Robust $\mathcal{H}_\infty$ Filtering for Markovian Jump Systems With Randomly Occurring Nonlinearities and Sensor Saturation: The Finite-Horizon Case

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***Abstract*—This paper addresses the robust $\mathcal{H}_\infty$ filtering problem for a class of discrete time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation. Two kinds of transition probability matrices for the Markovian process are considered, namely, the one with polytopic uncertainties and the one with partially unknown entries. The nonlinear disturbances are assumed to occur randomly according to stochastic variables satisfying the Bernoulli distributions. The main purpose of this paper is to design a robust filter, over a given finite-horizon, such that the $\mathcal{H}_\infty$ disturbance attenuation level is guaranteed for the time-varying Markovian jump systems in the presence of both the randomly occurring nonlinearities and the sensor saturation. Sufficient conditions are established for the existence of the desired filter satisfying the $\mathcal{H}_\infty$ performance constraint in terms of a set of recursive linear matrix inequalities. Simulation results demonstrate the effectiveness of the developed filter design scheme.**

***Index Terms*—Discrete time-varying systems, Markovian jumping parameters, randomly occurring nonlinearities, robust $\mathcal{H}_\infty$ filtering, sensor saturation.**

I. Introduction

As a class of hybrid system, Markovian jump systems (MJSs) have been attracting extensive research attention in the past years since MJSs are very appropriate to model the dynamic systems whose structure is subject to random abrupt variation mainly due to, for example, component failures or repairs, changing subsystem interconnections, sudden environmental disturbance and abrupt variations of the operating points of a nonlinear system; see, e.g., [12]–[14], [18] and the references therein. In most existing literature, the transition probabilities in the jumping process, which determine the system behavior to a great extent, have been assumed to be completely accessible. However, such an ideal assumption would inevitably limit the application of established results because of the difficulty and cost in obtaining precisely all the transition probabilities. Very recently, some initial results have been obtained in [22], [23], [29], and [30] for Markovian jumping systems with *partially* unknown transition probabilities.

For decades, filtering technique has been playing an important role in a variety of application areas including target tracking, image processing, signal processing and control engineering [14], [25], [28]. Among other existing filtering methods, the $\mathcal{H}_\infty$ filtering approach is closely related to many robustness problems such as stabilization and sensitivity minimization of uncertain systems, and has therefore gained persistent attention. Recently, much progress has been made in the study of the $\mathcal{H}_\infty$ filtering problem for Markovian jump systems; see, e.g., [1], [6], [7], [18], [20], and [21]. In particular, the $\mathcal{H}_\infty$ filters have been designed in [1] for Markovian jump linear systems and in [2] and [31] for nonlinear systems. When both the Markovian jump parameters and time-delays appear in the systems, the $\mathcal{H}_\infty$ filtering problems have been studied in [6], [12], and [13]. In [18], a delay-dependent approach has been developed to deal with the stochastic $\mathcal{H}_\infty$ filtering problem for a class of stochastic time-delay jumping systems subject to both sensor nonlinearities and exogenous nonlinear disturbances. It is noted that a common feature of the aforementioned $\mathcal{H}_\infty$ filtering results is that the complete knowledge is required for the transition probabilities of the jump process. Very recently, the problem of $\mathcal{H}_\infty$ filtering has been investigated in [30] for a class of discrete-time Markovian jump linear systems with partly unknown transition probabilities.

On another research front, networked control systems (NCSs) have become more and more popular for their successful industrial applications in aircrafts, manufacturing plants, automobiles, etc. The issues of random data packet dropouts [3], [8], [15], [20], random transmission delay [4], [9], [27], and signal quantization [11], which typically emerge in NCSs, have been well studied in the literature. Nevertheless, the randomly occurring nonlinearities (RONs), which also constitute an important class of network-induced phenomena, have been largely

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overlooked. As is well known, nonlinearities are widespread in practice, most of which can be described as the additive nonlinear disturbances caused by environmental circumstances. In the case of NCSs with communication constraints, such nonlinear disturbances may occur in a probabilistic way, that is, they may be randomly changeable in terms of their types and/or intensity. This gives rise to the so-called RONs that have started to receive some initial research interests in, for instance, [17] for complex networks.

It has been a well-recognized fact that, virtually, almost all models for real-time systems behave in a time-varying way, especially those after digital discretization. As such, time-varying stochastic systems have recently been paid much research attention; see, e.g., [5], [26], [28], [32] and the references therein. Unfortunately, in most existing literature, it is implicitly assumed that the measurements are always working under the linear condition such that the possible effect of amplitude saturation is ignored [10], [16], [19], [24]. Such an assumption is obviously unrealistic as sensors are all subject to physical saturations. Therefore, sensor saturation issue should have taken into account when designing filters. It should be pointed out that the robust $\mathcal{H}_\infty$ filtering problem for time-varying stochastic systems with sensor saturation is still an open and challenging issue, not to mention the case where the inaccessible Markovian jump parameter and RONs are also involved over a finite horizon. It is, therefore, the purpose of this paper to shorten such a gap.

The main contributions of this paper are listed as follows. 1) A distinguishing feature of the research problem addressed is that the time-varying system is considered and, accordingly, the finite-horizon filtering problem is investigated by developing an effective recursive linear matrix inequalities (RLMIs) approach. 2) In the plant under consideration, Markovian jumping parameters, time-varying parameters, randomly occurring nonlinearities, sensor saturation as well as parameter uncertainties exist simultaneously, which render more practical significance of our current research. Note that the sensor saturation issue has seldom been taken into account in filter design. 3) Randomly occurring nonlinearities are introduced to describe the phenomena of nonlinear disturbances appearing in a random way. 4) Two cases are considered for the transition probability matrix of the Markovian process with either polytopic uncertainties or partially unknown entries. 5) The proposed robust H_∞ filtering technique is dependent not only on the current available state estimate but also on the previous measurement, which serves as a recursive algorithm suitable for online application.

Notation: The notation used in the paper is fairly standard. The notation $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ will, respectively, mean expectation of x and expectation of x conditional on y. $\mathrm{Prob}\{\cdot\}$ represents the occurrence probability of the event "$\cdot$".

II. Problem Formulation

Let $r(k)$ $(k \in [0, N])$ be a Markov chain taking values in a finite state space $S = \{1, 2, \ldots, s\}$ with transition probability matrix $\hat{\Psi} = [\lambda_{ij}]$ given by

$$\mathrm{Prob}\{r(k+1) = j | r(k) = i\} = \lambda_{ij}, \ \forall i, j \in S$$

where $\lambda_{ij} \geq 0$ $(i, j \in S)$ is the transition probability from i to j and $\sum_{j=1}^{s} \lambda_{ij} = 1, \forall i \in S$.

In this paper, we consider the following two cases where the transition probability matrix $\hat{\Psi} = [\lambda_{ij}]$ is imperfectly known.

Case 1: The transition probability matrix $\hat{\Psi}$ belongs to a given polytope, namely, $\hat{\Psi} \in \Re$, where $\Re$ is a given convex bounded polyhedral domain described by ν vertices as follows:

$$\Re := \left\{ \hat{\Psi} \,\middle|\, \hat{\Psi} = \sum_{r=1}^{\nu} \psi_r \hat{\Psi}^{(r)}, \sum_{r=1}^{\nu} \psi_r = 1, \ \psi_r \geq 0, \quad r = 1, 2, \ldots, \nu \right\} \tag{1}$$

and $\hat{\Psi}^{(r)} = [\lambda_{ij}^{(r)}]$ $(i, j = 1, \ldots, s, r = 1, \ldots, \nu)$ are given transition probability matrices. It is easy to see that the convex combination of these transition probability matrices is also a possible transition probability matrix.

