On Some Queueing Systems with Server Vacations, Extended Vacations, Breakdowns, Delayed Repairs and Stand-bys

A thesis submitted for the degree of Doctor of Philosophy

By

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ABSTRACT

This research investigates a batch arrival queueing system with a Bernoulli scheduled vacation and random system breakdowns. It is assumed that the repair process does not start immediately after the breakdown. Consequently there maybe a delay in starting repairs. After every service completion the server may go on an optional vacation. When the original vacation is completed the server has the option to go on an extended vacation. It is assumed that the system is equipped with a stand-by server to serve the customers during the vacation period of the main server as well as during the repair process.

The service times, vacation times, repair times, delay times and extended vacation times are assumed to follow different general distributions while the breakdown times and the service times of the stand-by server follow an exponential distribution.

By introducing a supplementary variable we are able to obtain steady state results in an explicit closed form in terms of the probability generating functions. Some important performance measures including; the average length of the queue, the average number of customers in the system, the mean response time, and the value of the traffic intensity are presented.

The professional MathCad 2001 software has been used to illustrate the numerical results in this study.

DEDICATION

This thesis is dedicated with all my love to my husband

ALI M. ALZAMIL

His love, support and encouragement made this journey more comfortable.

Dear partner;

All the time you spent to take care of our three kids

Muhammed, Shaha and Abdul aziz

All the words you said telling me "you can do it"

All the time you spent traveling with me,

All the days you spent alone while I was very busy

All this Make it very difficult to express my appreciation and thanks

Now I know truly that

Beside every successful woman there is a great man

For this and more…I love you

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PROCLAMATION

During the course of this PhD research work, the following papers have been published

1. Khalaf, R.F., Madan, K. C. & Lucas, C.A. (2010), 'Steady-State Analysis of an $M^{[X]}/G/1$ Queue with Bernoulli Schedule,General Vacations, Random Breakdowns and General Delay Times for Repairs to Start'. *International Conference on Applied Mathematics and Numerical Analysis*. Paris, France, June 28-30, 2010. ISSN: 2070-3740 ISSUE 66. Article# 368 pages 2798-2812. This publication was based on the work of chapter three as a special case.

2. Khalaf, R.F., Madan, K. C. & Lucas, C.A. (2011a) 'An $M^{[X]}/G/1$ Queue with Bernoulli Schedule, General Vacation Times, Random Breakdowns, General Delay Times and General Repair Times', *Applied Mathematical Sciences*, 5(1), pp.35-51. This publication was based on the work of chapter three a special case.

3. Khalaf, R.F., Madan, K. C. & Lucas, C.A. (2011b) 'On a Batch Arrival Queuing System Equipped with a Stand-by Server During Vacation Periods or the Repairs Times of the Main Server' *Journal of Probability and Statistics,* Article ID 812726, doi:10.1155/2011/812726. This publication was based on the work of chapter four.

4. Khalaf, R.F., Madan, K. C. & Lucas, C.A. (2011c) 'An $M^{[X]}/G/1$ Queue with Bernoulli Schedule General Vacation Times, General Extended Vacations, Random Breakdowns, General Delay Times for Repairs to Start and General Repair Times', *Journal of Mathematics Research,* 3(4), pp. 8-20. This publication was based on the work of chapter three.

5. Khalaf, R.F., Madan, K. C. & Lucas, C.A. (2012) 'On an $M^{[X]}/G/1$ Queueing System with Random Breakdowns, Server Vacations, Delay Times and a Standby Server', *International Journal of Operational Research*, 15(1), pp. 30-47. This publication was based on the work of chapter five.

Table of Standard Notations for Queueing Systems

Notation	Definition
λ	The mean arrival rate (average number of customers arriving per unit time).
μ	The mean service rate (average number of customers served per unit time).
L	The expected number of customers in the system.
L_q	The expected number of customers in the queue.
W	The expected waiting time in the system.
W_q	The expected waiting time in the queue.
ρ	Traffic intensity denoting the fraction of time that the server is busy.

CHAPTER ONE: PRELIMINARIES

1.1 Introduction

Queueing systems are concerned with providing services. In such a system customers arrive at a service centre looking for service of some kind and depart after such service has been provided. It is a usual phenomenon that when a customer arrives the server may be busy providing service, therefore the arriving customers have to join the waiting line until they receive the service.

We identify three main features of the service centre: the customers' population, the waiting line and, the server(s).

The basic queueing model is shown in figure 1.1 below:

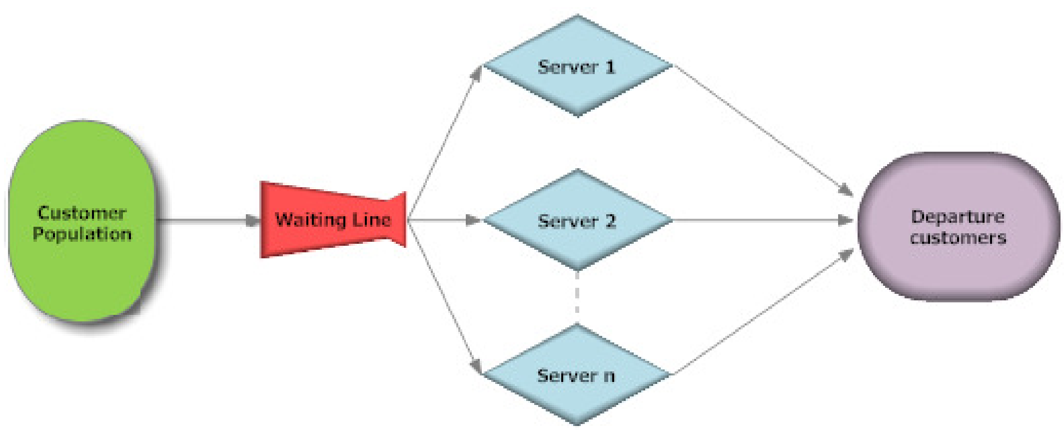


Figure 1.1: The basic queueing model

Queueing theory is an important branch of Mathematics in applied probability based on statistical distributions, calculus, matrix theory and complex analysis. It can also be classified as part of applied stochastic processes and decision science. A systematic analysis of a queueing system is crucial for the management to take efficient decisions. It helps to gain full utilization of a system, minimize its idle time, and streamline costs of operating the system.

Queueing Theory is a mathematical study of a queue, from it we can find specific answers for common questions such as how long the customer waits before service commences? How long should a customer spend in the system to complete their service? What is the average number of customers in the queue or/and in the system at any point of time? And how many servers should the service centre employ to get the best performance from the system.

Thus answers to questions about the mean waiting time in the queue, the mean system response time (waiting time in the queue plus service time), mean utilization of the service facility, distribution of the number of customers in the queue, distribution of the number of customers in the system and so forth need to be found.

There are a lot of situations in real life where waiting in a queue is required and essential. Queuing theory has applications in many fields; below we briefly describe some situations in which queueing theory is important.

- **The Internet Server**

  Many customers are connected to the main internet server. What is the acceptable number of customers connected to the internet per unit of time so that the internet provides a reasonable response rate? What happens when the number of customers increases? Is the capacity of the main frame computer sufficient?

- **Traffic Lights**

  We need to schedule traffic lights according to the time of the day such that the queues are acceptable.

- **Parking**

  In order to avoid overcrowding in front of a shopping mall, it is necessary to investigate what size and number of parking places to be provided.

- **Call Centers of any Company**

  The call center has a team structure to answer questions by phone, where each team helps customers from a specific region only. How long do customers have to wait before an operator becomes available? Is the number of incoming telephone lines enough? Are there enough operators? How many customers do not get their call answered?

1.2 Historical background

The history of queueing theory goes back more than a century. According to Bhat (2008), the first paper on the subject seems to be Johannsen's paper "Waiting Times and Number of Calls" (an article published in 1907 and reprinted in the Post Office Electrical Engineers Journal, London, October, 1910). From the point of view of an exact treatment, the method used in this paper was not mathematically exact, therefore, the paper that has historical importance is Erlang's (1909), "The Theory of Probabilities and Telephone Conversations". During the next 20 years Erlang's papers contain some of the most important techniques and concepts in queuing theory; for instance the notion of statistical equilibrium and the method of formulating the balance of state equations (later called Chapman-Kolmogorov equations).

"In 1915 Tore Olaus Engset developed the Engset formula before the breakthroughs of A. K. Erlang in 1917. The unpublished Engset's report "Om beregningen av vælgere i et automatisk telefonsystem" (1915) was published later in (1918)" for more details see (Myskja, 1998).

Molina (1927) published his paper "Application of the Theory of Probability to telephones Trunking Problems". One year later, Fry (1928) published his book "Probability and Its Engineering Uses" which expanded much of Erlang's earlier work.

During the next two decades several mathematicians became interested in these problems and developed general models which could be used in more complex situations.

The first use of the term "queueing system" occurred in 1951 in the *Journal of the Royal Statistical Society*, when D.C. Kendall published his article "Some Problems in the Theory of Queues". Of course, there were a huge number of articles on the subject much earlier (some used the word "queue" but not the word "queueing").

The same author who introduced the term queueing systems introduced an A/B/C queueing notation in 1953 and this has further been extended by Lee in 1966.

Finch (1958), studied the effect of the waiting room size on the performance measures of simple queues. The first textbook on queueing theory was published in 1958, "Queues, Inventories, and Maintenance" by P. M. Morse. In this year Haight (1958) introduced the concepts of bulking and parallel queues and White and Christie (1958) were the first to consider server breakdowns. The proof of the very famous formula in queueing theory, Little's formula (so called because it was first proved by John Little) was published in 1961.

Since Leonard (1961) published his first paper entitled "Information Flow in Large Communication Nets", the use of queueing for computer performance evaluation became an important class of queuing theory applications. Various studies have been published since that time including; Allen (1967), Fishman (1974), Forest, Mani, Richard and Fernando (1975), Leonard (1976), Stephen (1983), Lazowska, Zahorjan and Sevcik (1986) and (1996) and Walrand (1988) to mention some.

In 1967, Skinner considered what is now called the $M/G/1$ queueing system, but in a different context. Since that time several authers have studied the $M/G/1$ queue e.g., Jacob and Madhusoodanan (1987), Choi and Park (1990), Cao (1994), Madan (1994), Atencia, Fortes, Moreno and Sanchez (2006), to mention a few.

Markovian queueing systems, subject to more than one type of service, have been studied by Beja and Teller (1975). Further studies related to this area appeared later, we refer to Sen and Jain (1990), Madan (1991), Gail, Hantler and Taylor (1992), Whitt (1999), Hur and Paik (1999), Ke. (2003a), Bocharov, Manzo and Pechinkin(2005) and, Mishra and Yadav (2009).

Neuts (1981) introduced the matrix analytic method. As editor of communications in Statistics: Stochastic Models, he promoted a large variety of queueing models. Other workers in the matrix analytic area included; Neuts (1984) and (1989), Ramaswami (1990), Alfa (2002), Lothar and Dieter (2005) and, very recently Madan (2011).

During the last 30 years, a lot of important books on queueing theory and its applications have been published including Borovkov (1984), Kashyap and Chaudhry (1988), Nelson (1995), Bunday (1996), Gross and Harris (1998), Daigle (2005), Anisimov (2008) and, Mark (2010) to mention a few.

