Probability-Guaranteed H_∞ Finite-Horizon Filtering for A Class of Nonlinear Time-Varying Systems with Sensor Saturations

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Abstract

In this paper, the probability-guaranteed H_∞ finite-horizon filtering problem is investigated for a class of nonlinear time-varying systems with uncertain parameters and sensor saturations. The system matrices are functions of mutually independent stochastic variables that obey uniform distributions over known finite ranges. Attention is focused on the construction of a time-varying filter such that the prescribed H_∞ performance requirement can be guaranteed with probability constraint. By using the difference linear matrix inequalities (DLMIs) approach, sufficient conditions are established to guarantee the desired performance of the designed finite-horizon filter. The time-varying filter gains can be obtained in terms of the feasible solutions of a set of DLMIs that can be recursively solved by using the semi-definite programming method. A computational algorithm is specifically developed for the addressed probability-guaranteed H_∞ finite-horizon filtering problem. Finally, a simulation example is given to illustrate the effectiveness of the proposed filtering scheme.

Keywords

Discrete time-varying systems, probability performance, finite-horizon, H_∞ filtering, sensor saturation.

I. Introduction

Due to its clear engineering significance, the filtering problem has attracted a great deal of research attention in the past few decades. The filtering theory that has been successfully applied in many branches of engineering systems such as target tracking, mobile robot localization, and computer vision. A rich body of literature has appeared on the general filtering problem with a variety of performance requirements, see e.g. [1, 6, 8, 9, 11, 14, 16, 22–24]. It is well known that sensors may not always produce signals of unlimited amplitude due mainly to the physical constraints or technological restrictions. The sensor saturation, if not properly handled, will inevitably affect the implementation precision of the designed filtering/control algorithms and may even cause undesirable degradation of the filter/controller performance. Consequently, the sensor saturation problem has been gaining an increasing research interest, see e.g. [5, 12, 15, 20]. It is worth mentioning that, because of the mathematical complexity, most existing results concerning the sensor saturations have been concerned with *time-invariant systems over the infinite-horizon*. Unfortunately, in reality, almost all real-time systems should

This work was supported in part by the National Natural Science Foundation of China under Grants 61028008, 60825303 and 60834003, National 973 Project under Grant 2009CB320600, the Fok Ying Tung Education Fund under Grant 111064, the Special Fund for the Author of National Excellent Doctoral Dissertation of China under Grant 2007B4, the Key Laboratory of Integrated Automation for the Process Industry (Northeastern University) from the Ministry of Education of China, the Engineering and Physical Sciences Research Council (EPSRC) of the U.K. under Grant GR/S27658/01, the Royal Society of the U.K., and the Alexander von Humboldt Foundation of Germany.

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be time-varying especially those after digital discretization. Recently, motivated by the practical importance of the sensor saturation issues, the set-membership filtering problem has been investigated in [20] for a class of time-varying systems with saturated sensors.

In traditional control theory, the performance objectives of a controlled system are usually required to be met accurately. However, for many stochastic control problems, due to a variety of unpredictable disturbances, it is neither possible nor necessary to enforce the system performance with probability 1. Instead, it is quite common for practical control systems to attain their individual performance objective with certain satisfactory probability. These kinds of engineering problems have given rise to great challenges for the realization of multiple control objectives with respect to individual probability constraints. In particular, as a newly emerged research topic, the probability-guaranteed H_∞ controller design problem has been raised in [18] and then thoroughly investigated in [2–4, 19] in an elegant way. Despite the advances made on the research topic of probability-guaranteed design, there is still much room for further investigation on more comprehensive systems in order to cover more engineering practice. For example, in reality, most engineering systems are nonlinear and time-varying with saturated sensors, where the performances are usually evaluated over a finite-horizon for time-varying systems. It is, therefore, the purpose of this paper to address the probability-guaranteed H_∞ finite-horizon filtering problem for nonlinear time-varying systems with sensor saturation so as to complement the excellent results in [2–4, 19].

Motivated by the above discussion, in this paper, we aim to investigate the probability-guaranteed H_∞ finite-horizon filtering problem for a class of nonlinear discrete time-varying systems with sensor saturations. The considered uncertain parameters are governed by mutually independent stochastic variables that abide by uniform distributions over the known finite ranges. A parameter-box is sought for designing the time-varying H_∞ filter such that the H_∞ performance requirement is guaranteed with pre-specified probability constraints. A computational algorithm is presented to characterize the solution to the finite-horizon filtering problem based on the semi-definite programming method. A simulation example is given to show the effectiveness of the filtering scheme. The main contributions of this paper can be highlighted as follows: 1) the system model addressed is quite comprehensive that covers uncertain parameters, nonlinearities as well as sensor saturations, thereby better reflecting the reality; 2) the filtering problem addressed is dealt with over a finite-horizon with probability performance constraint; and 3) the algorithm developed is of recursive nature that is suitable for online applications.

The remainder of this paper is arranged as follows. Section II briefly introduces the problem under consideration. In Section III, the probabilistic performance requirement is expressed as a set of linear matrix inequalities (LMIs) and the H_∞ performance analysis is conducted by means of solving a set of difference linear matrix inequalities (DLMIs) [7, 13]. Moreover, a computational algorithm is presented to characterize the design of the probability-guaranteed robust H_∞ finite-horizon filter. An illustrative example is utilized in Section IV to show the effectiveness of the proposed approach. The paper is concluded in Section V.

Notations. The notations used throughout the paper are standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. For a matrix P, P^T and P^{-1} represent its transpose and inverse, respectively. The notation $P > 0$ $(P \geq 0)$ means that matrix P is real, symmetric and positive definite (positive semi-definite). $\mathrm{Prob}\{\cdot\}$ is used for the occurrence probability of the event "$\cdot$". $\|\cdot\|$ denotes the Euclidean norm of a vector. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. $l_2[0, N-1]$ is the space of square summable vector-value functions on an interval $[0, N-1]$ with the norm $\|\nu\|_{[0,N-1]} = \sqrt{\sum_{k=0}^{N-1} \|\nu(k)\|^2}$. I and 0 represent the identity matrix and the zero matrix with appropriate dimensions, respectively. In symmetric block matrices or long matrix expressions, we use a star "$*$" to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminaries

