ON THE TWO DIMENSIONAL GIERER-MEINHARDT SYSTEM WITH STRONG COUPLING

JUNCHENG WEI * AND MATTHIAS WINTER †

Abstract. We construct solutions with a single interior condensation point for the two-dimensional Gierer-Meinhardt system with strong coupling. The condensation point is located at a nondegenerate critical point of the diagonal part of the regular part of the Green's function for $-\Delta+1$ under the Neumann boundary condition. Our method is based on Liapunov-Schmidt reduction for a system of elliptic equations.

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1. Introduction. We study the Gierer-Meinhardt system (see [14]) which models biological pattern formation and can be written as follows (already suitably scaled)

$$(GM) \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}, \quad A > 0 \quad \text{in } \Omega, \\ \tau H_t = D \Delta H - H + \frac{A^r}{H^s}, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here, the unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations at a point $x \in \Omega \subset R^N$ and at time t of the biochemicals called activator and inhibitor, respectively; ϵ, τ, D are positive constants; $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^N; Ω is a smooth bounded domain in R^N; $\nu(x)$ is the outer normal at $x \in \partial\Omega$. The exponents p, q, r, s are assumed to satisfy the conditions

$$(A) \quad 1 < p < \left(\frac{N+2}{N-2}\right)_+, \; q > 0, \; r > 0, \; s \geq 0, \text{ and } 0 < \frac{p-1}{q} < \frac{r}{s+1}$$

where $(\frac{N+2}{N-2})_+ = \frac{N+2}{N-2}$ if $N \geq 3$; $= +\infty$ if $N = 1, 2$.

In numerical simulations of the activator-inhibitor system (GM), it is observed that, when the ratio ϵ^2/D is small, (GM) seems to have stable stationary solutions with the property that the activator concentration is localized around a finite number of points in $\overline{\Omega}$. Moreover, as $\epsilon \to 0$ the pattern exhibits a *"point condensation phenomenon"*. By this we mean that the activator concentration is localized in narrower and narrower regions around some points and eventually shrinks to a certain set of points as $\epsilon \to 0$. Hereby the maximum value of the inhibitor concentration diverges to $+\infty$.

*Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (wei@math.cuhk.edu.hk).

†Mathematisches Institut A, Universität Stuttgart, D-70511 Stuttgart, Germany (winter@mathematik.uni-stuttgart.de).

The stationary equation for (GM) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2\Delta A - A + \frac{A^p}{H^q} = 0, \quad A > 0 \quad \text{in } \Omega, \\ D\Delta H - H + \frac{A^r}{H^s} = 0, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Generally speaking system (1.1) is quite difficult to solve since it does neither have a variational structure nor a priori estimates. One way to study (1.1) is to examine the so-called *shadow system*. Namely, we let $D \to +\infty$ first. It is known (see [23], [31], [34], [39]) that the study of the shadow system amounts to the study of the following single equation:

$$\begin{cases} \epsilon^2\Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) has a variational struture and has been studied by numerous authors. It is known that equation (1.2) has both boundary spike solutions and interior spike solutions. For boundary spike solutions, see [5], [9], [15], [17], [22], [29], [30], [31], [39], [44], [46], and the references therein. (When $p = \frac{N+2}{N-2}, N \geq 3$, boundary spike solutions of (1.2) have been studied in [1], [2], [3], [12], [13], [27], etc.) For interior spike solutions, please see [4], [6], [18], [21], [40], [41], [45]. For stability of spike solutions, please see [7], [19], [32], [42] and [43].

In the case when D is finite and not large (this is the so-called *strong coupling* case), there are only very few results available. For $N = 1$, one can construct spike solutions for all $D \geq 1$. See [37]. In higher dimensions, as far as we know, there are no results, yet. (See [8], [28], and [34] for the study of related systems.) In this paper, we consider the case $N = 2$ since it has a particular asymptotic behavior.

Remark. Our approach does not work for dimensions $N \geq 3$ due to a different asymptotic behavior of the Green's function of $-\Delta + 1$ with the Neumann boundary condition.

From now on we suppose that $N = 2$. For simplicity we let $D = 1$.

We construct solutions with a single interior condensation point. It turns out that the condensation points in this case are different from those in the shadow system. We need to introduce some notation. Let $G(P, x)$ be the Green's function of $-\Delta + 1$ under the Neumann condition, i.e., G satisfies

$$\begin{cases} -\Delta G + G = \delta_P \quad \text{in } \Omega, \\ \frac{\partial G}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases}$$

where δ_P is the Dirac delta distribution at point P. It is also known that

$$G(P, x) = K(|x - P|) - H(P, x)$$

where $K(|x|)$ is the fundamental solution of $-\Delta + 1$ in R^2 with singularity at 0 and $H(P, x)$ is C^2 in Ω. It is known that

$$K(r) = -\log r - \mu + O(r) \quad \text{for } r \text{ small.} \tag{1.3}$$

We call $h(P) := H(P, P)$ the diagonal part of $H(P, x)$.

We have

THEOREM 1.1. *Let $P_0 \in \Omega$ be a nondegenerate critical point of $h(P)$. Then for ϵ sufficiently small, problem (1.1) has a solution (A_ϵ, H_ϵ) with the following properties:*

(1) $A_\epsilon(x) = \xi_\epsilon^{q/(p-1)}(w(\frac{x-P_\epsilon}{\epsilon}) + o(1))$ uniformly for $x \in \bar{\Omega}$ where $\xi_\epsilon > 0$ will be determined later, $P_\epsilon \to P_0$ as $\epsilon \to 0$, and w is the unique solution of the problem

$$(1.4) \qquad \begin{cases} \Delta w - w + w^p = 0, \quad w > 0 \quad \text{in } R^2, \\ w(0) = \max_{y \in R^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases}$$

(2) $H_\epsilon(x) = \xi_\epsilon(1 + O(\frac{1}{|\log \epsilon|}))$ uniformly for $x \in \bar{\Omega}$.

(3) $\xi_\epsilon^{s+1-\frac{qr}{p-1}} = (1 + o(1))\epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^r$.

Remark. It is known that the solution w to (1.4) is radial, unique and decays exponentially. (See [16], [24].)

We now outline the proof of Theorem 1.1.

Our method is based on Liapunov-Schmidt reduction which was used in [11], [35] and [36] to study semi-classical solutions of the nonlinear Schrödinger equation

$$(1.5) \qquad \frac{h^2}{2}\Delta U - (V - E)U + U^p = 0$$

in R^N where V is a potential function and E is a real constant. Namely, in [11], [35] and [36] solutions of (1.4) are constructed near a nondegenerate critical point of V provided that h is sufficiently small. Later this method was used in [17], [18], [41], [44], [45], [46] to construct spike solutions for (1.2).

Here we face a system of elliptic equations. Therefore the process is more complicated. To lay down the basic idea of our proof, we let

$$A_\epsilon = \xi_\epsilon^{q/(p-1)}\bar{A}_\epsilon, \quad H_\epsilon = \xi_\epsilon \bar{H}_\epsilon$$

where ξ_ϵ is to be chosen later. It is easy to see that system (1.1) is equivalent to the following

$$(1.6) \qquad \begin{cases} \epsilon^2 \Delta \bar{A}_\epsilon - \bar{A}_\epsilon + \bar{A}_\epsilon^p / \bar{H}_\epsilon^q = 0 \quad \text{in } \Omega, \\ \Delta \bar{H}_\epsilon - \bar{H}_\epsilon + c_\epsilon \bar{A}_\epsilon^r / \bar{H}_\epsilon^s = 0 \quad \text{in } \Omega, \\ \frac{\partial \bar{A}_\epsilon}{\partial \nu} = \frac{\partial \bar{H}_\epsilon}{\partial \nu} = 0 \ \text{ on } \Omega, \end{cases}$$

where

$$c_\epsilon = \xi_\epsilon^{\frac{qr}{p-1} - (s+1)}.$$

We fix a point $P \in \Omega$. We rescale

$$\tilde{A}_\epsilon(y) := \bar{A}_\epsilon(P + \epsilon y), \quad x = \epsilon y + P, \quad y \in \Omega_{\epsilon,P} := \{y | P + \epsilon y \in \Omega\}.$$

Then as $\epsilon \to 0$, if we assume that $\bar{H}_\epsilon(P + \epsilon y) \to 1$ in $L^\infty_{loc}(\Omega_{\epsilon,P})$, we have that $\tilde{A}_\epsilon \to V(y)$ where V satisfies

$$\begin{cases} \Delta V - V + V^p = 0, \quad V > 0 \quad \text{in } R^2, \\ V(0) = \max_{y \in R^2} V(y), \quad V \in H^1(R^2). \end{cases}$$

By a uniqueness result it is known that $V(y) = w(y)$ where w is the unique solution of (1.4). (See [16], [24].) Hence

$$\tilde{A}_\epsilon(y) \sim w(y).$$

(Here and thereafter $A \sim B$ means $A = (1 + o(1))B$ as $\epsilon \to 0$ in the corresponding norm.)