Case 2: Some elements in matrix $\hat{\Psi}$ are unknown, for example, the transition probability matrix $\hat{\Psi}$ may be

$$\hat{\Psi} = \begin{bmatrix} \lambda_{11} & ? & ? \\ ? & \lambda_{22} & ? \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

where "?" represents the unknown entries. For notation clarity, for any $i \in S$, we denote that

$$\hat{\Psi}_k^i := \{j : \lambda_{ij} \text{ is known}\}$$
$$\hat{\Psi}_{uk}^i := \{j : \lambda_{ij} \text{ is unknown}\}.$$

In this paper, we consider the following class of uncertain discrete stochastic nonlinear time-varying Markovian jump systems in the presence of sensor saturation defined on $k \in [0, N]$:

$$\begin{cases} x(k+1) = (A(k, r(k)) + \Delta A(k, r(k)))x(k) \\ \qquad\qquad +\alpha(k) f(k, x(k)) + D_1(k, r(k))w(k) \\ y(k) = \sigma(y_s(k, r(k))) + \beta(k) g(k, x(k)) \\ \qquad\quad +D_2(k, r(k))w(k) \\ z(k) = (L(k, r(k)) + \Delta L(k, r(k)))x(k) \\ y_s(k, r(k)) = C(k, r(k))x(k) \\ x(0) = \varphi_0 \end{cases} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ represents the state vector; $y(k) \in \mathbb{R}^r$ is the measurement output; $z(k) \in \mathbb{R}^m$ is a linear combination of the state variables to be estimated; $w(k) \in \mathbb{R}^p$ is the disturbance input which belongs to $l_2[0, N]$, φ_0 is a given real initial value. For the fixed system mode, $A(k, r(k)), D_1(k, r(k)), C(k, r(k)), D_2(k, r(k)), \quad L(k, r(k))$ are known, real, time-varying matrices with appropriate dimensions. $\Delta A(k, r(k))$ and $\Delta L(k, r(k))$ are unknown matrices representing the time-varying parameter uncertainties of the form

$$\begin{bmatrix} \Delta A(k, r(k)) \\ \Delta L(k, r(k)) \end{bmatrix} = \begin{bmatrix} H_1(r(k)) \\ H_2(r(k)) \end{bmatrix} F(k, r(k)) N(r(k))$$

where, for fixed system mode, $H_1(r(k))$, $H_2(r(k))$, and $N(r(k))$ are known, real, constant matrices of appropriate dimensions which characterize how the uncertain parameter in $F(k, r(k))$ enters the nominal matrices $A(k, r(k))$ and

$L(k, r(k))$, and $F(k, r(k))$ is an unknown time-varying matrix satisfying

$$F^T(k, r(k))F(k, r(k)) \leq I, \quad \forall r(k) \in S. \tag{3}$$

The nonlinear functions $f(k, x(k))$ and $g(k, x(k))$ satisfy the following conditions:

$$\begin{aligned} \|f(k, x(k))\|^2 &\leq \varepsilon_1(k)\|E_1(k)x(k)\|^2 \\ \|g(k, x(k))\|^2 &\leq \varepsilon_2(k)\|E_2(k)x(k)\|^2 \end{aligned} \tag{4}$$

for all $k \in [0, N]$, where $\varepsilon_1(k) > 0$ and $\varepsilon_2(k) > 0$ are known positive scalars, and $E_1(k)$ and $E_2(k)$ are known constant matrices.

The stochastic variables $\alpha(k)$ and $\beta(k)$ are two independent Bernoulli sequences which account for the phenomena of randomly occurred nonlinearities. A natural assumption on the sequences $\alpha(k)$ and $\beta(k)$ can be made as follows:

$$\begin{aligned} &\mathrm{Prob}\{\alpha(k) = 1\} = \bar{\alpha}, \quad \mathrm{Prob}\{\alpha(k) = 0\} = 1 - \bar{\alpha}, \\ &\mathrm{Prob}\{\beta(k) = 1\} = \bar{\beta}, \quad \mathrm{Prob}\{\beta(k) = 0\} = 1 - \bar{\beta} \end{aligned}$$

where $\bar{\alpha} \in [0\ 1]$ and $\bar{\beta} \in [0\ 1]$ are known constants. We also assume that the $r(k)$, $\alpha(k)$ and $\beta(k)$ are mutually independent.

Remark 1: As described in (2), the nonlinear functions $f(k, x(k))$ and $g(k, x(k))$ could occur independently and randomly according to individual probability distributions specified *a prior* through statistical tests.

The saturation function $\sigma(\cdot) : \mathbb{R}^r \mapsto \mathbb{R}^r$ is defined as

$$\sigma(v) = [\,\sigma_1^T(v_1) \quad \sigma_2^T(v_2) \quad \cdots \quad \sigma_r^T(v_r)\,]^T \tag{5}$$

with $\sigma_i(v_i) = \mathrm{sign}(v_i)\min\{V_{i,\max}, |V_i|\}$, where $V_{i,\max}$ is the ith element of the vector $V_{\max}$, the saturation level.

Definition 1: [24] A nonlinearity $\Psi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if

$$(\Psi(v) - \bar{H}_1 v)^T(\Psi(v) - \bar{H}_2 v) \leq 0, \forall v \in \mathbb{R}^r \tag{6}$$

for some real matrices $\bar{H}_1, \bar{H}_2 \in \mathbb{R}^{r \times r}$, where $\bar{H} = \bar{H}_2 - \bar{H}_1$ is a positive-definite symmetric matrix. In this case, we say that Ψ belongs to the sector $[\bar{H}_1\ \bar{H}_2]$.

If we assume that there exist two diagonal matrices K_1 and K_2 such that $0 \leq K_1 < I \leq K_2$, then the saturation function $\sigma(y_s(k, r(k)))$ in (2) can be decomposed into a linear and a nonlinear part as

$$\sigma(y_s(k, r(k))) = K_1 C(k, r(k))x(k) + \Psi(y_s(k, r(k))) \tag{7}$$

where $\Psi(y_s(k, r(k)))$ is a nonlinear vector-valued function satisfying a sector condition with $\bar{H}_1 = 0$, $\bar{H}_2 = K$, and can be described as follows:

$$\Psi^T(y_s(k, r(k)))(\Psi(y_s(k, r(k))) - Ky_s(k, r(k))) \leq 0 \tag{8}$$

where $K = K_2 - K_1$.

In this paper, the linear time-varying filter under consideration is of the following structure:

$$\begin{cases} \hat{x}(k+1) = A_f(k, r(k))\hat{x}(k) + B_f(k, r(k))y(k) \\ \hat{z}(k) = L_f(k, r(k))\hat{x}(k) \end{cases} \tag{9}$$

where $\hat{x}(k) \in \mathbb{R}^n$ represents the state estimate and $z(k) \in \mathbb{R}^m$ is the estimated output. For fixed system mode, the time-varying matrices $A_f(k, r(k)), B_f(k, r(k))$ and $L_f(k, r(k))$ are the filter parameters to be designed.

For presentation convenience, for each possible $r(k) = i$ ($i \in S$), a matrix $N(k, r(k))$ will be denoted by $N_i(k)$.

