SPIKES FOR THE GIERER-MEINHARDT SYSTEM IN TWO DIMENSIONS: THE STRONG COUPLING CASE

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ABSTRACT. Numerical computations often show that the Gierer-Meinhardt system has stable solutions which display patterns of multiple interior peaks (often also called spots). These patterns are also frequently observed in natural biological systems. It is assumed that the diffusion rate of the activator is very small and the diffusion rate of the inhibitor is finite (this is the so-called strong-coupling case). In this paper, we rigorously establish the existence and stability of such solutions of the full Gierer-Meinhardt system in two dimensions far from homogeneity. Green's function together with its derivatives plays a major role.

1. INTRODUCTION

In this paper, we continue our study of the Gierer-Meinhardt system (see [14]) which models biological pattern formation. Suitably rescaled, this system takes the form

$$(GM) \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{H}, & A > 0 \quad \text{in } \Omega, \\ \tau H_t = D \Delta H - H + \xi_\epsilon A^2, & H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\xi_\epsilon = \frac{2\pi}{\epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^2(y)\, dy} \qquad (1.1)$$

and w is the unique solution of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, & w > 0 \text{ in } R^2, \\ w(0) = \max_{y \in R^2} w(y), & w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \qquad (1.2)$$

The unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations of the biochemicals called activator and inhibitor at a point $x \in \Omega \subset R^2$ and

1

at a time $t > 0$, respectively; ϵ, τ, D are positive constants; $\Delta := \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^2; Ω is a smooth bounded domain in R^2; $\nu(x)$ is the outer normal at $x \in \partial\Omega$.

Let us first put the Gierer-Meinhardt system in its proper historical perspective. In 1957, Turing [43] proposed a mathematical model for morphogenesis, which describes the development of complex organisms from a single cell. He speculated that localized peaks (which are sometimes called spots) in the concentration of a chemical substance, known as an inducer or morphogen, could be responsible for a group of cells developing differently from the surrounding cells. He then demonstrated, with linear analysis around constant states, how a nonlinear reaction diffusion system could possibly generate such isolated peaks. Later in 1972, Gierer and Meinhardt [14] demonstrated the existence of such solutions numerically for what was later termed the Gierer-Meinhardt system, which is a simple system for explaining complex patterns and serves as a reasonably good model for many biological systems such as multicellular tissues or cells. See also the monography [29]. The theory has also very successfully been applied to beautiful patterns on sea shells [30].

In particular, numerical studies by Gierer and Meinhardt and more recently by Holloway [19] have revealed that when ϵ is small and D is finite, (GM) seems to have stable stationary solutions with the property that the activator concentrates around a finite number of points in $\overline{\Omega}$. Moreover, as $\epsilon \to 0$ the pattern exhibits a *"point condensation phenomenon"*. By this we mean that the activator concentrates in narrower and narrower regions of size $O(\epsilon)$ around these points and eventually shrinks to the set of points itself as $\epsilon \to 0$. Furthermore, the maximum of the inhibitor diverges to $+\infty$. Note that in contrast the typical size of structures for the inhibitor is of the order $\log \frac{1}{\epsilon}$. The presence of these two different length scales is the main reason why the analysis becomes difficult and we have to be very careful in choosing good approximations to the solution.

One issue in pattern formation has been pattern selection, in particular the issue of "stripes versus spots". Our result gives an example of a system where

spots are stable and therefore are a preferred pattern. There are some results based on nonlinear analysis close to homogeneous solutions [10], [25]. In this paper we present a nonlinear analysis close to solutions which are far from homogeneity. More precisely, we prove existence and stability of solutions with multiple spots. To the best of our knowledge, this is the first study of its kind for a full reaction-diffusion in a two-dimensional bounded domain. We point out that the main idea of the paper, namely to take $H \equiv 1$ to leading order in ϵ, simply does not work in higher space dimensions ($N > 2$).

The stationary equation for (GM) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{H} = 0, & A > 0 \quad \text{in } \Omega, \\ D\Delta H - H + \xi_\epsilon A^2 = 0, & H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Generally speaking system (1.3) is quite difficult to solve since it does neither have a variational structure nor a priori estimates. One way to study (1.3) is to examine the so-called *shadow system*. Namely, we let $D \to +\infty$ first. It is known (see [26], [36], [39], [45]) that the study of the shadow system amounts to the study of the following single equation for $p = 2$:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0, & u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Equation (1.4) has a variational structure and has been studied by numerous authors. It is known that equation (1.4) has both boundary spike solutions and interior spike solutions. For boundary spike solutions, see [5], [9], [15], [17], [24], [34], [35], [36], [45], [50], [52], and the references therein. (When $p = \frac{N+2}{N-2}, N \geq 3$, boundary spike solutions of (1.4) have been studied in [1], [2], [3], [12], [13], [32], etc.) For interior spike solutions, please see [4], [6], [18], [23], [46], [47], [51]. For stability of spike solutions, please see [20], [37], [48] and [49].

In the case when D is finite and not large (this is the so-called *strong coupling* case), there are only very few results available. For $N = 1$, one can construct spike solutions for all $D \geq 1$. See [42]. The stability problem has recently been solved for $N = 1$ [21]. (See [8], [33], and [39] for the study of

related systems.) In [53], we first constructed single interior spike solutions to (1.3) in the case $N = 2$ and $D = 1$. Note that $D = 1$ is set to simplify the presentation but that the proof works for any fixed positive constant D. Therefore for the rest of the paper we assume that $D = 1$. We establish the first rigorous result about existence and stability of multiple-spike solutions for the full Gierer-Meinhardt system (not the shadow system!) in higher dimensions. We would like to emphasize that our analysis is around the solutions which show the multiple-spot pattern and not just around constant solutions. To state the result, it is necessary to introduce the following notation.

Let $G(P, x)$ be Green's function of $-\Delta + 1$ under the Neumann boundary condition, i.e., G satisfies

$$\begin{cases} -\Delta G + G = \delta_P & \text{in } \Omega, \\ \frac{\partial G}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_P is the Dirac delta distribution at a point $P \in \Omega$. It is well-known that

$$G(P, x) = K(|x - P|) - H(P, x),$$

where $K(|x|)$ is the fundamental solution of $-\Delta + 1$ in R^2 with singularity at 0 and $H(P, x)$ is C^2 in Ω. It is also known that

$$K(r) = -\frac{1}{2\pi} \log r - \mu + O(r) \quad \text{as } r \to 0. \tag{1.5}$$

We denote by $h(P) := H(P, P)$ the Robin function.

In [53], the following theorem is proved, which gives existence of solutions with one spot.

Theorem A *Let $P_0 \in \Omega$ be a nondegenerate critical point of $h(P)$. Then for ϵ sufficiently small and $D = 1$, problem (1.3) has a solution (A_ϵ, H_ϵ) with the following properties:*

(1) $A_\epsilon(x) = w(\frac{x - P_\epsilon}{\epsilon}) + o(1)$ uniformly for $x \in \bar{\Omega}$, where $P_\epsilon \to P_0$ as $\epsilon \to 0$, w is the unique solution of the problem (1.2).

(2) $H_\epsilon(x) = 1 + O(\frac{1}{|\log \epsilon|})$ uniformly for $x \in \bar{\Omega}$.

(3) $\xi_\epsilon^{-1} = (\frac{1}{2\pi} + o(1))\epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^2$.

The main goals of this paper are twofold: first we construct equilibrium solutions with K interior peaks (interior $K-$peaked solutions), second we establish the stability of such solutions.

First let

$$\mathbf{P} = (P_1, ..., P_K) \in \Omega \times ... \times \Omega \cap \{|P_i - P_j| > \delta > 0 \quad \text{for } i \neq j\}.$$

Then we define

$$F(\mathbf{P}) = \sum_{k=1}^{K} H(P_k, P_k) - \sum_{i,j=1,...,K,i\neq j} G(P_i, P_j), \tag{1.6}$$

$$F_j(\mathbf{P}) = H(P_j, P_j) - \sum_{i=1,...,K,i\neq j} G(P_i, P_j), \quad j = 1, ..., K, \tag{1.7}$$

$$M(\mathbf{P}) = (\frac{\partial^2}{\partial \mathbf{P} \partial \mathbf{P}} F(\mathbf{P})), \tag{1.8}$$

where

$$\frac{\partial}{\partial P_{k,i}} H(P_k, P_k) := \frac{\partial}{\partial x_i} H(x, P_k)\Big|_{x=P_k}, \quad k = 1, \dots, K, i = 1, 2$$

in contrast with the usual definition.

(We arrange $\mathbf{P}$ such that $\mathbf{P} = (P_{1,1}, P_{1,2}, P_{2,1}, P_{2,2}, ..., P_{K,1}, P_{K,2})$).

Our first result is about existence of solutions with multiple spots.

Theorem 1.1. *Suppose that Ω is convex. Let $\mathbf{P}_0 = (P_1^0, ..., P_K^0) \in \Omega^K$ be a nondegenerate critical point of $F(\mathbf{P})$. Then for ϵ sufficiently small and $D = 1$, problem (1.3) has a solution (A_ϵ, H_ϵ) with the following properties:*

(1) $A_\epsilon(x) = \sum_{j=1}^{K} w(\frac{x-P_j^\epsilon}{\epsilon}) + o(1)$ uniformly for $x \in \bar{\Omega}$, $P_j^\epsilon \to P_j^0, j = 1, ...K$ as $\epsilon \to 0$, and w is the unique solution of the problem (1.2).

(2) $H_\epsilon(x) = 1 + O(\frac{1}{|\log \epsilon|})$ uniformly for $x \in \bar{\Omega}$.

(3) $\xi_\epsilon^{-1} = \frac{1}{2\pi} \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^2$.

Remark: It is a technical assumption that Ω is convex. In fact, from the proofs, it is easy to see that we just need that $F_j(\mathbf{P}_0) < 0, j = 1, ..., K$, which is satisfied when Ω is convex. (See Section 2 and the Appendix.)

Our second result is on stability:

Theorem 1.2. *Let* $\mathbf{P}_0$ *and* (A_ϵ, H_ϵ) *be defined as in Theorem 1.1. Then for* ϵ *and* τ *sufficiently small* (A_ϵ, H_ϵ) *is stable if all eigenvalues of the matrix* $M(\mathbf{P})$ *are negative.* (A_ϵ, H_ϵ) *is unstable if one of the eigenvalues of the matrix* $M(\mathbf{P})$ *is positive.*

Remark: In a general domain, the function $F(\mathbf{P})$ always has a global maximum point $\mathbf{P}_0$ in $\Omega \times ... \times \Omega$. (A proof of this fact can be found in the Appendix.) At such a point $\mathbf{P}_0$, the matrix $M(\mathbf{P}_0)$ is semi-negative definite. Thus our assumptions in Theorems 1.1 and Theorems 1.2 are reasonable ones.

Theorem 1.1 is proved by following the strategy in [53]. Namely, we use the Liapunov-Schmidt reduction method.

But in the multiple spot case great care is needed to handle their interaction. We shall frequently consult [53] and point out the new ideas and extensions which are needed.

Theorem 1.2 is completely new and can be proved by studying the *small* eigenvalues and the *large* eigenvalues of the linearized operator separately. The proof involves a lot of computations.

Now we lay down the basic ideas of the proof of Theorem 1.1.

As $\epsilon \to 0$, if we assume that $H_\epsilon(x) \to 1$ in $L^\infty_{loc}(\Omega)$, we have that $A_\epsilon(x) \sim \sum_{j=1}^K w\left(\frac{x - P_j^\epsilon}{\epsilon}\right)$ in $H^2_{loc}(R^2)$, where w satisfies (1.2). (Here and thereafter $A \sim B$ means $A = (1 + o(1))B$ as $\epsilon \to 0$ in the corresponding norm.)