To ensure that $\bar{H}_\epsilon(P + \epsilon y) \sim 1$, we note that

$$\bar{H}_\epsilon(P) = \int_\Omega G(P, x)\xi_\epsilon^{\frac{qr}{p-1}-(s+1)} \frac{\bar{A}_\epsilon^r(x)}{\bar{H}_\epsilon^s(x)} dx$$

$$= \epsilon^2 \int_{\Omega_{\epsilon,P}} G(P, P + \epsilon y)\xi_\epsilon^{\frac{qr}{p-1}-(s+1)} \frac{\tilde{A}^r(y)}{\bar{H}_\epsilon^s(P + \epsilon y)} dy$$

(by (1.3), $K(r) = -\log r - \mu + O(r)$ for r small)

$$\sim \xi_\epsilon^{\frac{qr}{p-1}-(s+1)} \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^r(y)\, dy.$$

This suggests that we take

$$\xi_\epsilon^{\frac{qr}{p-1}-(s+1)} \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^r(y)\, dy \sim 1.$$

Hence we should look for solutions of (1.1) with the following properties

$$A_\epsilon = \xi_\epsilon^{q/(p-1)} \bar{A}_\epsilon, \quad \bar{A}_\epsilon(y) = w(y) + \phi_\epsilon(y), \quad \phi_\epsilon \sim 0$$

$$\text{where} \quad y = \frac{x - P_\epsilon}{\epsilon} \quad \text{and} \quad |P_\epsilon - P_0| = o(1) \text{ as } \epsilon \to 0,$$

$$H_\epsilon = \xi_\epsilon \bar{H}_\epsilon, \quad \bar{H}_\epsilon(x) = 1 + \psi_\epsilon(x), \quad \psi_\epsilon \sim 0,$$

and

$$\xi_\epsilon^{\frac{qr}{p-1}-(s+1)} \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^r(y)\, dy \sim 1.$$

There are three main difficulties: First, $w(\frac{x-P_\epsilon}{\epsilon})$ does not satisfy the Neumann boundary condition. Second, the linearized problem arising from (1.4) has the N-dimensional kernel span$\{\frac{\partial w}{\partial y_1}, \ldots, \frac{\partial w}{\partial y_N}\}$. Therefore, if we linearize system (1.6) at $(w(\frac{x-P}{\epsilon}), 1)$ the linearized operator is not uniformly invertible with respect to ϵ. Third, we have two scales: $(\log \frac{1}{\epsilon})^{-1}$ and ϵ. They are simply incomparable.

The first difficulty can be overcome by introducing the following *projection*: Let $U \subset R^2$ be a smooth and open set. Suppose that $W \in H^1(R^2)$. The projection $\mathcal{P}_U W$ is defined by $\mathcal{P}_U W = W - \mathcal{Q}_U W$ where $\mathcal{Q}_U W$ satisfies

$$\begin{cases} \Delta \mathcal{Q}_U W - \mathcal{Q}_U W = 0 & \text{in } U, \\ \frac{\partial \mathcal{Q}_U W}{\partial \nu} = \frac{\partial W}{\partial \nu} & \text{on } \partial U. \end{cases} \tag{1.7}$$

The second difficulty is overcome by first "solving" (1.6) module approximate kernel and cokernel, respectively. Subsequently we use the nondegeneracy of the critical point of h at P_0 to choose P_ϵ near P_0 such that the finite-dimensional part lying in the approximate cokernel vanishes.

The third difficulty can be managed by choosing *suitable* approximate solutions.

From now on, we work with (1.6). The main points of the proof of Theorem 1.1 and the organization of this paper can be described as follows:

A)-Choose good approximate solutions.

We first study the solution $(A_{\epsilon,\mu}(x), H_{\epsilon,\mu}(x), c_{\epsilon,\mu})$ of the following problem

$$\begin{cases} \epsilon^2\Delta A - A + \frac{A^p}{(H(x)-\mu)^q} = 0, \quad x \in R^2, \\ \Delta H - H + c_{\epsilon,\mu}\frac{A^r}{(H(x)-\mu)^s} = 0, \quad x \in R^2, \\ H(0) = 1 + O(\frac{1}{\log\frac{1}{\epsilon}} + \mu) \end{cases} \tag{1.8}$$

where μ is small.

Next we choose $\mu := \mu_\epsilon(P)$ so that

$$\mu = \mathcal{Q}_\Omega(H_{\epsilon,\mu}(\cdot - P))(P). \tag{1.9}$$

Set

$$\hat{A}_{\epsilon,P}(x) := A_{\epsilon,\mu_\epsilon(P)}(x-P), \quad \hat{H}_{\epsilon,P}(x) := H_{\epsilon,\mu_\epsilon(P)}(x-P),$$

$$c_\epsilon = \xi^{\frac{qr}{p-1}-(s+1)}, \quad c_{\epsilon,P} := c_{\epsilon,\mu_\epsilon(P)}.$$

We now choose our approximate solutions:

$$A_{\epsilon,P}(y) := \mathcal{P}_{\Omega_{\epsilon,P}}\hat{A}_{\epsilon,P}(P+\epsilon y), \quad H_{\epsilon,P}(x) := \mathcal{P}_\Omega \hat{H}_{\epsilon,P}(x). \tag{1.10}$$

Set

$$\varphi_{\epsilon,P}(y) := \hat{A}_{\epsilon,P}(y) - A_{\epsilon,P}(y), \quad \psi_{\epsilon,P}(x) := \hat{H}_{\epsilon,P}(x) - H_{\epsilon,P}(x).$$

It will be proved that $\varphi_{\epsilon,P}(y) = O(e^{-d(P,\partial\Omega)/\epsilon})$ for a.e. $y \in \Omega_{\epsilon,P}$ and $\psi_{\epsilon,P} = \frac{1}{\log\frac{1}{\epsilon}}(H(P,x)+o(1))$ uniformly with respect to $x \in \Omega$.

We will analyze $A_{\epsilon,P}$ and $H_{\epsilon,P}$ in Section 2 and Section 3.

B)-The idea now is to look for a solution of (1.6) of the form

$$\bar{A}_\epsilon(P+\epsilon y) = A_{\epsilon,P}(y) + \phi(y), \quad \bar{H}_\epsilon(x) = H_{\epsilon,P}(x) + \psi(x).$$

We will show that, provided P is properly chosen, ϕ and ψ are expected to be insignificantly small.

We now write system (1.6) in operator form.

For any smooth and open set $U \subset R^2$, let

$$W_N^{2,t}(U) = \{u \in W^{2,t}(U) | \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U\}, \quad H_N^2(U) = W_N^{2,2}(U).$$

For $A(y) \in H^2_N(\Omega_{\epsilon,P})$, $H(x) \in W^{2,t}_N(\Omega)$ where $1 < t < 1.1$. (We need $t > 1$ so that the Sobolev embedding $W^{2,t}(\Omega) \subset L^\infty(\Omega)$ is continuous.) Set

$$S_\epsilon \begin{pmatrix} A \\ H \end{pmatrix} = \begin{pmatrix} S_1(A,H) \\ S_2(A,H) \end{pmatrix}$$

where $S_1(A,H) = \Delta_y A - A + A^p/H^q, \quad S_2(A,H) = \Delta_x H - H + c_{\epsilon,P} A^r/H^s$.

Then solving equation (1.6) is equivalent to

$$S_\epsilon \begin{pmatrix} A \\ H \end{pmatrix} = 0, \quad A \in H^2_N(\Omega_{\epsilon,P}), \quad H \in W^{2,t}_N(\Omega). \tag{1.11}$$

We now substitute $A = A_{\epsilon,P}(y) + \phi(y), \quad H = H_{\epsilon,P}(x) + \psi(x)$ into (1.11). The system determining ϕ and ψ can be written as

$$S'_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{pmatrix} E^1_{\epsilon,P} \\ E^2_{\epsilon,P} \end{pmatrix} + \begin{pmatrix} O(\|\phi\|^2_{L^2(\Omega_{\epsilon,P})} + \|\psi\|^2_{L^t(\Omega)}) \\ O(\|\phi\|^2_{L^2(\Omega_{\epsilon,P})} + \|\psi\|^2_{L_t(\Omega)}) \end{pmatrix} = 0,$$

where $E^i_{\epsilon,P}, i = 1,2$ denote the error terms and $E^1_{\epsilon,P} = S_1(A_{\epsilon,P}, H_{\epsilon,P}), E^2_{\epsilon,P} = S_2(A_{\epsilon,P}, H_{\epsilon,P})$. We will estimate the error terms in Section 3.

It is then natural to try to solve the equations for (ϕ,ψ) by a contraction mapping argument. The problem is that the linearized operator $S'_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix}$ is not uniformly invertible with respect to ϵ.

Thererefore, we now replace the above equation with

$$S'_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{pmatrix} E^1_{\epsilon,P} \\ E^2_{\epsilon,P} \end{pmatrix} + \begin{pmatrix} O(\|\phi\|^2_{L^2(\Omega_{\epsilon,P})} + \|\psi\|^2_{L^t(\Omega)}) \\ O(\|\phi\|^2_{L^2(\Omega_{\epsilon,P})} + \|\psi\|^2_{L_t(\Omega)}) \end{pmatrix} = \begin{pmatrix} v_{\epsilon,P} \\ 0 \end{pmatrix}$$

(1.12)

where $v_{\epsilon,P}$ lies in an appropriately chosen approximate cokernel of the linear operator

$$L_\epsilon := \Delta_y - 1 + pA^{p-1}_{\epsilon,P} H^{-q}_{\epsilon,P} - \frac{qr}{s+1} \frac{\int_{\Omega_{\epsilon,P}} A^{r-1}_{\epsilon,P} \cdot}{\int_{\Omega_{\epsilon,P}} A^r_{\epsilon,P}} A^p_{\epsilon,P},$$

$$L_\epsilon :\ H^2(\Omega_{\epsilon,P}) \to L^2(\Omega_{\epsilon,P})$$

and ϕ is orthogonal in $L^2(\Omega_{\epsilon,P})$ to the corresponding approximate kernel of L_ϵ.

C)-We solve (1.12) for (ϕ,ψ) module the approximate kernel. To this end, we need a detailed analysis of the operators L_ϵ and S'_ϵ. This together with the contraction mapping argument is done in Section 4.

D)-In the last step, we study a vector field $P \to W_\epsilon(P)$ such that $W_\epsilon(P) = 0$ implies $v_{\epsilon,P} = 0$ (and hence solutions of the system (1.6) can be found). To discuss the zeros of

$P \to W_\epsilon(P)$ we need very good estimates for the error terms $E^1_{\epsilon,P}$ and $E^2_{\epsilon,P}$. Much of Section 3 is devoted to this analysis. With a good estimate of $E^i_{\epsilon,P}, i = 1, 2$, we discover that under the geometric condition described in Theorem 1.1 there is a point P_ϵ in a small neighborhood of $P_0 \in \Omega$ such that $W_\epsilon(P_\epsilon) = 0$. This will complete the proof of Theorem 1.1 and is done in Section 5.