1.3 Characterization

In most cases, a queuing system is specified by stating the following six basic characteristics about it:

1.3.1 Input Pattern or Arrival Process of Customers

The arrival process means the manner in which the arrivals occur. It is specified by the interarrival time between any two consecutive arrivals. Usually the interarrival times are assumed to follow a common distribution and are independent of each other. The input pattern indicates the behaviour of the customers when arriving at the service system. Some customers may wait for a long time patiently, other customers are less patient and leave after a while. For instance, patients who visit the hospital to have an appointment with their doctor, if the doctor is not available then some of them will leave and possibly rebook their appointment. It is also very important to know if the customers arrive in batches or one by one.

1.3.2 Service Time Patterns.

The pattern of service times is the manner in which the service is rendered. It is specified by the time taken to complete a service. It is assumed usually that the service times follow a common distribution and are independent of each other and independent of the interarrival times. The most common distributions that the service times may have are deterministic and exponential distributions. Service times may also be dependent on the queue length.

1.3.3 Service Discipline

The service discipline indicates the manner in which the units are taken from the queue and allowed into service. Customers may be served in groups or one by one. The most known disciplines are:

- FIFO: (First in, First out): the usual queue discipline is first come first served (FCFS), a customer that finds the service centre busy goes to the end of the queue.

- LIFO: (Last in, First out): or last come first served (LCFS), a customer that finds the service centre busy proceeds immediately to the head of the queue, this customer will be served next, given that no further customers arrive.

- Random Service: also called (SIRO) the customers in the queue are served in a random order.

- Round Robin (RR): every customer gets a time slice. If the servicing of a customer is not completed at the end of this time then the customer is preempted and returned to the queue to be served according to the FCFS discipline.

- Priority Disciplines: every customer has a (static or dynamic) priority, the server selects the customers with the highest priority according to their time of arrival at the system. This scheme can use preemption or no preemption. In the preemption case the customer with the highest priority is allowed to enter service and stop the service of a customer with lower priority whose service is to be resumed after the higher priority customer is served. While in the case of no preemption the highest priority customer goes to the head of the queue and waits until the current service is completed.

1.3.4 Number of Servers.

A system may have a single server or a group of servers providing service to the customers. Increasing the number of service channels helps to decrease the waiting time. Given a number of service channels they may operate in parallel being able to serve customers simultaneously. It is generally assumed that the service mechanisms of the parallel channels operate independently of each other. An arrival who finds more than one free server may choose any one of them for receiving service. If he finds all the servers busy, he joins a queue common to all the servers, the first customer in the common queue goes to the server who becomes free first.

1.3.5 System Capacity.

This is the maximum number of customers allowed at any time in the system. A system may have an infinite capacity that is; the queue in front of the server(s) may grow to any length, in this case the system is called a delay system. In the case of a finite capacity this may be because of space or time limitation. The system has to be specified by the number of customers available, so that an arrival may not be able to join the system when the system is full, in this case the system is called a loss system.

1.3.6 Service Stages

The customers may proceed through one stage or several stages to complete their service before departing the system. In the case of multistage queuing systems, the customer enters a queue waits for service, gets served and departs the service station to enter a new queue for another service, and so on. In some multistage queuing systems recycling or feedback may be allowed, this case is common in manufacturing processes, where parts that do not meet quality standards are sent back for reprocessing.

1.4 Background Probability Theory

1.4.1 The concept of a Random Variable

Let T be the sample space associated with some experiment E. A random or stochastic variable, X, is a function that assigns a real number, $X(t)$, to each element $t \in T$.

Queuing systems provide many examples of random variables. For example, $X(t)$, may represent the number of customers in the system at time t.

1.4.2 Stochastic Processes

Let t be a parameter assuming values in a set T, and let $X(t)$ represent a random or stochastic variable for every $t \in T$. The family or collection of random variables

$\{X(t),\ t \in T\}$ is called a stochastic process. The parameter or index t is generally interpreted as time and the random variable, $X(t)$, as the state of the process at time t. The elements of T are time points, or epochs, and T is a linear set, countable or uncountable. (Methi 2003).

The stochastic process $\{X(t), t \in T\}$ is said to be a discrete-time process, if T is countable. If T is an interval of the real line (uncountable) then the stochastic process is said to be a continuous-time process.

The state space of the process denoted by S is the set of all possible values that the random variable $X(t)$ can assume; this set may be countable or uncountable. In general, a stochastic process may be put into one of four broad categories:

(i) discrete-time and discrete state space.

(ii) discrete-time and continuous state space.

(iii) continuous -time and discrete state space.

(iv) continuous -time and continuous state space.

In queuing systems many examples of stochastic processes can found. For example, the $X(t)$ might be the number of customers that arrive before a service counter by time t; then $\{X(t), t \geq 0\}$ is of the type (iii) above. If $X(n)$ represents the waiting time of the nth arrival; then $\{X(n),\ n = 0,1,2,...\}$ is of type (ii) above.

1.4.3 Markov Chains

A discrete state space process is often referred to as a chain. A process such as (i) above is a discrete-time chain, and a process such as (iii) is a continuous-time chain. A stochastic process $\{X(t),\ t \geq 0\}$ is called a Markov chain, if for every $x_i \in S$

$$\Pr\{X_n = x_n \mid X_{n-1} = x_{n-1}, ..., X_0 = x_0\}$$
$$= \Pr\{X_n = x_n \mid X_{n-1} = x_{n-1}\}$$

The transition probability from state j to state k can be defined as the conditional probability

$$P_{jk}(n) = \Pr\{X_n = k \mid X_{n-1} = j\} \quad k, j \in S$$

1.4.4 Birth-Death Process

The birth-death process is a special case of a continuous-time Markov process where the states represent the current size of a population and where the transitions are limited to births and deaths.

When a birth occurs, the process goes from state k to $k+1$. When a death occurs, the process goes from state k to state $k-1$. The process is specified by birth rates $\{\lambda_i\}_{i=0,1\ldots\infty}$ and death rates $\{\mu_i\}_{i=1,2\ldots\infty}$.

1.4.5 Transient and Steady-State Conditions of Birth-Death Processes

The variation in the probability of state k is the difference between the probabilities of moving into and out of state k.

The state-transition diagram of the birth-death process is shown in figure 1.2.

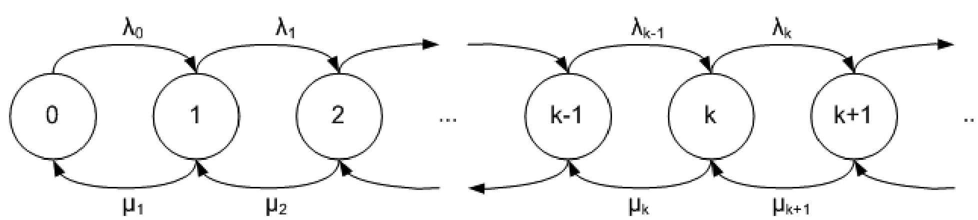


Figure 1.2: The state-transition diagram of the birth-death process

Assuming that $P_k(t)$ is the probability that at time t the process is in state k.

$$\text{Variation of flow} = \frac{dP_k(t)}{dt}$$

$$\text{Ingoing flow} = \lambda_{k-1}P_{k-1}(t) + \mu_{k+1}P_{k+1}(t)$$

$$\text{Outgoingflow} = \lambda_k P_k(t) + \mu_k P_k(t)$$

A steady state condition means that the state probabilities do not depend on the time any more. For $t \longrightarrow \infty$, the birth-death process may reach a steady-state (equilibrium) condition. If a steady-state solution exists, then all probabilities are constant and hence the steady-state solution is characterized by:

$$\lim_{t\to\infty} \frac{dP_k(t)}{dt} = 0 \qquad k = 0,1,2,...$$

We denote the steady state probability that the system is in state $k\,(k \in N)$ by p_k, which is defined by

$$p_k = \lim_{t\to\infty} P_k(t)$$

So we arrive at the following steady-state flow equations:

$$\begin{aligned} &0 = \mu_1 p_1 - \lambda_0 p_0 && k = 0 \\ &0 = -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} && k \geq 1 \end{aligned}$$

So, if $k = 0$, then $p_1 = \frac{\lambda_0}{\mu_1} p_0$. If $k = 1$, then $p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0$, and so on.

These equations can be recursively solved to obtain the following:

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}$$

Furthermore, since the p_k are probabilities, the normalization condition $\sum_{k=0}^{\infty} p_k = 1$ can be used to get

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}}$$

1.4.6 Relevant Probability Distributions

1.4.6.1 Bernoulli Distribution

The Bernoulli distribution, is a discrete probability distribution, which takes value 1 (success) with probability p and value 0 (failure) with probability $q = 1 - p$. Therefore it has a probability mass function,

$$P(X = i) = p^i (1-p)^{1-i}, \qquad i = 0, 1$$

The expected value of a Bernoulli random variable X is $E(X) = p$, and its variance is $\sigma^2(X) = p(1-p)$.

1.4.6.2 Exponential Distribution

A random variable X has an exponential distribution with parameter α, if and only if its probability density is given by

$$f(t) = \begin{cases} \alpha e^{-\alpha t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of an exponential random variable X is $E(X) = \frac{1}{\alpha}$, and its variance is, $\sigma^2(X) = \frac{1}{\alpha^2}$.

The exponential distribution is a continuous probability distribution, it is usually used to represent the time between events that happen at a constant average rate, for instance, arrivals in queueing theory.

An important property of an exponential random variable X with parameter α is the memoryless property. This property states that for all $x \ge 0$ and $t \ge 0$,

$$P(X > x+t \mid X > t) = P(X > x) = e^{-\alpha x}$$

The exponential distribution is the only continuous distribution with this property.

1.4.6.3 Poisson Distribution

A random variable *X* has a Poisson distribution with parameter λ, if and only if its probability distribution is given by

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \qquad n = 0,1,2,...$$

For the Poisson distribution

$$E(X) = \sigma^2(X) = \lambda.$$

Assuming the number of occurrences in some time interval to be a Poisson random variable is equivalent to assuming the time between successive arrivals follows an exponentially distributed random variable.

In queueing theory assuming that arrivals follow a Poisson distribution, is equivalent to assuming that the inter-arrival times (the time between arrivals) are exponentially distributed.

1.4.7 Laplace and Laplace-Stieltjes Transform

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Laplace-Stieltjes Transforms are a generalization of Laplace transforms to functions that are not necessarily Riemann integrable. This generalization is desirable when we are dealing with random variables that have a concentration of probability at a point.

Let $f(t)$ be a real valued function with domain $[0,\infty)$. The Laplace-Stieltjes transform of $f(t)$ denoted by $L(s)$, is defined by

$$L(s) = \int_0^\infty e^{-st} df(t)$$

We say that $L(s)$ exists if the above integral converges for at least one value of s.

1.4.8 Probability Generating Function

Suppose X is a random variable which assumes non-negative integer values 0, 1, 2,… and let $P_n = P(X = n),\ n = 0,1,2,...$ with $\sum_{n=0}^{\infty} P_n = 1$, then the probability generating function of X is defined as

$$P(t) = \sum_{n=0}^{\infty} P_n t^n = \sum_{n=0}^{\infty} P(X = n)t^n = E(t^x)$$

Generating functions are used in a manner similar to Laplace transform, but for functions of discrete variables.

1.5 Symbols, Terminologies and Queue Notations

Kendall (1953) introduced a shorthand notation to characterize a range of queueing models. It is a three-part code *A/B/C*, where:

- *A* denotes the distribution of the interarrival time.
- *B* denotes the distribution of the service times.
- *C* denotes the number of servers.

