44	1 7162(+1)
1/2	1/8	1/128	NC	6 6131(+2)	58	3 0785(+1)
1/4	1/16	1/64	NC	1 7025(+3)	34	1 0324(+1)
1/4	1/16	1/128	NC	2 1276(+3)	49	2 0169(+1)
1/4	1/16	1/256	NC	2 5831(+3)	63	3 2725(+1)

$\sigma = 1/18$, $w(i, \gamma, 1) = 1/|a(i, \gamma)|$, $1 \leq i \leq k$, $\gamma \in \mathcal{G}(\Gamma)$

continuous and non-periodic coefficients, we now apply the domain decomposition algorithms $\mathcal{ALG}$ 1, 2, 3 to a number of problems with randomly defined material coefficients, $a_{ijkl} \in \mathcal{L}_{\infty}(\Omega)$, $1 \leq i, j, k, l \leq 2$ We achieve this by defining Young's modulus to be a step function, constant in each subdomain Ω_i, $1 \leq i \leq k$, with the values obtained from the UNIX stdlib h random number generator functions srand48, drand48, i e ,

$$E(\underline{x}) \stackrel{\text{def}}{=} 1 + 100\,[\text{srand48}(i), \text{drand48}()] \in [1, 101), \quad \underline{x} \in \Omega_i, \quad 1 \leq i \leq k \qquad (5\,4\,13)$$

Thus, we first *seed* the random number generator using srand48(i) where $i \in \{1, \quad, k\}$ is the domain index and then obtain a uniformly distributed random number drand48() $\in [0, 1)$

Table 5 4 2f $a_{ijkl} \in \mathcal{PC}^\infty(\Omega)$, $1 \le i, j, k, l \le 2$

			$\mathcal{ALG}$ 2		$\mathcal{ALG}$ 3	
ε	H	h	n	$\kappa(N_h^{-1}S_h)$	n	$\kappa(M_h^{-1}S_h)$
1	1/4	1/16	NC	1 6800(+2)	NC	2 3370(+2)
1	1/4	1/32	NC	2 5683(+2)	NC	1 3773(+2)
1	1/4	1/64	NC	4 0085(+2)	NC	1 1971(+2)
1/2	1/8	1/32	NC	5 5316(+2)	NC	1 0075(+2)
1/2	1/8	1/64	NC	6 8480(+2)	NC	8 4559(+1)
1/2	1/8	1/128	NC	7 9447(+2)	NC	1 3063(+2)
1/4	1/16	1/64	NC	1 7003(+3)	77	5 9652(+1)
1/4	1/16	1/128	NC	2 1278(+3)	NC	9 0417(+1)
1/4	1/16	1/256	NC	2 5864(+3)	NC	1 7357(+2)

$\sigma = 1/114$, $w(i,\gamma,1) = 1/|a(i,\gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

Table 5 4 3a Random Young's Modulus values

Domain, i	1	2	3	4
$E(\underline{x})$, $\underline{x} \in \Omega_i$	18 0828	5 1630	92 2433	79 3235
Domain, i	5	6	7	8
$E(\underline{x})$, $\underline{x} \in \Omega_i$	66 4037	53 4840	40 5642	27 6444
Domain, i	9	10	11	12
$E(\underline{x})$, $\underline{x} \in \Omega_i$	14 7247	1 8049	88 8851	75 9653
Domain, i	13	14	15	16
$E(\underline{x})$, $\underline{x} \in \Omega_i$	63 0456	50 1258	37 2060	24 2863

The range, $E(\Omega)$, obtained in this way is presented in Table 5 4 3a

The material parameters $\nu, \mu, \lambda \in \mathbb{R}$ and the body force f are once again determined from relation (5 4 8), $\Omega \stackrel{\text{def}}{=} (0,1)^2$, and we employ the boundary conditions

$$\underline{u}(\underline{x}) \stackrel{\text{def}}{=} 0, \quad \underline{x} \in \partial\Omega_D, \qquad g(\underline{x}) \stackrel{\text{def}}{=} \begin{bmatrix} \sin(\pi x_2) \\ 0 \end{bmatrix}, \quad \underline{x} \in \partial\Omega_T \tag{5 4 14}$$