Finally, we remark that the stability of the solutions constructed in Theorem 1.1 should be related to the matrix $(\nabla_i \nabla_j h(P_0))$. This will be studied in a forthcoming paper.

Throughout this paper, we always assume that $P \in B_r(P_0)$ for some fixed small number $r > 0$. We shall frequently use the following technical lemma.

LEMMA 1.2. *Let u be a solution of*

$$\Delta u - u + f = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Suppose

$$|f(x)| \le \eta e^{-\frac{\alpha|x-P|}{\epsilon}}$$

for some $\alpha > 0$. Then if $\epsilon > 0$ is small enough we have

$$|u(P)| \le C_1 \eta \epsilon^2 \log \frac{1}{\epsilon}, \tag{1.13}$$

and

$$|u(P) - u(x)| \le C_2 \eta \epsilon^2 \log \left(\frac{|x-P|}{\epsilon} + 1 \right) \tag{1.14}$$

where $C_1 > 0, C_2 > 0$ are generic constants (which are independent of $\epsilon > 0$ and $\eta > 0$).

Proof. By the representation formula we calculate

$$u(x) = \int_\Omega G(x,z) f(z) dz$$

and

$$u(P) = \int_\Omega G(P,z) f(z) dz = \epsilon^2 \int_{\Omega_{\epsilon,P}} G(P, P+\epsilon y) f(P+\epsilon y) dy$$

$$\le C_1 \eta \epsilon^2 \log \frac{1}{\epsilon}.$$

Similarly we can obtain (1.14).

□

2. Study of the Approximate Solutions. In this section, we define a good approximate solution and study its properties. We will use the implicit function theorem and perturbation arguments. To this end, it is essential that we have the following important lemma.

LEMMA 2.1.

The operator

$$L := \Delta - 1 + p w^{p-1} - \frac{qr}{s+1} \frac{\int_{R^2} w^{r-1} \cdot}{\int_{R^2} w^r} w^p$$

with w defined in (1.4) is an invertible map from $H^2_r(R^2)$ to $L^2_r(R^2)$, where $H^2_r(R^2)$ ($L^2_r(R^2)$) is the subset of those functions of $H^2(R^2)$ ($L^2_r(R^2)$) which are radially symmetric.
Proof. We just need to prove that

$$\mathrm{kernel}(L) \cap H^2_r(R^2) = \{0\}, \quad \mathrm{kernel}(L^*) \cap H^2_r(R^2) = \{0\}$$

where L^* is the conjugate operator of L.

In fact, let $L\phi = 0$ for $\phi \in H^2_r(R^2)$. Then we have

$$L_0\Big(\phi - \frac{qr}{(p-1)(s+1)} \frac{\int_{R^2} w^{r-1}\phi}{\int_{R^2} w^r} w\Big) = 0$$

where $L_0 := \Delta - 1 + pw^{p-1}$. By Lemma 4.2 of [30], $\phi - \frac{qr}{(p-1)(s+1)} \frac{\int_{R^2} w^{r-1}\phi}{\int_{R^2} w^r} w = 0$. Multiplying this equation by w^{r-1} and integrating over R^2 we see that

$$\int_{R^2} w^{r-1}\phi = 0.$$

Since $\frac{qr}{(p-1)(s+1)} > 1$ we conclude $\phi = 0$.

Next we claim that $\mathrm{kernel}(L^*) \cap H^2_r(R^2) = \{0\}$. Let $\phi \in H^2_r(R^2)$ be such that $L^*\phi = 0$. Namely we have

$$L_0\phi - \frac{qr}{s+1} \frac{\int_{R^2} w^p\phi}{\int_{R^2} w^r} w^{r-1} = 0. \tag{2.1}$$

Multiplying (2.1) by w and integrating over R^2, we obtain

$$\Big(p - 1 - \frac{qr}{s+1}\Big) \int_{R^2} w^p\phi = 0$$

Since $p - 1 - \frac{qr}{s+1} < 0$ we get

$$\int_{R^2} w^p\phi = 0.$$

Hence $L_0\phi = 0$ and $\phi = 0$.

□

We now study the following system

$$\begin{cases} \epsilon^2\Delta A - A + \frac{A^p}{(H - \mathcal{Q}_\Omega H(P))^q} = 0, \quad x \in R^2, \\ \Delta H - H + c_{\epsilon,P} \frac{A^r}{(H - \mathcal{Q}_\Omega H(P))^s} = 0, \quad x \in R^2, \\ H(P) = 1 + O\big(\frac{1}{\log\frac{1}{\epsilon}}\big). \end{cases} \tag{2.2}$$

We have

Theorem 2.2. *For $\epsilon << 1$, there exists a unique solution $(\hat{A}_{\epsilon,P}(x), \hat{H}_{\epsilon,P}(x), c_{\epsilon,P})$ of (2.2) with the following properties:*

(1) $\hat{A}_{\epsilon,P}(x)$ and $\hat{H}_{\epsilon,P}(x)$ depend on $|x - P|$ only;

(2) $\hat{A}_{\epsilon,P} = (1 + o(1))w(\frac{|x-P|}{\epsilon})$;

(3) $\hat{H}_{\epsilon,P}(0) = 1 + O(\frac{1}{\log\frac{1}{\epsilon}})$;

(4) $\hat{H}_{\epsilon,P}(x) = \frac{\sigma_P}{\log\frac{1}{\epsilon}} K(|x-P|) + \frac{\epsilon}{\log\frac{1}{\epsilon}} J_{\epsilon,P}(|x-P|)$ *for* $|x| \geq \delta$ *where* $\sigma_P = 1 + o(1)$, $J_{\epsilon,P}(|x-P|)$, $\nabla_x J_{\epsilon,P}(|x-P|) = O(1)$.

Proof of Theorem 2.2. The proof is divided into the following steps:

STEP 1. We first look for radially symmetric solutions $(A_{\epsilon,\mu}, H_{\epsilon,\mu}, c_{\epsilon,\mu})$ of the following parametrized equation

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^p}{(H-\mu)^q} = 0, \quad x \in R^2, \\ \Delta H - H + c_{\epsilon,\mu} \frac{A^r}{(H-\mu)^s} = 0, \quad x \in R^2, \\ A(x) = A(|x|), \quad H(x) = H(|x|), \quad H(0) = 1 + O(\frac{1}{\log\frac{1}{\epsilon}} + \mu) \end{cases} \tag{2.3}$$

for $0 < \mu << 1$.

Problem (2.3) can be solved by the contraction mapping principle. We first need suitable approximate solutions. We note that the problem

$$\begin{cases} \Delta_y A - A + \frac{A^p}{(1-\mu)^q} = 0, \quad y \in R^2, \\ \Delta_x H - H + c_{\epsilon,\mu,0} \frac{A^r}{(H-\mu)^s} = 0, \quad x \in R^2, \\ x = \epsilon y, A(y) = A(|y|), H(x) = H(|x|), H(0) = 1 \end{cases} \tag{2.4}$$

has a unique solution $(A_{\epsilon,\mu,0}(y), H_{\epsilon,\mu,0}(x), c_{\epsilon,\mu,0})$ for $0 < \mu << 1$. In fact, it is well-known that (for given μ small) the first equation has the unique positive solution $A_{\epsilon,\mu,0}(y) = (1-\mu)^{q/(p-1)} w(y)$ with maximum at 0 and decaying to 0 at infinity (compare equation (1.4)). It is also easy to see that for given μ and $A \in H^2(R^2)$, the second equation has a unique solution $H_{\epsilon,\mu,0}(x) \in H^2(R^2)$ (note that the nonlinearity is concave). To ensure that $H_{\epsilon,\mu,0}(0) = 1$, we just need to choose $c_{\epsilon,\mu,0}$. In fact, by the standard representation formula

$$H_{\epsilon,\mu,0}(x) = \int_{R^2} K(|x-z|)\, c_{\epsilon,\mu,0}\, (1-\mu)^{rq/(p-1)} (H_{\epsilon,\mu,0} - \mu)^{-s}(z) w^r\left(\frac{z}{\epsilon}\right) dz.$$

Taking $x = 0$, we obtain

$$c_{\epsilon,\mu,0} = (1-\mu)^{s-rq/(p-1)} \left(\int_{R^2} K(|z|)(1 + O(\frac{1}{\log\frac{1}{\epsilon}} + \mu)) w^r\left(\frac{z}{\epsilon}\right) dz \right)^{-1}$$

$$= (1-\mu)^{s-rq/(p-1)} \left(\epsilon^2 (1 + O((\log\frac{1}{\epsilon})^{-1} + \mu) \int_{R^2} K(|\epsilon y|) w^r(y)\, dy \right)^{-1}$$

$$= (1-\mu)^{s-rq/(p-1)} \frac{1}{\epsilon^2 \log(1/\epsilon)} \left(\int_{R^2} w^r(y)\, dy \right)^{-1} + O\left(\frac{1/\log(1/\epsilon) + \mu}{\epsilon^2 \log(1/\epsilon)2} \right)$$

as $\epsilon \to 0$.

(Here we have used the fact that (by Lemma 1.2)

$$|H_{\epsilon,\mu,0}(x) - H_{\epsilon,\mu,0}(0)| \leq C \frac{1}{\log\frac{1}{\epsilon}} \log\left(\frac{|x-P|}{\epsilon} + 1 \right)$$

for some generic constant $C > 0$.)