To ensure that $H_\epsilon(P_j) \sim 1$ for $j = 1, \dots, K$ we note that

$$H_\epsilon(P_j^\epsilon) = \int_\Omega G(P_j^\epsilon, x)\xi_\epsilon A_\epsilon^2(x)dx$$

$$= \epsilon^2 \xi_\epsilon \int_{\Omega_\epsilon} G(P_j^\epsilon, P_j^\epsilon + \epsilon y)A_\epsilon^2(P_j^\epsilon + \epsilon y)dy$$

$$= \epsilon^2 \xi_\epsilon \sum_{k=1}^K \int_{\Omega_\epsilon} G(P_j^\epsilon, P_k^\epsilon + \epsilon y)w^2(y)dy(1 + o(1))$$

(by (1.5), $K(r) = -\frac{1}{2\pi}\log r - \mu + O(r)$ as $r \to 0$; $K(r)$ is bounded for $r \in [r_1, r_2]$ for $r_1, r_2 > 0$; see also Lemma 1.3 below)

$$= \frac{1}{2\pi}\xi_\epsilon \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^2(y)\, dy(1 + o(1)).$$

This suggests that we should choose ξ_ϵ as in (1.1). Hence we should look for solutions of (1.3) with the following properties

$$A_\epsilon(x) = \sum_{i=1}^{K} w\left(\frac{x - P_i^\epsilon}{\epsilon}\right) + \phi_\epsilon(y), \quad \phi_\epsilon \sim 0,$$

$$\text{where} \quad |P_i^\epsilon - P_i^0| = o(1) \ \text{as} \ \epsilon \to 0, \ i = 1, 2 \ldots, K,$$

$$H_\epsilon(x) = 1 + \psi_\epsilon(x), \quad \psi_\epsilon \sim 0.$$

We first recall the following definition from [53]: Suppose that $W \in H^1(R^2)$. The projection $\mathcal{P}_U W$ is defined by $\mathcal{P}_U W = W - \mathcal{Q}_U W$, where $\mathcal{Q}_U W$ satisfies

$$\begin{cases} \Delta \mathcal{Q}_U W - \mathcal{Q}_U W = 0 & \text{in} \ U, \\ \frac{\partial \mathcal{Q}_U W}{\partial \nu} = \frac{\partial W}{\partial \nu} & \text{on} \ \partial U \end{cases} \tag{1.9}$$

for an open set $U \subset R^2$.

The proof of Theorem 1.1 consists of the following steps:

A)-Choose good approximate solutions.

For ϵ small enough and $\mu < 0$ with $|\mu|$ small we first construct a particular radially symmetric solution $(A_{\epsilon,\mu}(x), H_{\epsilon,\mu}(x), \xi_{\epsilon,\mu})$ of the following problem:

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{(H-\mu)} = 0, & x \in R^2, \\ \Delta H - H + \xi_{\epsilon,\mu} A^2 = 0, & x \in R^2, \\ H(0) = 1. \end{cases} \tag{1.10}$$

Next we choose $\mu := \mu_{\epsilon,j}(\mathbf{P})$, where

$$\mu_{\epsilon,j}(\mathbf{P}) = \mathcal{Q}_\Omega(H_{\epsilon,\mu}(\cdot - P_j))(P_j) - \sum_{k \neq j} \mathcal{P}_\Omega(H_{\epsilon,\mu}(\cdot - P_k))(P_j), \quad j = 1, ..., K. \tag{1.11}$$

(The assumption that Ω is convex is needed to ensure that $\mu < 0$.)

Note that $\mu \sim \frac{1}{\log \frac{1}{\epsilon}}$. Therefore μ is small but not algebraically small in ϵ and for our approach to work we need to construct an approximation to (A, H) as in (1.10). It is simply not good enough to try the first guess which comes to mind: setting $\mu = 0$.

From this first approximation to the solution (A, H) in R^2 we construct an approximation to a K-spike solution in Ω in three steps: translation, projection, and superposition. Translation locates the j-th spike near P_j. Then projection produces Neumann boundary conditions, where the function

after projection is still very close to a solution. Finally superposition gives a multiple spike approximation out of a single spike approximation.

First we introduce the translation $(\hat{A}_{\epsilon,j}, \hat{H}_{\epsilon,j})$ to the point $P_j \in \Omega$ of the solution to (1.10):

$$\hat{A}_{\epsilon,j}(x) := A_{\epsilon,\mu_{\epsilon,j}(\mathbf{P})}(x - P_j), \quad \hat{H}_{\epsilon,j}(x) := H_{\epsilon,\mu_{\epsilon,j}(\mathbf{P})}(x - P_j).$$

Then we project the translated approximations

$$A_{\epsilon,j}(y) = \mathcal{P}_{\Omega_\epsilon} \hat{A}_{\epsilon,j}(\epsilon y)$$

and

$$H_{\epsilon,j}(x) = \mathcal{P}_\Omega \hat{H}_{\epsilon,j}(x),$$

where P_U was defined in (1.9) and

$$\Omega_\epsilon = \{y \in R^2 | \epsilon y \in \Omega\}.$$

Here we have used different scalings for activator and inhibitor, respectively, since then both resulting equations are independent of ϵ and the ϵ-dependence only appears in the scaling of the domain Ω_ϵ. Therefore one can formally pass to a limit in both equations. Note that also the approximate solution for fixed $\mathbf{P} \in \Omega^K$ converges to a limit as $\epsilon \to 0$ in the norm $H^2(\Omega_\epsilon) \times W^{2,t}(\Omega)$ for some $t > 1$. Later, in the derivation of Lemma 3.4 we will use these properties to construct a solution by applying the contraction mapping principle for a fixed operator in varying domains. We found that this is more transparent than using operators which do not have a limit. (See also Step B)- below).

Finally, we choose our approximate solutions by superposing the projected and translated approximations:

$$A_{\epsilon,\mathbf{P}}(y) := \sum_{j=1}^K A_{\epsilon,j}(y) \tag{1.12}$$

and

$$H_{\epsilon,\mathbf{P}}(x) := \sum_{j=1}^K H_{\epsilon,j}(x) \tag{1.13}$$

for

$$x \in \Omega, \quad y \in \Omega_\epsilon = \{y \in R^2 | \epsilon y \in \Omega\}.$$

the norm $H^2(\Omega_\epsilon) \times W^{2,t}(\Omega)$ for some $t > 1$. (See Step B)- below).

For later use we introduce the following notation: Translation plus superposition (without projection) is denoted by

$$\hat{A}_{\epsilon,\mathbf{P}}(x) := \sum_{j=1}^{K} \hat{A}_{\epsilon,j}(x), \quad \hat{H}_{\epsilon,\mathbf{P}}(x) := \sum_{j=1}^{K} \hat{H}_{\epsilon,j}(x),$$

$$\xi_{\epsilon,j} := \xi_{\epsilon,\mu_{\epsilon,j}}.$$

The error of the projection of the j-th translation is denoted by

$$\varphi_{\epsilon,j}(y) := \hat{A}_{\epsilon,j}\left(\frac{y}{\epsilon}\right) - A_{\epsilon,j}(y), \quad \psi_{\epsilon,j}(x) := \hat{H}_{\epsilon,j}(x) - H_{\epsilon,j}(x).$$

The sum of the errors of all K projections is denoted as follows:

$$\varphi_{\epsilon,\mathbf{P}}(y) := \hat{A}_{\epsilon,\mathbf{P}}\left(\frac{y}{\epsilon}\right) - A_{\epsilon,\mathbf{P}}(y), \quad \psi_{\epsilon,\mathbf{P}}(x) := \hat{H}_{\epsilon,\mathbf{P}}(x) - H_{\epsilon,\mathbf{P}}(x).$$

It will be proved that $\varphi_{\epsilon,\mathbf{P}}(y) = e.s.t.$ in $H^2(\Omega_\epsilon)$ and $\psi_{\epsilon,\mathbf{P}} = O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)$ in $L^\infty(\Omega)$.

We will analyze $A_{\epsilon,\mathbf{P}}$ and $H_{\epsilon,\mathbf{P}}$ in Section 2.

B)-The idea now is to look for a solution of (1.3) of the form

$$A_\epsilon(y) = A_{\epsilon,\mathbf{P}}(y) + \phi(y), \quad H_\epsilon(x) = H_{\epsilon,\mathbf{P}}(x) + \psi(x).$$

We will show that, provided $\mathbf{P}$ is properly chosen, ϕ and ψ are negligible.

We now write system (1.3) in operator form.

For any smooth and open set $U \subset R^2$, let

$$W_N^{2,t}(U) = \left\{ u \in W^{2,t}(U) \left| \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \right. \right\}, \quad H_N^2(U) = W_N^{2,2}(U).$$

For $A(y) \in H_N^2(\Omega_\epsilon)$, $H(x) \in W_N^{2,t}(\Omega)$, where $1 < t < 1.1$, we set

$$S_\epsilon \begin{pmatrix} A \\ H \end{pmatrix} = \begin{pmatrix} S_1(A,H) \\ S_2(A,H) \end{pmatrix},$$

where $S_1(A,H) = \Delta_y A - A + A^2/H$, $S_2(A,H) = \Delta_x H - H + \xi_\epsilon A^2$. (We need $t > 1$ so that the Sobolev embedding $W^{2,t}(\Omega) \subset L^\infty(\Omega)$ is continuous.)

Then solving equation (1.3) is equivalent to

$$S_\epsilon \begin{pmatrix} A \\ H \end{pmatrix} = 0, \quad A \in H_N^2(\Omega_\epsilon), \quad H \in W_N^{2,t}(\Omega). \tag{1.14}$$

We now substitute $A(y) = A_{\epsilon,\mathbf{P}}(y) + \phi(y), \quad H = H_{\epsilon,\mathbf{P}}(x) + \psi(x)$ into (1.14). The system determining ϕ and ψ can be written as

$$S_{\epsilon}' \left(\begin{array}{c} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{array} \right) \left[\begin{array}{c} \phi \\ \psi \end{array} \right] + \left(\begin{array}{c} E^1 \\ E^2 \end{array} \right) + \left(\begin{array}{c} O(\|\phi\|_{L^2(\Omega_\epsilon)}^2 + \|\psi\|_{L^t(\Omega)}^2) \\ O(\|\phi\|_{L^2(\Omega_\epsilon)}^2 + \|\psi\|_{L_t(\Omega)}^2) \end{array} \right) = 0,$$

where $E^i, i = 1, 2$ denote the error terms. For these we need very good estimates. Much of Section 2 is devoted to this analysis.

It is then natural to try to solve the equations for (ϕ, ψ) by a contraction mapping argument. The problem is that the linearized operator $S_{\epsilon}' \left(\begin{array}{c} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{array} \right)$ is not uniformly invertible with respect to ϵ.

Therefore, we now replace the equation above by

$$S_{\epsilon}' \left(\begin{array}{c} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{array} \right) \left[\begin{array}{c} \phi \\ \psi \end{array} \right] + \left(\begin{array}{c} E^1 \\ E^2 \end{array} \right) + \left(\begin{array}{c} O(\|\phi\|_{L^2(\Omega_\epsilon)}^2 + \|\psi\|_{L^t(\Omega)}^2) \\ O(\|\phi\|_{L^2(\Omega_\epsilon)}^2 + \|\psi\|_{L_t(\Omega)}^2) \end{array} \right) = \left(\begin{array}{c} v_{\epsilon,\mathbf{P}} \\ 0 \end{array} \right),$$

$$(1.15)$$

where $v_{\epsilon,\mathbf{P}}$ lies in an appropriately chosen approximate cokernel of the linear operator

$$L_\epsilon := \Delta_y - 1 + 2 A_{\epsilon,\mathbf{P}} H_{\epsilon,\mathbf{P}}^{-1} - 2 \frac{\int_{\Omega_\epsilon} A_{\epsilon,\mathbf{P}} \cdot}{\int_{\Omega_\epsilon} A_{\epsilon,\mathbf{P}}^2} A_{\epsilon,\mathbf{P}}^2,$$

$$L_\epsilon : \; H^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$$

and ϕ is orthogonal in $L^2(\Omega_\epsilon)$ to the corresponding approximate kernel of L_ϵ.

C)-We solve (1.15) for (ϕ, ψ) in the orthogonal complement of the approximate kernel. To this end, we need a detailed analysis of the operators L_ϵ and S_{ϵ}'. This together with the contraction mapping argument is done in Section 3.