For *A* and *B* the following abbreviations are very common:

- *M* (Markov): denotes the exponential interarrival and service time distribution with probability distribution function $A(t) = 1 - e^{-\lambda t}$ and probability density function $a(t) = \lambda e^{-\lambda t}$, where $\lambda > 0$ is a parameter.

- *D* (Deterministic): all values are from a deterministic "distribution" and equal a constant, i.e. (constant interarrival or service time).

- G (General): general distribution, not further specified. In most cases at least the mean and the variance are known.

- E_k (Erlang-k): Erlangian Distribution with k phases ($k \geq 1$). For the Erlang-k distribution which is usually used for modelling telephone call arrivals at a central office. The probability distribution function is

$$A(t) = 1 - e^{-k\mu t} \sum_{j=0}^{k-1} \frac{(k\mu t)^j}{j!}$$

  Where $\mu > 0$ is a parameter.

- H_k (Hyper-k): Hyperexponential distribution with k phases. Here the density function is

$$A(t) = \sum_{i=1}^{k} q_i (1 - e^{-\omega_i t})$$

  Where $\omega_i > 0$, $q_i > 0$, $i \in \{1...k\}$ are parameters and furthermore $\sum_{j=1}^{k} q_j = 1$ must hold.

Kendall's notation has been considerably extended by Lee (1966), to allow it to represent a wide variety of queueing systems, a queue then is represented by a sequence A/B/C/D/E, where D denotes the maximum size of the waiting line in the finite case (if $D = \infty$ then this letter is omitted) and E denotes the service discipline used (FIFO, LIFO and so forth). If E is omitted this means that the service discipline is FIFO.

Thus, the notation $M/M/1$ denotes a queue or model system with FIFO service, a single server, an infinite waiting line, the customer interarrival times are iid (independent and identically distributed) and exponentially distributed with parameter λ where the customer service times are also iid and exponentially distributed with parameter μ.

1.6 General Relationships in Queueing Theory

There are certain useful relationships in queueing theory that hold for *G*/*G*/*c* queues.

1.6.1 Traffic Intensity (Server Utilization)

Assuming λ, is the average rate of customers entering the system and μ, is the average rate of serving customers and *c* is the number of servers in the system, then the quantity $\rho = \frac{\lambda}{c\mu}$, is called the traffic intensity (also called the utilization factor or server utilization), ρ gives the fraction of time that the server is busy.

Obviously, in order for the steady-state conditions to exist it is required that $\lambda < c\mu$ ($\rho < 1$). This is the stability condition for the *M*/*M*/*c* systems. When the average number of arrivals in to the system is more than the maximum number of customers the system can serve, i.e. $\lambda > c\mu$ ($\rho > 1$) this means that the queue size never settles down, and there is no steady state.

When the arrival rate equals the maximum average service rate of the system, i.e $\rho = 1$, the randomness will prevent the queue from ever emptying out and allowing the server to catch up, and this causes the unbounded growth of the queue. In this case the steady state does not exist unless arrivals and service are deterministic and perfectly scheduled.

1.6.2 The Symbol $o(\Delta t)$

The notation $o(g(x))$ as $x \to x_0$ refers to any function that (as $x \to x_0$) decays to zero at least as rapidly as $g(x)$ [where $g(x)$>0], that is

$$\lim_{x \to x_0} \left| \frac{o(g(x))}{g(x)} \right| = k < \infty \qquad \text{(Leonard, 1975)}$$

As the standard mathematical "little *o*" notation will be used, this denotes

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

For example, if $o(\Delta t) = (\Delta t)^a$, $a \geq 2$ then $\lim_{\Delta t \to 0} \frac{(\Delta t)^a}{\Delta t} = \lim_{\Delta t \to 0} (\Delta t)^{a-1} = 0$.

By using this notation we ignore negligible terms which do not have any impact on the final results.

1.6.3 Little's Formula

As already pointed out, the number of customers queued in front of an arriving customer clearly gives an indication of the time the arriving customer has to wait in order to be served (Frode, 1998). In Little's law this fact is considerable, it establishes a relationship between the average number of customers in the system, the mean arrival rate and the mean customer response time in the steady state.

Little’s law is a general result holding even for *G*/*G*/1 queues; it also holds with other service disciplines other than FIFO. It states that "the average number of customers in a queueing system is equal to the average arrival rate of customers to that system, times the average time spent in that system". (Leonard, 1975).

Assuming that λ is the mean arrival rate, L is the expected number of customers in the system, L_q is the expected number of customers in the queue, W is the expected waiting time in the system and W_q is the expected waiting time in the queue. Little's formulas are; $L = \lambda W$ and $L_q = \lambda W_q$

It’s clear from the previous formulas that to evaluate the average waiting time in the system/queue, it’s enough to know the arrival rate and the mean number of customers in the system/queue, and vice versa.

Eilon's Proof of this formula is mentioned in (Methi, 2003).

1.7 Performance Measures

There are a number of performance indicators in the analysis of queueing models that measure the performance of the system, some of these measures are of interest to the customer looking for service at the queue, such as; mean response time and the mean number of customers in the queue. Other measures of interest to the service provider include; the server utilization and the service cost.

The most relevant performance measures in the analysis of queueing models are:

- The mean response time which is the mean time a customer spends in the system, i.e. the waiting time plus the service time, W.
- The mean number of customers in the system, L (including the one or those in service).
- The mean time spent in the queue, W_q.
- The mean number of customers in the queue, L_q.
- The mean utilization of system facility, ρ. The utilization gives the fraction of time that the server is busy.

Having such information about the system enables the service centre owner to determine the values of appropriate measures of effectiveness in the system and develop an optimal system (according to some criterion).

1.8 Server Vacations

A vacation in a queueing context is a period when the server is not available for providing service. Arrivals coming during the vacation can go into service only after the server returns from vacation. There are many situations that lead to a server vacation, i.e. machine breakdowns, systems maintenance and cyclic servers (where the server serves more than one queue in the system or serves more than one system).

Doshi (1986) discussed different types of vacation models:

- The single vacation model, in this model there is exactly one vacation after the end of each busy period. If the server comes back from this vacation, it does not go for another vacation even if the system is still empty at that time. This type of vacation may arise from cases like maintenance in production systems, the maintenance can be considered a vacation.

- The multiple vacation model, this type of vacation may arise from cases like maintenance in computer and communication systems where processors in a computer and communication systems do considerable testing and maintenance besides doing their primary functions (processing telephone calls, receiving and transmitting data, etc.). The maintenance work required is divided into short segments. Whenever the customers are absent, the processor does a segment of the maintenance work. When the system is idle, the server takes a vacation (works on a maintenance segment). Upon return from a vacation, the server starts service only if it finds K or more customers waiting in the queue, if the number waiting is less than K then it goes on another vacation (maintenance segment).

- The limited service vacation model in which the server takes a vacation on becoming idle or after having served m consecutive customers, or after time T.

The way that the server provides service in the system is related with the vacation type. In his survey Doshi (1986) mentioned some of the service models as the following:

- Gated service, in this case, as soon as the server returns from the vacation it places a gate behind the last waiting customer. It then begins to serve only customers who are within the gate, based on some rules of how many or for how long it could serve.

- Exhaustive service, in this case, the server serves customers until the system is emptied, then it leaves for a vacation.

- Limited service, in this case, a fixed limit of K is placed on the maximum number of customers that can be served before the server goes on vacation. The server leaves for vacation either: (i) when the system is empty, or (ii) when K customers have been served.

Since Doshi's survey many researchers have studied queueing systems with different kinds of vacations, e.g., Tian, Zhang and Cao (1989), LaMaire (1992), Igaki (1992), Madan and Saleh (2001), Tian and Zhang (2002), Chang, Takine, Chae and Lee (2002), Choi and Kim (2003), Arumuganathan and Jeyakumar (2005), Chang and Takine (2005), Madan and Choudhury (2005), Madan, Al-Rawi and Al-Nasser (2005) and Banik, Gupta and Pathak (2007).

1.9 Random Breakdowns

In queueing systems in which the server is a machine such as networks, communication systems, and computer systems, it is realistic to assume that the server may suddenly break down and hence it will not be able to provide service again until it is repaired. The breakdowns occur at random and the repair time could follow an exponential, general, deterministic etc. distribution.

Although, in his survey, Doshi (1986), considered machine breakdowns as a server vacation we can consider that vacations may take place when a human server in the system may like to take a pause or may leave the system for an uncertain period of time from time to time, the breakdowns occur suddenly when an electronic or a mechanical server is providing service. Obviously, vacations and breakdowns both affect a system's efficiency adversely.

Several authors have studied queueing systems subject to breakdowns. They produced mathematical results in terms of the queue size distribution at a random point of time, average queue length at a random point of time, average waiting time for a customer, waiting time cost for a customer, cost for the system's idle time and many other performance measures of the system's efficiency. We refer the reader to Avi-Itzhak and Naor (1963), Kulkarni and Choi (1990), Federgruen and So (1990), Jayawardene and Kella (1996), Aissani and Artalejo (1998), Wang, Cao and Li (2001), Madan, Abu-Dayyah and Gharaibeh (2003a), Wang, Chiang and Ke (2003) and, Wang (2004).

In recent years, a significant amount of work has been done on queues with random breakdowns by several authors, e.g., Vinck and Bruneel (2006), Senthil and Arumuganathan (2010) and, Jain and Jain (2010).

When a system suddenly stops functioning due to a failure, most of the papers available in the literature assume that the repair process on the system starts immediately. However, we will analyze a queueing model where there is a possible delay in starting the repair process with the aim of determining the effect of this delay on the efficiency of the system. This again is a very realistic assumption in real life.

1.10 The Supplementary Variable Technique

According to the exponentially distributed inter-arrival and service times of the $M/M/1$ queue, it is possible to model the queue size process $N(t)$ by a Markov process. If the arrival process and/or the service process fails to be memoryless then we can not model the queue size process $N(t)$ by a Markov process. The supplementary variable technique was introduced by Cox (1955). Keilson and Kooharian (1960) have indicated that this technique was used later by many researchers e.g., Henderson (1972), Choi, Hwang and Han (1998) and, Methi (2003).

This technique is an important one to obtain a transient solution of Non-Markovian systems. Inclusion of a supplementary variable enables one to write down the differential equations, as in the case of a Markovian system. To illustrate the supplementary variable technique we assume that the service times are distributed according to a general probability density function. Then the $N(t)$ process becomes intractable due to the missing Markov property. So we introduce a new random variable $X(t)$ denoting the elapsed service time or the remaining service time for the customer in service at time t. By augmenting the state description by the supplementary variable it can be shown that the compound two-dimensional stochastic process $(N(t); X(t))$ becomes a Markov process. More on supplementary variable method can be seen in (Methi, 2003).

The supplementary variable method is a simple and convenient way compared with the other methods such as the embedded Markov chain approach (Leonard, 1975) and residual life approach (Bose, 2002).

1.11 The $M^{[X]}/G/1$ Queueing System

$M^{[X]}/G/1$ represents a single-server queuing system, where the customers arrive in groups according to a compound Poisson process with the batch size iid random variable X. The service times of the individual customers are considered to be generally distributed. The queue discipline is service in the order of arrival between batches; that is, all customers in the n^{th} batch are served before the first customer in the $(n+1)^{\text{th}}$ batch and the service order within a batch is random or units within a batch are pre-arranged for the purpose of the service. (Methi, 2003).