Using the ansatz

$$A_{\epsilon,\mu}(y) = A_{\epsilon,\mu,0}(y) + a_{\epsilon,\mu}(y),$$

$$H_{\epsilon,\mu}(x) = H_{\epsilon,\mu,0}(x) + h_{\epsilon,\mu}(x),$$

and inserting it into (2.3) (with $c_{\epsilon,\mu} = c_{\epsilon,\mu,0}$) gives us

$$\Delta_y a_{\epsilon,\mu} - a_{\epsilon,\mu} = \frac{A^p_{\epsilon,\mu,0}}{(1-\mu)^q} - \frac{(A_{\epsilon,\mu,0} + a_{\epsilon,\mu})^p}{(H_{\epsilon,\mu,0} + h_{\epsilon,\mu} - \mu)^q},$$

$$\Delta_x h_{\epsilon,\mu} - h_{\epsilon,\mu} = c_{\epsilon,\mu,0}\frac{A^r_{\epsilon,\mu,0}}{(H_{\epsilon,\mu,0} - \mu)^s} - c_{\epsilon,\mu,0}\frac{(A_{\epsilon,\mu,0} + a_{\epsilon,\mu})^r}{(H_{\epsilon,\mu,0} + h_{\epsilon,\mu} - \mu)^s}.$$

The first equation can be rewritten as follows:

$$\Delta_y a_{\epsilon,\mu} - a_{\epsilon,\mu} + \frac{pA^{p-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{(1-\mu)^q} - \frac{qA^p_{\epsilon,\mu,0}h_{\epsilon,\mu}}{(1-\mu)^{q+1}} = e_1$$

where

$$e_1 = \frac{(A_{\epsilon,\mu,0} + a_{\epsilon,\mu})^p}{(1 + h_{\epsilon,\mu} - \mu)^q} - \frac{(A_{\epsilon,\mu,0} + a_{\epsilon,\mu})^p}{(H_{\epsilon,\mu,0} + h_{\epsilon,\mu} - \mu)^q}$$

$$-\frac{(A_{\epsilon,\mu,0} + a_{\epsilon,\mu})^p}{(1 + h_{\epsilon,\mu} - \mu)^q} + \frac{A^p_{\epsilon,\mu,0}}{(1-\mu)^q} + \frac{pA^{p-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{(1-\mu)^q} - \frac{qA^p_{\epsilon,\mu,0}h_{\epsilon,\mu}}{(1-\mu)^{q+1}}.$$

This implies

$$\|e_1(y)\|_{L^2(R^2)} = O\left(\|a_{\epsilon,\mu}(y)\|^2_{L^2(R^2)}\right) + O(\|h_{\epsilon,\mu}(x)\|^2_{L^\infty(\Omega)}) + O(\frac{1}{\log\frac{1}{\epsilon}} + \mu).$$

For a given $a_{\epsilon,\mu}$, we can solve the second equation directly since the nonlinearity is concave. Moreover, we have that $h_{\epsilon,\mu}$ satisfies

$$\Delta_x h_{\epsilon,\mu} - h_{\epsilon,\mu} + c_{\epsilon,\mu}\frac{rA^{r-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{(H_{\epsilon,\mu,0} - \mu)^s} - c_{\epsilon,\mu}\frac{sA^r_{\epsilon,\mu,0}h_{\epsilon,\mu}}{(H_{\epsilon,\mu,0} - \mu)^{s+1}} = e_2$$

where

$$e_2 = c_{\epsilon,\mu}\frac{A^r_{\epsilon,\mu,0}}{(H_{\epsilon,\mu,0} - \mu)^s} - c_{\epsilon,\mu}\frac{(A_{\epsilon,\mu,0} + a_{\epsilon,\mu})^r}{(H_{\epsilon,\mu,0} - \mu)^s}$$

$$+c_{\epsilon,\mu}\frac{rA^{r-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{(H_{\epsilon,\mu,0} - \mu)^s} - c_{\epsilon,\mu}\frac{sA^r_{\epsilon,\mu,0}h_{\epsilon,\mu}}{(H_{\epsilon,\mu,0} - \mu)^{s+1}}.$$

This implies

$$\|e_2\|_{L^2(R^2)} = O(\|a_{\epsilon,\mu}\|^2_{L^2(R^2)}) + O(\|h_{\epsilon,\mu}\|^2_{L^\infty(\Omega)}\|A^{r-1}_{\epsilon,\mu,0}\|_{L^2(R^2)}).$$

Thus by Lemma 1.2

$$h_{\epsilon,\mu}(x) = h_{\epsilon,\mu}(0) + O(\frac{1}{\log\frac{1}{\epsilon}})$$

and

$$h_{\epsilon,\mu}(0) = \int_{R^2} K(z)[c_{\epsilon,\mu}\frac{rA^{r-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{(H_{\epsilon,\mu,0}-\mu)^s} - c_{\epsilon,\mu}\frac{sA^r_{\epsilon,\mu,0}h_{\epsilon,\mu}}{(H_{\epsilon,\mu,0}-\mu)^{s+1}}] + O(\|a_{\epsilon,\mu}\|^2_{L^2(R^2)})$$

$$= c_{\epsilon,\mu}\int_{R^2} rA^{r-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}(1+O(\frac{1}{\log\frac{1}{\epsilon}}+\mu))$$

$$-c_{\epsilon,\mu}sh_{\epsilon,\mu}(0)\int_{R^2} A^r_{\epsilon,\mu,0}(1+O(\frac{1}{\log\frac{1}{\epsilon}}+\mu)) + O(\|a_{\epsilon,\mu}\|^2_{L^2(R^2)}).$$

So

$$h_{\epsilon,\mu}(0) = \frac{r}{s+1}\frac{\int A^{r-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{\int A^r_{\epsilon,\mu,0}} + O(\frac{1}{\log\frac{1}{\epsilon}}+\mu) + O(\|a_{\epsilon,\mu}\|^2_{L^2(R^2)}).$$

Substituting this into the first equation, the equation for $a_{\epsilon,\mu}$ becomes

$$\Delta_y a_{\epsilon,\mu} - a_{\epsilon,\mu} + \frac{pA^{p-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{(1-\mu)^q} - \frac{qrA^p_{\epsilon,\mu,0}}{(s+1)(1-\mu)^{q+1}}\frac{\int_{R^2} A^{r-1}_{\epsilon,\mu,0}a_{\epsilon,\mu}}{\int_{R^2} A^r_{\epsilon,\mu,0}}$$

$$= e_1 + O(\frac{1}{\log\frac{1}{\epsilon}}+\mu) + O(\|a_{\epsilon,\mu}\|^2_{L^2(R^2)})$$

in $L^2(R^2)$.

By Lemma 2.1 and a perturbation argument for $\epsilon << 1, \mu << 1$, the equation for $a_{\epsilon,\mu}$ can be solved and the solution is unique. Thus we have obtained a solution to (2.3).

Step 2. We choose μ such that

$$\mu = H_{\epsilon,\mu}(0) - \mathcal{P}_\Omega(H_{\epsilon,\mu}(\cdot - P))(P). \tag{2.5}$$

To this end, we note that this is equivalent to

$$\mu = \int_{R^2} (K(|z|) - G(P, P+z))\, c_{\epsilon,\mu}\,(H_{\epsilon,\mu}(z)-\mu)^{-s} A^r_{\epsilon,\mu}\left(\frac{z}{\epsilon}\right)\, dz$$

$$= \int_{R^2} H(P,P+z)c_{\epsilon,\mu}(H_{\epsilon,\mu}(z)-\mu)^{-s}A^r_{\epsilon,\mu}(\frac{z}{\epsilon})dz$$

$$= H(P,P)c_{\epsilon,\mu}\int_{R^2}(H_{\epsilon,\mu}(z)-\mu)^{-s}A^r_{\epsilon,\mu}(\frac{z}{\epsilon})dz$$

$$+O(\epsilon)\int_{R^2}|z|c_{\epsilon,\mu}(H_{\epsilon,\mu}(z)-\mu)^{-s}A^r_{\epsilon,\mu}(\frac{z}{\epsilon})dz.$$

Since $c_{\epsilon,\mu}\int_{R^2}(H_{\epsilon,\mu}(z)-\mu)^{-s}A^r_{\epsilon,\mu}(\frac{z}{\epsilon})dz=\frac{1+o(1)}{\log\frac{1}{\epsilon}}$, it is easy to see that by the contraction mapping principle, (1.9) has a unique solution $\mu=\mu_\epsilon(P)$.

We further calculate

$$\mu=\frac{1+o(1)}{\log\frac{1}{\epsilon}}[H(P,P)+O(\frac{1}{\log\frac{1}{\epsilon}})]$$

as $\epsilon\to 0$.

Let now

$$\hat{A}_{\epsilon,P}(x)=A_{\epsilon,\mu}(x-P),\quad \hat{H}_{\epsilon,P}(x)=H_{\epsilon,\mu}(x-P),\quad c_{\epsilon,P}=c_{\epsilon,\mu}$$

where $\mu:=\mu_\epsilon(P)$ is given by (1.9).

It is easy to see that (1), (2) and (3) of Thorem 1.1 are satisfied. It remains to prove (4). We have for $|x|\geq\delta$:

$$\hat{H}_{\epsilon,P}(x)=\frac{\epsilon^2\int_{R^2}K(|x-P-\epsilon y|)\frac{A^r_{\epsilon,\mu}(y)}{H^s_{\epsilon,\mu}}\,dy}{\int_{R^2}K(|\epsilon y|)\frac{A^r_{\epsilon,\mu}(y)}{H^s_{\epsilon,\mu}}\,dy}$$

$$=\frac{\sigma_P}{\log\frac{1}{\epsilon}}[K(|x-P|)+O(\epsilon)],\quad \sigma_P=1+o(1)$$

as $\epsilon\to 0$.

This implies Theoreom 2.2. □

3. Estimates of the Error Terms. In this section, we give some preliminary estimates. These will be used in the later sections.