where $\partial\Omega_D \stackrel{\text{def}}{=} \{\underline{x} \mid x_1 = 0,\ 0 \le x_2 \le 1\}$, and $\partial\Omega_T \stackrel{\text{def}}{=} \partial\Omega \backslash \partial\Omega_D$ The respective finite element triangulations, $\mathcal{T}_h(\Omega_i)$, $1 \le i \le k$, domain decompositions, $\overline{\Omega} = \cup_{i=1}^k \overline{\Omega}_i$, iteration parameters $\tau, n_{\max}$, and weights $w(i,\gamma,1)$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$ are constructed and defined as in problem 5 4 1 The computational results obtained with algorithms $\mathcal{ALG}$ 1, 2, 3 are presented in Table 5 4 3b

The asymptotic bound (5 3 45) is again confirmed by the results presented in Table 5 4 3b and, comparing these results with those in Table 5 4 3c, it is revealed that the constant, $C > 0$, which appears in (5 3 45), becomes dependent on the parameters α_i, $1 \le i \le k$

Table 5 4 3b $a_{ijkl} \in \mathcal{PC}^{\infty}(\Omega)$, $1 \le i, j, k, l \le 2$

h	H	$\mathcal{ALG}$ 1 n	$\kappa(S_h)$	$\mathcal{ALG}$ 2 n	$\kappa(N_h^{-1}S_h)$	$\mathcal{ALG}$ 3 n	$\kappa(M_h^{-1}S_h)$
1/16	1/2	38	1 3489(+3)	22	7 4894(+2)	8	1 8573(+0)
1/16	1/2	NC	2 8718(+3)	26	9 1959(+2)	9	2 2977(+0)
1/32	1/2	NC	5 8123(+3)	29	1 1149(+3)	10	2 8098(+0)
1/16	1/4	NC	2 6064(+3)	NC	6 0693(+3)	13	4 3405(+0)
1/32	1/4	NC	4 7821(+3)	NC	7 3067(+3)	17	6 3227(+0)
1/64	1/4	NC	2 9972(+3)	NC	8 6525(+3)	20	8 4442(+0)
1/32	1/8	NC	2 0358(+3)	NC	2 5722(+4)	18	5 6481(+0)
1/64	1/8	NC	4 1683(+3)	NC	2 4205(+4)	22	8 4329(+0)
1/128	1/8	NC	7 7705(+3)	NC	2 2588(+4)	27	1 1250(+1)

NC≡ No convergence after 80 iterations, $w(i,\gamma,1) = \alpha_i / \sum_{j\in a(i,\gamma)} \alpha_j$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

Table 5 4 3c $a_{ijkl} \in \mathcal{PC}^{\infty}(\Omega)$, $1 \le i, j, k, l \le 2$

h	H	$\mathcal{ALG}$ 2 n	$\kappa(N_h^{-1}S_h)$	$\mathcal{ALG}$ 3 n	$\kappa(M_h^{-1}S_h)$
1/16	1/2	39	4 5069(+2)	19	7 9592(+0)
1/16	1/2	48	5 2723(+2)	25	1 0679(+1)
1/32	1/2	54	6 0983(+2)	29	1 2915(+1)
1/16	1/4	NC	4 7172(+3)	39	3 1188(+1)
1/32	1/4	NC	5 8561(+3)	51	4 6471(+1)
1/64	1/4	NC	7 0365(+3)	59	6 1436(+1)
1/32	1/8	NC	1 7856(+4)	53	4 2239(+1)
1/64	1/8	NC	1 7347(+4)	66	6 3297(+1)
1/128	1/8	NC	1 6872(+4)	78	8 9413(+1)

$w(i,\gamma,1) = 1/|a(i,\gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$

when one defines $w(i,\gamma,p) \stackrel{\text{def}}{=} 1/|a(i,\gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$, $p \ge 1/2$ Furthermore, we point out that, based on the smaller values of $\kappa(N_h^{-1}S_h)$, $h > 0$ reported in Table 5 4 3c, one may expect more rapid convergence of algorithm $\mathcal{ALG}$ 2 when the weights are given by $w(i,\gamma,1) \stackrel{\text{def}}{=} 1/|a(i,\gamma)|$, $1 \le i \le k$, $\gamma \in \mathcal{G}(\Gamma)$ rather than (5 3 31), however, if the spectrum, $\sigma(N_h^{-1}S_h)$, $h > 0$, consists of a smaller number of compactly clustered groups of eigenvalues when the Neumann–Neumann preconditioner is defined in terms of the weights (5 3 30) then one should expect these results Indeed, we suggest that this is the explanation for the results obtained with the Neumann–Neumann preconditioner in Tables 5 4 3b,c