D)-In the last step, for $\mathbf{P} \in \Omega^K$ we study a vector field $\mathbf{P} \to W_\epsilon(\mathbf{P})$ such that $W_\epsilon(\mathbf{P}) = 0$ implies $v_{\epsilon,\mathbf{P}} = 0$ (and hence solutions of the system (1.3) can be found). To discuss the zeros of $\mathbf{P} \to W_\epsilon(\mathbf{P})$ we need the estimates for the error terms E^1 and E^2 given in Section 3.

We discover that under the geometric condition described in Theorem 1.1 there is a point $\mathbf{P}^\epsilon$ in a small neighborhood of $\mathbf{P}_0 \in \Omega^K$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$. This will complete the proof of Theorem 1.1 and is done in Section 4.

Throughout this paper, we always assume that $|\mathbf{P}-\mathbf{P}_0| < r$ for some fixed small number $r > 0$. We shall frequently use the following technical lemma.

Lemma 1.3. *Let u be a solution of*

$$\Delta u - u + f = 0 \quad in \ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad on \ \partial\Omega.$$

Suppose

$$|f(x)| \leq \eta e^{-\frac{\alpha|x-P|}{\epsilon}}$$

for some $\alpha > 0$. Then we have

$$|u(P)| \leq C_1 \eta \epsilon^2 \log \frac{1}{\epsilon} \tag{1.16}$$

and

$$|u(P) - u(x)| \leq C_2 \eta \epsilon^2 \log \left(\frac{|x-P|}{\epsilon} + 1 \right), \tag{1.17}$$

where $C_1 > 0, C_2 > 0$ are generic constants (independent of $\epsilon > 0$ and $\eta > 0$).

Proof: By the representation formula we calculate

$$u(x) = \int_\Omega G(x,z) f(z) dz$$

and

$$u(P) = \int_\Omega G(P,z) f(z) dz = \epsilon^2 \int_{\Omega_{\epsilon,P}} G(P, P+\epsilon y) \eta e^{-\alpha|y|} dy$$

$$\leq C_1 \eta \epsilon^2 \log \frac{1}{\epsilon}.$$

Similarly we can obtain (1.17).

$\square$

To establish stability and prove Theorem 1.2 the eigenvalues and eigenfunctions of the linearized operator of (1.3) have be calculated and their sign has to be determined.

For large eigenvalues by taking the limit $\epsilon \to 0$, we can reduce the problem to a nonlocal eigenvalue problem (NLEP) which has been studied by Wei [49]. This is done in Section 5.

For small eigenvalues fine calculations are needed as the interplay of the two equations of the Gierer-Meinhardt system enters into the analysis in a very intricate way. In particular, the different spots interact with each other

and with the boundary. By representing the eigenfunctions with respect to the new approximate kernel $\mathcal{K}^{new}_{\epsilon,\mathbf{P}^\epsilon}$ of the linearized operator we manage to reduce this problem to the positive definiteness of the matrix $M(\mathbf{P})$. This analysis is carried out in Section 6.

To simplify our notations, we use $e.s.t.$ to denote exponentially small terms in the corresponding norms, i.e. $e.s.t. = O(e^{-d/\epsilon})$ for some $d > 0$ (independent of ϵ).

2. Construction of the Approximate Solutions

In this section, we study the approximate solutions.

We first have

Lemma 2.1. *(Lemma 2.1 of* [53]*) The operator*

$$L := \Delta - 1 + 2w - 2\frac{\int_{R^2} w\cdot}{\int_{R^2} w^2}w^2$$

with w defined in (1.4) is an invertible map from $H^2_r(R^2)$ to $L^2_r(R^2)$, where $H^2_r(R^2)$ ($L^2_r(R^2)$) is the subset of those functions of $H^2(R^2)$ ($L^2(R^2)$) which are radially symmetric.

We next have

Lemma 2.2. *For $\epsilon << 1$ and $\mu < 0, |\mu| << 1$, there exists a unique radially symmetric solution $(A_{\epsilon,\mu}, H_{\epsilon,\mu}, \xi_{\epsilon,\mu})$ of the following parametrized equation*

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{H-\mu} = 0, & x \in R^2, \\ \Delta H - H + \xi_{\epsilon,\mu} A^2 = 0, & x \in R^2, \\ A(x) = A(|x|), \quad H(x) = H(|x|), \quad H(0) = 1. \end{cases} \tag{2.1}$$

Moreover, $(A_{\epsilon,\mu}, H_{\epsilon,\mu})$ is C^1 in μ with respect to the norm of $H^2(R^2) \times W^{2,t}(R^2)$.

Proof: A proof based on the contraction mapping principle is given as Step 1 in [53]. □

Remarks: 1. In Lemma 2.2, we need that $\mu < 0$, since otherwise $H - \mu$ may not be well-defined.

2. From the proof of Lemma 2.2 by the contraction mapping principle the following estimates are immediate:

$$A = w\left(1 + O(|\mu|) + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right) \quad \text{in } H^2_{loc}(R^2),$$

$$H = 1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right) \quad \text{in } W^{2,t}_{loc}(R^2),$$

$$\xi_{\epsilon,\mu} = \xi_\epsilon\left(1 + O(|\mu|) + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right)$$

as $\epsilon, \mu \to 0$ are immediate.

We now choose different μ for different $P_j, j = 1, ..., K$.

For each $j = 1, ..., K$, we define $\mu = \mu_{\epsilon,j}$ by

$$\mu = H_{\epsilon,\mu}(0) - \sum_{k=1}^{K} \mathcal{P}_\Omega(H_{\epsilon,\mu}(\cdot - P_k))(P_j), \tag{2.2}$$

which is equivalent to (1.11).

Note that, using Remark 2 after Lemma 2.2, this is also equivalent to

$$\mu = \int_{R^2}\left(K(|z|) - \sum_{k=1}^{K} G(P_k, P_j + z)\right)\xi_{\epsilon,\mu}A^2_{\epsilon,\mu}(z)\, dz$$

$$= \int_{R^2}\left(H(P_j, P_j + z) - \sum_{k\neq j} G(P_k, P_j + z)\right)\xi_{\epsilon,\mu}A^2_{\epsilon,\mu}(z)\, dz$$

$$= F_j(\mathbf{P})\xi_{\epsilon,\mu}\epsilon^2\int_{R^2}A^2_{\epsilon,\mu}(\epsilon y)dy + \xi_{\epsilon,\mu}\int_{R^2}O(\epsilon^3|y|)A^2_{\epsilon,\mu}(\epsilon y)dy$$

$$= F_j(\mathbf{P})\xi_{\epsilon,\mu}\epsilon^2\int_{R^2}w^2(y)\, dy(1 + O(|\mu| + \epsilon)).$$

By the implicit function theorem (2.2) has a unique solution $\mu_{\epsilon,j} < 0$ with $|\mu_{\epsilon,j}|$ small.

We further calculate

$$\mu_{\epsilon,j} = \frac{2\pi}{\log\frac{1}{\epsilon}}F_j(\mathbf{P})\left(1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right)$$

and

$$\xi_{\epsilon,j} = \xi_\epsilon \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}(|F_j(\mathbf{P})| + 1)\right)\right) \tag{2.3}$$

as $\epsilon \to 0$.

We have for $|x| \geq \delta$:

$$\hat{H}_{\epsilon,j}(x) = \frac{\int_{R^2} K(|x - \epsilon y|)\hat{A}_{\epsilon,\mu}^2(\epsilon y)\, dy}{\int_{R^2} K(|\epsilon y|)\hat{A}_{\epsilon,\mu}^2(\epsilon y)\, dy}$$

$$= \frac{1}{\log \frac{1}{\epsilon}}[K(|x - P_j|)\left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)] \quad \text{as } \epsilon \to 0,$$

where $\mu = \mu_{\epsilon,j}$.

We note that $\varphi_{\epsilon,j}(y) = \hat{A}_{\epsilon,j}(y) - \mathcal{P}_{\Omega_\epsilon}\hat{A}_{\epsilon,j}(y)$ satisfies

$$\Delta_y \varphi_{\epsilon,j} - \varphi_{\epsilon,j} = 0 \quad \text{in } \Omega_\epsilon,$$

$$\frac{\partial \varphi_{\epsilon,j}}{\partial \nu} = \frac{\partial \hat{A}_{\epsilon,j}}{\partial \nu} = O(e^{-d(P_j, \partial\Omega)/\epsilon}) \quad \text{in } L^2(\partial\Omega_\epsilon).$$

Hence,

$$\|\varphi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)} = O(e^{-d(P_j, \partial\Omega)/\epsilon}). \tag{2.4}$$

This implies

$$\|\varphi_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} = e.s.t.. \tag{2.5}$$

We further calculate for $|x - P_j| \geq \delta$:

$$\mathcal{P}_\Omega \hat{H}_{\epsilon,j}(x) = \frac{\int_{\Omega_{\epsilon,P}} G(x, P_j + \epsilon y)\hat{A}_{\epsilon,j}^2(\epsilon y)\, dy}{\int_{R^2} K(|\epsilon y|)\hat{A}_{\epsilon,\mu}^2(\epsilon y)\, dy}(1 + O(\epsilon))$$

$$= \frac{1}{\log \frac{1}{\epsilon}}[K(|x - P_j|) - H(x, P_j)]\left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right).$$

This implies

$$\psi_{\epsilon,\mathbf{P}}(x) = \frac{1}{\log \frac{1}{\epsilon}}\left[\sum_{j=1}^K H(x, P_j)\right]\left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right). \tag{2.6}$$

By (2.5) and (2.6), we see that the term involving $\varphi_{\epsilon,\mathbf{P}}$ is negligible in comparison with $\psi_{\epsilon,\mathbf{P}}$. We will use this in the later sections.

The reason for choosing $A_{\epsilon,\mu}$ and $H_{\epsilon,\mu}$ as we did lies in the following two estimates:

$$S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = \Delta_y A_{\epsilon,\mathbf{P}} - A_{\epsilon,\mathbf{P}} + \frac{A_{\epsilon,\mathbf{P}}^2}{H_{\epsilon,\mathbf{P}}}$$

$$= \frac{A_{\epsilon,\mathbf{P}}^2}{H_{\epsilon,\mathbf{P}}} - \sum_{j=1}^K \frac{\hat{A}_{\epsilon,j}^2}{\hat{H}_{\epsilon,j} - \mu_{\epsilon,j}}$$

$$= \frac{(\hat{A}_{\epsilon,\mathbf{P}} - \varphi_{\epsilon,\mathbf{P}})^2}{\hat{H}_{\epsilon,\mathbf{P}} - \psi_{\epsilon,\mathbf{P}}} - \sum_{j=1}^K \frac{\hat{A}_{\epsilon,j}^2}{\hat{H}_{\epsilon,j} - \mu_{\epsilon,j}}$$

$$= \frac{(\sum_{j=1}^K (\hat{A}_{\epsilon,j} - \varphi_{\epsilon,j}))^2}{\sum_{k=1}^K (\hat{H}_{\epsilon,k} - \psi_{\epsilon,k})} - \sum_{j=1}^K \frac{\hat{A}_{\epsilon,j}^2}{\hat{H}_{\epsilon,j} - \mathcal{Q}_\Omega \hat{H}_{\epsilon,j}(P_j) + \sum_{k \neq j} \mathcal{P}_\Omega \hat{H}_{\epsilon,k}(P_j)}$$

$$= \frac{(\sum_{j=1}^K (\hat{A}_{\epsilon,j} - \varphi_{\epsilon,j}))^2}{\sum_{k=1}^K (\hat{H}_{\epsilon,k} - \psi_{\epsilon,k})} - \sum_{j=1}^K \frac{\hat{A}_{\epsilon,j}^2}{\hat{H}_{\epsilon,j} - \psi_{\epsilon,j}(P_j) + \sum_{k \neq j} (\hat{H}_{\epsilon,k}(P_j) - \psi_{\epsilon,k}(P_j))}$$

$$= e.s.t. + \sum_{j=1}^K (\hat{A}_{\epsilon,j})^2$$

$$\times \left\{ \left[\sum_{k=1}^K (\hat{H}_{\epsilon,k} - \psi_{\epsilon,k}) \right]^{-1} - \left[\hat{H}_{\epsilon,j} - \psi_{\epsilon,j}(P_j) + \sum_{k \neq j} (\hat{H}_{\epsilon,k}(P_j) - \psi_{\epsilon,k}(P_j)) \right]^{-1} \right\}$$

$$= e.s.t. + \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}} \right) \right) \sum_{j=1}^K (\hat{A}_{\epsilon,j})^2$$

$$+ \sum_{j=1}^K \frac{(\hat{A}_{\epsilon,j})^2}{\sum_{k=1}^K (\hat{H}_{\epsilon,k}(P_j))^2} \left[\sum_{k=1}^K (\psi_{\epsilon,k} - \psi_{\epsilon,k}(P_j)) - \sum_{k \neq j} (\hat{H}_{\epsilon,k} - \hat{H}_{\epsilon,k}(P_j)) \right]$$

for $y \in \Omega_\epsilon$.