In this paper, we consider the following class of nonlinear uncertain time-varying systems defined on $k \in \{0, 1, \dots, N-1\}$:

$$\begin{cases} x(k+1) = A^{(\alpha)}(k)x(k) + B^{(\alpha)}(k)f(x(k)) + D^{(\alpha)}(k)\omega(k) \\ \quad y(k) = \sigma(C(k)x(k)) + E(k)\omega(k) \\ \quad z(k) = M(k)x(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^m$ is the measured output, $z(k) \in \mathbb{R}^r$ is the output vector to be estimated, $\omega(k) \in \mathbb{R}^p$ is the disturbance input belonging to $l_2[0, N-1]$, $f(x(k))$ is the nonlinear function, the initial state $x(0)$ is an unknown vector, $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_L \end{bmatrix}^T \in \mathbb{R}^L$ is the uncertain parameter vector, and all α_i $(i = 1, 2, \dots, L)$ are assumed to be mutually independent random variables. Each α_i is uniformly distributed over $[\beta_i, \delta_i]$ where β_i and δ_i are known endpoints of α_i $(i = 1, 2, \dots, L)$. The uncertain parameter vector α lies in an L-dimensional hyper-rectangle $\mathbf{B}$ with the vertices set denoted by

$$V_{\mathbf{B}} = \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_L \end{bmatrix}^T \middle| \ \alpha_i \in \{\beta_i, \delta_i\}, \ \ i = 1, 2, \dots, L \right\}. \tag{2}$$

Following [19], the uncertain matrices $A^{(\alpha)}(k)$, $B^{(\alpha)}(k)$ and $D^{(\alpha)}(k)$ in (1) are described by

$$A^{(\alpha)}(k) = A_0(k) + \sum_{i=1}^{L} \alpha_i A_i(k), \ B^{(\alpha)}(k) = B_0(k) + \sum_{i=1}^{L} \alpha_i B_i(k), \ C^{(\alpha)}(k) = C_0(k) + \sum_{i=1}^{L} \alpha_i C_i(k). \tag{3}$$

For a given sampling instant k, $A_i(k)$, $B_i(k)$, $D_i(k)$ $(i = 1, 2, \dots, L)$, $C(k)$, $E(k)$ and $M(k)$ are known constant matrices with appropriate dimensions. Accordingly, the uncertain matrices in (1) belong to the following general convex polytope

$$\Omega \triangleq \left\{ (A^{(\alpha)}(k), B^{(\alpha)}(k), D^{(\alpha)}(k)) \middle| (A^{(\alpha)}(k), B^{(\alpha)}(k), D^{(\alpha)}(k)) = \sum_{j=1}^{2^L} f_j \Omega^{(j)}(k), \ \ 0 \le f_j \le 1, \ \sum_{j=1}^{2^L} f_j = 1 \right\} \tag{4}$$

where $\Omega^{(j)}(k) = (A^{(j)}(k), B^{(j)}(k), D^{(j)}(k))$ $(j = 1, 2, \cdots, 2^L)$ are the vertex matrices. The relation between $\Omega^{(j)}(k)$ and $V_{\mathbf{B}}$ is given as follows:

$$A^{(j)}(k) = A(v_{\mathbf{B}}^{(j)}, k) = A(\alpha^{(j)}, k)$$

where $v_{\mathbf{B}}^{(j)}$ is the jth vertex of $\mathbf{B}$ generated by the parameter vector $\alpha^{(j)} = \begin{bmatrix} \alpha_1^{(j)} & \alpha_2^{(j)} & \dots & \alpha_L^{(j)} \end{bmatrix}^T$, $\alpha_i^{(j)} \in \{\beta_i, \delta_i\}$ $(i = 1, 2, \dots, L; j = 1, 2, \dots, 2^L)$. $B^{(j)}(k)$ and $D^{(j)}(k)$ have the similar expressions.

The saturation function $\sigma(\cdot)$ is defined as

$$\sigma(v) = \begin{bmatrix} \sigma_1(v_1) & \sigma_2(v_2) & \cdots & \sigma_m(v_m) \end{bmatrix}^T \tag{5}$$

with $\sigma_i(v_i) = \text{sign}(v_i) \min\{v_{i,\max}, |v_i|\}$, where $v_{i,\max}$ is the i-th element of the vector $v_{\max}$ with $v_{\max}$ being the saturation level.

To facilitate our development, we introduce the following definition.

Definition 1: [10] A nonlinearity $\Phi(\cdot)$ is said to satisfy the sector-bounded condition if

$$(\Phi(v) - V_1 v)^T(\Phi(v) - V_2 v) \leq 0 \tag{6}$$

for some real matrices V_1, V_2, where $V = V_2 - V_1$ is a symmetric positive-definite matrix. In this case, we say that $\Phi(\cdot)$ belongs to the sector $[V_1, V_2]$.

Assumption 1: The nonlinear function $f(x(k))$ in (1) belongs to the sector $[U_1(k), U_2(k)]$ where $U_1(k)$ and $U_1(k)$ are real matrices of appropriate dimensions.

Noting that there exist the diagonal matrices K_1 and K_2 such that $0 \leq K_1 < I \leq K_2$, the saturation function $\sigma(C(k)x(k))$ in (1) can be written as

$$\sigma(C(k)x(k)) = K_1 C(k)x(k) + \Phi_y(C(k)x(k)) \tag{7}$$

where $\Phi_y(C(k)x(k))$ is a nonlinear vector-valued function satisfying the sector-bounded condition with $V_1 = 0$ and $V_2 = K$. In this case, $\Phi_y(C(k)x(k))$ can be described as follows

$$\Phi_y^T(C(k)x(k))(\Phi_y(C(k)x(k)) - KC(k)x(k)) \leq 0 \tag{8}$$

where $K = K_2 - K_1$.

Remark 1: In (1), the state matrix $A^{(\alpha)}(k)$ includes the uncertainty induced by the parameter α distributed uniformly over a given interval, and the term $B^{(\alpha)}(k)f(x(k))$ involves both the uncertainties and the nonlinearities. In fact, in system modeling, it is usually the case that the system consists of both the linear part (e.g. $Ax(k)$) and the nonlinear part (e.g. $Bf(x(k))$) according to either the physical law or the linearization process. The coefficients A and B, however, might be inaccurate due to the unavoidable modeling error, and this leads to the possible parameter drifts obeying a uniform distribution law. In other words, some system parameters might be randomly perturbed within certain intervals due probably to the abrupt phenomena such as random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, modification of the operating point of a linearized model of nonlinear systems, etc. Such a kind of stochastic parameter systems can find many applications such as radar control, missile track estimation, satellite navigation, and digital control of chemical processes, see e.g. [21].