5 5 Conclusions

Our aim in Section 5 4 was to demonstrate through the use of computational examples that, for problems of heterogeneous linear elasticity, the inclusion of a globally defined coarse problem within the definition of a Neumann–Neumann type preconditioner leads to faster rates of convergence which do not vary significantly when the material properties change by large orders of magnitude and possess asymptotic properties which are similar to those first established in BREZINA & MANDEL (1993) as $H, h \to 0$ It was also our aim to implicitly demonstrate that the introduction, at the continuous level, of the bilinear forms b_i, $i \in S$ leads to an efficient and reliable approach to the solution of the undetermined problems (5 3 1) which are often treated in the literature with ad hoc modifications at the discrete level of the matrices $A_{i,h} = (a_i(\phi_r, \phi_s))_{r\,s=1}^{2N}$, $1 \leq i \leq k$

The results obtained in Section 5 4 show that, if one solves the domain decomposed interface problem (5 1 1) with the conjugate gradient algorithm using the preconditioner $M_h^{-1} \in \mathbb{R}^{2N,2N}$ then, as $H \to 0$, this leads to

(1) Dramatic increases in the convergence rate, $C(M_h^{-1} S_h)$, $h > 0$, compared with either the Neumann–Neumann preconditioner, N_h^{-1}, $h > 0$, (using any definition of $w(i, \gamma, p)$, $1 \leq i \leq k$, $\gamma \in \mathcal{G}(\Gamma)$, $p \geq 1/2$) or no preconditioner, i e , $P_h^{-1} = I$,

(2) Independence of the condition number, $\kappa(M_h^{-1} S_h)$, $h > 0$, and, therefore, the convergence rate of algorithm $\mathcal{ALG}$ 3 from material discontinuities and, thus, singularities,

(3) Logarithmic rate of growth $\log h^{-1}$ of $\kappa(M_h^{-1} S_h)$ as $h \to 0$ and, therefore, a slow decrease of the convergence rate, $C(M_h^{-1} S_h)$, as $h \to 0$

Finally, we observe that the coarse problem is required primarily for $H \approx 0$, i e , when the number of domains, k, is large It is cheap to implement because the coarse system matrix, B, is small compared to S_h, $h > 0$ and it is computed and factored only once

6 Discussion

Motivated by the need to devise reliable numerical methods for the treatment of elliptic equations and systems with coefficients which vary rapidly, discontinuously, and by large orders of magnitude, we have considered two different approaches In the first approach we have used homogenization concepts and Fourier series expansions to construct asymptotic expansions which can approximate the solutions of these problems in the case when the coefficients are periodic with period ε We have computed the asymptotic orders at which these approximations converge using extensive computational tests and analytical results In the second approach we have reformulated the Galerkin problem as a system of such problems using domain decomposition techniques and showed how these problems can be efficiently interfaced by constructing preconditioning operators which allow one to use conjugate gradient algorithms for the rapid iterative solution of the interface problem We have provided theoretical results which establish the preconditioning properties of this operator as $H, h \to 0$ and, using a number of computational results, demonstrated that these properties are fulfilled in practice

