HIGHER ORDER ENERGY EXPANSIONS FOR SOME SINGULARLY PERTURBED NEUMANN PROBLEMS

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Abstract- We consider the following singularly perturbed semilinear elliptic problem:

$$\epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $\epsilon > 0$ is a small constant and $p$ is a subcritical exponent. Let $J_{\epsilon}[u] := \int_{\Omega} (\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u^{p+1}) dx$ be its energy functional, with $u \in H^1(\Omega)$. Ni and Takagi ([15], [16]) proved that for a single boundary spike solution $u_\epsilon$, the following asymptotic expansion holds

$$J_{\epsilon}[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

where $c_1 > 0$ is a generic constant, $P_\epsilon$ is the unique local maximum point of $u_\epsilon$ and $H(P_\epsilon)$ is the boundary mean curvature function. In this paper, we obtain the following higher order expansion of $J_{\epsilon}[u_\epsilon]$

$$J_{\epsilon}[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],$$

where $c_2, c_3$ are generic constants and $R(P_\epsilon)$ is the Ricci scalar curvature at $P_\epsilon$. In particular $c_3 > 0$. Applications of this expansion will be given.

L’expansion de l’énergie de les solutions de les problèmes de la perturbation singulière

Résumé. Nous étudions le problème suivant de la perturbation singulière:

$$\epsilon^2 \Delta u - u + u^p = 0 \text{ dans } \Omega, \quad u > 0 \text{ dans } \Omega \quad \text{et} \quad \frac{\partial u}{\partial \nu} = 0 \text{ sur } \partial \Omega,$$

où $\Omega$ est un domaine ouvert dans $\mathbb{R}^N$, $\epsilon > 0$ est une constante petite et $p$ est un exposant subcritique. L’énergie s’écrit alors $J_{\epsilon}[u] := \int_{\Omega} (\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u^{p+1}) dx$, où $u \in H^1(\Omega)$. Ni et Takagi ([15], [16]) montrent que pour une solution $u_\epsilon$ avec une pique sur la frontière du domaine, l’existence de la expansion asymptotique suivant:

$$J_{\epsilon}[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

où $c_1 > 0$ est une constante générique, $P_\epsilon$ est le point unique du maximum local de $u_\epsilon$ et $H(P_\epsilon)$ est la fonction de la curvature moyenne sur la frontière. Nous dérivons de la expansion suivant de l’ordre plus élevé de $J_{\epsilon}[u_\epsilon]$

$$J_{\epsilon}[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],$$

où $c_2, c_3$ sont les constantes génériques et $R(P_\epsilon)$ est la curvature scalare de Ricci dans $P_\epsilon$. En particulier $c_3 > 0$. Nous présentons les applications de la expansion.
We consider the following singularly perturbed semilinear elliptic problem

$$\epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ denotes the Laplacian operator in $\mathbb{R}^N$, $\nu$ stands for the unit outer normal to $\partial \Omega$, $f(u) = u^p$ and $p$ satisfies $1 < p < \left(\frac{N+2}{N-2}\right)_+ = \frac{N+2}{N-2}$ when $N \geq 3; = +\infty$ when $N = 1, 2$.

Equation (1.1) arises in many branches of applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of Gierer-Meinhardt model in biological pattern formation ([7], [18]) or of parabolic equations in chemotaxis, population dynamics and phase transitions. Associated with (1.1) is the energy functional $J_\epsilon$ defined by

$$J_\epsilon[u] := \int_\Omega \left( \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - F(u) \right) \, dx \quad \text{for } u \in H^1(\Omega),$$

where $F(u) = \int_0^u f(s) \, ds$. 

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**Theorem 1.** For a solution $u_\epsilon$ of (I) with a peak on the boundary of the domain and with a unique local maximum of $u_\epsilon$, $H(P_\epsilon)$ is the function of the mean curvature on the boundary and $I[w]$ is the energy of the fundamental state in $\mathbb{R}^N$.