Remark 2: It is worth mentioning that the sensor saturation introduced in (1) reflects the reality more closely and, in turn, gives rise to additional difficulties in the design of probability-guaranteed H_∞ filters over a finite-time horizon. By using the sector-bounded approach developed in [17,20], a decomposition technique is utilized in (7) to facilitate the filter design in terms of DLMIs. In this case, the sector $[K_1, K_2]$ is employed to quantify the saturation-type nonlinearity. It will be shown later that the decomposition in (7) plays an important role in the development of the main results.

In this paper, the following time-varying filter is adopted for system (1):

$$\begin{cases} \hat{x}(k+1) = A_f(k)\hat{x}(k) + B_f(k)y(k) \\ \hat{z}(k) = M(k)\hat{x}(k), \quad \hat{x}(0) = \hat{x}_0 \end{cases} \tag{9}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the state estimate, $\hat{z}(k) \in \mathbb{R}^r$ is the estimated output, and $A_f(k)$, $B_f(k)$ $(0 \leq k \leq N-1)$ are filter parameters to be determined.

By defining $\eta(k) = \begin{bmatrix} x^T(k) & \hat{x}^T(k) \end{bmatrix}^T$ and letting filtering error be $\tilde{z}(k) = z(k) - \hat{z}(k)$, we have the following augmented system:

$$\begin{cases} \eta(k+1) = \bar{A}^{(\alpha)}(k)\eta(k) + \bar{B}^{(\alpha)}(k)f(H\eta(k), k) + \tilde{B}(k)\Phi_y(C(k)H\eta(k)) + \bar{D}^{(\alpha)}(k)\omega(k) \\ \tilde{z}(k) = \bar{M}(k)\eta(k) \end{cases} \tag{10}$$

where

$$\bar{A}^{(\alpha)}(k) = \begin{bmatrix} A^{(\alpha)}(k) & 0 \\ B_f(k)K_1C(k) & A_f(k) \end{bmatrix}, \quad \bar{B}^{(\alpha)}(k) = \begin{bmatrix} B^{(\alpha)}(k) \\ 0 \end{bmatrix}, \quad \bar{D}^{(\alpha)}(k) = \begin{bmatrix} D^{(\alpha)}(k) \\ B_f(k)E(k) \end{bmatrix},$$

$$\tilde{B}(k) = \begin{bmatrix} 0 \\ B_f(k) \end{bmatrix}, \quad H = \begin{bmatrix} I & 0 \end{bmatrix}, \quad \bar{M}(k) = \begin{bmatrix} M(k) & -M(k) \end{bmatrix}. \tag{11}$$

We are now in a position to formulate the probabilistic robust H_∞ *finite-horizon* filtering problem.

The Probabilistic Robust H_∞ Finite-Horizon Filtering Problem: For a given probability $0 < p < 1$, a specified disturbance attenuation level $\gamma > 0$ and a specified $R > 0$, our aim is to find a time-varying filter of the structure (9) satisfying

$$\text{Prob}\{J \leq 0\} \geq p \tag{12}$$

where

$$J := \|\tilde{z}\|^2_{[0,N-1]} - \gamma^2 \left(\|\omega\|^2_{[0,N-1]} + x^T(0)Rx(0) \right).$$

More specifically, we are interested in looking for the filter parameter matrices $A_f(k)$ and $B_f(k)$ in (9), and finding a parameter-box $\mathbf{B_T}$ ($\mathbf{B_T} \subseteq \mathbf{B}$) such that the following requirements are met simultaneously:

R1) the probability of $\alpha \in \mathbf{B_T}$ is not less than p,

R2) the H_∞ performance requirement $J \leq 0$ can be guaranteed in the parameter-box $\mathbf{B_T}$, where the parameter-box $\mathbf{B_T}$ is generated by $\alpha_i \in [a_i, b_i] \subseteq [\beta_i, \delta_i]$ ($i = 1, 2, \ldots, L$), and the set of the 2^L vertices $v_{\mathbf{B_T}}$ of $\mathbf{B_T}$ is given by

$$V_{\mathbf{B_T}} := \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_L \end{bmatrix}^T \,\middle|\, \alpha_i \in \{a_i, b_i\}, i = 1, 2, \cdots, L \right\}. \tag{13}$$

Remark 3: It should be pointed out that the requirements *R1* and *R2* are interconnected by means of the parameter-box $\mathbf{B_T}$. Accordingly, the endpoints a_i and b_i should be first determined for all α_i ($i = 1, 2, \cdots, L$), and then the vertices $v_{\mathbf{B_T}}$ of the parameter-box $\mathbf{B_T}$ in (13) can be obtained. Actually, the essential relationship between the requirements *R1* and *R2* is reflected by the L pairs $\{a_i, b_i\}$ ($i = 1, 2, \ldots, L$) of the parameter-box $\mathbf{B_T}$ in the requirement *R1*, which correspond to the vertices $A^{(j)}(k)$, $B^{(j)}(k)$ and $D^{(j)}(k)$ of the H_∞ performance requirement in *R2*.

III. Main Results

In this section, we deal with the probabilistic robust H_∞ filtering problem for the nonlinear time-varying system with a given disturbance attenuation level γ and a prescribed probability constraint p over a finite-time horizon. A sufficient condition is derived by using the DLMI approach such that the H_∞ performance requirement with probability constraint for system (10) can be guaranteed. Then, a computational algorithm is proposed to characterize the solution of the time-varying filter which can be readily obtained in a recursive way.

Let us first discuss the probability issue for requirement *R1*. Noting that all α_i are assumed to uniformly distribute over $[\beta_i, \delta_i]$ and they are mutually independent, as discussed in [4,19], the probability constraint of $\alpha \in \mathbf{B_T}$ can be expressed by

$$\prod_{i=1}^{L} (b_i - a_i) \geq \bar{p} \tag{14}$$

where $\bar{p} = p\prod_{i=1}^{L}(\delta_i - \beta_i)$ and the endpoints a_i, b_i $(i = 1, 2, \cdots, L)$ are the parameters to be determined which are associated with the parameter-box $\mathbf{B_T}$ in (13). Based on the algorithm presented in [19], by successively using the following Lemma, the probability constraint *R1* can be converted into another form that is easier to be handled.