Clearly, the asymptotic approach is only applicable for problems in which $\varepsilon \approx 0$ because it introduces errors of the order $O(\varepsilon^t)$ for some $t > 0$ which depend on the norm topology and the asymptotic approximations used An important property of these approximations is that the order, t, at which they converge does not vary with the level of regularity of the coefficients, thus, we expect identical rates of convergence for problems with either smooth or discontinuous material properties However, the regularity of the right hand side, f, of a problem is fundamental in this approach because it determines the rates, and the maximum possible rates, of convergence as $\varepsilon \to 0$ Furthermore, the level of regularity of f also determines how rapidly its Fourier series expansion converges Indeed, this latter property may cause practical difficulties, for example, if f is piecewise continuous then its Fourier series will converge slowly in the neighbourhoods of any discontinuities and many terms may be required to accurately represent the solution We observe that this difficulty also arises in BABUŠKA & MORGAN (1991) where, instead of a Fourier series, there is a Fourier transform and the task is to evaluate an integral over $\mathbb{R}^n$, $n \geq 1$ which may converge slowly We feel that one may attempt to treat this difficulty by using approximations, e g splines or mollifiers, which smooth the discontinuities of f in C and thus obtain more rapidly convergent Fourier series Clearly, the success of this approach would depend on how well one can control the magnitude of the additional errors which this process would introduce Unfortunately, we do not have sufficient time to explore this possibility

We have seen that the solutions, ϕ, of the elliptic problems of the type considered in Chapter 3 are holomorphic functions of ε and t everywhere in $\mathbb{R}^2 \backslash \mathcal{S}$ where

$$\mathcal{S} \stackrel{\text{def}}{=} \{(\varepsilon, t) \in \mathbb{R}^2 \mid \|(\epsilon, \tau) - (\varepsilon, t)\|_2 \to 0 \Rightarrow \|\phi(\bullet, \epsilon, \tau), H^1(\mathcal{P})\| \to \infty\} \qquad (6\ 1)$$

However, for $(\varepsilon, t) \in \mathcal{A} \stackrel{\text{def}}{=} \{\underline{x} \in \mathbb{R}^2 \mid (0 \to \underline{x}) \cap \mathcal{S} \neq \emptyset\}$, the asymptotic approximations ϕ_N, $N \geq 0$ fail to converge, i e ,

$$\|\phi(\bullet, \varepsilon, t) - \phi_N(\bullet, \varepsilon, t), H^1(\mathcal{P})\| \not\to 0 \quad (N \to \infty) \qquad (6\ 2)$$

Nevertheless, the good qualitative approximation properties illustrated in graphs 3 4 1–3 4 6 and 3 6 1–3 6 6 motivated our decision to use the asymptotic approximations ϕ_N, $N \geq 0$ at any point in $\mathcal{A}$ However, this differs from the elliptic problems studied in BABUŠKA & MORGAN (1991ii) which include the zero order term, $a\phi$, in their formulation the solutions, ϕ, of such problems are holomorphic *everywhere* in the (ε, t)–plane, i e ,

$$\|\phi_N(\bullet, \varepsilon, t) - \phi(\bullet, \varepsilon, t), H^1(\mathcal{P})\| \to 0 \quad (N \to \infty) \qquad \varepsilon, t \in \mathbb{R}, \qquad (6\ 3)$$

and the functions ϕ_N, $N \geq 0$ therefore provide valid asymptotic approximations everywhere in the (ε, t)–plane For $f_C \in H^m(C) \backslash H^{m+1}(C)$ the precise rate at which the asymptotic approximations $\tilde{u}^{\varepsilon}_{N,M,\ell,h}$, $N \geq m+2$, $M = m+2-p$, $\ell \in \mathbb{N}$, $h > 0$ converge to u^{ε} in the $H^p(C)$ norm topology as $\varepsilon \to 0$ remains an unsettled point, although we expect that more accurate estimates of these asymptotic rates of convergence can be determined by further reducing the discretization error through the use of more refined, perhaps, graded triangulations $\mathcal{T}_h(\Omega)$, $h > 0$ and/or adaptive techniques of approximation The task of attaining a given truncation error tolerance, e g , $\|f - f_\ell, \mathcal{L}_2(\Omega)\| < \tau$ for minimal $\ell \in \mathbb{N}$, provides a more difficult challenge, however, because the approximations $\chi_{\alpha,h}$, $|\alpha| \geq 1$, $h > 0$ and $\phi_{N,h}(\bullet, \varepsilon, \underline{n}\pi)$, $\underline{n} \in \mathcal{Z}^2 \backslash \{0\}$ are independent they can be computed in parallel efficiently on computers with parallel architectures