1.12 Literature Review and the Current Work

It is a realistic situation that the server is unavailable to serve the customers during occasional periods of time. If the server is human, it is normal that they may have to stop for a rest. The periods for which the server is unavailable to serve the customers according to a known schedule or pre-agreed policy is said to be the server vacation period.

Starting with Gaver (1962), vacation queues have been researched by a number of people including; Mitrani and Avi-Itzhak (1968), Fuhrmann (1984), Fuhrmann & Cooper (1985), Doshi (1985), Servi (1986), Blondia (1989), Chatterjee and Mukherjee (1990), Selvam and Sivasankaran (1994), Madan (2000a), Alfa (2003), Wang and Li (2008), to mention some.

An extensive survey on queueing systems with vacations can be found in Doshi (1986). Most of the previous studies are based on the well-known decomposition property discovered by Levy and Yechiali (1975) and further studied by Keilson and Servi (1987). This result is one of the most significant results of the research on vacation models and it states that the steady- state waiting time is the sum of two independent random variables. One is the waiting time and the other is the equilibrium residual time in a vacation. Madan (2000b), (2001) considered Bernoulli vacation models for two phase heterogeneous service as proposed by Keilson and Servi (1986) under certain modifications. Madan and Abu Al-Rub (2004) studied the single server queue with optional phase type server vacations based on exhaustive deterministic service and a single vacation policy.

Due to its wide applications the single arrival queueing systems $M/G/1$ have been studied by numerous researchers, we refer the reader to Madan and Baklizi (2002), Ke (2003b), Artalejo and Choudhury (2004) and Kella, Zwart and Boxma (2005).

It is more realistic to assume the arrivals occur in batches, rather than individuals, for example, the compound Poison arrival case. The single server $M^{[X]}/G/1$ queue with batch arrivals, where $M^{[X]}$ denotes a compound Poisson process, have been studied by numerous authors including, Lucantoni (1991), Choi, Kim, Shin and Pearce (2001), Al-Jararha and Madan (2003) and, Lee, Baek and Jeon (2005).

More recently, most of the studies have been devoted to batch arrival vacation models under different vacation policies because of its interdisciplinary character.

Numerous researchers have studied batch arrival queues with vacation time, we refer the reader to Baba (1986), Borthakur and Choudhury (1997), Frey and Takahashi (1999), Altman and Yechiali (2006) and, Choudhury (2007). Lee and Srinivasan (1989) considered a control policy on a $M^{[X]}/G/1$ with multiple vacations. In1994, Lee, Lee and Chae have dealt with $M^{[X]}/G/1$ with multiple vacations and N-policy. The N-policy vacation queue model means that the server is turned on when N or more customers are present, and off only when the system is empty. After the server is turned off, the server will not operate until at least N customers are present in the system. Choudhury (2000) has introduced the server setup period to the $M^{[X]}/G/1$ system and shown that the departure point queue size distribution is the convolution of the distribution of three independent random variables.

Chae, Lee and Ahn (2001) proposed an alternative approach, called the arrival time approach (ATA), to understand various $M/G/1$-type queues with generalized vacations. They showed, by an example, that the steady-state queue size distribution of an $M^{[X]}/G/1$ with multiple vacations at an arbitrary time can be decomposed into those of an ordinary $M^{[X]}/G/1$ and the number of customers during the vacation period.

Choudhury and Madan (2004) studied a batch arrival queue, where the concept of a Bernoulli schedule along with a vacation time are introduced for a two phase heterogeneous queueing system and obtained the queue size distribution at a departure epoch as a classical generalization of the well-known Pollaczek– Khinchine formula for this type of model. The same year**,** Madan, Al-Nasser and Al- Masri (2004), considered the batch arrivals queue with optional re-service.

Hur and Ahn (2005) studied a single server queueing system whose arrival stream is a compound Poisson process and service times are generally distributed. They considered three types of idle period, threshold, multiple vacations, and single vacation. For each model, they assumed after the idle period, the server needs a random amount of setup time before resuming service.

In recent years, a significant amount of work has been done on batch arrival queues with vacations and batch arrival queues with random breakdowns by several authors. We mention a few recent papers, Ke (2007a), Choudhury, Tadj and Paul (2007), Atencia, Bounza, and Moreno (2008), Maraghi, Madan and Darby-Dowman (2009 and 2010), Jain and Upadhyaya (2010) and finally the current researchers, Khalaf, Madan and Lucas (2010, 2011a, 2011b, 20011c and (2012).

1.13 Research Objectives

Vacations may take place when a human server in the system wishes to take a pause or leave the system for an uncertain period of time from time to time. Breakdowns may occur suddenly when an electronic or a mechanical server is providing service. Obviously, vacations and breakdowns both affect a system's efficiency adversely. Several authors have studied such systems and produced mathematical results in terms of the queue size distribution at a random point of time, average queue length at a random point of time, average waiting time for a customer, waiting time cost for a customer, cost for the system's idle time and many other performance measures of the system's efficiency.

In most of the systems mentioned in the literature, the server joins the system immediately after the completion of a vacation period. However, it is a realistic assumption that there is a delay in re-joining the system. In this research we study queueing systems in which the server takes an optional extended vacation before re-joining the system.

Similarly, when a system suddenly stops functioning due to a failure, most of the papers available in the literature assume that the repair process on the system starts immediately. However, we analyze a queueing model with delay in starting the repair process with the aim of determining the effect of delay on the efficiency of the system. This again is a very realistic assumption.

Recently Maraghi (2008) has studied some queueing systems with vacations and breakdowns. All these research papers assume no server delay in joining the system after completion of a vacation period and no delay in starting the repair process after a breakdown occurs. Our aim is to generalize not only some of the work done by Maraghi (2008) but also some other systems studied earlier by other authors.

In addition to extended vacations and delay in starting the repair process, we introduce the idea of a stand-by server in some of the systems. There are some systems in the queueing literature, e.g., Madan (1995), where a stand-by server is employed in the system when the main server is under repair. However, we study a new system which employs a stand-by server(s) not only during the repairs of a server but also during the period of vacation of the server.

Therefore, this research is conducted with the following objectives:

1. To determine the steady-state behavior of batch arrival queueing systems with Bernoulli schedule general vacations followed by a further optional extended vacation, random breakdowns general delay and general repairs.

2. To determine the steady-state behavior of batch arrival queueing systems with Bernoulli schedule general vacations, random breakdowns general repairs and

a stand-by server who's service time follows an exponential distribution during vacation periods and repair times of the main server.

3. To determine the steady-state behavior of batch arrival queueing systems with Bernoulli schedule general vacations, random breakdowns general repairs, general delay and a stand-by server who's service time follows an exponential distribution during vacation.

4. To determine the steady-state behavior of batch arrival queueing systems with Bernoulli schedule general vacations followed by a further optional extended vacation, random breakdowns, general repairs and an exponential stand-by service time distribution during repairs.

1.14 Research Methodology

The following are the commonly employed methods of solving a queueing model:

1. The method of recursive solution.
2. Generating function approach.
3. Laplace-Steiltjes transformation approach,
4. Integral equation approach.
5. Matrix-geometric method.
6. Supplementary variable technique.

We propose to use primarily methods 1, 2, 3 and 6 in our work.

1.15 Outline of Forthcoming Chapters

The following chapters represent the way the research progressed. In chapter two we present the basic queueing model which forms the starting point for later investigations in the dissertation which we develop for different queueing systems. The basic model is based on the work of Maraghi, et. al. (2010). However we give details of solving the equations and all the steps from the specification of the first probability equations to the final closed form solution of the queue size at a random epoch, the mean waiting time in the system and the mean waiting time in the queue, etc. Giving such details avoids showing repeatedly the same equations in later chapters. In the basic model we consider the batch arrival queueing system $M^{[X]}/G/1$ in which after any service completion the server has the option to leave for a vacation with probability p or continue service with probability $1 - p$. The server may suffer a random breakdown and if so the repair process on the system starts immediately. Although the customers arrive in batches, they are served one by one.

In chapter three we consider that the server may go on an extended vacation after the original vacation is completed with probability r or re-joins the system to serve the customers directly after the vacation with probability $1-r$. Moreover we assume that when the server breaks down, it does not enter the repair process immediately and there is a delay time waiting for the repair to start.

In chapter four a stand-by server has been added to the basic model. The stand-by server is assumed to serve the costumers during the vacation periods and repair periods of the main server.

Chapter five studies an $M^{[X]}/G/1$ queuing system with Bernoulli schedule server vacations. The server serves only one customer at a time and it may suffer from random breakdowns. It is assumed that there is a delay time before starting the repair process after the server's random breakdown. The system deploys a stand-by server during the vacation period.

In chapter six we study the $M^{[X]}/G/1$ queueing system with Bernoulli scheduled vacations. In this chapter the server may go on an extended vacation after the original vacation is completed with probability r or rejoin the system to serve the customers directly after the vacation with probability $1-r$. In addition to vacations and extended vacations, the system may suffer from random breakdowns from time to time. The repair process starts immediately after the breakdown. The system deploys a stand-by server only during the repair period.

For all models investigated, we assume that the service times, vacation times, extended vacation times, delay times and repair times have different general (arbitrary) distributions while the stand-by service times and the breakdown times follow exponential distributions.

The conclusions and contributions of this research with suggestions for further research are presented in chapter seven.

Glossary of Notations, Definitions and Abbreviations

In the rest of this dissertation we consider the following list of notations and definitions:

Notations	Definitions
$P_n(t,x)$	Probability that at time *t*, there are *n* $(n \geq 0)$ customers in the queue excluding the customer in service and the elapsed service time of this customer is *x*.
$P_n(t) = \int_0^\infty P_n(t,x)dx$	Probability that at time *t*, there are *n* $(n \geq 0)$ customers in the queue excluding the customer in service irrespective of the value of *x*.
$P_n(x) = \lim_{t \to \infty} P_n(t,x)$	The steady state probability corresponding to $P_n(t,x)$.
$P_n = \lim_{t \to \infty} P_n(t)$	The steady state probability corresponding to $P_n(t)$.
$P_q(x,z) = \sum_{n=0}^\infty z^n P_n(x), \quad P_q(z) = \sum_{n=0}^\infty z^n P_n$	The probability generating function of the queue size when the server is active.
$V_n(t,x)$	Probability that at time *t*, there are *n* $(n \geq 0)$ customers in the queue and the server is on vacation with elapsed vacation time *x*.
$V_n(t) = \int_0^\infty V_n(t,x)dx$	Probability that at time *t*, there are *n* $(n \geq 0)$ customers in the queue and the server is on vacation irrespective of the value of *x*.
$V_n(x) = \lim_{t \to \infty} V_n(t,x)$	The steady state probability corresponding to $V_n(t,x)$.