Recall that we choose our approximate solution as follows:

$$A_{\epsilon,P}(y)=\mathcal{P}_{\Omega_{\epsilon,P}}\hat{A}_{\epsilon,P},\quad H_{\epsilon,P}(x)=\mathcal{P}_\Omega\hat{H}_{\epsilon,P}(x).$$

Note that in this case

$$\mu=\mathcal{Q}_\Omega\hat{H}_{\epsilon,P}(P).$$

Also recall that

$$\varphi_{\epsilon,P}(y)=\mathcal{Q}_{\Omega_{\epsilon,P}}\hat{A}_{\epsilon,P}=\hat{A}_{\epsilon,P}-A_{\epsilon,P},\quad \psi_{\epsilon,P}(x)=\mathcal{Q}_\Omega\hat{H}_{\epsilon,P}=\hat{H}_{\epsilon,P}-H_{\epsilon,P}.$$

We note that $\varphi_{\epsilon,P}$ satisfies

$$\Delta_y\varphi_{\epsilon,P}-\varphi_{\epsilon,P}=0\quad\text{in }\Omega_{\epsilon,P},$$

$$\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}=\frac{\partial\hat{A}_{\epsilon,P}}{\partial\nu}=O(e^{-d(P,\partial\Omega)/\epsilon})\quad\text{on }\partial\Omega_{\epsilon,P}.$$

Hence

$$\|\varphi_{\epsilon,P}\|_{H^2(\Omega_{\epsilon,P})}=O(e^{-d(P,\partial\Omega)/\epsilon}).\tag{3.1}$$

By Theorem 2.2 we have

$$\mathcal{P}_\Omega \hat{H}_{\epsilon,P}(x) = \frac{\int_{\Omega_{\epsilon,P}} G(x, P+\epsilon y) \frac{\hat{A}^r_{\epsilon,P}(y)}{(\hat{H}_{\epsilon,P}-\mu_\epsilon(P))^s}\, dy}{\int_{R^2} K(|\epsilon y|) \frac{\hat{A}^r_{\epsilon,P}(y)}{(\hat{H}_{\epsilon,P}-\mu_\epsilon(P))^s}\, dy}$$

$$= \frac{1+o(1)}{\log\frac{1}{\epsilon}}[K(|x-P|) - H(x,P) + O(\epsilon)].$$

This implies

$$\psi_{\epsilon,P}(x) = \hat{H}_{\epsilon,P}(x) - \mathcal{P}_\Omega \hat{H}_{\epsilon,P}(x-P) = \frac{1+o(1)}{\log\frac{1}{\epsilon}}[H(x,P)+O(\epsilon)]$$

or, equivalently,

$$\psi_{\epsilon,P}(x) = \frac{1+o(1)}{\log\frac{1}{\epsilon}} H(P,x) + O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right). \tag{3.2}$$

By (3.1) and (3.2), we see that the term involving $\varphi_{\epsilon,P}$ can be neglected. This is what we will do in the later sections.

The reason for choosing $A_{\epsilon,\mu}$ and $H_{\epsilon,P}$ as we did lies in the two following estimates:

$$S_1(A_{\epsilon,P}, H_{\epsilon,P}) = \Delta_y A_{\epsilon,P} - A_{\epsilon,P} + \frac{A^p_{\epsilon,P}}{H^q_{\epsilon,P}}$$

$$= \frac{(\hat{A}_{\epsilon,P} - \varphi_{\epsilon,P})^p}{(\hat{H}_{\epsilon,P} - \psi_{\epsilon,P})^q} - \frac{(\hat{A}_{\epsilon,P})^p}{(\hat{H}_{\epsilon,P} - \psi_{\epsilon,P}(P))^q}$$

$$= O(e^{-d(P,\partial\Omega)/\epsilon}) + (\hat{A}_{\epsilon,P})^p[(\hat{H}_{\epsilon,P} - \psi_{\epsilon,P})^{-q} - (\hat{H}_{\epsilon,P} - \psi_{\epsilon,P}(P))^{-q}] \quad \text{(by (3.1))}$$

$$= O(e^{-d(P,\partial\Omega)/\epsilon}) - q(\hat{A}_{\epsilon,P})^p(\hat{H}_{\epsilon,P})^{-q-1}(\psi_{\epsilon,P}(x) - \psi_{\epsilon,P}(P)) + O((\frac{\epsilon}{\log\frac{1}{\epsilon}})^2 \hat{A}^p_{\epsilon,P})$$

for a.e. $y \in \Omega_{\epsilon,P}$. Similarly we have

$$S_2(A_{\epsilon,P}, H_{\epsilon,P}) = \Delta_x H_{\epsilon,P} - H_{\epsilon,P} + c_{\epsilon,P}\frac{A^r_{\epsilon,P}}{H^s_{\epsilon,P}}$$

$$= O(e^{-d(P,\partial\Omega)/\epsilon}) - sc_{\epsilon,P}(\hat{A}_{\epsilon,P})^r(\hat{H}_{\epsilon,P})^{-s-1}(\psi_{\epsilon,P}(x) - \psi_{\epsilon,P}(P)) + O(c_{\epsilon,P}(\frac{\epsilon}{\log\frac{1}{\epsilon}})^2 \hat{A}^r_{\epsilon,P})$$

for a.e. $x \in \Omega$.

We have thus obtained

LEMMA 3.1. *We have*

$$S_1(A_{\epsilon,P}, H_{\epsilon,P}) = O(e^{-d(P,\partial\Omega)/\epsilon}) - q(\hat{A}_{\epsilon,P})^p(\hat{H}_{\epsilon,P})^{-q-1}(\psi_{\epsilon,P}(x) - \psi_{\epsilon,P}(P)) + O((\frac{\epsilon}{\log\frac{1}{\epsilon}})^2 \hat{A}^p_{\epsilon,P}) \tag{3.3}$$

for a.e. $y \in \Omega_{\epsilon,P}$.

$$S_2(A_{\epsilon,P},H_{\epsilon,P}) = O(e^{-d(P,\partial\Omega)/\epsilon}) - sc_{\epsilon,P}(\hat{A}_{\epsilon,P})^r(\hat{H}_{\epsilon,P})^{-s-1}(\psi_{\epsilon,P}(x)-\psi_{\epsilon,P}(P)) + O(c_{\epsilon,P}(\frac{\epsilon}{\log\frac{1}{\epsilon}})^2\hat{A}^r_{\epsilon,P})$$

(3.4)

for a.e. $x \in \Omega$.

Hence

$$\|S_1(A_{\epsilon,P},H_{\epsilon,P})\|_{L^2(\Omega_{\epsilon,P})} = O(\frac{\epsilon}{\log\frac{1}{\epsilon}}), \tag{3.5}$$

$$\|S_2(A_{\epsilon,P},H_{\epsilon,P})\|_{L^t(\Omega)} = O(\epsilon^{2t^{-1}-1}(\frac{1}{\log\frac{1}{\epsilon}})^2) \tag{3.6}$$

for any $1 < t < 1.1$.

Proof. By direct computation. □

4. The Liapunov-Schmidt Reduction Method. This section is devoted to studying the linearized operator defined by

$$\tilde{L}_{\epsilon,P} := S'_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix},$$

$$\tilde{L}_{\epsilon,P} : H^2_N(\Omega_{\epsilon,P}) \times W^{2,t}_N(\Omega) \to L^2(\Omega_{\epsilon,P}) \times L^t(\Omega)$$

where $1 < t < 1.1$ is a fixed number.

Set

$$K_{\epsilon,P} := \text{span}\,\{\frac{\partial A_{\epsilon,P}}{\partial P_j}|j=1,...,N\} \subset H^2_N(\Omega_{\epsilon,P}),$$

$$C_{\epsilon,P} := \text{span}\,\{\frac{\partial A_{\epsilon,P}}{\partial P_j}|j=1,...,N\} \subset L^2(\Omega_{\epsilon,P}),$$

$$L_\epsilon := \Delta - 1 + pA^{p-1}_{\epsilon,P}H^{-q}_{\epsilon,P} - \frac{qr}{s+1}\frac{\int_{\Omega_{\epsilon,P}} A^{r-1}_{\epsilon,P}\cdot}{\int_{\Omega_{\epsilon,P}} A^r_{\epsilon,P}} A^p_{\epsilon,P}$$

and

$$L_{\epsilon,P} := \pi_{\epsilon,P} \circ L_\epsilon : K^\perp_{\epsilon,P} \to C^\perp_{\epsilon,P}$$

where $\pi_{\epsilon,P}$ is the projection in $L^2(\Omega_{\epsilon,P})$ onto $C^\perp_{\epsilon,P}$.

We remark that since $A_{\epsilon,P}(y) = (1+O(\frac{1}{\log\frac{1}{\epsilon}}))w(y)$, it is easy to see that

$$l_{\epsilon,P} := \pi_{\epsilon,P} \circ (\Delta - 1 + pA^{p-1}_{\epsilon,P}) : K^\perp_{\epsilon,P} \to C^\perp_{\epsilon,P}$$

is a one-to-one and surjective map. For the proof please see the proof of Propositions 6.1-6.2 in [41].

The following proposition is the key estimate in applying the Liapunov-Schmidt reduction method.

PROPOSITION 4.1. *For ϵ sufficiently small, the map $L_{\epsilon,P}$ is a one-to-one and surjective map. Moreover the inverse of $L_{\epsilon,P}$ exists and is bounded uniformly with respect to ϵ.*

Proof. We will follow the method used in [11], [35], [36], [41] and [44]. We first show that there exist constants $C>0, \bar{\epsilon}>0$ such that for all $\epsilon \in (0,\bar{\epsilon})$,

$$\|L_{\epsilon,P}\Phi\|_{L^2(\Omega_{\epsilon,P})} \geq C\|\Phi\|_{H^2(\Omega_{\epsilon,P})} \tag{4.1}$$

for all $\Phi \in K_{\epsilon,P}^{\perp}$.