The results which we have obtained are similar to those given by BOURGAT (1978) who uses the classical two–scale asymptotic expansions of BENSOUSSAN, LIONS, & PAPANICOLAOU (1978) Indeed, in BOURGAT (1978) it is claimed that the following error estimate is valid for solutions of the homogenized problem, u_0, satisfying $u_0 \in C^{6,\lambda}(\overline{\Omega})$

$$\|u^\varepsilon - u^\varepsilon_N, H^1(\Omega)\| \le C\,\varepsilon^{(N+2)/2}, \quad \varepsilon > 0, \quad 0 \le N \le 1 \tag{6 4}$$

where

$$u^\varepsilon_N(\underline{x}) \stackrel{\text{def}}{=} u_0(\underline{x}) + \varepsilon \sum_{|\alpha|=1} D^\alpha u_0(\underline{x})\,\eta_\alpha(\underline{x}/\varepsilon) + \quad + \varepsilon^N \sum_{|\alpha|=N} D^\alpha u_0(\underline{x})\,\eta_\alpha(\underline{x}/\varepsilon), \quad \varepsilon > 0, \quad N \ge 0 \tag{6 5}$$

and the functions η_α, $|\alpha| \ge 1$ are solutions of elliptic problems on $\mathcal{P}$, cf BAKHVALOV & PANASENKO (1989) Although the regularity assumption $u_0 \in C^{6,\lambda}(\overline{\Omega})$ is unlikely to be satisfied in practice, e g , $f \in \mathcal{L}_2(\Omega)\backslash H^1(\Omega)$ or Ω is nonconvex, this result shows that the approximations u^ε_N, $N \ge 0$ fulfill similar asymptotic rates of convergence as those observed in relations (4 5 1) and (4 5 2) Indeed, the analysis of Chapter 3 showed that these approaches are identical for the problems considered there The presence of the functions $D^\alpha u_0$, $|\alpha| \le N$ in the definition for u^ε_N, $N \ge 0$ causes a difficulty which does not arise in our approach the task of computing reliable numerical approximations of $D^\alpha u_0$, $|\alpha| \le N$ will often require special computational schemes, e g , gradient recovery techniques, and, depending on the form and regularity of f, these may introduce significant discretization errors

Thus, if the truncation errors, $\|\phi(\bullet, \varepsilon, \underline{n}\pi) - \phi_N(\bullet, \varepsilon, \underline{n}\pi), H^1(\mathcal{P})\|$, $\varepsilon > 0$, $\underline{n} \in \mathcal{Z}^2\backslash\{0\}$ and $\|f - f_\ell, \mathcal{L}_2(\Omega)\|$, $\ell \in \mathbb{N}$, can be made sufficiently small then this approach provides reliable numerical approximations Conversely, if the asymptotic truncation errors, $\|\phi(\bullet, \varepsilon, \underline{n}\pi) - \phi_N(\bullet, \varepsilon, \underline{n}\pi), H^1(\mathcal{P})\|$, $\underline{n} \in \mathcal{Z}^2\backslash\{0\}$, are too large for a given $\varepsilon > 0$ then, clearly, one must consider alternative methods of approximation for ϕ, e g , approximations of the form

$$\phi^\varepsilon_R(\underline{x}, \underline{t}) \stackrel{\text{def}}{=} \sum_{k=1}^{n} \zeta_k(\underline{x})\,\varrho_k(\underline{t}), \quad \underline{x} \in \mathcal{P} \subset \mathbb{R}^n, \quad \underline{t} \ne 0 \tag{6 6}$$

where ϱ_k, $1 \le k \le n$ are rational functions of $\underline{t}$ provide the basis for a different approach Indeed, the task is then to compute the approximations, ϕ^ε_R, such that the error, $\|\phi(\bullet, \varepsilon, \underline{n}\pi) - \phi^\varepsilon_R(\bullet, \underline{n}\pi), H^1(\mathcal{P})\|$, is small for $\|\underline{n}\|_\infty \le \ell$, $\ell \in \mathbb{N}$

The asymptotic approach can also be applied to problems of linear elastic or viscoelastic deformation, however, the difficulties described above become more pronounced because of the need to employ Fourier series expansions for each component of the body force $f = [f_1, f_2]$ Furthermore, the materials which exist in reality do not have perfectly periodic structures, in fact, the coefficients a_{ijkl}, $1 \le i, j, k, l \le 2$ can be considered as perturbations of periodic functions in the sense that, for almost all $\underline{x} \in \mathbb{R}^2$ and some $\tau > 0$,

$$|a_{ijkl}(\underline{x} + \underline{n}) - a_{ijkl}(\underline{x})| < \tau, \quad \underline{n} \in \mathcal{Z}^2, \quad 1 \le i, j, k, l \le 2 \tag{6 7}$$