$V_n = \lim_{t\to\infty} V_n(t)$	The steady state probability corresponding to $V_n(t)$.
$V_q(x,z) = \sum_{n=0}^{\infty} z^n V_n(x), \quad V_q(z) = \sum_{n=0}^{\infty} z^n V_n$	The probability generating function of the queue size when the server is on vacation.
$R_n(t,x)$	Probability that at time *t*, there are *n* ($n \ge 0$) customers in the queue, and the server is under repair with elapsed repair time *x*.
$R_n(t) = \int_0^{\infty} R_n(t,x)dx$	Probability that at time *t*, there are *n* ($n \ge 0$) customers in the queue and the server is under repair irrespective of the value of *x*.
$R_n(x) = \lim_{t\to\infty} R_n(t,x)$	The steady state probability corresponding to $R_n(t,x)$.
$R_n = \lim_{t\to\infty} R_n(t)$	The steady state probability corresponding to $R_n(t)$.
$R_q(x,z) = \sum_{n=0}^{\infty} z^n R_n(x), \quad R_q(z) = \sum_{n=0}^{\infty} z^n R_n$	The probability generating function of the queue size when the server is under repair.
$Q(t)$	Probability that at time *t*, there are no customers in the system and the server is idle but available in the system.
$\lim_{t\to\infty} Q(t) = Q$	The steady state probability corresponding to $Q(t)$.
$D_n(t,x)$	Probability that at time *t*, there are *n* ($n \ge 0$) customers in the queue, and the server is inactive due to a system breakdown and is waiting for repairs to start with elapsed delay time *x*.
$D_n(t) = \int_0^{\infty} D_n(t,x)dx$	Probability that at time *t*, there are *n* ($n \ge 0$) customers in the queue and the server is waiting for repairs to start irrespective of the value of *x*.

$D_n(x) = \lim_{t\to\infty} D_n(t,x)$	The steady state probability corresponding to $D_n(t,x)$.
$D_n = \lim_{t\to\infty} D_n(t)$	The steady state probability corresponding to $D_n(t)$.
$D_q(x,z) = \sum_{n=0}^{\infty} z^n D_n(x), \quad D_q(z) = \sum_{n=0}^{\infty} z^n D_n$	The probability generating function of the queue size when the server is waiting for a repair to start (on delay).
$E_n(t,x)$	Probability that at time *t*, there are *n* ($n \ge 0$) customers in the queue, and the server is on an extended vacation with elapsed extended vacation time *x*.
$E_n(t) = \int_0^{\infty} E_n(t,x)dx$	Probability that at time *t*, there are n ($n \ge 0$) customers in the queue and the server is on an extended vacation irrespective of the value of *x*.
$E_n(x) = \lim_{t\to\infty} E_n(t,x)$	The steady state probability corresponding to $E_n(t,x)$.
$E_n = \lim_{t\to\infty} E_n(t)$	The steady state probability corresponding to $E_n(t)$.
$E_q(x,z) = \sum_{n=0}^{\infty} z^n E_n(x), \quad E_q(z) = \sum_{n=0}^{\infty} z^n E_n$	The probability generating function of the queue size when the server is on an extended vacation.
$C(z) = \sum_{n=1}^{\infty} z^n c_n$	The probability generating function of the batch arrival size.
$S_q(z)$	The probability generating function of the queue length no matter what the state of the system is.
$G(x), \quad g(x)$	The distribution function and the density function respectively of the service times.
$B(x), \quad b(x)$	The distribution function and the density function respectively of the vacation times.

$H(x),\ \ h(x)$	The distribution function and the density function respectively of the repair times.
$W(x),\ \ w(x)$	The distribution function and the density function respectively of the delay times.
$F(x),\ \ f(x)$	The distribution function and the density function respectively of the extended vacation times.
$G^{*}(a)$	The Laplace-Stieltjes transform of the service times $G(x)$.
$B^{*}(b)$	The Laplace-Stieltjes transform of the vacation times $B(x)$.
$H^{*}(b)$	The Laplace-Stieltjes transform of the repair times $H(x)$.
$W^{*}(b)$	The Laplace-Stieltjes transform of the delay times $W(x)$.
$F^{*}(b)$	The Laplace-Stieltjes transform of the Extended vacation times $F(x)$.
$E(I)$	The average size of the batches of the arriving customers.
$E(V)$	The mean vacation time.
$E(R)$	The mean repair time.
$E(D)$	The mean delay time.
$E(eV)$	The mean extended vacation time.
$E(I(I-1))$	The second factorial moment of the batch size of arriving customers.

$E(V^2)$	The second moment of the vacation times.
$E(R^2)$	The second moment of the repair times.
$E(D^2)$	The second moment of the delay times.
$E(eV^2)$	The second moment of the extended vacation times.

Chapter Two: The Basic Mathematical Model: An $M^{[X]}/G/1$ Queue with Bernoulli Schedule, General Vacation Times, Random Breakdowns and General Repair Times

2.1 Introduction

Several research results have been published including a study of the $M^{[X]}/G/1$ queuing system with vacations and the $M^{[X]}/G/1$ queuing system with random breakdowns. We refer the reader to Ke (2001), Niu, Shu and Takahashi (2003), Choudhury (2003a) and, Xu, Bao and Tian (2007).

In recent years many authors have studied batch arrival queueing systems. Choudhury and Madan (2005) analyzed a two-stage batch arrival queueing system assuming that the server vacation is the modified Bernoulli schedule vacation under an N-policy**.** Kumar and Arumuganathan (2008) also studied the batch retrial queueing systems with general vacation time under a Bernoulli schedule but with two phases of heterogeneous service. Chang and Ke (2009) investigated an $M^{[X]}/G/1$ retrial queueing system with a modified vacation policy by applying the supplementary variable technique.

In this chapter we introduce the basic mathematical model of the batch arrival queueing system $M^{[X]}/G/1$ in which, after every service completion the server has the option to leave for a vacation with probability p or continue service with probability $1 - p$. Moreover, we assume that the server may breakdown randomly, and the repair process starts immediately after the breakdown. Although customers arrive at the service station in batches of variable size, they are served one by one. We assume that the service times, vacation times and repair times are generally distributed while the breakdown times are exponentially distributed.

This basic model will be used in further chapters to develop queueing systems with different assumptions which lead to new contributions of knowledge in queueing theory.

The general equations of this chapter are based on the work of Maraghi, et. al. (2010) but are suitably modified for use by us in our extensions to various queueing models. The procedure described for solving these problems is the work of the current author.

The rest of this chapter is organized as follows: the assumptions underlying the mathematical model are given in section 2.2. In section 2.3 all the steady state equations governing the basic mathematical system are formulated. In section 2.4 the supplementary variable technique is used to solve the equations of section 2.3 to find the queue size distribution at a random epoch. The average queue size and the average waiting time are given in section 2.5.

2.2 Assumptions

We consider a batch arrival queueing system, where customers arrive at the system according to a compound Poisson process in batches whose size is a random variable X with batch arrival rate λ . Let $c_i = \Pr[X = i]$, then $\lambda c_i \Delta t$ is the probability that during a short time interval $(t, t + \Delta t)$ a batch of size i $(i = 1, 2, 3, ...)$ arrives at the system. More details about the batch arrival queueing systems can be found in (Methi, 2003).

Although the customers arrive in groups, the single server can serve only one customer at a time based on the (FCFS) discipline. The service times are assumed to follow a general distribution. Let $G(x)$ and $g(x)$ be the distribution function and the density function of the service time respectively. The conditional probability of a service completion during the interval, $(x, x + \Delta x]$, given that the elapsed service time is x, is given by $\mu(x)\Delta x$, so that $\mu(x) = g(x)/(1 - G(x))$ and, therefore $g(x) = \mu(x)\exp(-\int_0^x \mu(s)ds)$.

The derivation of $g(x)$ is given in appendix A, (A.1).

Once the server completes a service it can go on a vacation of a random length of time with probability p, or stay in the system providing service with probability $1 - p$. The vacation times are assumed to follow a general distribution.

Let $B(x)$ and $b(x)$ be the distribution function and the density function of the vacation time respectively. The conditional probability of a vacation completion during the interval, $(x, x+\Delta x]$, given that the elapsed vacation time is x, is given by $\beta(x)\Delta x$, so that $\beta(x) = b(x)/(1-B(x))$ which implies $b(x) = \beta(x)\exp(-\int_0^x \beta(v)dv)$.

The system may breakdown at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted goes to the head of the queue. Once the system breaks down, its repairs start immediately. The duration of repairs follows a general (arbitrary) distribution with distribution function $H(x)$ and density function $h(x)$. Let $\gamma(x)\Delta x$ be the conditional probability of a completion of repair during the interval $(x, x+\Delta x]$ given that the elapsed repair time is x, so that $\gamma(x) = h(x)/(1-H(x))$ implies $h(x) = \gamma(x)\exp(-\int_0^x \gamma(t)dt)$.

We assume that all stochastic processes involved in the system are independent of each other.

2.3 Equations Governing the General Mathematical Model

According to the assumptions mentioned in the previous section, the following set of equations represent the queueing system we study in this chapter

$$\begin{aligned} P_n(t+\Delta t, x+\Delta x) &= (1-\lambda\Delta t)(1-\mu(x)\Delta x)(1-\alpha\Delta t)P_n(t,x) \\ &\quad + \lambda\sum_{i=1}^{n} c_i P_{n-i}(t,x)\Delta t \quad n \geq 1 \end{aligned} \tag{2.1}$$

$$P_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\mu(x)\Delta x)(1-\alpha\Delta t)P_0(t,x) \tag{2.2}$$

$$V_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\beta(x)\Delta x)V_n(t,x) + \lambda\sum_{i=1}^{n} c_i V_{n-i}(t,x)\Delta x \qquad n \geq 1 \tag{2.3}$$

$$V_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\beta(x)\Delta x)V_0(t,x) \tag{2.4}$$

$$R_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\gamma(x)\Delta x)R_n(t,x) + \lambda\sum_{i=1}^{n} c_i R_{n-i}(t,x)\Delta t \qquad n \geq 1 \tag{2.5}$$

$$R_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\gamma(x)\Delta x)R_0(t,x) \tag{2.6}$$

$$\begin{aligned} Q(t+\Delta t) = (1-\lambda\Delta t)Q(t) + (1-p)\int_0^\infty P_0(t,x)\mu(x)\Delta t dx \\ + \int_0^\infty R_0(t,x)\gamma(x)\Delta t dx + \int_0^\infty V_0(t,x)\beta(x)\Delta t dx \end{aligned} \tag{2.7}$$

A full explanation of equations (2.1) to (2.7) is given in appendix B, (B.1).