Let us now work on the system mode $r(k) = i, \forall\, i \in S$. Setting $\bar{x}(k) = [x^T(k) \quad \hat{x}^T(k)]^T$ and $\bar{z}(k) = z(k) - \hat{z}(k)$, we obtain an augmented system from (2) and (9) as follows:

$$\begin{cases} \bar{x}(k+1) = \bar{A}_i(k)\bar{x}(k) + \bar{B}_i(k)\Psi(y_{si}(k)) \\ \qquad\qquad +(G_i(k) + \bar{G}_i(k))h(k, x(k)) + \bar{D}_i(k)w(k) \\ \bar{z}(k) = \bar{L}_i(k)\bar{x}(k) \end{cases} \tag{10}$$

where

$$\begin{aligned} \bar{A}_i(k) &= \begin{bmatrix} A_i(k) + \Delta A_i(k) & 0 \\ B_{fi}(k)K_1C_i(k) & A_{fi}(k) \end{bmatrix} \\ \bar{B}_i(k) &= \begin{bmatrix} 0 \\ B_{fi}(k) \end{bmatrix}, \bar{D}_i(k) = \begin{bmatrix} D_{1i}(k) \\ B_{fi}(k)D_{2i}(k) \end{bmatrix} \\ G_i(k) &= \mathrm{diag}\{\bar{\alpha}I, \bar{\beta}B_{fi}(k)\} \\ \bar{G}_i(k) &= \mathrm{diag}\{(\alpha(k) - \bar{\alpha})I, (\beta(k) - \bar{\beta})B_{fi}(k)\} \\ h(k, x(k)) &= \begin{bmatrix} f(k, x(k)) \\ g(k, x(k)) \end{bmatrix} \\ \bar{L}_i(k) &= [\,L_i(k) + \Delta L_i(k) \quad -L_{fi}(k)\,]. \end{aligned} \tag{11}$$

Our aim in this paper is to design a finite-horizon filter in the form of (9) such that, for the given disturbance attenuation level $\gamma > 0$, positive definite matrices Q_i ($i = 1, 2, \ldots, s$) and the initial state $\bar{x}(0)$, the $\mathcal{H}_\infty$ performance index satisfies the following inequality:

$$\mathbb{E}\left\{\|\bar{z}(k)\|^2_{[0,N]}\right\} \leq \gamma^2\left(\mathbb{E}\left\{\|w(k)\|^2_{[0,N]}\right\} + e^T(0)Q_i e(0)\right) \tag{12}$$

where $e(0) = x(0) - \hat{x}(0)$.

III. Performance Analysis of Robust $\mathcal{H}_\infty$ Filter

Lemma 1 [24]: Let $Y_0(\eta), Y_1(\eta), \ldots, Y_p(\eta)$ be quadratic functions of $\eta \in \mathbb{R}^n$, $Y_i(\eta) = \eta^T T_i \eta$, $i = 0, 1, \ldots, p$, with $T_i = T_i^T$. Then, the implication $Y_1(\eta) \leq 0, \ldots, Y_p(\eta) \leq 0 \Rightarrow Y_0(\eta) \leq 0$ holds if there exist $\tau_1, \ldots, \tau_p > 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i \leq 0. \tag{13}$$

For presentation convenience, we denote

$$\begin{aligned}
\Omega_{11i}(k) &= 2\bar{A}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) - P_i(k) \\
&\quad + \bar{L}_i^T(k)\bar{L}_i(k) + \rho_i(k)\bar{E}(k) \\
\Omega_{21i}(k) &= \bar{B}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) + \frac{1}{2}\tau_i(k)\tilde{C}_i(k) \\
\Omega_{22i}(k) &= 2\bar{B}_i^T(k)\bar{P}_i(k+1)\bar{B}_i(k) - \tau_i(k)I \\
\Omega_{31i}(k) &= \bar{D}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) \\
\Omega_{32i}(k) &= \bar{D}_i^T(k)\bar{P}_i(k+1)\bar{B}_i(k) \\
\Omega_{33i}(k) &= 2\bar{D}_i^T(k)\bar{P}_i(k+1)\bar{D}_i(k) - \gamma^2 I \\
\bar{P}_i(k) &= \sum_{j=1}^{s}\lambda_{ij}P_j(k), \qquad i,j=1,\ldots,s \\
\hat{G}_i(k) &= \mathrm{diag}\{\sqrt{\bar{\alpha}(1-\bar{\alpha})}I, \sqrt{\bar{\beta}(1-\bar{\beta})}B_{fi}(k)\} \\
\tilde{C}_i(k) &= [KC_i(k)\ 0] \\
\bar{E}(k) &= \mathrm{diag}\{E_\varepsilon(k), 0\} \\
E_\varepsilon(k) &= \varepsilon_1(k)E_1^T(k)E_1(k) + \varepsilon_2(k)E_2^T(k)E_2(k). \quad (14)
\end{aligned}$$

Theorem 1: Consider system (2) subject to randomly occurring nonlinearities (4) and sensor saturation (5). Let the disturbance attenuation level $\gamma > 0$, sets of positive scalars $\{\rho_i(k) > 0, i \in S\}_{0\le k\le N}$, $\{\tau_i(k) > 0, i \in S\}_{0\le k\le N}$, positive definite matrices $Q_i > 0, i \in S$ and the filter parameters $\{A_{fi}(k)\}_{0\le k\le N}$, $\{B_{fi}(k)\}_{0\le k\le N}$ and $\{L_{fi}(k)\}_{0\le k\le N}$ $(i \in S)$ be given. The $\mathcal{H}_\infty$ performance index defined in (12) is achieved for all nonzero $w(k)$ if, with the initial condition $P_i(0) \le \gamma^2[I \quad -I]^T Q_i [I \quad -I]$, there exists a family of positive definite matrices $\{P_i(k)\}_{0\le k\le N+1}$ $(i \in S)$ satisfying the following recursive matrix inequalities:

$$\begin{bmatrix} -\rho_i(k)I & * & * \\ \hat{G}_i(k) & -\bar{P}_i^{-1}(k+1) & * \\ G_i(k) & 0 & -\frac{1}{4}\bar{P}_i^{-1}(k+1) \end{bmatrix} \le 0 \quad (15)$$

$$\begin{bmatrix} \Omega_{11i}(k) & * & * \\ \Omega_{21i}(k) & \Omega_{22i}(k) & * \\ \Omega_{31i}(k) & \Omega_{32i}(k) & \Omega_{33i}(k) \end{bmatrix} \le 0 \quad (16)$$

for all $0 \le k \le N$, where $\Omega_{11i}(k)$, $\Omega_{21i}(k)$, $\Omega_{22i}(k)$, $\Omega_{31i}(k)$, $\Omega_{32i}(k)$, $\Omega_{33i}(k)$, $\bar{P}_i(k)$, $\hat{G}_i(k)$, $\tilde{C}_i(k)$, $E_\varepsilon(k)$, $\bar{E}(k)$ are defined in (14).

Proof: For $r(k) = i$, define the following Lyapunov function

$$V(\bar{x}(k), r(k)) = \bar{x}^T(k)P_i(k)\bar{x}(k) \quad (17)$$

where $P_i(k) = \mathrm{diag}\{P_{1i}(k), P_{2i}(k)\} > 0$ are the solutions to (15) and (16). Then, for $r(k) = i$ and $r(k+1) = j$, one has from (10) that

$$\begin{aligned}
&\mathbb{E}\{\Delta V(\bar{x}(k), r(k))\} \\
&\quad \le \mathbb{E}\{\eta^T(k)\Lambda_k\eta(k) - \bar{z}^T(k)\bar{z}(k) + \gamma^2\omega^{\mathrm{T}}(k)\omega(k)\} \quad (18)
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_k &= \begin{bmatrix} \Omega_{11i}(k) & * & * \\ \bar{\Omega}_{21i}(k) & \bar{\Omega}_{22i}(k) & * \\ \Omega_{31i}(k) & \Omega_{32i}(k) & \Omega_{33i}(k) \end{bmatrix} \\
\eta(k) &= [\,\bar{x}^{\mathrm{T}}(k) \quad \Psi^T(y_{si}(k)) \quad \omega^{\mathrm{T}}(k)\,]^T \\
\bar{\Omega}_{21i}(k) &= \bar{B}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) \\
\bar{\Omega}_{22i}(k) &= 2\bar{B}_i^T(k)\bar{P}_i(k+1)\bar{B}_i(k). \quad (19)
\end{aligned}$$

Hence, the $\mathcal{H}_\infty$ performance index defined in (12) is given by

$$\begin{aligned}
&\mathbb{E}\left\{\|\bar{z}(k)\|^2_{[0,N]}\right\} \\
&\quad - \gamma^2\left(\mathbb{E}\left\{\|w(k)\|^2_{[0,N]}\right\} + e^T(0)Q_ie(0)\right) \\
&\le \mathbb{E}\left\{\sum_{k=0}^{N}\eta^T(k)\Lambda_k\eta(k)\right\} \\
&\quad - \mathbb{E}\{\bar{x}^T(N+1)\bar{P}_i(N+1)\bar{x}(N+1)\} \\
&\quad + \bar{x}^T(0)(P_i(0) - \gamma^2[I \quad -I]^T \\
&\quad \times Q_i[I \quad -I])\bar{x}(0). \quad (20)
\end{aligned}$$