In this case, the assumption of periodicity will introduce errors which need to be investigated

In the second approach our decision to use domain decomposition techniques as a method for developing practical parallel algorithms for the solution of large scale linear elastic problems was motivated by the opportunity to use the greater computational power provided by modern computers with parallel architectures

The computational results show that Algorithm $\mathcal{ALG}$ 3 provides a very robust approach for the solution of large scale elastic Galerkin problems However, the theoretical condition number bound provided in Theorem 5 7 requires that the boundaries, $\partial\Omega_i$, $1 \le i \le k$, of the subdomains, Ω_i, $1 \le i \le k$, should be aligned with the discontinuities of the coefficients a_{ijkl}, $1 \le i, j, k, l \le 2$ In some cases this assumption may be impractical or inconvenient and one may be compelled to construct decompositions (5 1 1 1) with the property $a_{ijkl} \notin C^0(\Omega_r)$, $1 \le i, j, k, l \le 2$, $1 \le r \le k$, i e , such that the discontinuities of a_{ijkl}, $1 \le i, j, k, l \le 2$ are not aligned with the boundaries, $\partial\Omega_i$, of the subdomains Ω_i, $1 \le i \le k$ Although, in this case, the condition number $\kappa(M_h^{-1} S_h)$ can again be bounded according to relation (5 3 45) the constant $C > 0$ will depend on the parameters α_i, $1 \le i \le k$, cf definition 5 2 Indeed, if the condition number increases with the magnitude of the coefficient discontinuities then the rate at which the iterates $\underline{u}_{\Gamma,h}^{(n)}$ converge to $\underline{u}_{\Gamma,h}$ as $n \to \infty$ will, correspondingly, decrease We feel that this is a shortcoming of the approach which is difficult to overcome, however, it is a difficulty which all domain decomposition methods share

For problems in three dimensions, $\Omega \subset \mathbb{R}^3$, one can also construct the preconditioning operator M_h, $h > 0$ for approximating spaces $S^h(\Omega) \subset H^1(\Omega)$ consisting of piecewise linear functions defined on tetrahedral triangulations $\mathcal{T}_h(\Omega)$, $h > 0$ We feel that Theorem 5 7 can be generalized to include problems of this type, however, because domain decomposition methods which use Steklov–Poincaré operators cause many more subdomains to be coupled than domain decomposition methods which use Lagrange multipliers to interface subdomains we expect that this approach will not compare favourably with Lagrange multiplier type approaches Finally, we feel that this approach would benefit from the use of approximating spaces other than $S^h(\Omega)$, $h > 0$ which can be employed, for example, to treat singularities

7 REFERENCES

Below we provide a list of the references used in the preceding Chapters The surnames of the authors of each reference in this list are ordered alphabetically with the year of publication given after the final authors name The list of references are ordered alphabetically according to the surnames of the authors of each reference Those references which cannot be distinguished purely by the authors' names and the year of publication include a roman numeral to resolve any ambiguity, e g , BABUŠKA & MORGAN (1991i) and BABUŠKA & MORGAN (1991ii)

Adams, R A (1975) *Sobolev Spaces*, Academic Press

Agoshkov, V I (1988) *Poincaré–Steklov's Operators and Domain Decomposition Methods in Finite Dimensional Spaces*, Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, pages 73–112

Akin, J E (1982) *Application and Implementation of Finite Element Methods*, Academic Press

Allaire, G (1992) *Homogenization and Two-Scale convergence*, SIAM J Math Anal, Volume 23, No 6, pages 1482–1518

Ashby, S , Manteuffel, T A , & Saylor, P E (1990) *A Taxonomy for Conjugate Gradient Methods*, SIAM J Numer Anal, Volume 27, pages 1542–1568

Axelsson, O (1994) *Iterative Solution Methods*, Cambridge University Press

Aziz, A K & Babuška, I (1972) *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Academic Press