In this work we derive the following asymptotic expansion:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

where $c_1 > 0$ is a constant genéric and $P_\epsilon \in \partial \Omega$ is the point of maximum local of $u_\epsilon$, $H(P_\epsilon)$ is the function of the mean curvature on the boundary and $I[w]$ is the energy of the fundamental state in $\mathbb{R}^N$.

Theorem 1.1. For a solution $u_\epsilon$ of (I) with a peak on the boundary of the domain and with a unique local maximum of $u_\epsilon$, we have

$$J_\epsilon = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 H(P_\epsilon)]^2 + c_3 R(P_\epsilon) + o(\epsilon^2) \right],$$

where $c_1, c_2, c_3$ are constants genéric.

As a consequence of the Theorem 1.1, we obtain an refinement of the results of [15] and [16].

**Corollary 2.** For a solution $u_\epsilon$ of the minimization problem (I) and for $\epsilon$ with $\epsilon > 0$ small enough we have

$$H(P_\epsilon) \to \max_{P \in \partial \Omega} H(P), R(P_\epsilon) \to \min_{Q \in \partial \Omega, H(Q) = \max_{P \in \partial \Omega} H(P)} R(Q).$$

They are two essential steps in the proof of the Theorem 1.1. In Step 1 we find a function $w_{\epsilon,P}$ with $\epsilon^2 \Delta \tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + w_{\epsilon,P} = O(\epsilon^2)$. In Step 2 we show that $u_\epsilon = \tilde{w}_{\epsilon,P} + O(\epsilon^\tau)$ for some $\tau > 1$.

1. **Introduction.** We consider the following singularly perturbed semilinear elliptic problem

$$\epsilon^2 \Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,$$ (1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ denotes the Laplace operator in $\mathbb{R}^N$, $\nu$ stands for the unit outer normal to $\partial \Omega$, $f(u) = u^p$ and $p$ satisfies $1 < p < \left(\frac{N+2}{N-2}\right)_+ = \frac{N+2}{N-2}$ when $N \geq 3; = +\infty$ when $N = 1, 2$.

Equation (1.1) arises in many branches of applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of Gierer-Meinhardt model in biological pattern formation ([7], [18]) or of parabolic equations in chemotaxis, population dynamics and phase transitions. Associated with (1.1) is the energy functional $J_\epsilon$ defined by

$$J_\epsilon[u] := \int_\Omega \left( \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - F(u) \right) \, dx \quad \text{for } u \in H^1(\Omega),$$

where $F(u) = \int_0^u f(s) \, ds$. 

In the pioneering papers [14], [15] and [16], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for \( \epsilon \) sufficiently small the least-energy solution has only one local maximum point \( P_\epsilon \) with \( P_\epsilon \in \partial \Omega \). Moreover, \( H(P_\epsilon) \to \max_{P \in \partial \Omega} H(P) \) as \( \epsilon \to 0 \), where \( H(P) \) is the mean curvature of \( \partial \Omega \) at \( P \). Since then, many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [1], [2], [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [15], [16], [17], [19], [20], [21], and the review article [18] and the references therein.

A common tool for proving the existence of spike solutions is by energy expansion: In [15] and [16], Ni and Takagi proved, among others, that for a single boundary spike solution \( u_\epsilon \) the following asymptotic expansion for \( J_\epsilon[u_\epsilon] \) holds true:

\[
J_\epsilon[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],
\]

where \( c_1 > 0 \) is a generic constant, \( P_\epsilon \) is the unique local maximum point of \( u_\epsilon \), \( H(P_\epsilon) \) is the mean curvature function at \( P_\epsilon \in \partial \Omega \), \( w \) is the unique solution of the following ground-state problem

\[
\Delta w - w + f(w) = 0, w > 0 \quad \text{in} \quad R^N, \quad w(0) = \max w(y), \quad \lim_{|y| \to +\infty} w(y) = 0,
\]

and \( I[w] \) is the ground-state energy \( I[w] = \frac{1}{2} \int_{R^N} \left( \nabla w \right)^2 + \frac{1}{2} w^2 - F(w) \) \( dy \). Based on (1.2), Ni and Takagi [16] concluded that the least energy solution must concentrate at a maximum point of the mean curvature function. However, if \( H(P) \) has more than one maximum point on \( \partial \Omega \), the asymptotic expansion (1.2) has to be refined to prove such a statement and the next order term in (1.2) becomes important. This is exactly the purpose of this paper.