Suppose that (4.1) is false. Then there exist sequences $\{\epsilon_k\}$, $\{P_k\}$, and $\{\phi_k\}$ with $P_k \in \Omega$, $\phi_k \in K_{\epsilon_k,P_k}^{\perp}$ such that

$$\|L_{\epsilon_k,P_k}\phi_k\|_{L^2(\Omega_{\epsilon_k,P_k})} \to 0, \tag{4.2}$$

$$\|\phi_k\|_{H^2(\Omega_{\epsilon_k,P_k})} = 1, \qquad k=1,\,2,\,\ldots\,. \tag{4.3}$$

Namely we have the following situation

$$\Delta_y\phi_k - \phi_k + pA_{\epsilon_k,P_k}^{p-1}H_{\epsilon_k,P_k}^{-q}\phi_k - \frac{qr}{s+1}\frac{\int_{\Omega_{\epsilon_k,P_k}} A_{\epsilon_k,P_k}^{r-1}\phi_k}{\int_{\Omega_{\epsilon_k,P_k}} A_{\epsilon_k,P_k}^{r}}A_{\epsilon_k,P_k}^{p} = f_k, \tag{4.4}$$

where

$$\|f_k\|_{L^2(\Omega_{\epsilon_k,P_k})} \to 0$$

$$\phi_k \in K_{\epsilon_k,P_k}^{\perp}, \quad \|\phi_k\|_{H^2(\Omega_{\epsilon_k,P_k})} = 1. \tag{4.5}$$

We now show that this is impossible. Set $A_k = A_{\epsilon_k,P_k}, \Omega_k = \Omega_{\epsilon_k,P_k}$.

Note that

$$H_{\epsilon_k,P_k} = 1 + o(1) \text{ in } L^{\infty}(\Omega),$$

$$(\Delta_y - 1 + pA_k^{p-1})\frac{A_k}{p-1} = A_k^p + o(1) \text{ in } L^2(\Omega_k).$$

Thus we have

$$(\Delta_y - 1 + pA_k^{p-1})(\phi_k - \frac{qr}{(s+1)(p-1)}\frac{\int_{\Omega_k} A_k^{r-1}\phi_k}{\int_{\Omega_k} A_k^r}A_k) = f_k + o(1) \text{ in } L^2(\Omega_k).$$

Since the projection of A_k into K_{ϵ_k,P_k} is $o(1)$ in $H^2(\Omega_k)$ and the operator

$$\Delta_y - 1 + pA_k^{p-1}$$

is an one-to-one and invertible map (with the inverse bounded uniformly with respect to ϵ) from $K_{\epsilon_k,P_k}^{\perp}$ to $C_{\epsilon_k,P_k}^{\perp}$, we have

$$\phi_k - \frac{qr}{(s+1)(p-1)}\frac{\int_{\Omega_k} A_k^{r-1}\phi_k}{\int_{\Omega_k} A_k^r}A_k = o(1) \text{ in } H^2(\Omega_k). \tag{4.6}$$

Since $\frac{qr}{(p-1)(s+1)} > 1$, (4.6) implies that

$$\|\phi_k\|_{H^2(\Omega_k)} = o(1).$$

A contradiction !

Thus (4.1) holds and $L_{\epsilon,P}$ is a one-to-one map.

Next we show that $L_{\epsilon,P}$ is also surjective. To this end, we just need to show that the conjugate of $L_{\epsilon,P}$ (denoted by $L^*_{\epsilon,P}$) is injective from $K^\perp_{\epsilon,P}$ to $C^\perp_{\epsilon,P}$.

Let $L^*_{\epsilon,P}\phi \in C^\perp_{\epsilon,P}, \quad \phi \in K^\perp_{\epsilon,P}$. Namely we have

$$\Delta_y \phi - \phi + pA^{p-1}_{\epsilon,P} H^{-q}_{\epsilon,P}\phi - \frac{qr}{s+1}\frac{\int_{\Omega_{\epsilon,P}} A^p_{\epsilon,P}\phi}{\int_{\Omega_{\epsilon,P}} A^r_{\epsilon,P}} A^{r-1}_{\epsilon,P} \in C_{\epsilon,P}. \tag{4.7}$$

We can assume that $\|\phi\|_{H^2(\Omega_{\epsilon,P})} = 1$.

Multiplying (4.7) by $A_{\epsilon,P}$ and integrating over $\Omega_{\epsilon,P}$, we obtain

$$(p-1-\frac{qr}{s+1})\int_{\Omega_{\epsilon,P}} A^p_{\epsilon,P}\phi = o(1)$$

or, equivalently,

$$\int_{\Omega_{\epsilon,P}} A^p_{\epsilon,P}\phi = o(1).$$

Hence ϕ satisfies

$$\Delta_y\phi - \phi + pA^{p-1}_{\epsilon,P}H^{-q}_{\epsilon,P}\phi + o(1) \in C_{\epsilon,P}, \quad \phi \in K^\perp_{\epsilon,P}$$

which implies that $\|\phi\|_{H^2(\Omega_{\epsilon,P})} = o(1)$. A contradiction !

Therefore $L_{\epsilon,P}$ is also surjective.

□

We now deal with system (1.6).

$\tilde{L}_{\epsilon,P}$ is not uniformly invertible in ϵ due to the approximate kernel

$$\mathcal{K}_{\epsilon,P} := K_{\epsilon,P} \oplus \{0\} \subset H^2_N(\Omega_{\epsilon,P}) \times W^{2,t}_N(\Omega).$$

We choose the approximate cokernel as follows:

$$\mathcal{C}_{\epsilon,P} := C_{\epsilon,P} \oplus \{0\} \subset L^2(\Omega_{\epsilon,P}) \times L^t(\Omega).$$

We then define

$$\mathcal{K}^\perp_{\epsilon,P} := K^\perp_{\epsilon,P} \oplus W^{2,t}_N(\Omega) \subset H^2_N(\Omega_{\epsilon,P}) \times W^{2,t}_N(\Omega),$$

$$\mathcal{C}^\perp_{\epsilon,P} := C^\perp_{\epsilon,P} \oplus L^t(\Omega) \subset L^2(\Omega_{\epsilon,P}) \times L^t(\Omega).$$

Let $\pi_{\epsilon,P}$ denote the projection in $L^2(\Omega_{\epsilon,P}) \times L^t(\Omega)$ onto $\mathcal{C}^\perp_{\epsilon,P}$. (Here the projection in the second component is the identity map.) We then show that the equation

$$\pi_{\epsilon,P} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix} = 0$$

has the unique solution $\Sigma_{\epsilon,P} = \begin{pmatrix} \Phi_{\epsilon,P}(y) \\ \Psi_{\epsilon,P}(x) \end{pmatrix} \in \mathcal{K}^{\perp}_{\epsilon,P}$ if ϵ is small enough.

As a preparation in the following two propositions we show the invertibility of the corresponding linearized operator.

PROPOSITION 4.2. *Let $\mathcal{L}_{\epsilon,P} = \pi_{\epsilon,P} \circ \tilde{L}_{\epsilon,P}$. There exist positive constants $\overline{\epsilon}, \lambda$ such that for all $\epsilon \in (0, \overline{\epsilon})$*

$$\|\mathcal{L}_{\epsilon,P}\Sigma\|_{L^2(\Omega_{\epsilon,P})\times L^t(\Omega)} \geq \lambda \|\Sigma\|_{H^2(\Omega_{\epsilon,P})\times W^{2,t}(\Omega)} \tag{4.8}$$

for all $\Sigma \in \mathcal{K}^{\perp}_{\epsilon,P}$.

PROPOSITION 4.3. *There exists a positive constant $\overline{\overline{\epsilon}}$ such that for all $\epsilon \in (0, \overline{\overline{\epsilon}})$ the map*

$$\mathcal{L}_{\epsilon,P} = \pi_{\epsilon,P} \circ \tilde{L}_{\epsilon} : \mathcal{K}^{\perp}_{\epsilon,P} \to \mathcal{C}^{\perp}_{\epsilon,P}$$

is surjective.

Proof of Proposition 4.2. This proposition follows from Proposition 4.1. In fact, suppose that (4.8) is false. Then there exist sequences $\{\epsilon_k\}$, $\{P_k\}$, and $\{\Sigma_k\}$ with $P_k \in \Omega$, $\Sigma_k = \begin{pmatrix} \phi_k(y) \\ \psi_k(x) \end{pmatrix} \in \mathcal{K}^{\perp}_{\epsilon_k,P_k}$ such that

$$\|\mathcal{L}_{\epsilon_k,P_k}\Sigma_k\|_{L^2(\Omega_{\epsilon_k,P_k})\times L^t(\Omega)} \to 0, \tag{4.9}$$

$$\|\Sigma_k\|_{H^2(\Omega_{\epsilon_k,P_k})\times W^{2,t}(\Omega)} = 1, \qquad k = 1,\, 2,\, \ldots. \tag{4.10}$$

Namely we have the following situation

$$\Delta_y \phi_k - \phi_k + pA^{p-1}_{\epsilon_k,P_k} H^{-q}_{\epsilon_k,P_k}\phi_k - qA^{p}_{\epsilon_k,P_k} H^{-q-1}_{\epsilon_k,P_k}\psi_k = f_k, \quad \|f_k\|_{L^2(\Omega_{\epsilon_k,P_k})} \to 0, \tag{4.11}$$

$$\Delta_x \psi_k - \psi_k + rc_{\epsilon_k,P_k} A^{r-1}_{\epsilon_k,P_k} H^{-s}_{\epsilon_k,P_k}\phi_k - sc_{\epsilon_k,P_k} A^{r}_{\epsilon_k,P_k} H^{-s-1}_{\epsilon_k,P_k}\psi_k = g_k \tag{4.12}$$

where

$$\|g_k\|_{L^t(\Omega)} \to 0,$$

$$\phi_k \in K^{\perp}_{\epsilon_k,P_k}, \tag{4.13}$$

$$\|\phi_k\|^2_{H^2(\Omega_{\epsilon_k,P_k})} + \|\psi_k\|^2_{W^{2,t}(\Omega)} = 1. \tag{4.14}$$

We now show that this is impossible. Set $A_k = A_{\epsilon_k,P_k}, \Omega_k = \Omega_{\epsilon_k,P_k}$.