Babuška, I (1974i) *Solution of Problems with Interfaces and Singularities*, Technical Note BN–789, University of Maryland

Babuška, I (1974ii) *Solution of Interface Problems by Homogenization I*, Technical Note BN–782, University of Maryland

Babuška, I (1974iii) *Solution of the Interface problem by Homogenization II*, Technical Note BN–787, University of Maryland

Babuška, I & Morgan, R C (1985) *Composites with a periodic structure Mathematical analysis and numerical treatment*, Comp & Maths with Appls , Volume 11, No 10, pages 995–1005

Babuška, I & Morgan, R C. (1991i) *An approach for constructing families of homogenized equations for periodic media I An integral representation and its consequences*, SIAM J Math Anal , Volume 22, No 1, pages 1–15

Babuška, I & Morgan, R C (1991ii) *An approach for constructing families of homogenized equations for periodic media II Properties of the kernel*, SIAM J Math Anal , Volume 22, No 1, pages 16–33

Babuška, I & Osborn, J.E (1985) *Finite element methods for the solution of problems with rough input data*, Singularities and constructive methods for their treatment, Lecture Notes in Mathematics, Springer–Verlag, Volume 1121, pages 1–27

Bakhvalov, N S & Panasenko, G P (1989) *Homogenization Averaging Processes in Periodic Media*, Kluwer Academic Publishers, Dordrecht

Bensoussan, A , Lions, J L , & Papanicolaou, G (1978) *Asymptotic methods in periodic structures*, North–Holland, Amsterdam

Bjorstad, P E & Hvidsten, A (1987) *Iterative Methods for Substructured Elasticity Problems in Structural Analysis*, SIAM J Numer Anal , Volume 23, pages 301–312

Bjorstad, P E & Widlund, O B (1986) *Iterative Methods for Solving Elliptic Problems on Regions Partitioned into Substructures*, SIAM J Numer Anal , Volume 23, pages 1097–1120

Blumenfeld, M (1985) *The regularity of interface problems on corner regions*, Singularities and constructive methods for their treatment, Lecture Notes in Mathematics Volume 1121, Springer–Verlag, pages 38–54

Bourgat, J F , Glowinski, R , LeTallec, P , & Vidrascu, M (1989) *Variational Formulation and Algorithm for Trace Operator in Domain Decomposition Calculations*, Proceedings of the Second International Symposium on Domain Decomposition Methods, SIAM, pages 3–16

Bourgat, J F (1978) *Numerical experiments of the homogenization method for operators with periodic coefficients*, Laboratoire de recherche en informatique et automatique, Rapport de Recherche No 277

Bramble, J H , Pasciak, J E , & Schatz, A H (1986) *The Construction of Preconditioners for Elliptic problems by Substructuring, I*, Math Comp , Volume 47, pages 103–134

Brenner, S C & Ridgway Scott, L (1994) *The Mathematical Theory of Finite Element Methods*, Springer–Verlag

Brezina, M. & Mandel, J (1993) *Balancing Domain Decomposition Theory and Performance in Two and Three Dimensions*, Technical Report, Computational Mathematics Group, University of Colorado at Denver, March

Brezzi, F. (1974) *On the Existence, Uniqueness and Approximation of Saddle Point Problems arising for Lagrangian Multipliers*, R A I R O Anal Numer , Volume 8, pages 129–151

Briggs, H A & Hwang, K. (1986) *Computer Architecture and Parallel Processing*, McGraw–Hill

Brown, A L & Page, A (1970) *Elements of Functional Analysis*, Van Nostrand Reinhold Company

Champeney, D C (1987) *A Handbook of Fourier theorems*, Cambridge University Press

Ciarlet, P G (1978) *The Finite Element Method for Elliptic Problems*, North–Holland

DeRoeck, Y.H & LeTallec, P. (1991) *Analysis and Test of a Local Domain Decomposition Preconditioner*, Proceedings of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, pages 112–128

DeRoeck, Y H , LeTallec, P & Vidrascu, M (1991) *Domain Decomposition Methods for Large Linearly Elliptic Three Dimensional Problems*, J Comput Appl Math , Volume 34, pages 93–117

Dryja, M (1991) *A Method of Domain Decomposition for Three-Dimensional Finite Element Elliptic Problems*, Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, pages 43–61