We now state our main theorem. First, we introduce boundary deformations. Let \( P \in \partial \Omega \). After rotation and translation of the coordinate system we may assume that the inward normal to \( \partial \Omega \) at \( P \) points in the direction of the positive \( x_N \)-axis, that \( P = 0 \), and that there exists a constant \( \delta > 0 \) and a smooth function \( \rho \) such that \( \Omega \cap B_\delta(P) = \{(x', x_N) | x_N > \rho(x')\} \). Moreover, we may assume that

\[
\rho(x') = \frac{1}{2} \sum_{i=1}^{N-1} k_i x_i^2 + O(|x'|^3), \quad x' = (x_1, ..., x_{N-1}),
\]

where \( k_i, i = 1, ..., N - 1 \) are the principal curvatures at \( P \). (Note that \( H(P) = \frac{1}{N-1} \sum_{i=1}^{N-1} k_i \) is the mean curvature.) For \( N \geq 3 \), we also need to define \( R(P) = \sum_{i \neq j} k_i k_j \), which is called Ricci scalar curvature at \( P \). When \( N = 2 \), we let \( R(P) = 0 \).

Now we can state the main result of this paper.

**Theorem 1.** Let \( u_\epsilon \) be a single boundary spike solution of (1.1) with a unique local maximum point \( P_\epsilon \in \partial \Omega \). Then, for \( \epsilon \) sufficiently small, we have

\[
J_\epsilon = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],
\]

where \( c_1 = \frac{N-1}{N+1} \int_{R^N} (w')^2 y_N dy > 0 \), and \( c_2, c_3 \) are generic constants. Moreover, we have \( c_3 > 0 \). Here \( R_N^+ = \{(y', y_N) | y_N > 0\} \).

As a corollary, we give a refinement of the results of [15] and [16].

**Corollary 2** Let \( u_\epsilon \) be a least energy solution of (1.1). Then, for \( \epsilon \) sufficiently small, we have

\[
H(P_\epsilon) \to \max_{P \in \partial \Omega} H(P), \quad R(P_\epsilon) \to \min_{Q \in \partial \Omega, H(Q) = \max_{P \in \partial \Omega} H(P)} R(Q).
\]
Proof: We sketch the main ideas of the proof. For details, see Section 5 of [22]. Substituting \( u_\epsilon = \tilde{w}_{\epsilon,P} + \epsilon^\sigma \phi_\epsilon \) into (1.1), we see from (2.1) that \( \phi_\epsilon \) satisfies

\[
e^2 \Delta \phi_\epsilon - \phi_\epsilon + f'(\tilde{w}_{\epsilon,P}) \phi_\epsilon = O(\epsilon^{\sigma/2}) + N_\epsilon[\phi_\epsilon] \text{ in } \Omega, \quad \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \text{ on } \partial \Omega, \tag{2.5}
\]

where \( N_\epsilon[\phi_\epsilon] = -\epsilon^{-\tau}[f(\tilde{w}_{\epsilon,P} + \epsilon^\sigma \phi_\epsilon) - f(\tilde{w}_{\epsilon,P}) - \epsilon^{\sigma/2} f'(\tilde{w}_{\epsilon,P}) \phi_\epsilon] = o(1)|\phi_\epsilon|, \) by the mean value theorem.