Now we calculate

$$S_2(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = \Delta_x H_{\epsilon,\mathbf{P}} - H_{\epsilon,\mathbf{P}} + \xi_\epsilon A_{\epsilon,\mathbf{P}}^2$$

$$= \xi_\epsilon (\hat{A}_{\epsilon,\mathbf{P}} - \varphi_{\epsilon,\mathbf{P}})^2 - \xi_\epsilon (\hat{A}_{\epsilon,\mathbf{P}})^2$$

$$= e.s.t.$$

for $x \in \Omega$.

We have thus obtained

Lemma 2.3. *The following estimates hold:*

$$S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) \tag{2.7}$$

$$= e.s.t. + \left(1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right)\sum_{j=1}^{K}(\hat{A}_{\epsilon,j})^2$$

$$+ \sum_{j=1}^{K}\frac{(\hat{A}_{\epsilon,j})^2}{\sum_{k=1}^{K}(\hat{H}_{\epsilon,k}(P_j))^2}\left[\sum_{k=1}^{K}(\psi_{\epsilon,k} - \psi_{\epsilon,k}(P_j)) - \sum_{k\neq j}(\hat{H}_{\epsilon,k} - \hat{H}_{\epsilon,k}(P_j))\right]$$

for $y \in \Omega_\epsilon$ and

$$S_2(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = e.s.t. \tag{2.8}$$

for $x \in \Omega$.

  Hence,

$$\|S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}})\|_{L^2(\Omega_\epsilon)} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right), \tag{2.9}$$

$$\|S_2(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}})\|_{L^t(\Omega)} = e.s.t. \tag{2.10}$$

for any $1 < t < 1.1$.

Proof: By direct computation. (See before the statement of Lemma 2.3).
□

3. The Liapunov-Schmidt Reduction Method

This section is devoted to studying the linearized operator defined by

$$\tilde{L}_{\epsilon,\mathbf{P}} := S'_\epsilon\begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix},$$

$$\tilde{L}_{\epsilon,\mathbf{P}} : H_N^2(\Omega_\epsilon) \times W_N^{2,t}(\Omega) \to L^2(\Omega_\epsilon) \times L^t(\Omega),$$

where $1 < t < 1.1$ is a fixed number.

  Set

$$K_{\epsilon,\mathbf{P}} := \text{span}\left\{\frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K, \, l = 1, \ldots, 2\right\} \subset H_N^2(\Omega_\epsilon),$$

$$C_{\epsilon,\mathbf{P}} := \text{span}\left\{\frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K, \, l = 1, \ldots, 2\right\} \subset L^2(\Omega_\epsilon),$$

$$L_\epsilon := \Delta - 1 + 2A_{\epsilon,\mathbf{P}}H_{\epsilon,\mathbf{P}}^{-1} - 2\frac{\int_{\Omega_\epsilon} A_{\epsilon,\mathbf{P}}\cdot}{\int_{\Omega_\epsilon} A_{\epsilon,\mathbf{P}}^2}A_{\epsilon,\mathbf{P}}^2$$

and

$$L_{\epsilon,\mathbf{P}} := \hat{\pi}_{\epsilon,\mathbf{P}} \circ L_\epsilon : K_{\epsilon,\mathbf{P}}^\perp \to C_{\epsilon,\mathbf{P}}^\perp,$$

where $\hat{\pi}_{\epsilon,\mathbf{P}}$ is the projection in $L^2(\Omega_\epsilon)$ onto $C_{\epsilon,\mathbf{P}}^\perp$.

We remark that since $A_{\epsilon,\mathbf{P}}(y) = \sum_{j=1}^K \left(1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right) w\left(y - \frac{P_j^\epsilon}{\epsilon}\right)$, it is easy to see that

$$l_{\epsilon,\mathbf{P}} := \hat{\pi}_{\epsilon,\mathbf{P}} \circ (\Delta - 1 + 2A_{\epsilon,\mathbf{P}}) : K_{\epsilon,\mathbf{P}}^\perp \to C_{\epsilon,\mathbf{P}}^\perp$$

is an injective and surjective map. For the proof please see the proof of Propositions 6.1–6.2 in [47].

The following proposition is the key estimate in applying the Liapunov-Schmidt reduction method.

Proposition 3.1. *For ϵ sufficiently small, the map $L_{\epsilon,\mathbf{P}}$ is an injective and surjective map. Moreover the inverse of $L_{\epsilon,\mathbf{P}}$ exists and is bounded uniformly with respect to ϵ.*

Proof: We will follow the method used in [11], [40], [41], [47] and [50]. We first show that there exist constants $C > 0, \bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$,

$$\|L_{\epsilon,\mathbf{P}}\Phi\|_{L^2(\Omega_\epsilon)} \geq C\|\Phi\|_{H^2(\Omega_\epsilon)} \tag{3.1}$$

for all $\Phi \in K_{\epsilon,\mathbf{P}}^\perp$.

Suppose that (3.1) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\mathbf{P}_k\}$, and $\{\phi_k\}$ with $\mathbf{P}_k \in \Omega^K$, $\phi_k \in K_{\epsilon_k,\mathbf{P}_k}^\perp$ such that

$$\|L_{\epsilon_k,\mathbf{P}_k}\phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0, \tag{3.2}$$

$$\|\phi_k\|_{H^2(\Omega_{\epsilon_k})} = 1, \qquad k = 1, 2, \dots. \tag{3.3}$$

Namely, we have the following situation

$$\Delta_y\phi_k - \phi_k + 2A_{\epsilon_k,\mathbf{P}_k}H_{\epsilon_k,\mathbf{P}_k}^{-1}\phi_k - 2\frac{\int_{\Omega_{\epsilon_k}} A_{\epsilon_k,\mathbf{P}_k}\phi_k}{\int_{\Omega_{\epsilon_k}} A_{\epsilon_k,\mathbf{P}_k}^2}A_{\epsilon_k,\mathbf{P}_k}^2 = f_k, \tag{3.4}$$

where

$$\|f_k\|_{L^2(\Omega_{\epsilon_k})} \to 0,$$

$$\phi_k \in K_{\epsilon_k,\mathbf{P}_k}^\perp, \quad \|\phi_k\|_{H^2(\Omega_{\epsilon_k})} = 1. \tag{3.5}$$

We now show that this is impossible. Set $A_k = A_{\epsilon_k, \mathbf{P}_k}, \Omega_k = \Omega_{\epsilon_k}$.

Note that

$$H_{\epsilon_k, \mathbf{P}_k} = 1 + o(1) \text{ in } L^\infty(\Omega),$$

$$(\Delta_y - 1 + 2A_k)A_k = A_k^2 + o(1) \text{ in } L^2(\Omega_k).$$

Thus we have

$$(\Delta_y - 1 + 2A_k)(\phi_k - 2\frac{\int_{\Omega_k} A_k \phi_k}{\int_{\Omega_k} A_k^2} A_k) = f_k + o(1) \text{ in } L^2(\Omega_k).$$

Since the projection of A_k onto $K_{\epsilon_k, \mathbf{P}_k}$ is $o(1)$ in $H^2(\Omega_k)$ and the operator

$$\Delta_y - 1 + 2A_k$$

is a one-to-one map (with the inverse bounded uniformly with respect to ϵ) from $K_{\epsilon_k, \mathbf{P}_k}^\perp$ to $C_{\epsilon_k, \mathbf{P}_k}^\perp$, we have

$$\phi_k - 2\frac{\int_{\Omega_k} A_k \phi_k}{\int_{\Omega_k} A_k^2} A_k = o(1) \text{ in } H^2(\Omega_k). \tag{3.6}$$

Multiplying (3.6) by A_k and integrating implies that

$$\int_{\Omega_k} A_k \phi_k = 0$$

and therefore

$$\|\phi_k\|_{H^2(\Omega_k)} = o(1).$$

A contradiction !

Thus (3.1) holds and $L_{\epsilon, \mathbf{P}}$ is a one-to-one map.

Next we show that $L_{\epsilon, \mathbf{P}}$ is also surjective. To this end, we just need to show that the conjugate of $L_{\epsilon, \mathbf{P}}$ (denoted by $L_{\epsilon, \mathbf{P}}^*$) is injective from $K_{\epsilon, \mathbf{P}}^\perp$ to $C_{\epsilon, \mathbf{P}}^\perp$.

Let $L_{\epsilon, \mathbf{P}}^* \phi \in C_{\epsilon, \mathbf{P}}^\perp$, $\phi \in K_{\epsilon, \mathbf{P}}^\perp$. Namely, we have

$$\Delta_y \phi - \phi + 2A_{\epsilon, \mathbf{P}} H_{\epsilon, \mathbf{P}}^{-1} \phi - 2\frac{\int_{\Omega_{\epsilon, \mathbf{P}}} A_{\epsilon, \mathbf{P}}^2 \phi}{\int_{\Omega_\epsilon} A_{\epsilon, \mathbf{P}}^2} A_{\epsilon, \mathbf{P}} \in C_{\epsilon, \mathbf{P}}. \tag{3.7}$$

We can assume that $\|\phi\|_{H^2(\Omega_\epsilon)} = 1$.

Multiplying (3.7) by $A_{\epsilon, \mathbf{P}}$ and integrating over Ω_ϵ, we obtain

$$\int_{\Omega_\epsilon} A_{\epsilon, \mathbf{P}}^2 \phi = o(1)$$

Hence ϕ satisfies

$$\Delta_y \phi - \phi + 2A_{\epsilon, \mathbf{P}} H_{\epsilon, \mathbf{P}}^{-1} \phi + o(1) \in C_{\epsilon, \mathbf{P}}, \quad \phi \in K_{\epsilon, \mathbf{P}}^\perp$$

which implies that $\|\phi\|_{H^2(\Omega_\epsilon)} = o(1)$. A contradiction !

Therefore $L_{\epsilon,\mathbf{P}}$ is also surjective.

$\square$

We now deal with system (1.14).

The operator $\tilde{L}_{\epsilon,\mathbf{P}}$ is not uniformly invertible in ϵ due to the approximate kernel

$$\mathcal{K}_{\epsilon,\mathbf{P}} := K_{\epsilon,\mathbf{P}} \oplus \{0\} \subset H_N^2(\Omega_\epsilon) \times W_N^{2,t}(\Omega).$$

We choose the approximate cokernel as follows:

$$\mathcal{C}_{\epsilon,\mathbf{P}} := C_{\epsilon,\mathbf{P}} \oplus \{0\} \subset L^2(\Omega_\epsilon) \times L^t(\Omega).$$

We then define

$$\mathcal{K}_{\epsilon,\mathbf{P}}^\perp := K_{\epsilon,\mathbf{P}}^\perp \oplus W_N^{2,t}(\Omega) \subset H_N^2(\Omega_\epsilon) \times W_N^{2,t}(\Omega),$$

$$\mathcal{C}_{\epsilon,\mathbf{P}}^\perp := C_{\epsilon,\mathbf{P}}^\perp \oplus L^t(\Omega) \subset L^2(\Omega_\epsilon) \times L^t(\Omega).$$

Let $\pi_{\epsilon,\mathbf{P}}$ denote the projection in $L^2(\Omega_\epsilon) \times L^t(\Omega)$ onto $\mathcal{C}_{\epsilon,\mathbf{P}}^\perp$. (Here the second component of the projection is the identity map.) We then show that the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = 0$$

has the unique solution $\Sigma_{\epsilon,\mathbf{P}} = \begin{pmatrix} \Phi_{\epsilon,\mathbf{P}}(y) \\ \Psi_{\epsilon,\mathbf{P}}(x) \end{pmatrix} \in \mathcal{K}_{\epsilon,\mathbf{P}}^\perp$ if ϵ is small enough.