Lemma 1: [19] Let the probability constraint $p > 0$ be given. The inequality (14) is equivalent to

$$\prod_{j=1}^{m_1} s_{1,j} \geq \sqrt{\bar{p}} \tag{15}$$

where $s_{1,j}$ $(j = 1, 2, \ldots, m_1)$ are the positive scalars to be determined. When L is even, we let $m_1 = \frac{L}{2}$ and then have

$$\begin{bmatrix} b_{2j-1} - a_{2j-1} & s_{1,j} \\ * & b_{2j} - a_{2j} \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, m_1. \tag{16}$$

When L is odd, we set $m_1 = \frac{L-1}{2} + 1$ and it follows that (16) holds for $j = 1, 2, \ldots, m_1 - 1$ and

$$\begin{bmatrix} b_L - a_L & s_{1,m_1} \\ * & 1 \end{bmatrix} \geq 0. \tag{17}$$

Remark 4: Both the conditions (16) and (17) in Lemma 1 are related to the H_∞ performance requirement due to the effect of the uncertain parameters α_i $(i = 1, 2, \ldots, L)$. In the implementation, the conditions (16) and (17) should be concurrently solved with the H_∞ performance requirement. Then, the endpoints a_i and b_i for all α_i $(i = 1, 2, \ldots, L)$ can be determined and the filter parameter matrices $\{A_f(k)\}_{0\leq k\leq N-1}$ and $\{B_f(k)\}_{0\leq k\leq N-1}$ can be designed.

To this end, we introduce the following lemma that will be used in deriving our main results.

Lemma 2: (S-Procedure) Let $W_0(x)$, $W_1(x)$, $\ldots$, $W_l(x)$ be quadratic functions of $x \in \mathbb{R}^n$, i.e. $W_i(x) = x^T Q_i x$ with $Q_i = Q_i^T$ $(i = 0, 1, \ldots, l)$. If there exist scalars $\tau_1 \geq 0$, $\tau_2 \geq 0$, $\ldots$, $\tau_l \geq 0$ such that

$$Q_0 - \sum_{i=1}^{l} \tau_i Q_i \leq 0, \tag{18}$$

then the following implication is true

$$W_1(x) \leq 0, W_2(x) \leq 0, \ldots, W_l(x) \leq 0 \Longrightarrow W_0(x) \leq 0. \tag{19}$$

A. H_∞ Performance Analysis

Having analyzed the requirement *R1*, we are ready to deal with the requirement *R2*. The following theorems present sufficient conditions under which the augmented system governed by (10) satisfies the H_∞ performance requirement.

Theorem 1: Let the disturbance attenuation level $\gamma > 0$, the probability constraint $p > 0$, the initial matrix $R > 0$ and the filter parameters $\{A_f(k)\}_{0\leq k\leq N-1}$, $\{B_f(k)\}_{0\leq k\leq N-1}$ be given. The H_∞ performance requirement $J \leq 0$ holds if there exist two families of positive scalars $\{\tau_i(k)\}_{0\leq k\leq N-1}$ $(i = 1, 2)$ and a family of positive definite matrices $\{P(k)\}_{0\leq k\leq N}$ satisfying the initial condition

$$\eta^T(0)P(0)\eta(0) \leq \gamma^2 \eta^T(0)\hat{R}\eta(0) \tag{20}$$

and the matrix inequality

$$\Xi(k)=\begin{bmatrix}\Xi_{11}(k) & \Xi_{12}(k) & \Xi_{13}(k) & \bar{A}^{(\alpha)T}(k)P(k+1)\bar{D}^{(\alpha)}(k)\\ * & \Xi_{22}(k) & \bar{B}^{(\alpha)T}(k)P(k+1)\tilde{B}(k) & \bar{B}^{(\alpha)T}(k)P(k+1)\bar{D}^{(\alpha)}(k)\\ * & * & \tilde{B}^{T}(k)P(k+1)\tilde{B}(k)-\tau_2(k)I & \tilde{B}^{T}(k)P(k+1)\bar{D}^{(\alpha)}(k)\\ * & * & * & \bar{D}^{(\alpha)T}(k)P(k+1)\bar{D}^{(\alpha)}(k)-\gamma^2 I\end{bmatrix}\leq 0 \qquad (21)$$

where

$$\begin{aligned}\Xi_{11}(k) &= \bar{A}^{(\alpha)T}(k)P(k+1)\bar{A}^{(\alpha)}(k)-P(k)+\bar{M}^{T}(k)\bar{M}(k)-\tau_1(k)\frac{\bar{U}_1^T(k)\bar{U}_2(k)+\bar{U}_2^T(k)\bar{U}_1(k)}{2},\\ \Xi_{12}(k) &= \bar{A}^{(\alpha)T}(k)P(k+1)\bar{B}^{(\alpha)}(k)+\tau_1(k)\frac{\bar{U}_1^T(k)+\bar{U}_2^T(k)}{2},\\ \Xi_{13}(k) &= \bar{A}^{(\alpha)T}(k)P(k+1)\tilde{B}(k)+\tau_2(k)\frac{H^TC^T(k)K^T}{2},\\ \Xi_{22}(k) &= \bar{B}^{(\alpha)T}(k)P(k+1)\bar{B}^{(\alpha)}(k)-\tau_1(k)I,\\ \bar{U}_1(k) &= \begin{bmatrix} U_1(k) & 0\end{bmatrix},\ \ \bar{U}_2(k)=\begin{bmatrix} U_2(k) & 0\end{bmatrix},\ \ \hat{R}=\mathrm{diag}\{R,0\}. \qquad (22)\end{aligned}$$

Proof: By defining

$$J(k)=\eta^T(k+1)P(k+1)\eta(k+1)-\eta^T(k)P(k)\eta(k), \qquad (23)$$

it can be obtained from (10) that

$$J(k) = \xi^T(k)\Upsilon^T(k)P(k+1)\Upsilon(k)\xi(k)-\eta^T(k)P(k)\eta(k) \qquad (24)$$

where

$$\begin{aligned}\xi(k) &= \begin{bmatrix}\eta^T(k) & f^T(H\eta(k),k) & \Phi_y^T(C(k)H\eta(k)) & \omega^T(k)\end{bmatrix}^T,\\ \Upsilon(k) &= \begin{bmatrix}\bar{A}^{(\alpha)}(k) & \bar{B}^{(\alpha)}(k) & \tilde{B}(k) & \bar{D}^{(\alpha)}(k)\end{bmatrix}.\end{aligned}$$