Now we can prove (2.3). Suppose not, then there exists a sequence \( \epsilon_k \to 0 \) such that \( M_\epsilon := ||\phi_{\epsilon_k}||_{L^\infty(\Omega)} \to +\infty. \) For simplicity of notation, we still denote \( \epsilon_k \) by \( \epsilon \). Let \( M_\epsilon = |\phi_\epsilon(x_\epsilon)|, \) where \( x_\epsilon \in \Omega. \) Without loss of generality, we may assume that \( x_\epsilon \) is a maximum point of \( \phi_\epsilon. \) We proceed by proving two claims.

2. Theorem 1 holds true if we replace \(-u + u^p\) with more general nonlinearities. See [22].

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2. Two Important Lemmas. In this section we present two main lemmas needed to prove Theorem 1. We begin with the following on good approximate functions.

Lemma 3. For each \( P \in \partial \Omega, \) there exists a smooth function \( \tilde{w}_{\epsilon,P} \) such that

\[
e^2 \Delta \tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + f(\tilde{w}_{\epsilon,P}) = O(\epsilon^{\sigma+1}), \tag{2.1} \]

\[
J_\epsilon[\tilde{w}_{\epsilon,P}] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P) + \epsilon^2 [c_2 (H(P))^2 + c_3 R(P)] + o(\epsilon^2) \right], \tag{2.2}
\]

where \( \sigma = \min(1, p - 1) \) and \( c_1, c_2, c_3 \) are generic constants. In particular,

\[
c_3 = \frac{1}{16} \int_{\Omega} \left[ |\nabla \Psi_0|^2 + |\Psi_0|^2 - f'(w) \Psi_0^2 \right] dy > 0, \tag{2.3}
\]

where \( \Psi_0 \) satisfies \( \Delta \Psi_0 - \Psi_0 + f'(w) \Psi_0 = 0 \) in \( R_+^N, \) \( \frac{\partial \Psi_0}{\partial \nu} = \frac{w}{|y|} (y_1^2 - y_2^2) \) on \( \partial R_+^N. \)

The proof of Lemma 3 is technical and we refer to Section 2 and Section 3 of [22].

Our next lemma is about the expansion of \( u_\epsilon \) which is a single boundary spike solution of (1.1). Let \( P_0 \) be its local maximum point. The key observation is that by using \( \tilde{w}_{\epsilon,P} \) as our approximating function, we just need to expand \( u_\epsilon \) up to \( O(\epsilon^\tau) \) for some \( \tau > 1. \) In fact, we do not even need to know the exact asymptotic expansion in \( O(\epsilon^\tau). \) We now choose \( \tau = 1 + \frac{\sigma}{2}. \) Thus we get

Lemma 4: For \( \epsilon \) sufficiently small, we have \( u_\epsilon = \tilde{w}_{\epsilon,P} + \epsilon^\sigma \phi_\epsilon, \) where \( \phi_\epsilon \) satisfies

\[
||\phi_\epsilon||_{L^\infty(\Omega)} \leq C, \tag{2.4}
\]

\[
\epsilon^{-N} \int_{\Omega} (\epsilon^2 |\nabla \phi_\epsilon|^2 + |\phi_\epsilon|^2) dx \leq C. \tag{2.3}
\]

Proof: We sketch the main ideas of the proof. For details, see Section 5 of [22]. Substituting \( u_\epsilon = \tilde{w}_{\epsilon,P} + \epsilon^\sigma \phi_\epsilon \) into (1.1), we see from (2.1) that \( \phi_\epsilon \) satisfies

\[
e^2 \Delta \phi_\epsilon - \phi_\epsilon + f'(\tilde{w}_{\epsilon,P}) \phi_\epsilon = O(\epsilon^{\sigma/2}) + N_\epsilon[\phi_\epsilon] \text{ in } \Omega, \quad \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \text{ on } \partial \Omega, \tag{2.5}
\]

where \( N_\epsilon[\phi_\epsilon] = -\epsilon^{-\tau}[f(\tilde{w}_{\epsilon,P} + \epsilon^\sigma \phi_\epsilon) - f(\tilde{w}_{\epsilon,P}) - \epsilon^{\sigma/2} f'(\tilde{w}_{\epsilon,P}) \phi_\epsilon] = o(1)|\phi_\epsilon|, \) by the mean value theorem.