As a preparation in the following two propositions we show the invertibility of the corresponding linearized operator.

Proposition 3.2. *Let $\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}}$. There exist positive constants $\bar{\epsilon}, \lambda$ such that for all $\epsilon \in (0, \bar{\epsilon})$*

$$\|\mathcal{L}_{\epsilon,\mathbf{P}}\Sigma\|_{L^2(\Omega_\epsilon) \times L^t(\Omega)} \geq \lambda \|\Sigma\|_{H^2(\Omega_\epsilon) \times W^{2,t}(\Omega)} \tag{3.8}$$

for all $\Sigma \in \mathcal{K}_{\epsilon,\mathbf{P}}^\perp$.

Proposition 3.3. *There exists a positive constant $\bar{\bar{\epsilon}}$ such that for all $\epsilon \in (0, \bar{\bar{\epsilon}})$ the map*

$$\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon\mathbf{P}} : \mathcal{K}_{\epsilon,\mathbf{P}}^\perp \to \mathcal{C}_{\epsilon,\mathbf{P}}^\perp$$

is surjective.

Proof of Proposition 3.2: This proposition follows from Proposition 3.1. In fact, suppose that (3.8) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\mathbf{P}_k\}$, and $\{\Sigma_k\}$ with $\mathbf{P}_k \in \Omega^K$, $\Sigma_k = \begin{pmatrix} \phi_k(y) \\ \psi_k(x) \end{pmatrix} \in \mathcal{K}_{\epsilon_k,\mathbf{P}_k}^{\perp}$ such that

$$\|\mathcal{L}_{\epsilon_k,\mathbf{P}_k}\Sigma_k\|_{L^2(\Omega_{\epsilon_k}) \times L^t(\Omega)} \to 0, \tag{3.9}$$

$$\|\Sigma_k\|_{H^2(\Omega_{\epsilon_k}) \times W^{2,t}(\Omega)} = 1, \qquad k = 1, 2, \dots . \tag{3.10}$$

Namely, we have the following situation

$$\Delta_y \phi_k - \phi_k + 2A_{\epsilon_k,\mathbf{P}_k} H_{\epsilon_k,\mathbf{P}_k}^{-1} \phi_k - A_{\epsilon_k,\mathbf{P}_k}^2 H_{\epsilon_k,\mathbf{P}_k}^{-2} \psi_k = f_k, \quad \|f_k\|_{L^2(\Omega_{\epsilon_k})} \to 0, \tag{3.11}$$

$$\Delta_x \psi_k - \psi_k + 2\xi_{\epsilon_k} A_{\epsilon_k,\mathbf{P}_k} \phi_k = g_k, \tag{3.12}$$

where

$$\|g_k\|_{L^t(\Omega)} \to 0,$$

$$\phi_k \in K_{\epsilon_k,\mathbf{P}_k}^{\perp}, \tag{3.13}$$

$$\|\phi_k\|_{H^2(\Omega_{\epsilon_k})}^2 + \|\psi_k\|_{W^{2,t}(\Omega)}^2 = 1. \tag{3.14}$$

We now show that this is impossible. Set $A_k = A_{\epsilon_k,\mathbf{P}_k}$, $\Omega_k = \Omega_{\epsilon_k}$, $\mathbf{P}_k = (P_1^k, P_2^k, \dots, P_K^k)$, $\xi_k = \xi_{\epsilon_k}$.

We first note that by (3.12) we have

$$\|\psi_k\|_{L^\infty(\Omega)} \le C$$

and hence by Lemma 1.3 and Sobolev embedding,

$$|\psi_k(x) - \psi_k(P_j^k)| \le C|x - P_j^k|^\alpha + \frac{1}{\log \frac{1}{\epsilon}} \log \left(1 + \frac{|x - P_j^k|}{\epsilon} \right)$$

for some $\alpha > 0$ since $t > 1$. Thus

$$\|A_k^2(\psi_k - \psi_k(P_j^k))\|_{L^2(\Omega_k)} \to 0 \quad \text{in} \quad L^2(\Omega_k) \quad \text{as } k \to \infty \tag{3.15}$$

for every $j = 1, 2, \dots, K$. Moreover by (3.12),

$$\psi_k(P_j^k) = \int_{\Omega_k} G(P_j^k, z) 2\xi_k(A_{j,k}\phi_k - g_k)$$

$$= (2 + o(1))\xi_k \log \frac{1}{\epsilon_k} \int_{\Omega_k} A_{j,k}\phi_k + o(1)$$

and so

$$\psi_k(P_j^k) = 2\frac{\int_{\Omega_k} A_{j,k}\phi_k}{\int_{\Omega_k} A_{j,k}^2} + o(1) \qquad \text{for } j = 1, 2, \dots, K.$$

Thus we have

$$\mathcal{L}_{\epsilon_k,\mathbf{P}_k}\phi_k = o(1) \quad \text{in} \quad L^2(\Omega_k), \quad \phi_k \in K_{\epsilon_k,\mathbf{P}_k}^{\perp}. \tag{3.16}$$

By Proposition 3.1, $\|\phi_k\|_{H^2(\Omega_k)} = o(1)$. Hence $\psi_k(\mathbf{P}_k) = o(1)$ and by elliptic estimates $\|\psi_k\|_{W^{2,t}(\Omega)} = o(1)$.

This contradicts the assumption (3.14) and the proof of Proposition 3.2 is completed. $\square$

Proof of Proposition 3.3: We just need to show that the conjugate operator of $\mathcal{L}_{\epsilon,\mathbf{P}}$ (denoted by $\mathcal{L}_{\epsilon,\mathbf{P}}^*$) is injective from $\mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$ to $\mathcal{C}_{\epsilon,\mathbf{P}}^{\perp}$. Suppose not. Then there exist $\phi \in K_{\epsilon,\mathbf{P}}^{\perp}$, $\psi \in W^{2,t}(\Omega)$ such that

$$\Delta_y\phi - \phi + 2A_{\epsilon,\mathbf{P}}H_{\epsilon,\mathbf{P}}^{-1}\phi + 2\xi_\epsilon A_{\epsilon,\mathbf{P}}\psi \in C_{\epsilon,\mathbf{P}}^{\perp},$$

$$\Delta_x\psi - \psi - A_{\epsilon,\mathbf{P}}^2 H_{\epsilon,\mathbf{P}}^{-2}\phi = 0,$$

$$\|\phi\|_{H^2(\Omega_\epsilon)}^2 + \|\psi\|_{W^{2,t}(\Omega)}^2 = 1.$$

Similar to the proof of Proposition 3.2, we have

$$\psi(P_j) = -(1 + o(1))\xi_\epsilon \frac{\int_{\Omega_\epsilon} A_{\epsilon,\mathbf{P}}^2\phi}{\int_{\Omega_\epsilon} A_{\epsilon,\mathbf{P}}^2}$$

and substituting into the equation for ϕ we obtain

$$L_{\epsilon,\mathbf{P}}\phi + o(1) \in C_{\epsilon,\mathbf{P}}^{\perp}, \quad \phi \in K_{\epsilon,\mathbf{P}}^{\perp}.$$

By Proposition 3.1, $\|\phi\|_{H^2(\Omega_\epsilon)} = o(1)$ and hence $\|\psi\|_{W^{2,t}(\Omega)} = o(1)$. A contradiction !

$\square$

Now we are in a position to solve the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \phi \\ H_{\epsilon,\mathbf{P}} + \psi \end{pmatrix} = 0. \tag{3.17}$$

Since $\mathcal{L}_{\epsilon,\mathbf{P}}|_{\mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}}$ is invertible (call the inverse $\mathcal{L}_{\epsilon,\mathbf{P}}^{-1}$) we can rewrite (3.17) as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = M_{\epsilon,\mathbf{P}} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{3.18}$$

where

$$M_{\epsilon,\mathbf{P}}\begin{pmatrix}\phi\\\psi\end{pmatrix} = -(\mathcal{L}_{\epsilon,\mathbf{P}}^{-1}\circ\pi_{\epsilon,\mathbf{P}})(S_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}\\H_{\epsilon,\mathbf{P}}\end{pmatrix})-(\mathcal{L}_{\epsilon,\mathbf{P}}^{-1}\circ\pi_{\epsilon,\mathbf{P}})N_{\epsilon,\mathbf{P}}\begin{pmatrix}\phi\\\psi\end{pmatrix}$$

for $\begin{pmatrix}\phi\\\psi\end{pmatrix}\in H_N^2(\Omega_\epsilon)\times W^{2,t}(\Omega)$ and

$$N_{\epsilon,\mathbf{P}}\begin{pmatrix}\phi\\\psi\end{pmatrix} = S_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}+\phi\\H_{\epsilon,\mathbf{P}}+\psi\end{pmatrix}-S_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}\\H_{\epsilon,\mathbf{P}}\end{pmatrix}-S_\epsilon'\begin{pmatrix}A_{\epsilon,\mathbf{P}}\\H_{\epsilon,\mathbf{P}}\end{pmatrix}\begin{bmatrix}\phi\\\psi\end{bmatrix}.$$

We now use introduce the shorthand

$$\Sigma = \begin{pmatrix}\phi\\\psi\end{pmatrix}.$$

We are going to show that the operator $M_{\epsilon,\mathbf{P}}$ is a contraction on

$$B_{\epsilon,\delta}\equiv\{\Sigma\in H^2(\Omega_\epsilon)\times W^{2,t}(\Omega)|\|\Sigma\|_{H^2(\Omega_\epsilon)\times W^{2,t}(\Omega)}<\delta\}$$

if δ is small enough. We have by Lemma 2.3, Propositions 3.2 and 3.3

$$\|M_{\epsilon,\mathbf{P}}(\Sigma)\|_{H^2(\Omega_\epsilon)\times W^{2,t}(\Omega)}\le\lambda^{-1}(\|\pi_{\epsilon,\mathbf{P}}\circ N_{\epsilon,\mathbf{P}}(\Sigma)\|_{L^2(\Omega_\epsilon)\times L^t(\Omega)}$$

$$+\left\|\pi_{\epsilon,\mathbf{P}}\circ S_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}\\H_{\epsilon,\mathbf{P}}\end{pmatrix}\right\|_{L^2(\Omega_\epsilon)\times L^t(\Omega)})$$

$$\le\lambda^{-1}C(c(\delta)\delta+\frac{\epsilon}{\log\frac{1}{\epsilon}}),$$

where $\lambda>0$ is independent of $\delta>0$ and $c(\delta)\to 0$ as $\delta\to 0$. Similarly we show

$$\|M_{\epsilon,\mathbf{P}}(\Sigma)-M_{\epsilon,\mathbf{P}}(\Sigma')\|_{H^2(\Omega_\epsilon)\times W^{2,t}(\Omega)}\le\lambda^{-1}c(\delta)\delta\|\Sigma-\Sigma'\|_{H^2(\Omega_\epsilon)\times W^{2,t}(\Omega)},$$

where $c(\delta)\to 0$ as $\delta\to 0$. If we choose δ small enough, then $M_{\epsilon,\mathbf{P}}$ is a contraction on $B_{\epsilon,\delta}$. The existence of a fixed point $\Sigma_{\epsilon,\mathbf{P}}$ now follows from the contraction mapping principle and $\Sigma_{\epsilon,\mathbf{P}}$ is a solution of (3.18).

We have thus proved

Lemma 3.4. *There exists $\bar{\epsilon}>0$ such that for every pair of $\epsilon,\mathbf{P}$ with $0<\epsilon<\bar{\epsilon}$ there exists a unique $(\Phi_{\epsilon,\mathbf{P}},\Psi_{\epsilon,\mathbf{P}})\in\mathcal{K}_{\epsilon,\mathbf{P}}^\perp$ satisfying $S_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}\\H_{\epsilon,\mathbf{P}}+\Psi_{\epsilon,\mathbf{P}}\end{pmatrix}\in$ $\mathcal{C}_{\epsilon,\mathbf{P}}$ and*

$$\|(\Phi_{\epsilon,\mathbf{P}},\Psi_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)\times W^{2,t}(\Omega)}\le C\frac{\epsilon}{\log\frac{1}{\epsilon}}. \tag{3.19}$$

We can improve the estimates in Lemma 3.4.