Adding the zero term $\tilde{z}^T(k)\tilde{z}(k)-\gamma^2\omega^T(k)\omega(k)-\tilde{z}^T(k)\tilde{z}(k)+\gamma^2\omega^T(k)\omega(k)$ to the right side of (24) yields

$$J(k) = \xi^T(k)\Theta(k)\xi(k)-\tilde{z}^T(k)\tilde{z}(k)+\gamma^2\omega^T(k)\omega(k) \qquad (25)$$

where

$$\begin{aligned}\Theta(k) &= \begin{bmatrix}\Theta_{11}(k) & \Theta_{12}(k) & \bar{A}^{(\alpha)T}(k)P(k+1)\tilde{B}(k) & \bar{A}^{(\alpha)T}(k)P(k+1)\bar{D}^{(\alpha)}(k)\\ * & \Theta_{22}(k) & \bar{B}^{(\alpha)T}(k)P(k+1)\tilde{B}(k) & \bar{B}^{(\alpha)T}(k)P(k+1)\bar{D}^{(\alpha)}(k)\\ * & * & \tilde{B}^{T}(k)P(k+1)\tilde{B}(k) & \tilde{B}^{T}(k)P(k+1)\bar{D}^{(\alpha)}(k)\\ * & * & * & \bar{D}^{(\alpha)T}(k)P(k+1)\bar{D}^{(\alpha)}(k)-\gamma^2 I\end{bmatrix},\\ \Theta_{11}(k) &= \bar{A}^{(\alpha)T}(k)P(k+1)\bar{A}^{(\alpha)}(k)-P(k)+\bar{M}^T(k)\bar{M}(k),\\ \Theta_{12}(k) &= \bar{A}^{(\alpha)T}(k)P(k+1)\bar{B}^{(\alpha)}(k),\\ \Theta_{22}(k) &= \bar{B}^{(\alpha)T}(k)P(k+1)\bar{B}^{(\alpha)}(k).\end{aligned}$$

Subsequently, summing up (25) on both sides from 0 to $N-1$ with respect to k leads to

$$\begin{aligned}\sum_{k=0}^{N-1}J(k) &= \eta^T(N)P(N)\eta(N)-\eta^T(0)P(0)\eta(0)\\ &= \sum_{k=0}^{N-1}\xi^T(k)\Theta(k)\xi(k)-\sum_{k=0}^{N-1}[\tilde{z}^T(k)\tilde{z}(k)-\gamma^2\omega^T(k)\omega(k)] \qquad (26)\end{aligned}$$

and therefore

$$\begin{aligned} J &= \sum_{k=0}^{N-1}[\tilde{z}^T(k)\tilde{z}(k) - \gamma^2\omega^T(k)\omega(k)] - \gamma^2 x^T(0)Rx(0) \\ &= \sum_{k=0}^{N-1}\xi^T(k)\Theta(k)\xi(k) - \eta^T(N)P(N)\eta(N) + \eta^T(0)(P(0) - \gamma^2\hat{R})\eta(0). \end{aligned} \tag{27}$$

Noting $P(N) > 0$ and the initial condition (20), we know that $J \leq 0$ is true if the following inequality

$$\xi^T(k)\Theta(k)\xi(k) \leq 0 \tag{28}$$

holds. On the other hand, it follows from the sector-bounded condition of the nonlinear function $f(x(k))$ that

$$\xi^T(k)\Phi_f(k)\xi(k) \leq 0 \tag{29}$$

where

$$\Phi_f(k) = \begin{bmatrix} \frac{\bar{U}_1^T(k)\bar{U}_2(k)+\bar{U}_2^T(k)\bar{U}_1(k)}{2} & -\frac{\bar{U}_1^T(k)+\bar{U}_2^T(k)}{2} & 0 & 0 \\ * & I & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

with $\bar{U}_1(k)$ and $\bar{U}_2(k)$ defined in (22).

Similarly, considering the sensor saturation constraint (8), we have

$$\xi^T(k)\Psi_y(k)\xi(k) \leq 0 \tag{30}$$

where

$$\Psi_y(k) = \begin{bmatrix} 0 & 0 & -\frac{1}{2}H^TC^T(k)K^T & 0 \\ * & 0 & 0 & 0 \\ * & * & I & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Let us now prove that, with the conditions (29)-(30), (28) is true. For this purpose, we rewrite inequality (21) into the following form:

$$\Theta(k) - \tau_1(k)\Phi_f(k) - \tau_2(k)\Psi_y(k) \leq 0. \tag{31}$$

By applying Lemma 2, (28) follows from (29)-(30) immediately. The proof of this theorem is now complete. ■

After the H_∞ performance analysis conducted in Theorem 1, we proceed to address the design problem of the finite-horizon H_∞ filter for the time-varying system with sensor saturation by employing the DLMI approach.

Theorem 2: Let the disturbance attenuation level $\gamma > 0$, the probability constraint $p > 0$ and the initial matrix $R > 0$ be given. The H_∞ performance requirement $J \leq 0$ holds if there exist two families of positive scalars $\{\tau_i(k)\}_{0 \leq k \leq N-1}$ $(i = 1, 2)$, positive definite matrix $P(0) = \mathrm{diag}\{P_1(0), P_2(0)\} > 0$, families of positive definite matrices $\{Q_1(k)\}_{1\le k\le N}$ and $\{Q_2(k)\}_{1\le k\le N}$, and families of real-valued matrices $\{A_f(k)\}_{0\le k\le N-1}$ and $\{B_f(k)\}_{0\le k\le N-1}$ satisfying the following initial condition

$$\eta^T(0)P(0)\eta(0) \le \gamma^2\eta^T(0)\hat{R}\eta(0) \tag{32}$$

and the recursive linear matrix inequalities

$$\Omega(k) = \begin{bmatrix} \Omega_{11}(k) & -M^T(k)M(k) & \Omega_{13}(k) & \Omega_{14}(k) & 0 & A^{(j)T}(k) & C^T(k)K_1^T B_f^T(k) \\ * & \Omega_{22}(k) & 0 & 0 & 0 & 0 & A_f^T(k) \\ * & * & -\tau_1(k)I & 0 & 0 & B^{(j)T}(k) & 0 \\ * & * & * & -\tau_2(k)I & 0 & 0 & B_f^T(k) \\ * & * & * & * & -\gamma^2 I & D^{(j)T}(k) & E^T(k)B_f^T(k) \\ * & * & * & * & * & -Q_1(k+1) & 0 \\ * & * & * & * & * & * & -Q_2(k+1) \end{bmatrix} \le 0,$$
$$j = 1, 2, \ldots, 2^L \tag{33}$$

where

$$\begin{aligned} \Omega_{11}(k) &= -P_1(k) + M^T(k)M(k) - \tau_1(k)\frac{U_1^T(k)U_2(k) + U_2^T(k)U_1(k)}{2}, \\ \Omega_{13}(k) &= \tau_1(k)\frac{U_1^T(k) + U_2^T(k)}{2}, \\ \Omega_{14}(k) &= \tau_2(k)\frac{C^T(k)K^T}{2}, \\ \Omega_{22}(k) &= -P_2(k) + M^T(k)M(k), \end{aligned}$$

and the parameters are updated by

$$P_1(k+1) = Q_1^{-1}(k+1), \quad P_2(k+1) = Q_2^{-1}(k+1). \tag{34}$$

Here, $\hat{R}$ is defined in (22), $A^{(j)}(k)$, $B^{(j)}(k)$ and $D^{(j)}(k)$ are the j-th vertex matrices of the polytope Ω corresponding to the parameter-box $\mathbf{B_T}$.