Noting that $\bar{P}_i(N+1) > 0$ and the initial condition $P_i(0) \le \gamma^2[I \quad -I]^T Q_i[I \quad -I]$, we can get (12) when the following inequality holds:

$$\eta^T(k)\Lambda_k\eta(k) \le 0 \quad (21)$$

In terms of the sensor saturation constraint in (8), we have

$$\Psi^T(y_{si}(k))(\Psi(y_{si}(k)) - \tilde{C}_i(k)\bar{x}(k)) \le 0 \quad (22)$$

which can be written in $\eta(k)$ as

$$\eta^T(k)\Phi_k\eta(k) \le 0 \quad (23)$$

where

$$\Phi_k = \frac{1}{2}\begin{bmatrix} 0 & * & * \\ -\tilde{C}_i(k) & 2I & * \\ 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

Now, it suffices to find a condition such that (21) holds subject to the sensor saturation constraints (23). By using Lemma 1, the sufficient condition such that the inequalities (23) imply (21) is that there exist positive scalars $\tau_i(k)$ such that

$$\Lambda_k - \tau_i(k)\Phi_k \le 0 \quad (25)$$

and then the rest of the proof follows from the statement of Theorem 1 immediately. The proof is complete. ■

IV. Design of Robust $\mathcal{H}_\infty$ Filters

In this section, given the imperfect transition probability matrix described in *Case 1* and *Case 2*, we shall discuss the robust

$\mathcal{H}_\infty$ filter design problem for the discrete time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation. Before presenting the theorem, let us denote

$$
\begin{aligned}
\bar{P}_{1i}^{(r)}(k) &= \sum_{j=1}^{s}\lambda_{ij}^{(r)}P_{1j}(k)\\
\bar{P}_{2i}^{(r)}(k) &= \sum_{j=1}^{s}\lambda_{ij}^{(r)}P_{2j}(k)\\
&\qquad i,j=1,\ldots,s,\ r=1,\ldots,\nu.\\
\Phi_{11i}(k) &= \mathrm{diag}\{-\rho_i(k)I,-\rho_i(k)I\}\\
\Phi_{22i}^{(r)}(k) &= \mathrm{diag}\Big\{-\bar{P}_{1i}^{(r)}(k+1),-\bar{P}_{2i}^{(r)}(k+1),\\
&\qquad -\frac{1}{4}\bar{P}_{1i}^{(r)}(k+1),-\frac{1}{4}\bar{P}_{2i}^{(r)}(k+1)\Big\}\\
\Phi_{21i}^{(r)}(k) &= \begin{bmatrix}\tilde{\alpha}\bar{P}_{1i}^{(r)}(k+1) & 0\\ 0 & \tilde{\beta}\bar{N}_i^{(r)}(k)\\ \bar{\alpha}\bar{P}_{1i}^{(r)}(k+1) & 0\\ 0 & \bar{\beta}\bar{N}_i^{(r)}(k)\end{bmatrix}\\
\tilde{\alpha} &= \sqrt{\bar{\alpha}(1-\bar{\alpha})},\tilde{\beta}=\sqrt{\bar{\beta}(1-\bar{\beta})}\\
\Upsilon_{11i}(k) &= \begin{bmatrix}\Xi_{1i}(k) & *\\ \Xi_{2i}(k) & \Xi_{3i}(k)\end{bmatrix}\\
\Upsilon_{21i}^{(r)}(k) &= \begin{bmatrix}\Xi_{4i}^{(r)}(k) & \Xi_{5i}^{(r)}(k)\\ \Xi_{6i}^{(r)}(k) & 0\end{bmatrix}\\
\Upsilon_{22i}^{(r)}(k) &= \mathrm{diag}\Big\{-\bar{P}_{1i}^{(r)}(k+1),-\bar{P}_{2i}^{(r)}(k+1)\\
&\qquad -\bar{P}_{1i}^{(r)}(k+1),-\bar{P}_{2i}^{(r)}(k+1)\ -\bar{P}_{1i}^{(r)}(k+1)\Big\}\\
\Upsilon_{31i}^{(r)}(k) &= \begin{bmatrix}0 & \Xi_{7i}^{(r)}(k)\\ \Xi_{8i} & 0\end{bmatrix}\\
\Upsilon_{32i} &= \begin{bmatrix}0 & 0\\ \Xi_{9i} & 0\end{bmatrix}\\
\Upsilon_{33i}^{(r)}(k) &= \mathrm{diag}\Big\{-\bar{P}_{2i}^{(r)}(k+1),-\bar{P}_{1i}^{(r)}(k+1)\\
&\qquad -\bar{P}_{2i}^{(r)}(k+1),-\xi_iI,-\xi_iI\Big\}\\
\Xi_{1i}(k) &= \mathrm{diag}\{\Pi_{1i}(k),-P_{2i}(k)\}\\
\Xi_{3i}(k) &= \mathrm{diag}\{-\tau_i(k)I,-\gamma^2I,-I\}\\
\Pi_{1i}(k) &= -P_{1i}(k)+\rho_i(k)\\
&\quad\times\left(\varepsilon_1(k)E_1^T(k)E_1(k)+\varepsilon_2(k)E_2^T(k)E_2(k)\right)\\
\Xi_{2i}(k) &= \begin{bmatrix}\frac{1}{2}\tau_i(k)KC_i(k) & 0\\ 0 & 0\\ L_i(k) & -L_{fi}(k)\end{bmatrix}\\
\Xi_{4i}^{(r)}(k) &= \begin{bmatrix}\bar{P}_{1i}^{(r)}(k+1)A_i(k) & 0\\ \bar{N}_i^{(r)}(k)K_1C_i(k) & \bar{M}_i^{(r)}(k)\end{bmatrix}\\
\Xi_{5i}^{(r)}(k) &= \begin{bmatrix}0 & \bar{P}_{1i}^{(r)}(k+1)D_{1i}(k) & 0\\ \bar{N}_i^{(r)}(k) & \bar{N}_i^{(r)}(k)D_{2i}(k) & 0\end{bmatrix}\\
\Xi_{6i}^{(r)}(k) &= \begin{bmatrix}\bar{P}_{1i}^{(r)}(k+1)A_i(k) & 0\\ \bar{N}_i^{(r)}(k)K_1C_i(k) & \bar{M}_i^{(r)}(k)\\ 0 & 0\end{bmatrix}\\
\Xi_{7i}^{(r)}(k) &= \begin{bmatrix}\bar{N}_i^{(r)}(k) & 0 & 0\\ 0 & \bar{P}_{1i}^{(r)}(k+1)D_{1i}(k) & 0\\ 0 & \bar{N}_i^{(r)}(k)D_{2i}(k) & 0\\ 0 & 0 & H_{2i}^T\end{bmatrix}\\
\Xi_{8i} &= \begin{bmatrix}\xi_iN_i & 0\end{bmatrix}\\
\Xi_{9i} &= \begin{bmatrix}H_{1i}^T & 0 & H_{1i}^T\\ 0 & 0 & 0\end{bmatrix}.
\end{aligned}
$$