Lemma 3.5. *Let $(\Phi_{\epsilon,\mathbf{P}}, \psi_{\epsilon,\mathbf{P}})$ be given by Lemma 3.4. Then we have*

$$\|\Phi_{\epsilon,\mathbf{P}}\|_{L^\infty(\Omega_\epsilon)} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right), \quad \|\Psi_{\epsilon,\mathbf{P}}\|_{L^\infty(\Omega)} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right) \tag{3.20}$$

and

$$|\Psi_{\epsilon,\mathbf{P}}(x) - \Psi_{\epsilon,\mathbf{P}}(P_j)| \leq C\frac{\epsilon}{(\log\frac{1}{\epsilon})^2}\log\left(1 + \frac{|x - P_j|}{\epsilon}\right)$$

$$\text{for } x \neq P_j, \quad j = 1, 2, \ldots, K. \tag{3.21}$$

Proof:

By Sobolev embedding it follows that

$$\|\Psi_{\epsilon,\mathbf{P}}\|_{L^\infty(\Omega)} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right).$$

Then we note that by a cut-off argument

$$\|\Phi_{\epsilon,\mathbf{P}}\|_{L^\infty(\Omega_\epsilon)} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right). \tag{3.22}$$

Finally, by Lemma 1.3

$$|\Psi_{\epsilon,\mathbf{P}}(x) - \Psi_{\epsilon,\mathbf{P}}(P_j)| = O\left(\frac{\epsilon}{(\log\frac{1}{\epsilon})^2}\log\left(1 + \frac{|x - P_j|}{\epsilon}\right)\right), \quad i = 1, \ldots, K.$$

Lemma 3.5 is proved.

$\square$

4. The reduced problem

In this section we solve the reduced problem and prove our existence theorem.

By Lemma 3.4 there exists a unique solution $(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}_{\epsilon,\mathbf{P}}^\perp$ such that

$$S_\epsilon\left(\begin{array}{c} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{array}\right) = \left(\begin{array}{c} v_{\epsilon,\mathbf{P}} \\ 0 \end{array}\right) \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Our idea is to find $\mathbf{P}$ such that also

$$S_\epsilon\left(\begin{array}{c} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{array}\right) \perp \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Let

$$W_{\epsilon,j,i}(\mathbf{P}) := \frac{\log\frac{1}{\epsilon}}{\epsilon^2}\int_\Omega S_1(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}})\frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}},$$

$$W_\epsilon(\mathbf{P}) := (W_{\epsilon,1,1}(\mathbf{P}), \ldots, W_{\epsilon,K,2}(\mathbf{P})).$$

Note that $P_{j,i}$ denotes the i-th component of the j-th point ($i = 1, \ldots, 2$, $j = 1, \ldots, K$).

Then $W_\epsilon(\mathbf{P})$ is a map which is continuous in $\mathbf{P}$ and our problem is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{P})$.

Let us now calculate $W_\epsilon(\mathbf{P})$.

By Lemma 3.5,

$$|\Psi_{\epsilon,\mathbf{P}}(x) - \Psi_{\epsilon,\mathbf{P}}(P_j)| = O\left(\frac{\epsilon}{(\log \frac{1}{\epsilon})^2} \log\left(1 + \left|\frac{x - P_j}{\epsilon}\right|\right)\right), \qquad (4.1)$$

$j = 1, \ldots, K$.

By (2.7) and (2.8), we have

$$\int_\Omega S_1(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}} + \psi_{\epsilon,\mathbf{P}}) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$= \epsilon^2 \int_{\Omega_{\epsilon,\mathbf{P}}} (\Delta_y \Phi_{\epsilon,\mathbf{P}} - \Phi_{\epsilon,\mathbf{P}} + 2 A_{\epsilon,\mathbf{P}} H_{\epsilon,\mathbf{P}}^{-1} \Phi_{\epsilon,\mathbf{P}} - A_{\epsilon,\mathbf{P}} H_{\epsilon,\mathbf{P}}^{-2} \Psi_{\epsilon,\mathbf{P}}) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$-\epsilon^2 \int_{\Omega_\epsilon} (\hat{A}_{\epsilon,\mathbf{P}})^2 (\hat{H}_{\epsilon,\mathbf{P}})^{-2}$$

$$\left[\sum_{k=1}^K (\psi_{\epsilon,k}(P_j + \epsilon y) - \psi_{\epsilon,k}(P_j)) - \sum_{k \neq j}(\hat{H}_{\epsilon,k}(P_j + \epsilon y) - \hat{H}_{\epsilon,k}(P_j))\right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}(y)\, dy$$

$$+O\left(\epsilon^3 \left(\frac{1}{\log \frac{1}{\epsilon}}\right)^2\right) + e.s.t.$$

$$= I_1 + I_2 + O\left(\epsilon^3 \left(\frac{1}{\log \frac{1}{\epsilon}}\right)^2\right),$$

where I_1, I_2 are defined by the last equality.

For I_1, we note that $\|\Psi_{\epsilon,P}\|_{L^\infty(\Omega_\epsilon)} = O(\frac{\epsilon}{\log \frac{1}{\epsilon}})$, $\frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} = -\frac{1+o(1)}{\epsilon} \frac{\partial w}{\partial y_i}$ and hence

$$I_1 = \epsilon \int_{\Omega_\epsilon} (A_{\epsilon,\mathbf{P}} \Psi_{\epsilon,\mathbf{P}}) \frac{\partial w}{\partial y_i} + O\left(\epsilon^2 \left(\frac{1}{\log \frac{1}{\epsilon}}\right)^2\right)$$

$$= \epsilon \int_{\Omega_\epsilon} (w \Psi_{\epsilon,\mathbf{P}}) \frac{\partial w}{\partial y_i} + O\left(\epsilon^2 \left(\frac{1}{\log \frac{1}{\epsilon}}\right)^2\right)$$

$$= \epsilon \int_{\Omega_\epsilon} w(y) [\Psi_{\epsilon,\mathbf{P}}(P_j + \epsilon y) - \Psi_{\epsilon,\mathbf{P}}(P_j)] \frac{\partial w(y)}{\partial y_i} + O\left(\epsilon^2 \left(\frac{1}{\log \frac{1}{\epsilon}}\right)^2\right)$$

$$= O\left(\epsilon^2 \left(\frac{1}{\log \frac{1}{\epsilon}}\right)^2\right)$$

by (4.1).

For I_2, we have

$$I_2 = C\epsilon \int_{\Omega_\epsilon} \left[\sum_{k=1}^{K} (\psi_{\epsilon,k}(P_j + \epsilon y) - \psi_{\epsilon,j}(P_j)) - \sum_{k \neq j} (\hat{H}_{\epsilon,k}(P_j + \epsilon y) - \hat{H}_{\epsilon,k}(P_j)) \right]$$

$$\frac{\partial w}{\partial y_i} dy \left(1 + O \left(\frac{1}{\log \frac{1}{\epsilon}} \right) \right)$$

$$= C \frac{\epsilon}{\log \frac{1}{\epsilon}} \int_{R^2} -[(H(P_j, P_j + \epsilon y) - H(P_j, P_j)) - \sum_{k \neq j} (G(P_k, P_j) - G(P_k, P_j + \epsilon y))]$$

$$w'(|y|) \frac{y_i}{|y|} dy \left(1 + O \left(\frac{1}{\log \frac{1}{\epsilon}} \right) \right)$$

$$= -C \frac{\epsilon^2}{\log \frac{1}{\epsilon}} \frac{\partial}{\partial P_{j,i}} F(\mathbf{P}) \int_{R^2} w'(|y|)|y| \, dy \left(1 + O \left(\frac{1}{\log \frac{1}{\epsilon}} \right) \right)$$

as $\epsilon \to 0$ uniformly in $\mathbf{P}$, where $w'(|y|) = \frac{d}{dr} w(r)$ for $r = |y|$ and $C \neq 0$ denotes a generic constant.

Combining I_1 and I_2, we have

$$W_\epsilon(\mathbf{P}) = c_0 \nabla_{\mathbf{P}} F(\mathbf{P}) \left(1 + O \left(\frac{1}{\log \frac{1}{\epsilon}} \right) \right),$$

where $c_0 \neq 0$ is a generic constant.

Suppose at $\mathbf{P}_0$, we have $\nabla_{\mathbf{P}} F(\mathbf{P}) = 0, \det(\nabla_j \nabla_k (F(\mathbf{P}_0)) \neq 0$, then standard Brouwer's fixed point theorem shows that for $\epsilon << 1$ there exists a $\mathbf{P}_\epsilon$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$ and $\mathbf{P}^\epsilon \to \mathbf{P}_0$.

Thus we have proved the following proposition.

Proposition 4.1. *For ϵ sufficiently small there exist points $\mathbf{P}_\epsilon$ with $\mathbf{P}^\epsilon \to \mathbf{P}_0$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$.*

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1: By Proposition 4.1, there exists $\mathbf{P}^\epsilon \to \mathbf{P}_0$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$. In other words, $S_1(A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}, H_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}) = 0$. and therefore $S_\epsilon(A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}, H_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}) = 0$. Let $A_\epsilon = (A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}), H_\epsilon = (H_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon})$. It is easy to see that $H_\epsilon = 1 + O(\frac{1}{\log \frac{1}{\epsilon}}) > 0$ and hence $A_\epsilon \geq 0$. By the Maximum Principle, $A_\epsilon > 0$. Moreover A_ϵ, H_ϵ satisfy Theorem 1.1.

$\square$

5. Stability Analysis: Large Eigenvalues

In this section, we study the eigenvalues with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.

The key is the following theorem, whose proof can be found in Theorem 1.4 of [49].

Consider the following eigenvalue problem

$$L\phi := \Delta\phi - \phi + 2w\phi - 2\frac{\int_{R^N} w\phi}{\int_{R^N} w^2}w^2 = \alpha_0\phi, \ \phi \in H^2(R^N), \tag{5.1}$$

where w is the unique solution of (1.2).

We then have

Theorem 5.1. *Let $\alpha_0 \neq 0$ be an eigenvalue of L. Then we have $Re(\alpha_0) \leq -c_1$ for some $c_1 > 0$.*

We need to analyze the following eigenvalue problem

$$\epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + 2\frac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \tag{5.2}$$

$$\Delta\psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon A_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon, \tag{5.3}$$

where λ_ϵ is some complex number and

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon), \ \psi_\epsilon \in H_N^2(\Omega). \tag{5.4}$$

In this section, we study the large eigenvalues, i.e., we assume that $|\lambda_\epsilon| \geq c > 0$ for ϵ small and c small. If $Re(\lambda_\epsilon) \leq -c$, we are done. (So λ_ϵ is a stable large eigenvalue.) Therefore we may also assume that $Re(\lambda_\epsilon) \geq -c$. The analysis of (5.2), (5.3) will be presented for the case $\tau = 0$. By a straightforward perturbation argument using the implicit function theorem all the steps and therefore also all the results hold true for $\tau > 0$ small enough.

Let us assume that

$$\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} < +\infty.$$

We cut off ϕ_ϵ as follows: Let $r_0 > 0$ be so small that $B_{6r_0}(P_i) \subset \Omega, B_{3r_0}(P_i) \cap B_{3r_0}(P_j) = \emptyset, i \neq j, i, j = 1, ..., K$. Introduce

$$\phi_{\epsilon,j}(x) = \phi_\epsilon\chi\left(\frac{x - P_j^\epsilon}{\epsilon r_0}\right), \qquad x \in \Omega,$$

where χ is a smooth cut-off function which is equal to 1 in $B_1(0)$ and which is equal to 0 in $R^2 \setminus \overline{B_2(0)}$.

From (5.2) and the fact that $\mathrm{Re}(\lambda_\epsilon) \geq -c$ and that A_ϵ has exponential decay, we have that

$$\phi_\epsilon = \sum_{j=1}^{K} \phi_{\epsilon,j} + e.s.t.$$

Then we extend $\phi_{\epsilon,j}$ to a function defined on R^2 such that

$$\|\phi_{\epsilon,j}\|_{H^1(R^2)} \leq C\|\phi_{\epsilon,j}\|_{H^1(\Omega_\epsilon)}, \qquad j = 1, \ldots, K.$$

Without loss of generality we may assume that $\|\phi_\epsilon\|_\epsilon = \|\phi_\epsilon\|_{H^1(\Omega_\epsilon)} = 1$. Then $\|\phi_{\epsilon,j}\|_\epsilon \leq C$. By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,j} \to \phi_j$ as $\epsilon \to 0$ in $H^1(R^2)$ for $j = 1, \ldots, K$.