We first note that by (4.12) we have

$$\|\psi_k\|_{L^\infty(\Omega)} \leq C$$

and hence by Lemma 1.2 and Sobolev embedding,

$$|\psi_k(x) - \psi_k(P_k)| \leq C|x - P_k|^\alpha + \frac{1}{\log\frac{1}{\epsilon}} \log\left(\frac{|x-P|}{\epsilon} + 1\right)$$

for some $\alpha > 0$ since $t > 1$. Thus

$$\|A_k^p(\psi_k - \psi_k(P_k))\|_{L^2(\Omega_k)} \to 0, \quad k = 1, 2, \ldots \quad \text{in} \quad L^2(\Omega_k). \tag{4.15}$$

Moreover by (4.12),

$$\psi_k(P_k) = \int_{\Omega_k} G(P, z)(rc_{\epsilon_k, P_k} A_k^{r-1} H_{\epsilon_k, P_k}^{-s} \phi_k - sc_{\epsilon_k, P_k} A_k^r H_{\epsilon_k, P_k}^{-s-1} \psi_k - g_k)$$

$$= (1 + o(1)) rc_{\epsilon_k, P_k} \log \frac{1}{\epsilon_k} \int_{\Omega_k} A_{\epsilon_k, P_k}^{r-1} \phi_k - (1 + o(1)) s\psi_k(P_k) c_{\epsilon_k, P_k} \int_{\Omega_k} A_k^r + o(1).$$

So

$$\psi_k(P_k) = \frac{r}{s+1} \frac{\int_{\Omega_k} A_k^{r-1} \phi_k}{\int_{\Omega_k} A_k^r} + o(1).$$

Thus we have

$$L_{\epsilon_k, P_k} \phi_k = o(1) \quad \text{in} \quad L^2(\Omega_k), \quad \phi_k \in K_{\epsilon_k, P_k}^{\perp}. \tag{4.16}$$

By Proposition 4.1, $\|\phi_k\|_{H^2(\Omega_k)} = o(1)$. Hence $\psi_k(P_k) = o(1)$ and by elliptic estimates $\|\psi_k\|_{W^{2,t}(\Omega)} = o(1)$.

This contradicts the assumption (4.14) and the proof of Proposition 4.2 is completed. □

Proof of Proposition 4.3. We just need to show that the conjugate operator of $\mathcal{L}_{\epsilon,P}$ (denoted by $\mathcal{L}_{\epsilon,P}^*$) is injective from $\mathcal{K}_{\epsilon,P}^{\perp}$ to $\mathcal{C}_{\epsilon,P}^{\perp}$. Suppose not. Then there exist $\phi \in K_{\epsilon,P}^{\perp}$, $\psi \in W^{2,t}(\Omega)$ such that

$$\Delta_y \phi - \phi + pA_{\epsilon,P}^{p-1} H_{\epsilon,P}^{-q} \phi + rc_{\epsilon,P} A_{\epsilon,P}^{r-1} H_{\epsilon,P}^{-s} \psi \in C_{\epsilon,P}^{\perp},$$

$$\Delta_x \psi - \psi - sc_{\epsilon,P} A_{\epsilon,P}^r H_{\epsilon,P}^{-s-1} \psi - qA_{\epsilon,P}^p H_{\epsilon,P}^{-q-1} \phi = 0,$$

$$\|\phi\|_{H^2(\Omega_{\epsilon,P})}^2 + \|\psi\|_{W^{2,t}(\Omega)}^2 = 1.$$

Similar to the proof of Proposition 4.2, we have

$$\psi(P) = -(1 + o(1)) c_{\epsilon,P} \frac{q}{s+1} \frac{\int_{\Omega_{\epsilon,P}} A_{\epsilon,P}^p \phi}{\int_{\Omega_{\epsilon,P}} A_{\epsilon,P}^r}$$

and substituting into the equation for ϕ we obtain

$$L_{\epsilon,P} \phi + o(1) \in C_{\epsilon,P}^{\perp}, \quad \phi \in K_{\epsilon,P}^{\perp}.$$

By Proposition 4.1, $\|\phi\|_{H^2(\Omega_{\epsilon,P})} = o(1)$ and hence $\|\psi\|_{W^{2,t}(\Omega)} = o(1)$. A contradiction ! □

Now we are in a position to solve the equation

$$\pi_{\epsilon,P} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \phi \\ H_{\epsilon,P} + \psi \end{pmatrix} = 0. \tag{4.17}$$

Since $\mathcal{L}_{\epsilon,P}|_{\mathcal{K}^{\perp}_{\epsilon,P}}$ is invertible (call the inverse $\mathcal{L}^{-1}_{\epsilon,P}$) we can rewrite

$$\Sigma = -(\mathcal{L}^{-1}_{\epsilon,P} \circ \pi_{\epsilon,P})(S_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix}) - (\mathcal{L}^{-1}_{\epsilon,P} \circ \pi_{\epsilon,P}) N_{\epsilon,P}(\Sigma) \equiv M_{\epsilon,P}(\Sigma) \tag{4.18}$$

where

$$N_{\epsilon,P}(\Sigma) = S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \phi \\ H_{\epsilon,P} + \psi \end{pmatrix} - S_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} - S'_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}$$

and the operator $M_{\epsilon,P}$ is defined by the last equation for $\Sigma \in H^2_N(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega)$. We are going to show that the operator $M_{\epsilon,P}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\Sigma \in H^2(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega) | \|\Sigma\|_{H^2(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega)} < \delta\}$$

if δ is small enough. We have by Lemma 3.1, Propositions 4.2 and 4.3,

$$\|M_{\epsilon,P}(\Sigma)\|_{H^2(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega)} \leq \lambda^{-1}(\|\pi_{\epsilon,P} \circ N_{\epsilon,P}(\Sigma)\|_{L^2(\Omega_{\epsilon,P}) \times L^t(\Omega)}$$

$$+ \left\| \pi_{\epsilon,P} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} \right\|_{L^2(\Omega_{\epsilon,P}) \times L^t(\Omega)})$$

$$\leq \lambda^{-1} C (c(\delta)\delta + \epsilon^{2t^{-1}-1} \frac{1}{\log \frac{1}{\epsilon}}) \qquad \text{(by Lemma 3.1)}$$

where $\lambda > 0$ is independent of $\delta > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly we show

$$\|M_{\epsilon,P}(\Sigma) - M_{\epsilon,P}(\Sigma')\|_{H^2(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega)} \leq \lambda^{-1} C (\epsilon^{1/2} + c(\delta)\delta) \|\Sigma - \Sigma'\|_{H^2(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega)}$$

where $c(\delta) \to 0$ as $\delta \to 0$. If we choose δ small enough, then $M_{\epsilon,P}$ is a contraction on $B_{\epsilon,\delta}$. The existence of a fixed point $\Sigma_{\epsilon,P}$ now follows from the Contraction Mapping Principle and $\Sigma_{\epsilon,P}$ is a solution of (4.18).

We have thus proved

LEMMA 4.4. *There exists $\bar{\epsilon} > 0$ such that for every pair of ϵ, P with $0 < \epsilon < \bar{\epsilon}$ there exists a unique $(\Phi_{\epsilon,P}, \Psi_{\epsilon,P}) \in \mathcal{K}^{\perp}_{\epsilon,P}$ satisfying $S_\epsilon(\begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix}) \in \mathcal{C}_{\epsilon,P}$ and*

$$\|(\Phi_{\epsilon,P}, \Psi_{\epsilon,P})\|_{H^2(\Omega_{\epsilon,P}) \times W^{2,t}(\Omega)} \leq C \epsilon^{2t^{-1}-1}. \tag{4.19}$$

We can improve the estimates in Lemma 4.4.

LEMMA 4.5.

Let $(\Phi_{\epsilon,P}, \psi_{\epsilon,P})$ be given by Lemma 4.4. Then we have

$$\|\Phi_{\epsilon,P}\|_{L^\infty(\Omega_{\epsilon,P})} = O(\frac{\epsilon}{\log \frac{1}{\epsilon}}), \quad \|\Psi_{\epsilon,P}\|_{L^\infty(\Omega)} = O(\frac{\epsilon}{\log \frac{1}{\epsilon}}) \tag{4.20}$$

and

$$|\Psi_{\epsilon,P}(x)-\Psi_{\epsilon,P}(P)|\le C\frac{\epsilon}{(\log\frac{1}{\epsilon})^2}\log\left(\frac{|x-P|}{\epsilon}+1\right)\quad \textit{for } x\neq P. \tag{4.21}$$

Proof. The proof is divided into several steps.