Theorem 2: Consider system (2) with unknown transition probability matrix described in *Case 1*. Let $\gamma > 0$ be a given disturbance attenuation level. For given positive definite matrices $Q_i > 0$ $(i \in S)$, if there exist families of positive definite matrices $\{P_{1i}(k)\}_{0\le k\le N+1}$, $\{P_{2i}(k)\}_{0\le k\le N+1}$ $(i=1,2,\ldots,s)$, families of positive scalars $\{\rho_i(k)\}_{0\le k\le N}$, $\{\tau_i(k)\}_{0\le k\le N}$ $(i=1,2,\ldots,s)$ and $\{\xi_i\}_{i=1,2,\ldots,s}$, and families of real-valued matrices $\{\bar{M}_i^{(r)}(k)\}_{0\le k\le N}$, $\{\bar{N}_i^{(r)}(k)\}_{0\le k\le N}$ and $\{L_{fi}(k)\}_{0\le k\le N}$ satisfying the following RLMIs:

$$\begin{bmatrix}\Phi_{11i}(k) & *\\ \Phi_{21i}^{(r)}(k) & \Phi_{22i}^{(r)}(k)\end{bmatrix}<0,\quad i=1,2,\ldots,s,\ r=1,2,\ldots,\nu. \tag{26}$$

$$\begin{bmatrix}\Upsilon_{11i}(k) & * & *\\ \Upsilon_{21i}^{(r)}(k) & \Upsilon_{22i}^{(r)}(k) & *\\ \Upsilon_{31i}^{(r)}(k) & \Upsilon_{32i} & \Upsilon_{33i}^{(r)}(k)\end{bmatrix}\le 0,\quad i=1,2,\ldots,s,\ r=1,2,\ldots,\nu \tag{27}$$

with the initial condition

$$\begin{bmatrix}P_{1i}(0)-\gamma^2Q_i & \gamma^2Q_i\\ \gamma^2Q_i & P_{2i}(0)-\gamma^2Q_i\end{bmatrix}\le 0 \tag{28}$$

where $\bar{P}_{1i}^{(r)}(k)$, $\Phi_{11i}(k)$, $\Phi_{22i}^{(r)}(k)$, $\Phi_{21i}^{(r)}(k)$, $\tilde{\alpha}$, $\Upsilon_{11i}(k)$, $\Upsilon_{21i}^{(r)}(k)$, $\Upsilon_{22i}^{(r)}(k)$, $\Upsilon_{31i}^{(r)}(k)$, Υ_{32i}, $\Upsilon_{33i}^{(r)}(k)$, $\Xi_{1i}(k)$, $\Xi_{3i}(k)$, $\Pi_{1i}(k)$, $\Xi_{2i}(k)$, $\Xi_{4i}^{(r)}(k)$, $\Xi_{5i}^{(r)}(k)$, $\Xi_{6i}^{(r)}(k)$, $\Xi_{7i}^{(r)}(k)$, Ξ_{8i} and Ξ_{9i} are defined previously, then there exists an nth-order filter of the form (9) which ensures the $\mathcal{H}_\infty$ performance constraint in (12), where $L_{fi}(k)$ is given as part of the RLMI solution and the other two filter parameters are given by

$$
\begin{aligned}
A_{fi}(k) &= \bar{P}_{2i}^{(r)-1}(k+1)\bar{M}_i^{(r)}(k)\\
B_{fi}^{(r)}(k) &= \bar{P}_{2i}^{(r)-1}(k+1)\bar{N}_i^{(r)}(k).
\end{aligned}
$$

Proof: Since the transition probability matrix $\hat{\Psi}=[\lambda_{ij}]$ belongs to the convex polyhedral set $\Re$, there always exist scalars $\psi_r \ge 0$ $(r=1,2,\ldots,\nu)$ such that $\hat{\Psi}=\sum_{r=1}^{\nu}\psi_r\hat{\Psi}^{(r)}$, $\sum_{r=1}^{\nu}\psi_r=1$, where $\hat{\Psi}^{(r)}=[\lambda_{ij}^{(r)}]$ $(r=1,2,\ldots,\nu)$ are ν vertexes of the polytope. Hence, considering (15) and (16) in Theorem 1, we have

$$\begin{bmatrix}-\rho_i(k)I & * & *\\ \hat{G}_i(k) & -\bar{P}_i^{(r)-1}(k+1) & *\\ G_i(k) & 0 & -\frac{1}{4}\bar{P}_i^{(r)-1}(k+1)\end{bmatrix}\le 0 \tag{29}$$

$$\begin{bmatrix}\Omega_{11i}^{(r)}(k) & * & *\\ \Omega_{22i}^{(r)}(k) & \Omega_{22i}^{(r)}(k) & *\\ \Omega_{31i}^{(r)}(k) & \Omega_{32i}^{(r)}(k) & \Omega_{33i}^{(r)}(k)\end{bmatrix}\le 0 \tag{30}$$

where

$$\begin{aligned}
\Omega_{11i}^{(r)}(k) &= 2\bar{A}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{A}_i(k)\\
&\quad - P_i(k) + \bar{L}_i^T(k)\bar{L}_i(k) + \rho_i(k)\bar{E}(k)\\
\Omega_{21i}^{(r)}(k) &= \bar{B}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{A}_i(k) + \frac{1}{2}\tau_i(k)\tilde{C}_i(k)\\
\Omega_{22i}^{(r)}(k) &= 2\bar{B}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{B}_i(k) - \tau_i(k)I\\
\Omega_{31i}^{(r)}(k) &= \bar{D}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{A}_i(k)\\
\Omega_{32i}^{(r)}(k) &= \bar{D}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{B}_i(k)\\
\Omega_{33i}^{(r)}(k) &= 2\bar{D}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{D}_i(k) - \gamma^2 I\\
\bar{P}_i^{(r)}(k) &= \sum_{j=1}^{s}\lambda_{ij}^{(r)}P_j(k), \qquad i,j=1,\ldots,s.
\end{aligned}$$

Note that $P_i(k) = \mathrm{diag}\{P_{1i}(k), P_{2i}(k)\}$ where $P_{1i}(k) \in \mathbb{R}^{n\times n}$ and $P_{2i}(k) \in \mathbb{R}^{n\times n}$. Noticing (29)–(30), by using Schur complement, S-procedure and some algebraic manipulations, we can obtain (26)–(27) and this completes the proof of the theorem. ■

Remark 2: Now, let us consider the uncertain discrete-time Markovian system (2) with known transition probability matrix $\hat{\Psi}$. In this case, the inequalities (26) and (27) reduce to

$$\Phi_i(k) = \begin{bmatrix} \Phi_{11i}(k) & * \\ \Phi_{21i}(k) & \Phi_{22i}(k) \end{bmatrix} < 0 \tag{31}$$

$$\Upsilon_i(k) = \begin{bmatrix} \Upsilon_{11i}(k) & * & * \\ \Upsilon_{21i}(k) & \Upsilon_{22i}(k) & * \\ \Upsilon_{31i}(k) & \Upsilon_{32i} & \Upsilon_{33i}(k) \end{bmatrix} \le 0. \tag{32}$$

Theorem 2 provides a design scheme for a time-varying filter in the presence of unknown transition probability matrix in Case 1. Now we are going to consider the similar problem with unknown transition probability matrix in Case 2, and the following theorem is established along a similar line. Before stating the following theorem, let us denote

$$\hat{\Phi}_{21i}(k) = \begin{bmatrix} \bar{f}\hat{P}_{1i}(k+1) & 0 \\ 0 & \sqrt{\bar{\beta}(1-\bar{\beta})}\bar{N}_i(k) \\ \bar{\alpha}\hat{P}_{1i}(k+1) & 0 \\ 0 & \bar{\beta}\bar{N}_i(k) \end{bmatrix}$$

$$\begin{aligned}
\hat{\Phi}_{22i}(k) &= \mathrm{diag}\Big\{-\hat{P}_{1i}(k+1), -\hat{P}_{2i}(k+1)\\
&\qquad -\frac{1}{4}\hat{P}_{1i}(k+1), -\frac{1}{4}\hat{P}_{2i}(k+1)\Big\}\\
\hat{\Upsilon}_{21i}(k) &= \begin{bmatrix} \hat{\Xi}_{4i}(k) & \hat{\Xi}_{5i}(k) \\ \hat{\Xi}_{6i}(k) & 0 \end{bmatrix}\\
\hat{\Upsilon}_{22i}(k) &= \mathrm{diag}\Big\{-\hat{P}_{1i}(k+1), -\hat{P}_{2i}(k+1)\\
&\qquad - \hat{P}_{1i}(k+1), -\hat{P}_{2i}(k+1)\ -\hat{P}_{1i}(k+1)\Big\}\\
\hat{\Upsilon}_{31i}(k) &= \begin{bmatrix} 0 & \hat{\Xi}_{7i}(k) \\ \Xi_{8i} & 0 \end{bmatrix}\\
\bar{f} &= \sqrt{\bar{\alpha}(1-\bar{\alpha})}\\
\hat{\Upsilon}_{33i}(k) &= \mathrm{diag}\{-\hat{P}_{2i}(k+1), -\hat{P}_{1i}(k+1)\\
&\qquad - \hat{P}_{2i}(k+1), -\xi_i I, -\xi_i I\}
\end{aligned}$$