Now we can prove (2.3). Suppose not, then there exists a sequence \( \epsilon_k \to 0 \) such that \( M_\epsilon := ||\phi_\epsilon||_{L^\infty(\Omega)} \to +\infty. \) For simplicity of notation, we still denote \( \epsilon_k \) by \( \epsilon \). Let \( M_\epsilon = |\phi_\epsilon(x_\epsilon)|, \) where \( x_\epsilon \in \Omega. \) Without loss of generality, we may assume that \( x_\epsilon \) is a maximum point of \( \phi_\epsilon. \) We proceed by proving two claims.
Claim 1: $|\frac{x_\epsilon-P_\epsilon}{\epsilon}| \leq C$. Suppose not, that is $|\frac{x_\epsilon-P_\epsilon}{\epsilon}| \to +\infty$. Then $-1 + \hat{f}(\hat{w}_{\epsilon,P_\epsilon}(x_\epsilon)) \leq -\frac{1}{\epsilon}$ for $\epsilon$ small. Since $\frac{\partial \phi}{\partial \nu} = 0$, by the Hopf boundary Lemma, $x_\epsilon \not\in \partial \Omega$. So $x_\epsilon \in \Omega$, which implies $\Delta \phi_\epsilon(x_\epsilon) \leq 0$. From (2.5) we then deduce that

$$ (1 - \hat{f}(\hat{w}_{\epsilon,P_\epsilon}(x_\epsilon)))M_\epsilon + o(1)M_\epsilon + O(\epsilon^{-1}) \leq 0 $$

and hence $M_\epsilon$ is bounded, a contradiction. Let $\hat{\phi}_\epsilon(y) = \frac{\phi(x)}{M_\epsilon}$, where $\epsilon y = x - P$.

Claim 2: $\hat{\phi}_\epsilon(y) \to 0$ in $C^1_{\text{loc}}(R_+^N)$, as $\epsilon \to 0$. In fact, from the equation for $\hat{\phi}_\epsilon$, we see that as $\epsilon \to 0$, $\hat{\phi}_\epsilon \to \hat{\phi}_0$, where $\Delta \hat{\phi}_0 - \hat{\phi}_0 + \hat{f}(w)\hat{\phi}_0 = 0$, $|\hat{\phi}_0| \leq 1$, in $R_+^N$, $\frac{\partial \hat{\phi}_0}{\partial y} = 0$ on $\partial R_+^N$. By the nondegeneracy of $w$, there exist $N-1$ constants $a_1, ..., a_{N-1}$ such that $\hat{\phi}_0 = \sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}$. On the other hand, we know that $\nabla_x u_\epsilon(P_\epsilon) = 0, k = 1, ..., N - 1$ and hence

$$ 0 = \nabla_x (\hat{w}_{\epsilon,P_\epsilon}(P_\epsilon)) = O(\epsilon) + \epsilon^{-1}M_\epsilon \nabla_y \hat{\phi}_\epsilon(0). $$

Thus we have $\nabla_y \hat{\phi}_\epsilon(0) \to 0$ which shows that $\nabla_y \hat{\phi}_\epsilon(0) = 0$. This implies $\nabla_y (\sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}) y = 0, k = 1, ..., N - 1$. Thus $a_1 = ... = a_{N-1} = 0$. This proves Claim 2.

Equation (2.3) now follows from Claim 1 and Claim 2: Let $y_\epsilon = \frac{x_\epsilon-P_\epsilon}{\epsilon}$. Then by Claim 1, $|y_\epsilon| \leq C$. So we may assume that $y_\epsilon \to y_0$ as $\epsilon \to 0$. Since $\hat{\phi}_\epsilon(y_\epsilon) = 1$, we have $\hat{\phi}_\epsilon(y_0) = 1$, which contradicts Claim 2.