First we note that by the equation for $\Phi_{\epsilon,P}$ and Lemmas 3.1 and 4.4

$$\Delta_y\Phi_{\epsilon,P}-\Phi_{\epsilon,P}+pA_{\epsilon,P}^{p-1}H_{\epsilon,P}^{-q}-qA_{\epsilon,P}^{p}H_{\epsilon,P}^{-q-1}\Psi_{\epsilon,P}+f_1\in C_{\epsilon,P}$$

where $\|f_1\|_{L^2(\Omega_{\epsilon,P})}=O(\frac{\epsilon}{\log\frac{1}{\epsilon}})$. Hence we obtain

$$\|\Phi_{\epsilon,P}\|_{H^2(\Omega_{\epsilon,P})}\le C\|A_{\epsilon,P}^{p}H_{\epsilon,P}^{-q-1}\Psi_{\epsilon,P}\|_{L^2(\Omega_{\epsilon,P})}+O(\frac{\epsilon}{\log\frac{1}{\epsilon}})$$

$$\le C\|\Psi_{\epsilon,P}(x)\|_{L^\infty(\Omega)}+O(\frac{\epsilon}{\log\frac{1}{\epsilon}}). \tag{4.22}$$

Next $\Psi_{\epsilon,P}$ satisfies

$$\Delta_x\Psi_{\epsilon,P}-\Psi_{\epsilon,P}=f_2:=c_{\epsilon,P}\frac{\hat{A}_{\epsilon,P}^r}{(\hat{H}_{\epsilon,P}-\psi_{\epsilon,P}(P))^s}-c_{\epsilon,P}\frac{(\hat{A}_{\epsilon,P}+\Phi_{\epsilon,P})^r}{(\hat{H}_{\epsilon,P}-\psi_{\epsilon,P}(x)+\Psi_{\epsilon,P})^s}$$

We have

$$|f_2(x)|\le Cc_{\epsilon,P}(w(y)^{r-1}|\Phi_{\epsilon,P}(y)|+w^r(y)|\Psi_{\epsilon,P}(x)|)+O(\frac{\epsilon}{\log\frac{1}{\epsilon}}c_{\epsilon,P}w^r(y)) \tag{4.23}$$

for a.e. $x\in\Omega$.

Therefore we have by Lemma 1.2 and (4.22)

$$\Psi_{\epsilon,P}(x)=O(\frac{\epsilon}{\log\frac{1}{\epsilon}})+O(\frac{1}{\log\frac{1}{\epsilon}}\|\Psi_{\epsilon,P}\|_{L^\infty(\Omega)})$$

and so

$$\|\Psi_{\epsilon,P}\|_{L^\infty(\Omega)}=O(\frac{\epsilon}{\log\frac{1}{\epsilon}}),$$

or, equivalently,

$$\|\Psi_{\epsilon,P}\|_{L^\infty(\Omega_{\epsilon,P})}=O(\frac{\epsilon}{\log\frac{1}{\epsilon}})$$

where $y=(x-P)/\epsilon$. Lemma 4.5 is proved. $\square$

Moreover by Lemma 1.2 and (4.23),

$$\Psi_{\epsilon,P}(x)-\Psi_{\epsilon,P}(P)=O(\frac{\epsilon}{(\log\frac{1}{\epsilon})^2}|\log|y||).$$

5. The reduced problem. In this section we solve the reduced problem and prove our main theorem.

By Lemma 4.4 there exists a unique solution $(\Phi_{\epsilon,P}, \psi_{\epsilon,P}) \in \mathcal{K}^{\perp}_{\epsilon,P}$ such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix} = \begin{pmatrix} v_{\epsilon,P} \\ 0 \end{pmatrix} \in \mathcal{C}_{\epsilon,P}.$$

Our idea is to find P such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix} \perp \mathcal{C}_{\epsilon,P}.$$

Let

$$W_{\epsilon,j}(P) := \frac{\log \frac{1}{\epsilon}}{\epsilon^2} \int_\Omega (S_1(A_{\epsilon,P} + \Phi_{\epsilon,P}, H_{\epsilon,P} + \Psi_{\epsilon,P}) \frac{\partial A_{\epsilon,P}}{\partial P_j}),$$

$$W_\epsilon(P) := (W_{\epsilon,1}(P), ..., W_{\epsilon,N}(P)).$$

Then $W_\epsilon(P)$ is a continuous map in P and our problem is reduced to finding a zero of the vector field $W_\epsilon(P)$.

Let us now calculate $W_\epsilon(P)$.

By Lemma 4.5,

$$\Psi_{\epsilon,P}(x) - \Psi_{\epsilon,P}(P) = O\left(\frac{\epsilon}{(\log \frac{1}{\epsilon})^2} \log\left(\frac{|x-P|}{\epsilon} + 1\right)\right). \tag{5.1}$$

By (3.3) and (3.4), we have

$$\int_\Omega (S_1(A_{\epsilon,P} + \Phi_{\epsilon,P}, H_{\epsilon,P} + \psi_{\epsilon,P}) \frac{\partial A_{\epsilon,P}}{\partial P_j})$$

$$= \epsilon^2 \int_{\Omega_{\epsilon,P}} (\Delta_y \Phi_{\epsilon,P} - \Phi_{\epsilon,P} + pA^{p-1}_{\epsilon,P} H^{-q}_{\epsilon,P} \Phi_{\epsilon,P} - qA^{p-1}_{\epsilon,P} H^{-q-1}_{\epsilon,P} \Psi_{\epsilon,P}) \frac{\partial A_{\epsilon,P}}{\partial P_j}$$

$$+ O(\epsilon^3 (\frac{1}{\log \frac{1}{\epsilon}})^2)$$

$$+ \epsilon^2 \int_{\Omega_{\epsilon,P}} -q(\hat{A}_{\epsilon,P})^p (\hat{H}_{\epsilon,P})^{-q-1} [\psi_{\epsilon,P}(P+\epsilon y) - \psi_{\epsilon,P}(P)] \frac{\partial A_{\epsilon,P}}{\partial P_j}(y)\, dy$$

$$+ O(e^{-d(P,\partial\Omega)/\epsilon}) = I_1 + I_2$$

where I_1, I_2 are defined by the last equality.

For I_1, we note that $\|\Psi_{\epsilon,P}\|_{L^\infty(\Omega_{\epsilon,P}} = O(\frac{\epsilon}{\log\frac{1}{\epsilon}})$, $\frac{\partial A_{\epsilon,P}}{\partial P_j} = -\frac{1+o(1)}{\epsilon}\frac{\partial w}{\partial y_j}$ and hence

$$I_1 = \epsilon\int_{\Omega_{\epsilon,P}}(qA_{\epsilon,P}^{p-1}H_{\epsilon,P}^{-q-1}\Psi_{\epsilon,P})\frac{\partial w}{\partial y_j} + O(\epsilon^2(\frac{1}{\log\frac{1}{\epsilon}})^2)$$

$$= \epsilon\int_{\Omega_{\epsilon,P}}(qw^{p-1}\Psi_{\epsilon,P})\frac{\partial w}{\partial y_j} + O(\epsilon^2(\frac{1}{\log\frac{1}{\epsilon}})^2)$$

$$= \epsilon\int_{\Omega_{\epsilon,P}}(qw^{p-1}(y)H_{\epsilon,P}^{-q-1}(\Psi_{\epsilon,P}(P+\epsilon y)-\Psi_{\epsilon,P}(P)))\frac{\partial w}{\partial y_j} + O(\epsilon^2(\frac{1}{\log\frac{1}{\epsilon}})^2)$$

$$= O(\epsilon^2\frac{1}{(\log\frac{1}{\epsilon})^2})$$

by (5.1).

For I_2 we have

$$I_2 = C\epsilon\int_{\Omega_{\epsilon,P}}[\psi_{\epsilon,P}(P+\epsilon y)-\psi_{\epsilon,P}(P)]\frac{\partial w}{\partial y_j}]\,dy(1+O(\frac{1}{\log\frac{1}{\epsilon}}))$$

$$= C\frac{\epsilon}{\log\frac{1}{\epsilon}}\int_{R^2} -[H(P,P+\epsilon y)-H(P,P)]w'(|y|)\frac{y_i}{|y|}\,dy(1+O(\frac{1}{\log\frac{1}{\epsilon}}))$$

$$= -C\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\frac{\partial}{\partial P_j}H(P,P)\int_{R^2} w'(|y|)|y|\,dy + O\left(\frac{\epsilon^N}{(\log\frac{1}{\epsilon})^2}\right)$$

as $\epsilon\to 0$ uniformly in P, where $w'(|y|) = \frac{d}{dr}w(r)$ for $r=|y|$ and $C\neq 0$ denotes a generic constant.

Combining I_1 and I_2, we have

$$W_\epsilon(P) = c_0\nabla_P H(P,P) + O(\frac{1}{\log\frac{1}{\epsilon}}),$$

where $c_0\neq 0$ is a generic constant.

Suppose at P_0, we have $\nabla_P H(P_0,P_0)=0, \det(\nabla_j\nabla_k H(P_0,P_0))\neq 0$ then the standard Brouwer's fixed point theorem shows that for $\epsilon << 1$ there exists a P_ϵ such that $W_\epsilon(P_\epsilon)=0$ and $P_\epsilon\to P_0$.

Thus we have proved the following proposition.

PROPOSITION 5.1. *For ϵ sufficiently small there exist points P_ϵ with $P_\epsilon\to P_0$ such that $W_\epsilon(P_\epsilon)=0$.*

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 5.1, there exists $P_\epsilon\to P_0$ such that $W_\epsilon(P_\epsilon)=0$. In other words, $S_1(A_{\epsilon,P_\epsilon}+\Phi_{\epsilon,P_\epsilon}, H_{\epsilon,P_\epsilon}+\Psi_{\epsilon,P_\epsilon})=0$. Let $\xi_\epsilon^{\overline{\frac{qr}{(p-1)(s+1)-qr}}} = c_{\epsilon,P_\epsilon}, A_\epsilon = \xi_\epsilon^{q/(p-1)}(A_{\epsilon,P_\epsilon}+\Phi_{\epsilon,P_\epsilon}), H_\epsilon = \xi_\epsilon(H_{\epsilon,P}+\Psi_{\epsilon,P_\epsilon})$. It is easy to see that $H_\epsilon = 1+O(\frac{1}{\log\frac{1}{\epsilon}})>0$ and hence $A_\epsilon\geq 0$. By the Maximum Principle, $A_\epsilon>0$. Moreover A_ϵ, H_ϵ satisfy Theorem 1.1.

□

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