We have by (5.3)

$$\psi_\epsilon(x) = \xi_\epsilon \int_\Omega 2G(x, x') A_\epsilon(x') \phi_\epsilon(x') \, dx'. \tag{5.5}$$

At each $x = P_j^\epsilon$, $j = 1, \ldots, K$, we get

$$\psi_\epsilon(P_j^\epsilon) = 2\xi_\epsilon \int_\Omega G(P_j^\epsilon, x) \sum_{l=1}^{K} w\left(\frac{x - P_l^\epsilon}{\epsilon}\right) \phi_{\epsilon,l}\left(\frac{x}{\epsilon}\right) dx \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)$$

$$= \frac{2\pi}{\epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^2(y) \, dy} \frac{1}{2\pi} \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w(y) \phi_{\epsilon,j}(y) \, dy \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)$$

$$= \frac{\int_{R^2} w(y) \phi_{\epsilon,j}(y) \, dy}{\int_{R^2} w^2(y) \, dy} \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right), \quad j = 1, \ldots, K.$$

Substituting this into (5.2) implies

$$\Delta\phi_{\epsilon,j} - \phi_{\epsilon,j} + 2w\phi_{\epsilon,j} - 2\frac{\int_\Omega w\phi_{\epsilon,j}}{\int_{R^2} w^2} w^2 = \lambda_\epsilon \phi_{\epsilon,j}\left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)$$

in $H^1(\Omega_\epsilon)$. Sending $\epsilon \to 0$ with $\lambda_\epsilon \to \lambda_0$, this implies

$$\Delta\phi_j - \phi_j + 2w\phi_j - 2\frac{\int_{R^2} w\phi_j}{\int_{R^2} w^2} w^2 = \lambda_0 \phi_j. \tag{5.6}$$

By Theorem 5.1, the eigenvalue of (5.2), (5.3) satisfies $\mathrm{Re}(\lambda_0) \leq -c_1 < 0$ if $\lambda_0 \neq 0$. So the non-zero eigenvalues of (5.2), (5.3) all have strictly negative real parts. This means they are all stable. We conclude that all eigenvalues λ_ϵ of (5.2), (5.3), for which $|\lambda_\epsilon| \geq c > 0$ holds, satisfy $\mathrm{Re}(\lambda_\epsilon) \leq -c < 0$ for ϵ small enough. They are all stable.

In the next section we shall study the eigenvalues λ_ϵ which tend to zero as $\epsilon \to 0$.

6. Stability Analysis: Small Eigenvalues

We now study (5.2), (5.3) for small eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. This part of the analysis is very involved and we shall need some new calculations to carry it through.

Let

$$\bar{A}_\epsilon = A_{\epsilon, \mathbf{P}^\epsilon} + \Phi_{\epsilon, \mathbf{P}^\epsilon}, \bar{H}_\epsilon = H_{\epsilon, \mathbf{P}^\epsilon} + \Psi_{\epsilon, \mathbf{P}^\epsilon}.$$

The system (5.2), (5.3) becomes

$$\epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + 2 \frac{\bar{A}_\epsilon}{\bar{H}_\epsilon} \phi_\epsilon - \frac{(\bar{A}_\epsilon)^2}{(\bar{H}_\epsilon)^2} \psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \qquad (6.1)$$

$$\Delta \psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon \bar{A}_\epsilon \phi_\epsilon = \tau \lambda_\epsilon \psi_\epsilon. \qquad (6.2)$$

We take $\tau = 0$ for simplicity.

Let us define

$$\tilde{A}_{\epsilon,j}(x) = \chi(\frac{x - P_j^\epsilon}{r_0}) \bar{A}_\epsilon(x), j = 1, ..., K.$$

Then it is easy to see that

$$\bar{A}_\epsilon(x) = \sum_{j=1}^{K} \tilde{A}_{\epsilon,j}(x) + e.s.t.$$

Note that $\tilde{A}_{\epsilon,j}(x) \sim w(\frac{x - P_j^\epsilon}{\epsilon})$ in $H_{loc}^2(\Omega)$ and $\tilde{A}_{\epsilon,j}$ satisfies

$$\epsilon^2 \Delta \tilde{A}_{\epsilon,j} - \tilde{A}_{\epsilon,j} + \frac{(\tilde{A}_{\epsilon,j})^2}{\bar{H}_\epsilon} + e.s.t. = 0$$

Thus $\frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k}$ satisfies

$$\epsilon^2 \Delta \frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k} - \frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k} + \frac{2 \tilde{A}_{\epsilon,j}}{\bar{H}_\epsilon} \frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k} - \frac{(\tilde{A}_{\epsilon,j})^2}{\bar{H}_\epsilon^2} \frac{\partial \bar{H}_\epsilon}{\partial x_k} + e.s.t. = 0$$

Setting $\lambda_0 = 0$ in (5.6) gives

$$\Delta(\phi_j - c(\phi_j)w) - (\phi_j - c(\phi_j)w) + 2w(\phi_j - c(\phi_j)w) = 0$$

where $c_j(\phi) = 2 \frac{\int_{R^2} w \phi_j}{\int_{R^2} w^2}$, which implies that $\phi_j \in span\{\frac{\partial w}{\partial y_k}, k = 1, 2\}$.

This suggests that we decompose

$$\phi_\epsilon = \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \epsilon \frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k} + \phi_\epsilon^\perp \qquad (6.3)$$

with real numbers $a_{j,k}^\epsilon$, where

$$\phi_\epsilon^\perp \perp \mathcal{K}_{\epsilon,\mathbf{P}^\epsilon}^{new} = \text{span}\,\{\frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k}|j=1,\ldots,K,\,k=1,2\} \subset H_N^2(\Omega_\epsilon).$$

Accordingly, we have

$$\psi_\epsilon(x) = \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \psi_{\epsilon,j,k} + \psi_\epsilon^\perp,$$

where $\psi_{\epsilon,j,k}$ is the unique solution of the problem

$$\Delta \psi_{\epsilon,j,k} - \psi_{\epsilon,j,k} + \xi_\epsilon \epsilon \frac{\partial(\tilde{A}_{\epsilon,j}^2)}{\partial x_k} = 0 \qquad \text{in } \Omega,$$

$$\frac{\partial \psi_{\epsilon,j,k}}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$

and $\psi_\epsilon^\perp$ satisfies

$$\Delta \psi_\epsilon^\perp - \psi_\epsilon^\perp + 2\xi_\epsilon \bar{A}_\epsilon \phi_\epsilon^\perp = 0 \qquad \text{in } \Omega,$$

$$\frac{\partial \psi_\epsilon^\perp}{\partial \nu} = 0 \qquad \text{on } \partial\Omega.$$

Suppose that $\|\phi_{\epsilon,j}\|_\epsilon = 1$. Then $|a_{j,k}^\epsilon| \leq C$.

We divide our proof into two steps.

Step 1: Estimates of $\phi_\epsilon^\perp$.

Substituting the decompositions of ϕ_ϵ and ψ_ϵ into (5.2) we have

$$\epsilon \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \frac{(\tilde{A}_{\epsilon,j})^2}{(\bar{H}_\epsilon)^2} \left[-\psi_{\epsilon,j,k} + \frac{\partial \bar{H}_\epsilon}{\partial x_k}\right] + e.s.t.$$

$$+\epsilon^2 \Delta \phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\frac{\bar{A}_\epsilon}{\bar{H}_\epsilon}\phi_\epsilon^\perp - \frac{(\bar{A}_\epsilon)^2}{(\bar{H}_\epsilon)^2}\psi_\epsilon^\perp - \lambda_\epsilon \phi_\epsilon^\perp$$

$$= \lambda_\epsilon \epsilon \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k}. \qquad (6.4)$$

Set

$$I_1 = \epsilon \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \frac{(\tilde{A}_{\epsilon,j})^2}{(\bar{H}_\epsilon)^2} \left[-\psi_{\epsilon,j,k} + \frac{\partial \bar{H}_\epsilon}{\partial x_k}\right]$$

and
$$I_2 = \epsilon^2 \Delta \phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\frac{\bar{A}_\epsilon}{\bar{H}_\epsilon}\phi_\epsilon^\perp - \frac{(\bar{A}_\epsilon)^2}{(\bar{H}_\epsilon)^2}\psi_\epsilon^\perp - \lambda_\epsilon \phi_\epsilon^\perp.$$

Since $\phi_\epsilon^\perp \perp \mathcal{K}_{\epsilon,\mathbf{P}^\epsilon}^{new}$, then similar to the proof of Proposition 3.2 it follows that
$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\|I_1\|_{L^2(\Omega_\epsilon)}.$$

Let us now compute I_1.

We calculate that for $x \in B_{r_0}(P_l^\epsilon)$
$$\frac{\partial \bar{H}_\epsilon}{\partial x_k}(x) = \frac{2\pi}{\epsilon^2 \log\frac{1}{\epsilon}\int_{R^2} w^2}\int_\Omega \frac{\partial}{\partial x_k}G(x,x')(\bar{A}_\epsilon(x')^2\,dx'$$
$$= \frac{2\pi}{\epsilon^2 \log\frac{1}{\epsilon}\int_{R^2} w^2}$$

$$\times \left(\int_\Omega \frac{\partial}{\partial x_k}(K(|x-x'|) - H(x,x'))(\tilde{A}_{\epsilon,l}(x'))^2\,dx' + \int_\Omega \sum_{s \ne l}\frac{\partial}{\partial x_k}G(x,x')(\tilde{A}_{\epsilon,s}(x'))^2\,dx'\right)$$
$$\left(1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right)$$

and
$$\psi_{\epsilon,l,k}(x) = \frac{2\pi}{\epsilon^2 \log\frac{1}{\epsilon}\int_{R^2} w^2}\int_\Omega (K(|x-x'|) - H(x,x'))\frac{\partial}{\partial x_k'}(\tilde{A}_{\epsilon,l})^2\,dx'\left(1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right).$$

Thus for $x \in B_{r_0}(P_l^\epsilon)$, we have
$$\frac{\partial \bar{H}_\epsilon}{\partial x_k}(x) - \psi_{\epsilon,l,k}(x)$$

$$= \left[\frac{2\pi}{\epsilon^2 \log\frac{1}{\epsilon}\int_{R^2} w^2}\left(\int_\Omega [\frac{\partial}{\partial x_k}K(|x-x'|)(\tilde{A}_{\epsilon,l}(x'))^2 - K(|x-x'|)\frac{\partial}{\partial x_k'}(\tilde{A}_{\epsilon,l}(x'))^2]dx'\right)\right.$$

$$- \frac{2\pi}{\epsilon^2 \log\frac{1}{\epsilon}\int_{R^2} w^2}\int_\Omega [\frac{\partial}{\partial x_k}H(x,x'))(\tilde{A}_{\epsilon,l}(x'))^2 - H(x,x')\frac{\partial}{\partial x_k'}(\tilde{A}_{\epsilon,l}(x'))^2]dx'$$

$$\left.+ \frac{2\pi}{\epsilon^2 \log\frac{1}{\epsilon}\int_{R^2} w^2}\int_\Omega \sum_{s \ne l}\frac{\partial}{\partial x_k}G(x,x')(\tilde{A}_{\epsilon,s}(x'))^2\,dx'\right]\left(1 + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right)\right).$$

Using the fact that
$$\frac{\partial}{\partial x_k}K(|x-x'|) + \frac{\partial}{\partial x_k'}K(|x-x'|) = 0 \quad \text{for } x \ne x'$$

and integrating by parts we get
$$\frac{\partial \bar{H}_\epsilon}{\partial x_k}(x) - \psi_{\epsilon,l,k}(x)$$

$$= \frac{2\pi}{\log \frac{1}{\epsilon}} (-\frac{\partial}{\partial x_k} F_l(x)) + O((\log \frac{1}{\epsilon})^{-2}), \tag{6.5}$$

where

$$F_l(x) = H(x, P_l^\epsilon) - \sum_{j \neq l} G(x, P_j^\epsilon). \tag{6.6}$$

Observe that

$$\frac{\partial}{\partial x_m} F_l(x)|_{x=P_l^\epsilon} = o(1)$$

since $\mathbf{P}^\epsilon \to \mathbf{P}_0$ and $\mathbf{P}_0$ is a critical point of $F(\mathbf{P})$. Furthermore,

$$I_1(x) = O(\epsilon^2 \frac{1}{(\log \frac{1}{\epsilon})^2}) \quad \text{for} \quad x \in \cup_{l=1}^K B_{r_0}(P_l^\epsilon).$$

Hence we have

$$\|I_1\|_{L^2(\Omega_\epsilon)} = o(\frac{\epsilon}{\log \frac{1}{\epsilon}} \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|)$$

and

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = o(\frac{\epsilon}{\log \frac{1}{\epsilon}} \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|). \tag{6.7}$$

It is easy to show that

$$\int_\Omega (I_2 \epsilon \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m}) dx' = \int_\Omega (\frac{\tilde{A}_{\epsilon,l}^2}{\bar{H}_\epsilon^2} (\epsilon \frac{\partial \bar{H}_\epsilon}{\partial x_m} \phi_\epsilon^\perp - \epsilon \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m} \psi_\epsilon^\perp)) dx'$$

$$= o(\frac{\epsilon^4}{\log \frac{1}{\epsilon}} \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|)$$

since

$$\frac{\partial \bar{H}}{\partial x_m} = O(\frac{1}{\log \frac{1}{\epsilon}}) \quad \text{in } \Omega.$$

Step 2: Algebraic equations for $a_{j,k}^\epsilon$.