Multiplying (2.5) by $\phi_\epsilon$, integrating over $\Omega$ and using (2.3), we obtain (2.4).

3. Proofs of Theorem 1 and Corollary 2.

We prove Theorem 1 by using Lemma 3 and Lemma 4.

Proof of Theorem 1: Since both $\hat{w}_{\epsilon,P_\epsilon}$ and $\phi_\epsilon$ satisfy the Neumann boundary condition, we get

$$ J_\epsilon[u_\epsilon] = J_\epsilon[\hat{w}_{\epsilon,P_\epsilon}] + \epsilon^2 \int_{\Omega} (\epsilon^2 \nabla \hat{w}_{\epsilon,P_\epsilon} \nabla \phi_\epsilon + \hat{w}_{\epsilon,P_\epsilon} \phi_\epsilon - f(\hat{w}_{\epsilon,P_\epsilon}) \phi_\epsilon) \, dx $$

$$ + \frac{\epsilon^2}{2} \left( \int_{\Omega} (\epsilon^2 |\nabla \phi_\epsilon|^2 + |\phi_\epsilon|^2) \, dx - \int_{\Omega} \hat{f}(\hat{w}_{\epsilon,P_\epsilon}) \phi_\epsilon^2 \, dx \right) $$

$$ - \int_{\Omega} [F(\hat{w}_{\epsilon,P_\epsilon} + \epsilon^2 \phi_\epsilon) - F(\hat{w}_{\epsilon,P_\epsilon}) - \epsilon^2 f(\hat{w}_{\epsilon,P_\epsilon}) \phi_\epsilon - \frac{\epsilon^2}{2} \hat{f}(\hat{w}_{\epsilon,P_\epsilon}) \phi_\epsilon^2] \, dx. $$

By Lemma 4, the last two terms are $o(\epsilon^{N+2})$. Now integrating by parts and using (2.1) we obtain

$$ \epsilon^2 \int_{\Omega} (\epsilon^2 \nabla \hat{w}_{\epsilon,P_\epsilon} \nabla \phi_\epsilon + \hat{w}_{\epsilon,P_\epsilon} \phi_\epsilon - f(\hat{w}_{\epsilon,P_\epsilon}) \phi_\epsilon) \, dx = \epsilon^2 \int_{\Omega} S_\epsilon[\hat{w}_{\epsilon,P_\epsilon}] \phi_\epsilon \, dx = O(\epsilon^{N+1+\tau+\sigma}). $$

Hence $J_\epsilon[u_\epsilon] = J_\epsilon[\hat{w}_{\epsilon,P_\epsilon}] + o(\epsilon^{N+2})$ which, by Lemma 3, finishes the proof of Theorem 1.

Next, we prove Corollary 2.

Proof of Corollary 2: Let $u_\epsilon$ be a least energy solution of (1.1). By Theorem 1, we have

$$ c_\epsilon := J_\epsilon[u_\epsilon] = \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 (c_2 H(P_\epsilon))^2 + c_3 R(P_\epsilon) \right] + o(\epsilon^2). $$

On the other hand, by using $\hat{w}_{\epsilon,Q}$ as test function, we see that

$$ c_\epsilon \leq \epsilon^N \left[ \frac{1}{2} I[w] - c_1 \epsilon H(Q) + \epsilon^2 (c_2 H(Q))^2 + c_3 R(Q) \right] + o(\epsilon^2). $$


where we take $Q$ such that $H(Q) = \max_{P \in \partial \Omega} H(P)$. Comparing (3.1) with (3.2), we arrive at
\[
c_1 H(Q) - \epsilon (c_2 H(Q)^2 + c_3 R(Q)) + o(\epsilon) \leq c_1 H(P_\epsilon) - \epsilon (c_2 H(P_\epsilon)^2 + c_3 R(P_\epsilon)) + o(\epsilon).
\]
Since $c_1 > 0, c_3 > 0$, we obtain (1.5).

References


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