$$\begin{aligned}
\hat{\Xi}_{4i}(k) &= \begin{bmatrix} \hat{P}_{1i}(k+1)A_i(k) & 0 \\ \bar{N}_i(k)K_1C_i(k) & \bar{M}_i(k) \end{bmatrix}\\
\hat{\Xi}_{5i}(k) &= \begin{bmatrix} 0 & \hat{P}_{1i}(k+1)D_{1i}(k) & 0 \\ \bar{N}_i(k) & \bar{N}_i(k)D_{2i}(k) & 0 \end{bmatrix}\\
\hat{\Xi}_{6i}(k) &= \begin{bmatrix} \hat{P}_{1i}(k+1)A_i(k) & 0 \\ \bar{N}_i(k)K_1C_i(k) & \bar{M}_i(k) \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Theorem 3: Consider system (2) with unknown transition probability matrix described in Case 2. Let $\gamma > 0$ be a given disturbance attenuation level. For given positive definite matrices $Q_i > 0$ $(i \in S)$, assume that there exist families of positive definite matrices $\{P_{1i}(k)\}_{0\le k\le N+1}, \{P_{2i}(k)\}_{0\le k\le N+1}$ $(i = 1,2,\ldots,s)$, families of positive scalars $\{\rho_i(k)\}_{0\le k\le N}$, $\{\tau_i(k)\}_{0\le k\le N}, (i = 1,2,\ldots,s)$, $\{\xi_i\}_{i=1,2,\ldots,s}$ and families of real-valued matrices $\{\bar{M}_i(k)\}_{0\le k\le N}$, $\{\bar{N}_i(k)\}_{0\le k\le N}$ and $\{L_{fi}(k)\}_{0\le k\le N}$ satisfying the following RLMIs:

$$\hat{\Phi}_i(k) = \begin{bmatrix} \Phi_{11i}(k) & * \\ \hat{\Phi}_{21i}(k) & \hat{\Phi}_{22i}(k) \end{bmatrix} < 0, \quad i = 1,2,\ldots,s,\ r = 1,2,\ldots,\nu. \tag{33}$$

$$\hat{\Upsilon}_i(k) = \begin{bmatrix} \Upsilon_{11i}(k) & * & * \\ \hat{\Upsilon}_{21i}(k) & \hat{\Upsilon}_{22i}(k) & * \\ \hat{\Upsilon}_{31i}(k) & \Upsilon_{32i} & \hat{\Upsilon}_{33i}(k) \end{bmatrix} \le 0, \quad i = 1,2,\ldots,s,\ r = 1,2,\ldots,\nu \tag{34}$$

with the initial condition

$$\begin{bmatrix} P_{1i}(0) - \gamma^2 Q_i & \gamma^2 Q_i \\ \gamma^2 Q_i & P_{2i}(0) - \gamma^2 Q_i \end{bmatrix} \le 0 \tag{35}$$

where $\hat{\Phi}_{21i}(k)$, $\hat{\Phi}_{22i}(k)$, $\hat{\Upsilon}_{21i}(k)$, $\hat{\Upsilon}_{22i}(k)$, $\hat{\Upsilon}_{31i}(k)$, $\hat{\Upsilon}_{33i}(k)$, $\hat{\Xi}_{4i}(k)$, $\hat{\Xi}_{5i}(k)$, and $\hat{\Xi}_{6i}(k)$ are defined previously, and $\Phi_{11i}(k)$, $\Upsilon_{11i}(k)$ and Υ_{32i} are the same as defined in Theorem 2, and if $\hat{\Psi}_k^i = \emptyset$, we take

$$\hat{P}_{1i}(k) = P_{1i}(k), \quad \hat{P}_{2i}(k) = P_{2i}(k), \quad i = 1,\ldots,s,$$

otherwise

$$\begin{aligned}
\hat{P}_{1i}(k) &= \Big(\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}\Big)^{-1}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{1j}(k)\\
\hat{P}_{1i}(k) &= P_{1i}(k),\ j \in \hat{\Psi}_{uk}^i\\
\hat{P}_{2i}(k) &= \Big(\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}\Big)^{-1}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{2j}(k)\\
\hat{P}_{2i}(k) &= P_{2i}(k), \quad j \in \hat{\Psi}_{uk}^i.
\end{aligned}$$

Then, there exists an nth order filter of the form (9) which ensures the $\mathcal{H}_\infty$ performance constraint in (12), where $L_{fi}(k)$ is given as part of the RLMI solution and the other two filter parameters are given by

$$\begin{aligned}
A_{fi}(k) &= \hat{P}_{2i}^{-1}(k+1)\bar{M}_i(k)\\
B_{fi}(k) &= \hat{P}_{2i}^{-1}(k+1)\bar{N}_i(k).
\end{aligned}$$

Proof: Denote

$$\zeta_{ki} = \left(\sum_{j\in\hat{\Psi}_k^i} \lambda_{ij}\right)^{-1} \tag{36}$$

It is clear that, for any $i \in S$, (31) and (32) can be rewritten as shown in the equation at the bottom of the page. Therefore, if

$$\begin{aligned}
&\hat{\Phi}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{1j}(k),\\ \hat{P}_{2i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{2j}(k)\end{array}\right. < 0\\
&\hat{\Phi}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=P_{1i}(k),\\ \hat{P}_{2i}(k)=P_{2i}(k)\end{array}\right. < 0, j\in\hat{\Psi}_{uk}^i\\
&\hat{\Upsilon}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{1j}(k)\\ \hat{P}_{2i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{2j}(k)\end{array}\right. < 0\\
&\hat{\Upsilon}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=P_{1i}(k),\\ \hat{P}_{2i}(k)=P_{2i}(k)\end{array}\right. < 0, j\in\hat{\Psi}_{uk}^i
\end{aligned}$$

we have $\Phi_i(k) < 0$ and $\Upsilon_i(k) < 0$ for any $i \in S$. This completes the proof. ■

Remark 3: Theorem 3 provides feasible solutions to the filter design problem for time-varying Markovian jump system (2) under partially unknown transition probabilities. Note that if $\hat{\Psi}_{uk}^i = \emptyset$ holds for any $i \in S$, i.e., all the transition probabilities are accessible, the corresponding results in (33)–(34) reduce to (31) and (32). Similarly, when $\hat{\Psi}_k^i = \emptyset$ holds for any $i \in S$, i.e., all the transition probabilities are inaccessible, Theorem 3 is still valid at the cost of the incremental conservatism. More specifically, the more known entries in the transition probability matrix, the less conservatism of the results we would have.

Based on the Theorem 2 and Theorem 3, we suggest the following Robust H_∞ Filter Design (RHFD) algorithm involving recursive LMIs conditions.

Algorithm RHFD:

Step 1. Given the $\mathcal{H}_\infty$ performance index γ, positive definite matrices Q_i $(i = 1, 2, \ldots, s)$ and the state initial conditions $x(0)$ and $\hat{x}(0)$. Select the initial values for matrices $\{P_{1i}(0)$ and $P_{2i}(0)\}$ which satisfy the condition (28) and set $k = 0$.

Step 2. Obtain the positive matrices $P_{1i}(k+1)$ and $P_{2i}(k+1)$, and matrices $\bar{M}_i(k), \bar{N}_i(k), L_{fi}(k)$ for the sampling instant k by solving the RLMIs (26)–(27) or (33)–(34), respectively, with known parameters $P_{1i}(k)$ and $P_{2i}(k)$.