Dryja, M , & Widlund, O B (1992) *Additive Schwarz Methods for Elliptic Finite Element Problems in Three Dimensions*, Proceedings of the Fifth International Symposium on Domain Decomposition methods for Partial Differential Equations, pages 3–18

Edmunds, D E & Evans, D (1989) *Spectral Theory and Differential Operators*, Oxford University Press

Farhat, C. (1991) *A Saddle-Point Principle Domain Decomposition Method for the Solution of Solid Mechanics Problems*, Proceedings of the Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, pages 271–292

Farhat, C & Roux, F–X (1991) *A Method of Finite Element Tearing and Interconnecting and its Parallel Solution Algorithm*, Int J Numer Methods Engrg , Volume 32, pages 1205–1227

Farhat, C., Mandel, J , & Roux, F–X (1994) *Optimal Convergence Properties of the FETI Domain Decomposition Method*, Comput Methods Appl. Mech. Engrg , Volume 115, pages 365–385

Golub, G E & Van Loan, C F. (1989) *Matrix Computations*, Johns and Hopkins

Grisvard, P (1985) *Elliptic Problems in Nonsmooth Domains*, Pitman

Hackbusch, W (1992) *Elliptic Differential Equations, Theory and Numerical Treatment*, Springer–Verlag

Hildebrand, F B (1987) *Introduction to Numerical Analysis*, Dover

Horn, R A. & Johnson, C R (1985) *Matrix Analysis*, Cambridge University Press

Kantorovich, L V & Krylov, V I (1964) *Approximate Methods in Higher Analysis*, Interscience Publishers, New York

Kellogg, R B (1971) *Singularities in Interface Problems*, Numerical Solution of Partial Differential Equations–II, Academic Press, pages 351–400

Kellogg, R B (1972) *Higher Order Singularities for Interface Problems*, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Academic Press, pages 589–602

Knops, R J. & Payne, L E. (1971) *Uniqueness Theorems in Linear Elasticity*, Springer Tracts in Natural Philosophy, Volume 19

Kreyszig, E. (1978) *Introductory Functional Analysis with Applications*, John Wiley & Sons

LeTallec, P (1994) *Domain Decomposition Methods in Computational Mechanics*, Computational Mechanics Advances, Volume 1, No 2, North-Holland

Lions, J L. & Magenes, E. (1972) *Non-Homogeneous Boundary Value Problems Volume I*, Springer-Verlag

Mandel, J. (1993) *Balancing Domain Decomposition*, Commun Numer Methods Engrg , Volume 9, pages 233–241

Marsden, J E & Hughes, T.J R. (1987) *Mathematical Foundations of Elasticity*, Prentice Hall

Murat, F. & Tartar, L. (1994) *H-Convergence*, Publications du laboratoire d'analyse numerique, A94001

Nitsche, J.A. & Schatz, A H. (1974) *Interior Estimates for Ritz-Galerkin Methods*, Math Comp , Volume 28, pages 937-958

Oden, J.T & Reddy, J N. (1976) *An Introduction to the Mathematical Theory of Finite Elements*, Wiley–Interscience

Quarteroni, A. & Valli, A. *Theory and Application of Steklov-Poincaré Operators for Boundary Value Problems*, Industrial and Applied Mathematics

Riensch, C. & Bauer, F L. (1968) *Rational QR Transformations with Newton's Shift for Symmetric Tridiagonal Matrices*, Numer Math , Volume 11, pages 264–272

Riesz, F & SZ.–Nagy, B. (1965) *Functional Analysis*, Frederick Ungar Publishing Company

Sanchez–Palencia, E. (1980) *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Volume 127, Springer–Verlag

Schwarz, H A (1890) *Gesammelte Mathematische Abhandlungen*, Volume 2, Springer

Spencer, A J M (1980) *Continuum Mechanics*, Longman

Smirnov, V.I (1964) *A Course of Higher Mathematics, Volume V Integration and Functional Analysis*, Pergamon Press

Tartar, L (1980) *Convergence of the Homogenization Process*, Appendix of SANCHEZ–PALENCIA (1980)

Wloka, J (1987) *Partial Differential Equations*, Cambridge University Press