Proof: Considering (4) and replacing $\mathbf{B}$ with $\mathbf{B_T}$, we can see that (21) is true if the following matrix inequalities hold

$$\hat{\Xi}^{(j)}(k) = \begin{bmatrix} \hat{\Xi}_{11}(k) & \hat{\Xi}_{12}(k) & \hat{\Xi}_{13}(k) & 0 & \bar{A}^{(j)T}(k) \\ * & -\tau_1(k)I & 0 & 0 & \bar{B}^{(j)T}(k) \\ * & * & -\tau_2(k)I & 0 & \tilde{B}^T(k) \\ * & * & * & -\gamma^2 I & \bar{D}^{(j)T}(k) \\ * & * & * & * & -P^{-1}(k+1) \end{bmatrix} \le 0, \quad j = 1, 2, \ldots, 2^L \tag{35}$$

where

$$\begin{aligned} \hat{\Xi}_{11}(k) &= -P(k) + \bar{M}^T(k)\bar{M}(k) - \tau_1(k)\frac{\bar{U}_1^T(k)\bar{U}_2(k) + \bar{U}_2^T(k)\bar{U}_1(k)}{2}, \\ \hat{\Xi}_{12}(k) &= \tau_1(k)\frac{\bar{U}_1^T(k) + \bar{U}_2^T(k)}{2}, \quad \hat{\Xi}_{13}(k) = \tau_2(k)\frac{H^T C^T(k)K^T}{2}, \\ \bar{A}^{(j)}(k) &= \begin{bmatrix} A^{(j)}(k) & 0 \\ B_f(k)K_1C(k) & A_f(k) \end{bmatrix}, \quad \bar{B}^{(j)}(k) = \begin{bmatrix} B^{(j)}(k) \\ 0 \end{bmatrix}, \quad \bar{D}^{(j)}(k) = \begin{bmatrix} D^{(j)}(k) \\ B_f(k)E(k) \end{bmatrix}, \end{aligned}$$

and $\bar{U}_i(k)$ $(i = 1, 2)$ are defined in (22).

Noticing (11) and setting

$$P(k) = \text{diag}\{P_1(k), P_2(k)\}, \quad P^{-1}(k) = \text{diag}\{Q_1(k), Q_2(k)\},$$

it follows that (35) is implied by (33). To this end, the proof of this theorem follows readily from Theorem 1 together with the initial condition (32). ■

Remark 5: In Theorem 2, a sufficient condition is proposed to ensure the existence of the desired filter gains by verifying the feasibility of a set of DLMIs (33). It is worth emphasizing that, as discussed in Remark 3, the vertices $A^{(j)}(k)$, $B^{(j)}(k)$ and $D^{(j)}(k)$ in (33) are unknown that depend on the parameters a_i and b_i $(i = 1, 2, \ldots, L)$. It should be mentioned that, according to (15)-(17) in Lemma 1, the probability constraint p is implicitly reflected in (14)-(17) (in Lemma 1) and (35) (in Theorem 2). More specifically, the determined endpoints a_i and b_i are essentially related to the probability constraint p in (14), which have close connection with the requirements *R1* and *R2*. In this sense, the probability constraint p has been reflected in the main results.

B. Computational Algorithm

According to Lemma 1 and Theorem 2, a recursive algorithm can be given to obtain the time-varying filter matrices $\{A_f(k)\}_{0\le k\le N-1}$ and $\{B_f(k)\}_{0\le k\le N-1}$. Since the inequalities (16), (17) and (33) are linear with respect to all unknown variables, they can be easily solved by using the semi-definite programming method. The following algorithm shows how to design the time-varying filter parameters.

Algorithm 1: (Probability-guaranteed H_∞ finite-horizon filtering recursive algorithm)

Step 1. Set the H_∞ performance index γ, the required probability p, the positive definite matrix R, the initial state $\eta(0)$ and the recursive time N. Select the initial values for matrices $P_1(0)$ and $P_2(0)$ satisfying the initial condition (32). For $k = 0$, solve (16), (17) and (33) to obtain the values of a_i and b_i $(i = 1, 2, \ldots, L)$ for $V_{\mathbf{B_T}}$, $Q_1(1)$ and $Q_2(1)$ as well as the desired filter parameter matrices $A_f(0)$ and $B_f(0)$. Compute $P_1(1)$, $P_2(1)$ by using the parameter update formula (34) and set $k = 1$.

Step 2. Solve (33) at $V_{\mathbf{B_T}}$ to obtain the values of matrices $Q_1(k+1)$, $Q_2(k+1)$ as well as the filter parameter matrices $A_f(k)$, $B_f(k)$.

Step 3. Set $k = k + 1$ and obtain $P_1(k + 1)$, $P_2(k + 1)$ by using the parameter update formula (34).

Step 4. If $k = N$, then stop, else go to *Step 2*.

Remark 6: Based on the recursive algorithm developed above, we can obtain the filter gains $\{A_f(k)\}_{0\le k\le N-1}$ and $\{B_f(k)\}_{0\le k\le N-1}$ step by step at every sampling instant k. The proposed scheme is of a form suitable for recursive computation in online applications. In the case that the algorithm is not feasible in some sampling instant k, we can adjust the initial pre-specified parameters' values and repeat the iterative algorithm. On the other hand, it can be observed from (14) that, the less the probability constraint p, the better the feasibility of (14), and therefore the easier the addressed probability-constrained filter design problem is feasible. Moreover, the bigger the disturbance attenuation level γ, the better the feasibility of (33).

Remark 7: The system (1) under consideration is quite comprehensive that covers uncertain stochastic variables, time-varying nature, nonlinearities, sensor saturations and external disturbances. Furthermore, two performance indices (p and γ) are used for the finite-horizon filter design problems to guarantee that the H_∞-requirement can be achieved with a pre-specified probability. Note that the main results established in Theorem 2 contain all the information of the addressed general systems including the physical parameters, the probability constraint, the H_∞ attenuation level, sector-bounds of the nonlinearities and the amplitudes of the sensor saturations. In the next section, a simulation example is provided to show the usefulness of the proposed time-varying filtering technique.