Multiplying both sides of (6.4) by $-\epsilon \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m}$ and integrating over Ω, we obtain

$$r.h.s. = \epsilon^2 \lambda_\epsilon \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \int_\Omega \frac{\partial \tilde{A}_{\epsilon,j}}{\partial x_k} \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m}$$

$$= \epsilon^2 \lambda_\epsilon \sum_{j,k} a_{j,k}^\epsilon \delta_{jl} \delta_{km} \int_{R^2} \left(\frac{\partial w}{\partial y_1}\right)^2 dy \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)$$

$$= \epsilon^2 \lambda_\epsilon a_{l,m}^\epsilon \int_{R^2} \left(\frac{\partial w}{\partial y_1}\right)^2 \left(1 + O\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)$$

and

$$l.h.s. = \epsilon^2 \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \int_\Omega \frac{(\tilde{A}_{\epsilon,j})^2}{(\bar{H}_\epsilon)^2} \left[-\psi_{\epsilon,j,k} + \frac{\partial \bar{H}_\epsilon}{\partial x_k} \right] \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m} + e.s.t.$$

$$+ \int_\Omega (I_2 \epsilon \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m}) dx'$$

$$= \epsilon^2 \sum_{k=1}^{2} a_{l,k}^\epsilon \int_\Omega \frac{(\tilde{A}_{\epsilon,l})^2}{(\bar{H}_\epsilon)^2} \left[-\psi_{\epsilon,l,k} + \frac{\partial \bar{H}_\epsilon}{\partial x_k} \right] \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m}$$

$$+ o(\frac{\epsilon^4}{\log \frac{1}{\epsilon}} \sum_{j=1}^{K} \sum_{k=1}^{2} |a_{j,k}^\epsilon|).$$

Using (6.5), we obtain

$$l.h.s. = \epsilon^2 \frac{2\pi}{\log \frac{1}{\epsilon}} \sum_{k=1}^{2} a_{l,k}^\epsilon$$

$$\times \int_\Omega \frac{(\tilde{A}_{\epsilon,l})^2}{(\tilde{H}_\epsilon)^2} (-\frac{\partial}{\partial x_k} F_l(x)) \frac{\partial \tilde{A}_{\epsilon,l}}{\partial x_m}$$

$$+ o(\frac{\epsilon^4}{\log \frac{1}{\epsilon}} \sum_{j=1}^{K} \sum_{k=1}^{2} |a_{j,k}^\epsilon|)$$

$$= \epsilon^4 \frac{2\pi}{\log \frac{1}{\epsilon}} \int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m \sum_{k=1}^{2} a_{l,k}^\epsilon \left(-\frac{\partial}{\partial P_{l,m}^\epsilon} \frac{\partial}{\partial P_{l,k}^\epsilon} F(\mathbf{P}^\epsilon) \right)$$

$$+ o(\frac{\epsilon^4}{\log \frac{1}{\epsilon}} \sum_{j=1}^{K} \sum_{k=1}^{2} |a_{j,k}^\epsilon|).$$

Note that

$$\int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m = \int_{R^2} w^2 w' \frac{y_m^2}{|y|}$$

$$= \frac{1}{2} \int_{R^2} w^2 w' |y| < 0.$$

Thus we have

$$l.h.s. = \epsilon^4 \frac{\pi}{\log \frac{1}{\epsilon}} (-\int_{R^2} w^2 w' |y|) \sum_{k=1}^{2} a_{l,k}^\epsilon \left(\frac{\partial}{\partial P_{l,m}^\epsilon} \frac{\partial}{\partial P_{l,k}^\epsilon} F(\mathbf{P}^\epsilon) \right)$$

$$+ o(\frac{\epsilon^4}{\log \frac{1}{\epsilon}} \sum_{j=1}^{K} \sum_{k=1}^{2} |a_{j,k}^\epsilon|).$$

Combining the *l.h.s.* and *r.h.s*, we have

$$\epsilon^2 \frac{\pi}{\log \frac{1}{\epsilon}} (-\int_{R^2} w^2 w' |y|) \sum_{k=1}^{2} a_{l,k}^\epsilon \left(\frac{\partial}{\partial P_{l,m}^\epsilon} \frac{\partial}{\partial P_{l,k}^\epsilon} F(\mathbf{P}^\epsilon) \right)$$

$$+o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^{\epsilon}|\right)$$

$$=\lambda_{\epsilon}a_{l,m}^{\epsilon}\int_{R^2}\left(\frac{\partial w}{\partial y_1}\right)^2.$$

This implies that the the small eigenvalues with $\lambda_{\epsilon}\to 0$ satisfy $|\lambda_{\epsilon}|\sim C\frac{\epsilon^2}{\log\frac{1}{\epsilon}}$ with some $C>0$. Furthermore, (asymptotically) they are eigenvalues of the matrix $(\frac{\partial^2}{\partial\mathbf{P}^2}F(\mathbf{P})|_{\mathbf{P}=\mathbf{P}_0})$ and the coefficients $a_{j,k}^{\epsilon}$ are the corresponding eigenvectors. If the matrix $(\frac{\partial^2}{\partial\mathbf{P}^2}F(\mathbf{P})|_{\mathbf{P}=\mathbf{P}_0}$ is strictly negative definite, it follows that $\lambda_0<0$. Therefore the small eigenvalues λ_{ϵ} are stable if ϵ is small enough. The implicit function theorem tells us that ϕ_{ϵ} together with a suitable ψ_{ϵ} actually is a solution of (5.2), (5.3). This finishes the proof of Theorem 1.2. $\qquad\square$

Our analysis is a rigorous derivation of the frequently numerically observed fact that the two-dimensional Gierer-Meinhardt system for a finite diffusion rate of the inhibitor have stable solutions which show a pattern of multiple interior spots.

APPENDIX: STUDY OF THE FUNCTION $F(\mathbf{P})$

In this appendix, we collect some facts about the functions $F_j(P), F(\mathbf{P})$.

First Fact: If Ω is convex, then $F_j(P)<0, j=1,...,K$ for $P\in\Omega$.

Proof: In fact in this case, $G(P_i,P_j)>0$ for $i\neq j$. Moreover, $H(x,P)$ satisfies

$$\Delta_x H - H = 0 \text{ in } \Omega$$

and

$$\frac{\partial H(x,P)}{\partial\nu_x}=\frac{\partial K(|x-P|)}{\partial\nu_x}=K'(|x-P|)\frac{<x-P,\nu_x>}{|x-P|}<0$$

on $\partial\Omega$. By the Maximum Principle, $H(x,P)<0$ in Ω and $G(x,P)>0$ in Ω. Hence $F_j(\mathbf{P})<0$. $\square$

Second Fact: The function $F(\mathbf{P})$ admits a global maximum point.

Proof: For $\delta > 0$ small, let

$$\Lambda := \{(P_1, ..., P_K)|P_i \in \Omega, d(P_i, \partial\Omega) \geq \delta, \min_{i \neq j} |P_i - P_j| \geq \delta\}$$

Then we consider the following maximization problem

$$\max_{\mathbf{P} \in \Lambda} F(\mathbf{P}).$$

Since $F(\mathbf{P})$ is a continuous function, there exists a point $\mathbf{P}_0 \in \Lambda$ such that $F(\mathbf{P}_0) = \max_{\mathbf{P} \in \Lambda} F(\mathbf{P})$. We now prove that $\mathbf{P}_0$ is in the interior of Λ. Assume not. Then (i) $d(P_i, \partial\Omega) = \delta$ for some i, or, (ii) $|P_i - P_j| = \delta$ for some i, j.

In case (i): We calculate

$$F(\mathbf{P}) \leq H(P_i, P_i),$$

where $H(x, P_i)$ solves

$$\Delta_x H - H = 0 \text{ in } \Omega,$$

$$\frac{\partial H(x, P_i)}{\partial \nu_x} = \frac{\partial K(|x - P_i|)}{\partial \nu_x} = K'(|x - P_i|)\frac{< x - P_i, \nu_x >}{|x - P_i|}$$

for $x \in \partial\Omega$. We estimate

$$\left|\frac{\partial H(x, P_i)}{\partial \nu_x}\right| = \left|\frac{\partial K(|x - P_i|)}{\partial \nu_x}\right|$$

$$\geq C\left|\frac{1}{|x - P_i|}\frac{< x - P_i, \nu_x >}{|x - P_i|}\right| \geq C\left|\frac{1}{|x - P_i|}\right|.$$

Let $Q_i \in \partial\Omega$ be a point with $|P_i - Q_i| = d(P_i, \partial\Omega)$. If $\delta > 0$ is small enough, then Q_i is unique. Then for $x \in \partial\Omega$.

$$|x - P_i| \leq |x - Q_i| + \delta.$$

The standard representation formula implies

$$H(P_i, P_i) = \int_{\partial\Omega} G(P_i, x)\frac{\partial}{\partial \nu}K(|x - P_i|) \, dx.$$

Parametrizing $\partial\Omega$ by arclength (with $s = 0$ corresponding to Q_i) and using the following estimates for δ small and $s < \delta$

$$|G(P_i, x)| \geq C \log \frac{1}{s + \delta}, \quad \left|\frac{\partial K(|x - P_i|)}{\partial \nu_x}\right| \geq C\frac{1}{s + \delta}$$

we calculate

$$|H(P_i, P_i)| \geq C \int_0^{s_0} \log\left(\frac{1}{s + \delta}\right) \frac{1}{s + \delta} \, ds$$

and setting $s_0 = \delta$ we conclude

$$|H(P_i, P_i)| \geq C \log \frac{1}{\delta}$$

$$\rightarrow -\infty \quad \text{as } \delta \rightarrow 0.$$

Thus there exists $\mathbf{P_1} \in \Lambda$ with $F(\mathbf{P_1}) > F(\mathbf{P})$ if δ is small enough. This is a contradiction.

In case (ii): We estimate

$$F(\mathbf{P}) \leq -G(P_i, P_j) + O(1)$$

$$\leq K(|P_i - P_j|) + O(1)$$

$$= -\frac{1}{2\pi} \log \frac{1}{2\pi} + O(1)$$

$$= -\frac{1}{2\pi} \log \frac{1}{\delta} + O(1)$$

$$\rightarrow -\infty \quad \text{as } \delta \rightarrow 0.$$

Therefore there exists $\mathbf{P_1} \in \Lambda$ with $F(\mathbf{P_1}) > F(\mathbf{P})$ if δ is small enough. This is the desired contradiction. $\square$

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