Subtracting and adding a term $P_n(t, x+\Delta x)$ to the LHS in equation (2.1), then dividing by $\Delta t(\Delta x)$ and taking limits as $\Delta t \to 0\,(\Delta x \to 0)$, we get

$$\frac{\partial}{\partial t}P_n(t,x) + \frac{\partial}{\partial x}P_n(t,x) = -(\lambda + \mu(x) + \alpha)P_n(t,x) + \lambda\sum_{i=1}^{n} c_i P_{n-i}(t,x) \tag{2.8}$$

taking limit as $t \to \infty$, this yields

$$\frac{\partial}{\partial x}P_n(x) = -(\lambda + \mu(x) + \alpha)P_n(x) + \lambda\sum_{i=1}^{n} c_i P_{n-i}(x) \qquad n \geq 1 \tag{2.9}$$

Subtracting and adding a term $P_0(t, x+\Delta x)$ to the LHS in equation (2.2), then dividing by $\Delta t(\Delta x)$ and taking limits as $\Delta t \to 0\,(\Delta x \to 0)$, we get

$$\frac{\partial}{\partial x}P_0(x) = -(\lambda + \mu(x) + \alpha)P_0(x) \tag{2.10}$$

By following the same process we set out to get equations (2.9) from equations (2.1), and equation (2.10) from (2.2). From equations (2.3) to (2.6) we get respectively

$$\frac{\partial}{\partial x}V_n(x) = -(\lambda + \beta(x))V_n(x) + \lambda\sum_{i=1}^{n} c_i V_{n-i}(x) \qquad n \geq 1 \tag{2.11}$$

$$\frac{\partial}{\partial x}V_0(x) = -(\lambda + \beta(x))V_0(x) \tag{2.12}$$

$$\frac{\partial}{\partial x}R_n(x)+(\lambda+\gamma(x))R_n(x)=\lambda\sum_{i=1}^{n}c_iR_{n-i}(x)\quad n\geq 1 \tag{2.13}$$

$$\frac{\partial}{\partial x}R_0(x)=-(\lambda+\gamma(x))R_0(x)=0 \tag{2.14}$$

Dividing equation (2.7) by Δt and taking limit as $\Delta t\rightarrow 0$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}Q(t)&=-\lambda Q+\int_0^\infty R_0(t,x)\gamma(x)dx\\&+\int_0^\infty V_0(t,x)\beta(x)dx+(1-p)\int_0^\infty P_0(t,x)\mu(x)dx\end{aligned} \tag{2.15}$$

taking limits as $t\rightarrow\infty$, we get

$$0=-\lambda Q+\int_0^\infty R_0(x)\gamma(x)dx+\int_0^\infty V_0(x)\beta(x)dx+(1-p)\int_0^\infty P_0(x)\mu(x)dx \tag{2.16}$$

The following boundary conditions are used to solve the above equations

$$\begin{aligned}P_n(0)&=(1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx+\int_0^\infty V_{n+1}(x)\beta(x)dx\\&+\int_0^\infty R_{n+1}(x)\gamma(x)dx+\lambda c_{n+1}Q\qquad n\geq 0\end{aligned} \tag{2.17}$$

$$V_n(0)=p\int_0^\infty P_n(x)\mu(x)dx,\qquad n\geq 0 \tag{2.18}$$

$$R_n(0)=\alpha\int_0^\infty P_{n-1}(x)dx=\alpha P_{n-1}\qquad n\geq 1 \tag{2.19}$$

$$R_0(0)=0 \tag{2.20}$$

A full explanations of the boundary conditions (2.17) to (2.20) are given in appendix B, (B.2).

2.4 Distribution of the Queue Length at any Point of Time

To solve equations (2.9), to (2.14), for a closed form solution we follow the procedure set out below.

We multiply equation (2.9) by z^n, and sum over n from 1 to ∞, and add it to equation (2.10) resulting in the following equation

$$\frac{\partial}{\partial x}P_q(x,z)+(\lambda-\lambda C(z)+\mu(x)+\alpha)P_q(x,z)=0 \tag{2.21}$$

Following a similar process, from equations (2.11) and (2.12), (2.13) and (2.14), we get respectively

$$\frac{\partial}{\partial x}V_q(x,z)+(\lambda-\lambda C(z)+\beta(x))V_q(x,z)=0 \tag{2.22}$$

$$\frac{\partial}{\partial x}R_q(x,z)+(\lambda-\lambda C(z)+\gamma(x))R_q(x,z)=0 \tag{2.23}$$

Multiplying equation (2.17) by z^{n+1}, and summing over n from 0 to ∞, results in

$$\begin{aligned} zP_q(0,z)&= \\ (1-p)\int_0^\infty P_q(x,z)\mu(x)dx&+\int_0^\infty V_q(x,z)\beta(x)dx+\int_0^\infty R_q(x,z)\gamma(x)dx+\lambda C(z)Q \\ &-\left((1-p)\int_0^\infty P_0(x)\mu(x)dx+\int_0^\infty V_0(x)\beta(x)dx+\int_0^\infty R_0(x)\gamma(x)dx\right) \end{aligned} \tag{2.24}$$

Using equation (2.16) to replace

$-\left((1-p)\int_0^\infty P_0(x)\mu(x)dx+\int_0^\infty V_0(x)\beta(x)dx+\int_0^\infty R_0(x)\gamma(x)dx\right)$ by $-\lambda Q$, we have

$$\begin{aligned} zP_q(0,z)=(1-p)\int_0^\infty P_q(x,z)\mu(x)dx+\int_0^\infty V_q(x,z)\beta(x)dx \\ +\int_0^\infty R_q(x,z)\gamma(x)dx+\lambda Q(C(z)-1) \end{aligned} \tag{2.25}$$

Next we multiply equation (2.18) by z^n and sum over n from 0 to ∞, to obtain

$$V_q(0,z) = p\int_0^\infty P_q(x,z)\mu(x)dx \tag{2.26}$$

Similarly, multiplying (2.19) by z^n and summing over n from 0 to ∞, adding to (2.20), we obtain

$$R_q(0,z) = \alpha z\int_0^\infty P_q(x,z)dx = \alpha z P_q(z) \tag{2.27}$$

Integrating equation (2.21) from 0 to x yields

$$P_q(x,z) = P_q(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu(t)dt} \tag{2.28}$$

Where $P_q(0,z)$ is given by equation (2.25).

Let $a = \lambda - \lambda C(z) + \alpha$. Integrating equation (2.28) by parts with respect to x yields

$$P_q(z) = P_q(0,z)\left(\frac{1-G^*(a)}{a}\right) \tag{2.29}$$

Where $G^*(a) = \int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x}dG(x)$ is the Laplace-Stieltjes transform of the service times $G(x)$.

The details of integrating equation (2.28) by parts is given in appendix A, (A.2).

Multiplying both sides of equation (2.28) by $\mu(x)$ and integrating over x we get

$$\int_0^\infty P_q(x,z)\mu(x)dx = P_q(0,z)G^*(a) \tag{2.30}$$

Using equations (2.30) and (2.26) we get

$$V_q(0,z) = pP_q(0,z)G^*(a) \tag{2.31}$$

likewise, integrating equation (2.22) from 0 to x, we obtain

$$V_q(x,z) = V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \tag{2.32}$$

Substituting for $V_q(0,z)$ from (2.31) in equation (2.32) we get

$$V_q(x,z) = pP_q(0,z)G^*(a)\ e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \tag{2.33}$$

Let $m = \lambda - \lambda C(z)$. Integrating equation (2.33) by parts with regard to x yields

$$V_q(z) = \frac{pP_q(0,z)G^*(a)\ \left(1-B^*(m)\right)}{m} \tag{2.34}$$

Where $B^*(m) = \int_0^\infty e^{-(\lambda-\lambda C(z))x} dB(x)$ is the Laplace-Stieltjes transform of the vacation times $B(x)$.

Multiplying equation (2.33) by $\beta(x)$ and integrating over x we get

$$\int_0^\infty V_q(x,z)\beta(x)dx = pP_q(0,z)G^*(a)B^*(m) \tag{2.35}$$

Integrating equation (2.23) from 0 to x, yields

$$R_q(x,z) = R_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{2.36}$$

Substituting for $R_q(0,z)$ from (2.27) and (2.29) in equation (2.36) we obtain

$$R_q(x,z) = \frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)}{a} e^{-(\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{2.37}$$

Integrating equation (2.37) by parts with respect to x we obtain

$$R_q(z) = \frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)\left(1-H^*(m)\right)}{a\ m} \tag{2.38}$$

Where $H^*(m)=\int_0^\infty e^{-(\lambda-\lambda C(z))x}dH(x)$ is the Laplace-Stieltjes transform of the repair times $H(x)$.

Multiplying both sides of equation (2.37) by $\gamma(x)$ and integrating over x we obtain

$$\int_0^\infty R_q(x,z)\gamma(x)dx=\frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)H^*(m)}{a} \tag{2.39}$$

Now using equations (2.30), (2.35) and (2.39) in equation (2.25) yields

$$P_q(0,z)=\frac{-amQ}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \tag{2.40}$$

Substituting for $P_q(0,z)$ in equation (2.29), (2.34) and (2.38) we get

$$P_q(z)=\frac{-mQ\left(1-G^*(a)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \tag{2.41}$$

$$V_q(z)=\frac{-apQG^*(a)\left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \tag{2.42}$$

$$R_q(z)=\frac{-Q\alpha z\left(1-G^*(a)\right)\left(1-H^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \tag{2.43}$$

Let $S_q(z)$ be the p.g.f (probability generating function) of the queue length no matter what the state of the system is, i.e. $S_q(z)=P_q(z)+V_q(z)+R_q(z)$.

Then adding equations (2.41), (2.42) and (2.43) we obtain

$$S_q(z)=\frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-H^*(m)\right)\right)-apQG^*(a)\left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \tag{2.44}$$

The normalization condition $S_q(1)+Q=1$, is used in order to determine Q.

Because of the indeterminate form (0/0 form) of $S_q(z)$, when $z = 1$, then L'Hopital's rule is applied on equation (2.44) to obtain

$$\lim_{z\to 1} S_q(z) = \frac{\lambda E(I)Q\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+p\alpha\lambda E(I)QE(V)G^*(\alpha)}{\alpha\left(1-pG^*(\alpha)\lambda E(I)E(V)\right)-\lambda E(I)\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)-\alpha\left(1-G^*(\alpha)\right)} \tag{2.45}$$

Where $C(1)=1$, $C'(1)=E(I)$ is the average size of the batches of the arriving customers, $B^*(0)=1$, $B^{*\prime}(0)=\lambda E(I)E(V)$, $H^*(0)=1$, and $H^{*\prime}(0)=\lambda E(I)E(R)$.

Hence, adding Q to the right hand side of equation (2.45) and equating to 1 we obtain

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{2.46}$$

Equation (2.46) gives the probability that the server is idle. From equation (2.46) the utilization factor, ρ of the system is given by

$$\rho = \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{2.47}$$

The details of applying L'Hopital's rule on equation (2.44) and the steps to obtain Q are given in appendix A, (A.3).

2.5 The Mean Length of the Queue and the Mean Waiting Time

To find L_q, the steady state average queue length, where $L_q = \frac{d}{dz}S_q(z)\Big|_{z=1}$, we note that this formula is of the 0/0 form, we write $S_q(z)$ given in (2.44) as $S_q(z)=N(z)/D(z)$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (2.44) respectively. Then using L'Hopital's rule twice we obtain

$$L_q = \lim_{z\to 1}\frac{D'(z)N''(z)-N'(z)D''(z)}{2\left(D'(z)\right)^2} \tag{2.48}$$

The details of obtaining equation (2.48) is given in appendix A, (A.4).

Finding the required derivatives at $z=1$ we have

$$N'(1) = Q\lambda E(I)\left(\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+\alpha p E(V)G^*(\alpha)\right) \tag{2.49}$$

$$\begin{aligned} &N''(1) = \\ &Q(\lambda E(I))^2\left(\alpha\left(1-G^*(\alpha)\right)E(R^2)\right. \\ &\left.+2G^{*'}(\alpha)\left(1+\alpha E(R)\right)+\alpha p G^*(\alpha)E(V^2)-2pE(V)\left(G^*(\alpha)+\alpha G^{*'}(\alpha)\right)\right) \\ &+Q\lambda E(I(I-1))\left(\alpha p G^*(\alpha)E(V)+\left(1-G^*(\alpha)\right)\left((1+\alpha E(R)\right)\right)\right) \\ &+2\alpha\lambda E(I)Q\left(1-G^*(\alpha)\right)E(R) \end{aligned} \tag{2.50}$$

$$D'(1) = -\lambda E(I)\left(\alpha p(E(V)G^*(\alpha)+\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)\right)+\alpha G^*(\alpha) \tag{2.51}$$

$$\begin{aligned} D''(1) = &-(\lambda E(I))^2 G^*(\alpha)\left(\alpha p E(V^2)-\alpha E(R^2)-2p(V)\right) \\ &-\lambda E(I(I-1))\left(\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)+\alpha p G^*(\alpha)E(V)\right)\right) \\ &-2(\lambda E(I))^2 G^{*'}(\alpha)\left(1+\alpha E(R)-\alpha p E(V)\right) \\ &-2\lambda E(I)\left(1+\alpha E(R)\left(1-G^*(\alpha)\right)+\alpha G^{*'}(\alpha)\right)-\alpha(\lambda E(I))^2 E(R^2) \end{aligned} \tag{2.52}$$

Where $E(V^2)$ is the second moment of the vacation times, $E(R^2)$ is the second moment of the repair times, $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers, and Q has been found in (2.46).