Step 3. Derive the other two filter parameter matrices $A_{fi}(k)$ and $B_{fi}(k)$ by solving (29), and set $k = k + 1$.

Step 4. If $k < N$, then go to Step 2, otherwise exit.

V. Illustrative Examples

In this section, we present two simulation examples to illustrate the usefulness and flexibility of the time-varying filter design method developed in this paper. Consider a class of uncertain discrete stochastic nonlinear time-varying Markovian jump systems with sensor saturation in the form (2).

Example 1: Consider *Case 1* where the transition probability matrix of the Markov process is unknown but it resides in a polytope with the following two vertices:

$$\hat{\Psi}^{(1)} = \begin{bmatrix} 0.5 & 0.5\\ 0.3 & 0.7\end{bmatrix}, \quad \hat{\Psi}^{(2)} = \begin{bmatrix} 0.6 & 0.4\\ 0.5 & 0.5\end{bmatrix}.$$

Suppose that the system involves two modes, and the system data are given as follows:

Mode 1:

$$\begin{aligned}
A_1(k) &= \begin{bmatrix} 0.2 & 0.2\sin(k)\\ 1.1\sin(5k) & 0.5\end{bmatrix}\\
D_{11}(k) &= \begin{bmatrix} 0.1\sin(3k)\\ -0.3\end{bmatrix}\\
H_{11} &= \begin{bmatrix} 0.1\\ 0.3\end{bmatrix}\\
D_{21}(k) &= -0.3\sin(3k)\\
H_{21} &= 0.2,\\
L_1(k) &= [\,0.3\sin(2k) \quad 0.7\,]\\
N_1 &= [0 \quad 0.5]\\
C_1(k) &= [\,0.9 \quad 0.5\sin(5k)\,], \quad |F_1(k)| \le 1
\end{aligned}$$

$\sigma(y_{s1}(k))$ is a saturation function described as follows:

$$\sigma(y_{s1}(k)) = \begin{cases} \sigma(y_{s1}(k)) = y_{s1}(k),\\ \text{if } -V_{ys1j,\max} \le y_{s1}(k) \le V_{ys1j,\max};\\ \sigma(y_{s1}(k)) = V_{ys1j,\max},\\ \text{if } y_{s1}(k) > V_{ys1j,\max};\\ \sigma(y_{s1}(k)) = -V_{ys1j,\max},\\ \text{if } y_{s1}(k) < -V_{ys1j,\max};\end{cases}$$

Mode 2:

$$\begin{aligned}
A_2(k) &= \begin{bmatrix} 0.3\sin(k) & 0.1\\ 1.3 & 0.5\sin(5k)\end{bmatrix},\\
D_{12}(k) &= \begin{bmatrix} 0.1\\ 0.4\sin(3k)\end{bmatrix}, H_{12} = \begin{bmatrix} 0.2\\ 0.1\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Phi_i(k) &= \sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}\hat{\Phi}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{1j}(k)\\ \hat{P}_{2i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{2j}(k)\end{array}\right. + \sum_{j\in\hat{\Psi}_{uk}^i}\lambda_{ij}\hat{\Phi}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=P_{1i}(k)\\ \hat{P}_{2i}(k)=P_{2i}(k)\end{array}\right.\\
\Upsilon_i(k) &= \sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}\hat{\Upsilon}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{1j}(k),\\ \hat{P}_{2i}(k)=\zeta_{ki}\sum_{j\in\hat{\Psi}_k^i}\lambda_{ij}P_{2j}(k)\end{array}\right. + \sum_{j\in\hat{\Psi}_{uk}^i}\lambda_{ij}\hat{\Upsilon}_i(k)\left|\begin{array}{l}\hat{P}_{1i}(k)=P_{1i}(k)\\ \hat{P}_{2i}(k)=P_{2i}(k)\end{array}\right.
\end{aligned}$$

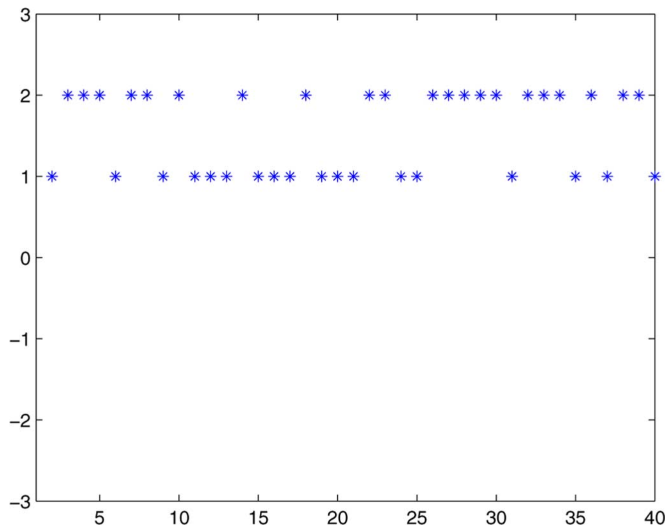

Fig. 1. Random mode r(k).

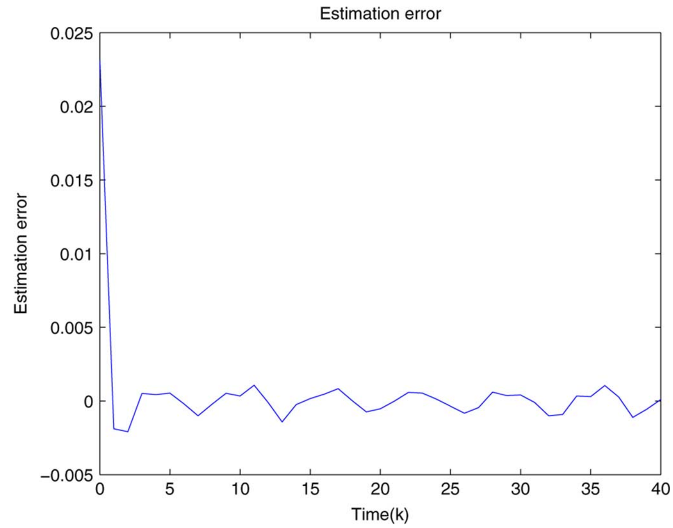

Fig. 2. Estimation error.

$$D_{22}(k) = -0.2\sin(4k), \quad H_{22} = 0.1,$$
$$L_2(k) = [0.4\sin(2k) \quad 0.2], \quad N_2 = [0 \quad 0.5],$$
$$C_2(k) = [1.3 \quad 0.2\sin(k)], \quad |F_2(k)| \le 1$$
$$\sigma(y_{s2}(k)) = \begin{cases} \sigma(y_{s2}(k)) = y_{s2}(k), \\ \text{if } -V_{ys2j,\max} \le y_{s2}(k) \le V_{ys2j,\max}; \\ \sigma(y_{s2}(k)) = V_{ys2j,\max}, \\ \text{if } y_{s2}(k) > V_{ys2j,\max}; \\ \sigma(y_{s2}(k)) = -V_{ys2j,\max}, \\ \text{if } y_{s2}(k) < -V_{ys2j,\max}; \end{cases}$$

and the nonlinear functions $f(k, x(k))$ and $g(k, x(k))$ are selected as

$$f(k, x(k)) = \left[\tfrac{0.2x_1(k)}{2x_2^2(k)+1} \quad 0.1x_1(k)\sin(x_2(k)) \right]^T$$
$$g(k, x(k)) = 0.2x_1(k)\sin(x_2(k)).$$

It is easy to see that the constraint (4) can be met with $\varepsilon_1(k) = \varepsilon_2(k) = 1$ and $E_1(k) = E_2(k) = \mathrm{diag}\{0.2, 0.15\}$. In this example, the saturation values are taken as $V_{ys11} = V_{ys21} = 0.06$ and $K = 0.2$, $K_1 = 0.8$. The state initial value $x(0) = [0.2 - 0.5]^T$, $\hat{x}(0) = [-0.2 - 0.16]^T$. The exogenous disturbance input $w(k)$ is supposed to be a random noise uniformly distributed over $[-0.5\ 0.5]$ and the probabilities are assumed to be $\bar{\alpha} = \bar{\beta} = 0.9$. Set $\gamma = 0.5$ and let $Q_1 = Q_2 = \mathrm{diag}\{1, 1\}$. Choose the parameters' initial values to satisfy (28).