Remark 8: In this paper, we endeavor to answer the following three questions. 1) How to establish a model that is as comprehensive as possible to reflect the engineering practice? 2) How to evaluate the system performance with *probability* constraint for time-varying system? 3) How to develop an effective yet easy-to-implement algorithm to achieve the main objectives? In the end, the designed time-varying filter gains can be obtained in terms of the feasible solutions of a set of DLMIs that can be recursively solved by using the semi-definite programming method, and a computational algorithm is specifically developed for the addressed probability-guaranteed H_∞ finite-horizon filtering problem.

IV. A Numerical Example

In this section, a simulation example is presented to illustrate the effectiveness of the time-varying filter developed in this paper.

Consider the discrete time-varying nonlinear system (1) with the system parameters as follows:

$$\begin{aligned} A_0(k) &= \begin{bmatrix} -0.6 & 0.38 \\ 0.2\sin(3k) & -0.5 \end{bmatrix}, \ B_0(k) = \begin{bmatrix} 0.05 & -0.28 \\ 0.3\sin(2k) & -0.6 \end{bmatrix}, \ D_0(k) = \begin{bmatrix} 0.2\sin(3k) \\ 0.3 \end{bmatrix}, \\ A_1(k) &= B_1(k) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ D_1(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C(k) = \begin{bmatrix} -1.3 & 2\sin(3k) \end{bmatrix}, \\ M(k) &= \begin{bmatrix} 0.15 & 0.13 \end{bmatrix}, \ E(k) = 0.28. \end{aligned}$$

The sensor saturation function $\sigma(\cdot)$ is described as

$$\begin{cases} \sigma(C(k)x(k)) = V_{yj,\max}, & \text{if } C(k)x(k) > V_{yj,\max}; \\ \sigma(C(k)x(k)) = C(k)x(k), & \text{if } -V_{yj,\max} \le C(k)x(k) \le V_{yj,\max}; \\ \sigma(C(k)x(k)) = -V_{yj,\max}, & \text{if } C(k)x(k) < -V_{yj,\max}. \end{cases} \tag{36}$$

The nonlinear function $f(x(k))$ is chosen as

$$f(x(k)) = \begin{bmatrix} -0.1x_1(k) + 0.15x_2(k) + \frac{0.1x_2(k)\sin(x_1(k))}{\sqrt{x_1^2(k)+x_2^2(k)+10}} \\ -0.05x_1(k) + 0.05x_2(k) \end{bmatrix},$$

which belongs to the sector $[U_1(k), U_2(k)]$ with

$$U_1(k) = \begin{bmatrix} -0.4 & 0 \\ -0.2 & -0.3 \end{bmatrix}, \ U_2(k) = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}.$$

In this example, let $V_{yj,\max} = 0.03$, $K = 1.8$, $K_1 = 0.01$ and $p = 0.90$. The uncertain parameter α obeys the uniform distribution over $[-0.05, 0.15]$. The external disturbance input is selected as $\omega(k) = \frac{0.3\sin(2k)}{k+1}$. Setting $\gamma = 1.2$, $\eta(0) = \begin{bmatrix} -0.22 & 0.08 & 0.04 & -0.12 \end{bmatrix}^T$ and $R = \mathrm{diag}\{1.8, 1.8\}$, we can find the initial matrices $P_1(0) = \mathrm{diag}\{1.2, 1.2\}$ and $P_2(0) = \mathrm{diag}\{0.52, 0.53\}$ satisfying the initial condition (32). The desired filter parameters can be solved recursively according to Algorithm 1, and the results are listed in Table I from $k = 0$ to $k = 6$.

The corresponding simulation results are presented in Figs. 1-6, where the actual states $x_1(k)$, $x_2(k)$ and their estimates $\hat{x}_1(k)$, $\hat{x}_2(k)$ are given in Fig. 1 and Fig. 2, respectively. The output $z(k)$ and its estimate

$\hat{z}(k)$ are plotted in Fig. 3. The estimation error $\tilde{z}(k)$ is shown in Fig. 4. The ideal measurements and actual measurements are depicted in Fig. 5. For the clarity, here, only the case of noise-free sensor is presented. Moreover, the l_2 norms of the estimation error $\tilde{z}(k)$ and the external disturbance $\omega(k)$ can be calculated, respectively. Accordingly, the actual l_2-gain from the external disturbance to the estimation error can be obtained. The actual H_∞ performance is plotted in Fig. 6, which is significantly lower than the given performance level $\gamma = 1.2$. Note that the traditional H_∞ problem can be recovered by setting the probability constraint p as 1. Under the same conditions, the traditional H_∞ problem (i.e., the case when $p = 1$) is infeasible, which further shows the advantage of our algorithm. The simulation results have demonstrated the effectiveness of the time-varying H_∞ filter strategy presented in this paper.

V. Conclusions

In this paper, we have studied the probability-guaranteed robust H_∞ finite-horizon filtering problem for a class of nonlinear time-varying systems with sensor saturation. The uniform distribution has been used to characterize the statistical characteristics of the uncertain parameters. A time-varying filter has been designed and a parameter-box has been sought such that the disturbance attenuation level and the required probability are simultaneously guaranteed. A computational algorithm has been proposed for the design of the robust H_∞ time-varying filter. Finally, the effectiveness of the developed filtering approach has been illustrated by a simulation example.

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TABLE I

FILTER PARAMETERS

k	0	1	2	3
$A_f(k)$	$\begin{bmatrix} -0.2265 & -0.1034 \\ -0.1132 & -0.0517 \end{bmatrix}$	$\begin{bmatrix} -2.7797 & -1.3822 \\ -1.3899 & -0.6911 \end{bmatrix}$	$\begin{bmatrix} -0.2064 & -0.0948 \\ -0.1032 & -0.0474 \end{bmatrix}$	$\begin{bmatrix} -3.1076 & -1.5494 \\ -1.5538 & -0.7747 \end{bmatrix}$
$B_f(k)$	$\begin{bmatrix} 0.0060 \\ 0.0030 \end{bmatrix}$	$\begin{bmatrix} 0.0188 \\ 0.0094 \end{bmatrix}$	$\begin{bmatrix} 0.0004 \\ 0.0002 \end{bmatrix}$	$\begin{bmatrix} 0.0227 \\ 0.0114 \end{bmatrix}$
k	4	5	6	$\cdots$
$A_f(k)$	$\begin{bmatrix} -0.1851 & -0.0853 \\ -0.0925 & -0.0427 \end{bmatrix}$	$\begin{bmatrix} -3.4455 & -1.7175 \\ -1.7228 & -0.8587 \end{bmatrix}$	$\begin{bmatrix} -0.1661 & -0.0761 \\ -0.0831 & -0.0381 \end{bmatrix}$	$\cdots$
$B_f(k)$	$\begin{bmatrix} 0.0024 \\ 0.0012 \end{bmatrix}$	$\begin{bmatrix} 0.0175 \\ 0.0087 \end{bmatrix}$	$\begin{bmatrix} 0.0018 \\ 0.0009 \end{bmatrix}$	$\cdots$