Substituting for $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ from (2.49), (2.50), (2.51), and (2.52) into (2.48) we obtain L_q in a closed form. Further, the mean waiting time of a customer can be found using Little's law $W_q = L_q/\lambda$.

The work detailed here is now further extended in the forthcoming chapters.

Chapter Three: An $M^{[X]}/G/1$ Queue with Bernoulli Schedule, General Vacation Times, General Extended Vacation Times, Random Breakdowns, General Delay Times and General Repair Times

3.1 Introduction

Queuing systems with server vacations and/or random system breakdowns have been studied by numerous researchers as we mentioned earlier. For more papers in this area we refer the reader to Choudhury and Borthakur (2000), Takine (2001), Choudhury (2002), Madan, Abu-Dayyah and Saleh (2002), Anabosi and Madan (2003), Madan and Al-Rawwash (2005), Katayama and Kobayashi (2006), Madan and Choudhury (2006), Liu, Xu and Tian (2007), Ke (2007b), and Wang and Li (2010), to mention a few.

In most of the papers including the ones mentioned above the authors assume that whenever the system breaks down the repair process starts immediately. It is quite common that as a result of a sudden breakdown, the system may have to wait for repairs to start. We term the time the server spends waiting for repairs to start as 'delay time'.

Also most of the analyses in the past have assumed that just after the vacation period is over, the server immediately rejoins the system and starts providing service to the waiting customers. However, in many real life situations, the server may have to take an extended vacation due to a variety of reasons including illness, personal engagement or the need to attend to some other task.

As in the basic model, we study an $M^{[X]}/G/1$ queuing system with Bernoulli schedule server vacations. The server serves one customer at a time and it may suffer from random breakdowns.

The first new assumption in this chapter is that the repair process does not necessarily start immediately after a breakdown, thus there may be a delay before starting repairs. The second new assumption in this chapter is that the server may go on an extended vacation after the original vacation is completed with probability r or rejoins the system to serve customers directly after the vacation with probability $1-r$.

We assume that the service times, vacation times, extended vacation times, repair times and delay times each have a general distribution while the breakdown times are exponentially distributed.

It is the first study of a queueing system with five general distributions. From our literature review we find that the maximum number of general distributions considered in most queueing systems to be three.

This chapter is organized as follows: the mathematical model is given in section 3.2. In section 3.3 all the equations representing the mathematical system in its steady state are formulated. The supplementary variable technique is used in this section to obtain the closed form of the p.g.f of the queue length. The mean size of the queue and the mean waiting time in the queue are given in section 3.4. This along with three particular cases are given in section 3.5. The three cases are used to show the relationship between this work and previous works done by other researchers. Numerical and graphical illustrations are given in section 3.6.

3.2 Mathematical Model

We now extend the basic model of chapter two to account for delay times and extended vacation times. When the system breaks down, there is a potential delay before the repairs start. Let $W(x)$ be the distribution function of the delay time which follows a general distribution, and $w(x)$ its density function. Let $\varphi(x)\Delta x$ be the conditional probability of a completion of a delay during the interval $(x, x+\Delta x]$ given that the elapsed delay time is x, so that $\varphi(x) = w(x)/(1-W(x))$ and, therefore $w(x) = \varphi(x)\exp(-\int_0^x \varphi(t)dt)$.

After a vacation period the server has the option of taking an extended vacation. We assume that with probability r the server takes an extended vacation and with probability $1-r$ the server rejoins the system immediately after completion of a vacation. Let $F(x)$ and $f(x)$ be the distribution function and the probability density function respectively of the extended vacation time which follows a general distribution. Let $\theta(x)\Delta x$ be the conditional probability of a completion of an extended vacation during the interval $(x, x+\Delta x]$ given that the elapsed extended vacation time is x, so that $\theta(x) = f(x)/(1-F(x))$ and, therefore $f(x) = \theta(x)\exp(-\int_0^x \theta(t)dt)$.

3.3 Equations Governing the System and the Distribution of Length of the Queue at any Point of Time

According to the assumptions mentioned in the previous section we now introduce the following new equations to account for delays and extended vacations,

$$D_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\varphi(x)\Delta x)D_n(t,x) + \lambda\sum_{i=1}^{n} c_i D_{n-i}(t,x)\Delta t \quad n \geq 1 \tag{3.1}$$

$$D_0(t+\Delta t, x+\Delta x) = 0 \tag{3.2}$$

$$E_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\theta(x)\Delta x)E_n(t,x) + \lambda\sum_{i=1}^{n} c_i E_{n-i}(t,x)\Delta t \quad n \geq 1 \tag{3.3}$$

$$E_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\theta(x)\Delta x)E_0(t,x) \tag{3.4}$$

$$\begin{aligned} Q(t+\Delta t) = {} & (1-\lambda\Delta t)Q(t) + (1-p)\int_0^\infty P_0(t,x)\mu(x)\Delta t dx + \int_0^\infty R_0(t,x)\gamma(x)\Delta t dx \\ & + (1-r)\int_0^\infty V_0(t,x)\beta(x)\Delta t dx + \int_0^\infty E_0(t,x)\theta(x)\Delta t dx \end{aligned} \tag{3.5}$$

A full explanation of equations (3.1) to (3.5) is given in appendix B, (B.3).

Subtracting and adding a term $D_n(t, x+\Delta x)$ to the LHS in equation (3.1), then dividing by Δt and taking limits as $\Delta t \to 0$, then taking limit as $t \to \infty$, this yields

$$\frac{\partial}{\partial x} D_n(x) + (\lambda + \varphi(x))D_n(x) = \lambda\sum_{i=1}^{n} c_i D_{n-i}(x) \quad n \geq 1 \tag{3.6}$$

From equation (3.2) we get

$$\frac{\partial}{\partial x} D_0(x) = 0 \tag{3.7}$$

Following the same process from equations (3.3), (3.4) and (3.5) we obtain respectively

$$\frac{\partial}{\partial x} E_n(x) + (\lambda + \theta(x)) E_n(x) = \lambda \sum_{i=1}^{n} c_i E_{n-i}(x) \quad n \geq 1 \tag{3.8}$$

$$\frac{\partial}{\partial x} E_0(x) = -(\lambda + \theta(x)) E_0(x) \tag{3.9}$$

$$\begin{aligned} \lambda Q &= \int_0^\infty R_0(x)\gamma(x)dx + (1-p)\int_0^\infty P_0(x)\mu(x)dx \\ &+ (1-r)\int_0^\infty V_0(x)\beta(x)dx + \int_0^\infty E_0(x)\theta(x)dx \end{aligned} \tag{3.10}$$

Thus the equations governing this system are (3.6) to (3.10), and (2.9) to (2.16).

The following boundary conditions are used to solve the above differential equations

$$\begin{aligned} P_n(0) &= (1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx + (1-r)\int_0^\infty V_{n+1}(x)\beta(x)dx \\ &+ \int_0^\infty E_{n+1}(x)\theta(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + \lambda c_{n+1}Q \qquad n \geq 0 \end{aligned} \tag{3.11}$$

$$V_n(0) = p\int_0^\infty P_n(x)\mu(x)dx, \qquad n \geq 0 \tag{3.12}$$

$$E_n(0) = r\int_0^\infty V_n(x)\beta(x)dx \qquad n \geq 0 \tag{3.13}$$

$$D_n(0) = \alpha\int_0^\infty P_{n-1}(x)dx = \alpha P_{n-1} \qquad n \geq 1 \tag{3.14}$$

$$R_n(0) = \int_0^\infty D_n(x)\varphi(x)dx, \qquad n \geq 0 \tag{3.15}$$

$$D_0(0) = R_0(0) = 0 \tag{3.16}$$

A full explanations of the boundary conditions (3.11) to (3.16) are given in appendix B, (B.4).

We multiply equation (3.6) by z^n, and sum over n from 1 to ∞, add this to equation (3.7) and after simplification we obtain the following equation

$$\frac{\partial}{\partial x} D_q(x,z) + (\lambda - \lambda C(z) + \varphi(x)) D_q(x,z) = 0 \tag{3.17}$$

Following the same process from equations (3.8) and (3.9) we obtain

$$\frac{\partial}{\partial x} E_q(x,z) + (\lambda - \lambda C(z) + \theta(x)) E_q(x,z) = 0 \tag{3.18}$$

Multiplying equation (3.11) by z^{n+1}, and summing over n from 0 to ∞, we obtain

$$zP_q(0,z) = (1-p)\int_0^\infty P_q(x,z)\mu(x)dx + (1-r)\int_0^\infty V_q(x,z)\beta(x)dx + \int_0^\infty R_q(x,z)\gamma(x)dx + \int_0^\infty E_q(x,z)\theta(x)dx$$
$$+\lambda C(z)Q - \left((1-p)\int_0^\infty P_0(x)\mu(x)dx + +(1-r)\int_0^\infty V_0(x,z)\beta(x)dx + \int_0^\infty E_0(x)\theta(x)dx + \int_0^\infty R_0(x,z)\gamma(x)dx \right)$$

Using equation (3.10) from the above equation we obtain

$$\begin{aligned} zP_q(0,z) &= (1-p)\int_0^\infty P_q(x,z)\mu(x)dx + (1-r)\int_0^\infty V_q(x,z)\beta(x)dx \\ &+ \int_0^\infty R_q(x,z)\gamma(x)dx + \int_0^\infty E_q(x,z)\theta(x)dx + \lambda Q(C(z)-1) \end{aligned} \tag{3.19}$$

Multiplying equation (3.12) by z^n and summing over n from 0 to ∞, we obtain

$$V_q(0,z) = p\int_0^\infty P_q(x,z)\mu(x)dx \tag{3.20}$$

Following the same process with equations (3.13), (3.14) and (3.15) we get respectively

$$E_q(0,z) = r\int_0^\infty V_q(x,z)\beta(x)dx \tag{3.21}$$

$$D_q(0,z) = \alpha z P_q(z) \tag{3.22}$$

$$R_q(0,z) = \int_0^\infty D_q(x,z)\varphi(x)dx \tag{3.23}$$

Integrating equation (3.17) from 0 to *x*, yields

$$D_q(x,z) = D_q(0,z)e^{-(\lambda - \lambda C(z))x - \int_0^x \varphi(t)dt} \tag{3.24}$$

Substituting for $D_q(0,z)$ from (3.22) in equation (3.24) we get

$$D_q(x,z) = \alpha z P_q(z)e^{-(\lambda - \lambda C(z))x - \int_0^x \varphi(t)dt} \tag{3.25}$$

Integrating equation (3.25) by parts with respect to *x* we obtain

$$D_q(z) = \frac{\alpha z P_q(z)\left(1 - W^*(m)\right)}{m} \tag{3.26}$$

Where $W^*(m) = \int_0^\infty e^{-(\lambda - \lambda C(z))x} dW(x)$ is the Laplace-Stieltjes transform of the delay time $W(x)$.