Consider the real transition probability matrix as

$$\hat{\Psi} = \begin{bmatrix} 0.56 & 0.44 \\ 0.42 & 0.58 \end{bmatrix}$$

which means that $\psi_1 = 0.4$ and $\psi_2 = 0.6$ in (1). According to the robust H_∞ filter design algorithm (RHFD), the RLMIs in Theorem 2 can be solved recursively subject to given initial conditions and prespecified performance indices.

The simulation results are shown in Figs. 1–3, where Fig. 1 plots one of the possible realizations of the Markovian jumping mode $r(k)$. Under this mode sequence, the estimation error $\bar{z}(k)$ is depicted in Fig. 2. Fig. 3 shows the sensor output. Note that

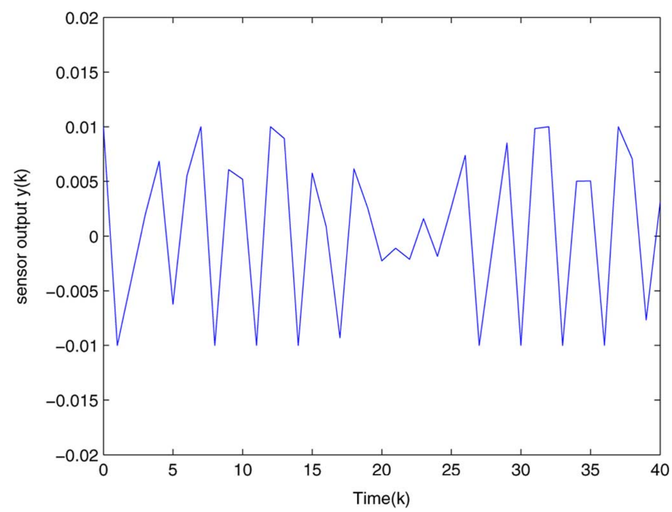

Fig. 3. Sensor output.

the sensor outputs is saturated. The simulation has confirmed that the designed filter performs very well.

Example 2: Consider *Case 2* where some elements in the transition probability matrix of the Markov process are unknown and the possible three cases for $\hat{\Psi}$ are given as follows:

$$\hat{\Psi}_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix},$$
$$\hat{\Psi}_2 = \begin{bmatrix} 0.6 & 0.4 \\ ? & ? \end{bmatrix}, \quad \hat{\Psi}_3 = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

where $\hat{\Psi}_1$ (respectively, $\hat{\Psi}_2, \hat{\Psi}_3$) shows that the elements in transition probabilities matrix are completely known (respectively, partially known or completely unknown), and the other parameters of the discrete stochastic nonlinear time-varying system (2) are the same as in *Example 1*. Similarly, according to the robust H_∞ filter design algorithm (RHFD), the RLMIs in Theorem 3 can be solved recursively subject to given initial conditions and prespecified performance indices. The corresponding simulation results for the estimation error in these three cases

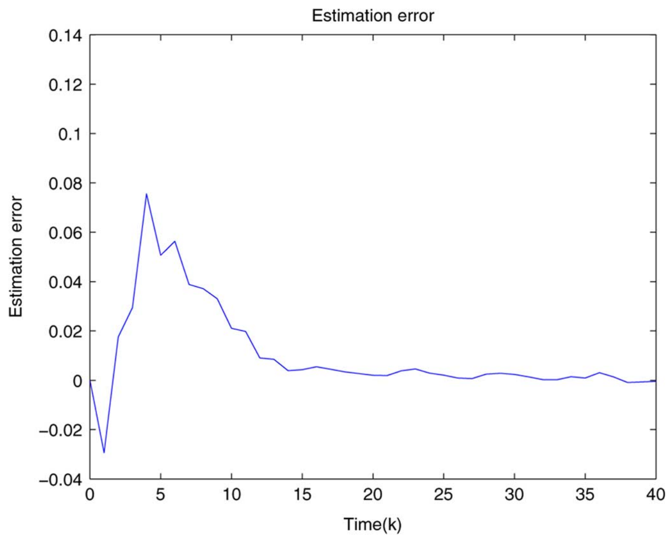

Fig. 4. Estimation error of $\hat{\Psi}_1$ case.

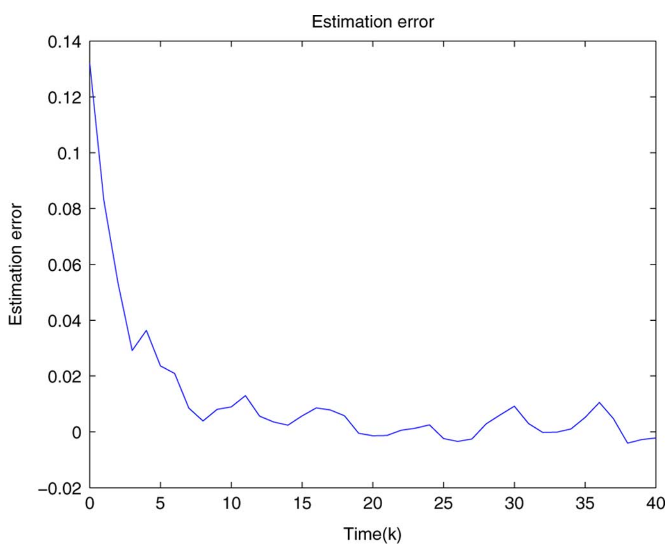

Fig. 5. Estimation error of $\hat{\Psi}_2$ case.

are given in Figs. 4–6, respectively. Again, it can be seen that the more known entries in the transition probability matrix we have, the less conservatism of the condition there would be.

VI. Conclusion

In this paper, the robust $\mathcal{H}_\infty$ filtering problem has been investigated for time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation. The considered transition probability matrix includes the case with polytopic uncertainties and the case with partially unknown transition probabilities, respectively. Also, the case with completely known or completely unknown transition probabilities have been studied as two special cases. The randomly occurring nonlinearities have been modeled by the Bernoulli distributed white sequences with known conditional probabilities. Sufficient conditions have been derived for the filtering augmented system under consideration to satisfy the $\mathcal{H}_\infty$ performance constraint. The corresponding robust $\mathcal{H}_\infty$ filters have been designed by solving sets of RLMIs. Two numerical simulation

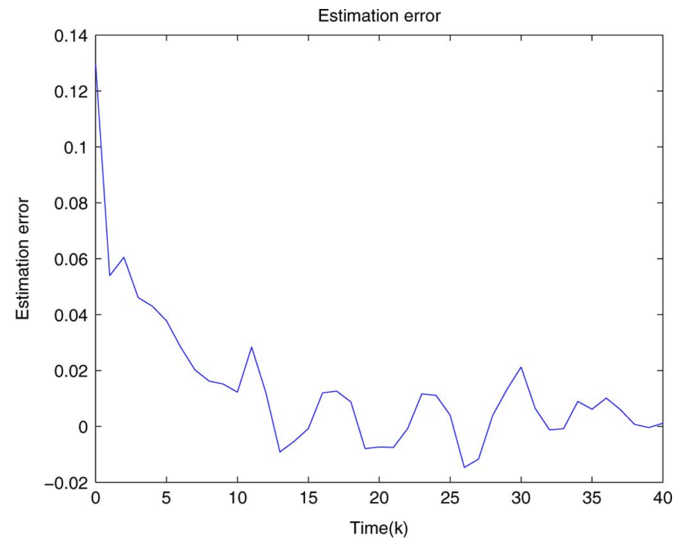

Fig. 6. Estimation error of $\hat{\Psi}_3$ case.

examples have been used to demonstrate the effectiveness of the filtering technology presented in this paper.

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