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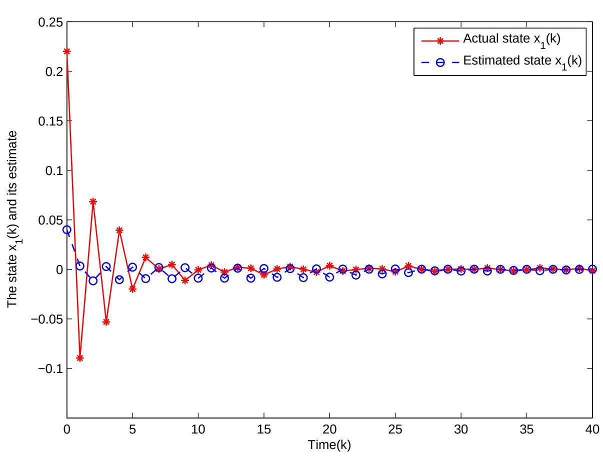

Fig. 1. The state $x_1(k)$ and its estimate $\hat{x}_1(k)$

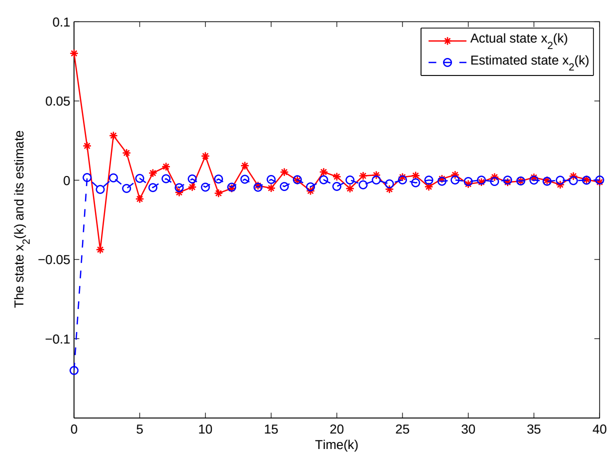

Fig. 2. The state $x_2(k)$ and its estimate $\hat{x}_2(k)$

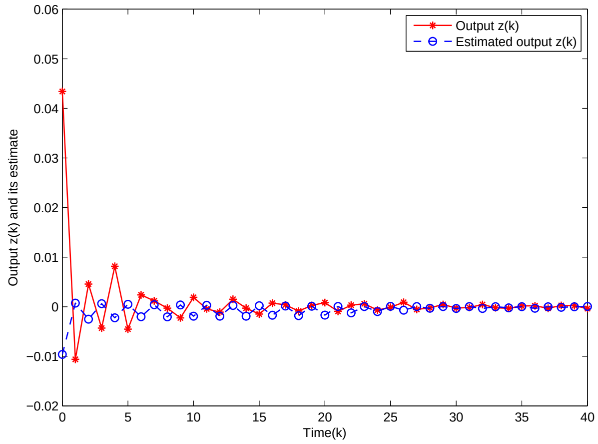

Fig. 3. Output $z(k)$ and its estimate $\hat{z}(k)$

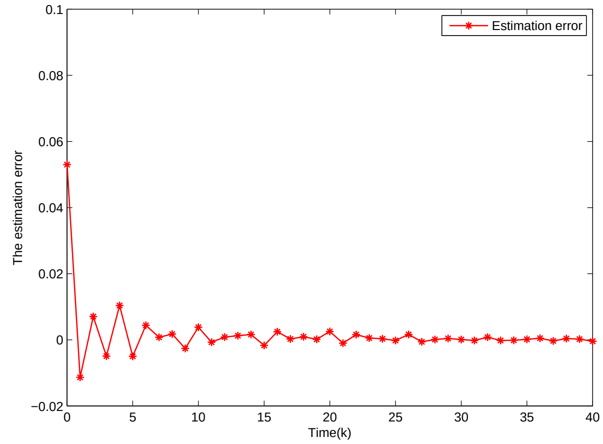

Fig. 4. The estimation error $\tilde{z}(k)$

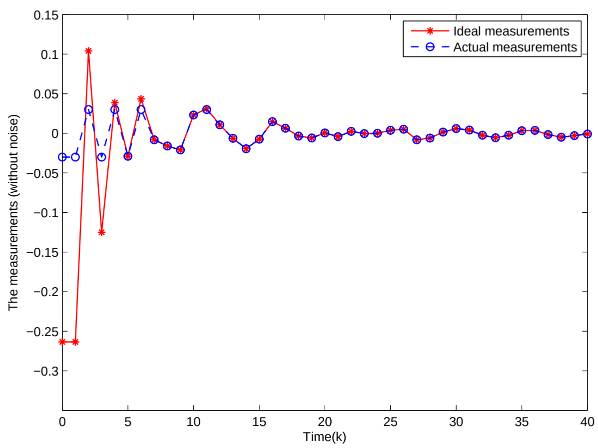

Fig. 5. The measurements (without noise)

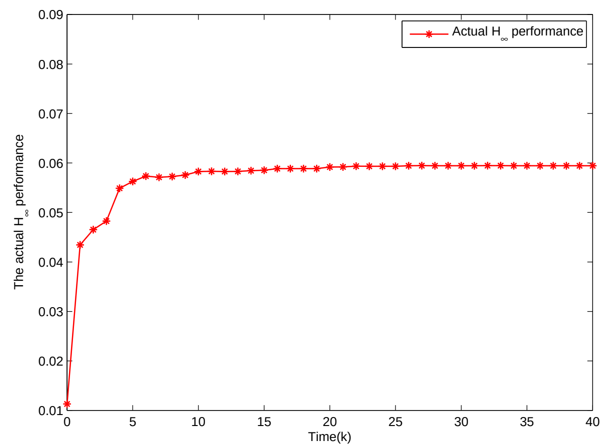

Fig. 6. The actual H_∞ performance