Substituting for $P_q(z)$ from equation (2.29) yields

$$D_q(z) = \frac{\alpha z P_q(0,z)\left(1 - G^*(a)\right)\left(1 - W^*(m)\right)}{am} \tag{3.27}$$

Multiplying equation (3.25) by $\varphi(x)$ and integrating over *x* then substituting for $P_q(z)$ from equation (2.29) we obtain

$$\int_0^\infty D_q(x,z)\varphi(x)dx = \frac{\alpha z P_q(0,z)\left(1 - G^*(a)\right)W^*(m)}{a} \tag{3.28}$$

From equations (3.23) and (3.28) we obtain

$$R_q(0,z)=\frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)W^*(m)}{a} \tag{3.29}$$

Note that $R_q(0,z)$ given in equation (3.29) is different from the one given in equation (2.27), and will affect equation (2.36), where substitution for $R_q(0,z)$ from (3.29) in (2.36) gives

$$R_q(x,z)=\frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)W^*(m)}{a}e^{-(\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{3.30}$$

Integrating equation (3.30) by parts with respect to x we obtain

$$R_q(z)=\frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)W^*(m)\left(1-H^*(m)\right)}{a\,m} \tag{3.31}$$

Multiplying equation (3.30) by $\gamma(x)$ and integrating over x we obtain

$$\int_0^\infty R_q(x,z)\gamma(x)dx=\frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)W^*(m)H^*(m)}{a} \tag{3.32}$$

Integrating equation (3.18) from 0 to x, yields

$$E_q(x,z)=E_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \theta(t)dt} \tag{3.33}$$

Substituting for $E_q(0,z)$ from (3.21) and (2.35) we obtain

$$E_q(x,z)=rpP_q(0,z)G^*(a)B^*(m)e^{-(\lambda-\lambda C(z))x-\int_0^x \theta(t)dt} \tag{3.34}$$

Integrating equation (3.34) by parts with regard to x results in

$$E_q(z)=\frac{rpP_q(0,z)G^*(a)B^*(m)\left(1-F^*(m)\right)}{m} \tag{3.35}$$

Where $F^*(m) = \int_0^\infty e^{-(\lambda-\lambda C(z))x} dF(x)$ is the Laplace-Stieltjes transform of the

Extended vacation time $F(x)$.

Multiplying equation (3.34) by $\theta(x)$ and integrating over x we get

$$\int_0^\infty E_q(x,z)\theta(x)dx = rpP_q(0,z)G^*(a)B^*(m)F^*(m) \tag{3.36}$$

Using equations (2.30), (2.35), (3.32) and (3.36) in equation (3.19) we obtain

$$P_q(0,z) = \frac{-amQ}{a\left(z - G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.37}$$

Substituting for $P_q(0,z)$ from equation (3.37) in equations (2.29), (2.34), (3.27), (3.31) and (3.35), we obtain

$$P_q(z) = \frac{-mQ\left(1-G^*(a)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.38}$$

$$V_q(z) = \frac{-aQpG^*(a)\left(1-B^*(m)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.39}$$

$$D_q(z) = \frac{-Q\alpha z\left(1-G^*(a)\right)\left(1-W^*(m)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.40}$$

$$R_q(z) = \frac{-Q\alpha z\left(1-G^*(a)\right)W^*(m)\left(1-H^*(m)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.41}$$

$$E_q(z) = \frac{-aQrpG^*(a)B^*(m)\left(1-F^*(m)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.42}$$

3.4 The Distribution of the Queue Length at any point of time

In this chapter the probability generating function p.g.f $S_q(z)$ is given by

$$S_q(z) = P_q(z) + V_q(z) + E_q(z) + D_q(z) + R_q(z)$$

Then adding equations (3.38) to (3.42) we obtain

$$S_q(z) = \frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-F^*(m)H^*(m)\right)\right) - aQpG^*(a)\left(1-B^*(m)\left(1-r+rW^*(m)\right)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \quad (3.43)$$

The normalization condition $S_q(1) + Q = 1$ is used in order to determine Q.

Because of the indeterminate form of $S_q(1)$, L'Hopital's rule is applied twice on equation (3.43), to achieve

$$\lim_{z\to 1} S_q(z) = \frac{Q\lambda E(I)\left(\left(1-G^*(\alpha)\right)\left(1+\alpha(E(D)+E(R)\right)+\alpha p\, G^*(\alpha)\left(E(V)+rE(eV)\right)\right)}{-\alpha p\lambda E(I)G^*(\alpha)\left(E(V)+rE(eV)\right) - \lambda E(I)\left(1-G^*(\alpha)\right)\left(1+\alpha\left(E(D)+E(R)\right)\right) + \alpha G^*(\alpha)}$$

(3.44)

Now adding Q to $\lim_{z\to 1} S_q(z)$ given in equation (3.44) and equating to 1 and simplifying we obtain

$$Q = 1 - \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{E(D)}{G^*(\alpha)} + \frac{E(R)}{G^*(\alpha)} - \frac{1}{\alpha} - E(D) - E(R) + p\left(E(V) + rE(eV)\right)\right) \quad (3.45)$$

From equation (3.45) the traffic intensity ρ is given by

$$\rho = \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{E(D)}{G^*(\alpha)} + \frac{E(R)}{G^*(\alpha)} - \frac{1}{\alpha} - E(D) - E(R) + p\left(E(V) + rE(eV)\right)\right) \quad (3.46)$$

We substitute the value of Q from equation (3.45) into equation (3.43), which enables us to determinate $S_q(z)$.

3.5 The Average Queue Size and the Average Waiting Time

Following the procedure as set out in chapter two, we carry out the derivatives of the numerator and denominator of the right hand side of (3.46) at $z = 1$, we have

$$\begin{aligned}&N'(1) = \\ &Q\Phi\big((1 - G^*(\alpha))(1 + \alpha(E(D) + E(R))) + \alpha p Q\, G^*(\alpha)\big(E(V) + rE(eV)\big)\big)\end{aligned} \tag{3.47}$$

$$\begin{aligned}&N''(1) = \\ &\Phi^2 Q\big(\alpha(1 - G^*(\alpha))\big(E(D^2) + E(R^2) + 2E(D)E(R)\big) \\ &+ 2G^{*'}(\alpha)(1 + \alpha(E(D) + E(R))) + \alpha p G^*(\alpha)\big(E(V^2) + rE(eV^2) + 2rE(V)E(eV)\big) \\ &- 2p\big(E(V) + rE(eV)\big)\Big(\alpha G^{*'}(\alpha) + G^*(\alpha)\Big)\Big) \\ &+ \Lambda Q\big((1 - G^*(\alpha))(1 + \alpha(E(D) + E(R))) + \alpha p G^*(\alpha)\big(E(V) + rE(eV)\big)\big) \\ &+ 2\alpha Q\Phi(1 - G^*(\alpha))\big(E(D) + E(R)\big)\end{aligned} \tag{3.48}$$

$$\begin{aligned}&D'(1) = \\ &-\Phi\big((1 - G^*(\alpha))(1 + \alpha(E(D) + E(R))) + \alpha p G^*(\alpha)\big(E(V) + rE(eV)\big)\big) \\ &+ \alpha G^*(\alpha)\end{aligned} \tag{3.49}$$

$$
\begin{aligned}
&D''(1) = \\
&-\Phi^2 G^*(\alpha)\big(\alpha p\big(E(V^2)+rE(eV^2)+2rE(V)E(eV)\big)-2p\big(E(V)+rE(eV)\big) \\
&-\alpha\big(E(D^2)+E(R^2)+2E(D)E(R)\big)\big)-\alpha\Phi^2\big(E(D^2)+E(R^2)+2E(D)E(R)\big) \\
&-\Lambda\big(\big(1-G^*(\alpha)\big)\big(1+\alpha(E(D)+E(R)\big)+\alpha p G^*(\alpha)\big(E(V)+rE(eV)\big)\big) \\
&-2\Phi^2 G^{*'}(\alpha)\big(1+\alpha(E(D)+E(R))-\alpha p\big(E(V)+rE(eV)\big)\big) \\
&-2\Phi\left(1+\alpha G^{*'}(\alpha)+\alpha\big(1-G^*(\alpha)\big)\big(E(D)+E(R)\big)\right)
\end{aligned}
\tag{3.50}
$$

Where $\Lambda = \lambda E(I(I-1))$ and $\Phi = \lambda \mathrm{E(I)}$.

The mean waiting time of a customer can be found using Little's Law $W_q = L_q/\lambda$. The average size of the system can be found using the known relation $L = L_q + \rho$. The average time the customer spends in the system, namely the mean response time (the time in the queue plus the service time) can be found by the other version of Little's Law, $W = L/\lambda$.

3.6 Particular Cases

3.6.1 Case 1: No Delay for Repairs to Start

Once the system breaks down, its repairs start immediately and there is no delay time. We let $E(D)=0$ and $W^*(m)=1$ then from the main results we obtain

$$
S_q(z) = \frac{-Q\big(1-G^*(a)\big)\big(m+\alpha z\big(1-H^*(m)\big)\big)-a\,QpG^*(a)\,\big(1-B^*(m)\big(1-r+rF^*(m)\big)\big)}{a\big(z-G^*(a)\big(1-p+pB^*(m)\big(1-r+rF^*(m)\big)\big)\big)-\alpha z\big(1-G^*(a)\big)H^*(m)}
\tag{3.51}
$$

$$
Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)+p\big(E(V)+rE(eV)\big)\right)
\tag{3.52}
$$

$$
N'(1) = Q\Phi\big(\big(1-G^*(\alpha)\big)\big(1+\alpha E(R)\big)+\alpha p\,G^*(\alpha)\,\big(E(V)+rE(eV)\big)\big)
\tag{3.53}
$$

$$N''(1) =$$
$$\Phi^2 Q\left(\alpha\left(1-G^*(\alpha)\right)E(R^2)+2G^{*'}(\alpha)\left(1+\alpha\left(E(D)+E(R)\right)\right)\right.$$
$$+\alpha p G^*(\alpha)\left(E(V^2)+rE(eV^2)+2rE(V)E(eV)\right) \tag{3.54}$$
$$\left.-2p\left(E(V)+rE(eV)\right)\left(\alpha G^{*'}(\alpha)+G^*(\alpha)\right)\right)+2\alpha Q\Phi\left(1-G^*(\alpha)\right)E(R)$$
$$+\Lambda Q\left(\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+\alpha p G^*(\alpha)\left(E(V)+rE(eV)\right)\right)$$

$$D'(1) =$$
$$-\Phi\left(\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+\alpha p G^*(\alpha)\left(E(V)+rE(eV)\right)\right)+\alpha G^*(\alpha) \tag{3.55}$$

$$D''(1) = -\Phi^2 G^*(\alpha)\left(\alpha p\left(E(V^2)+rE(eV^2)+2rE(V)E(eV)\right)\right.$$
$$\left.-2p\left(E(V)+rE(eV)\right)-\alpha E(R^2)\right)-\alpha\Phi^2 E(R^2)$$
$$-\Lambda\left(\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+\alpha p G^*(\alpha)\left(E(V)+rE(eV)\right)\right) \tag{3.56}$$
$$-2\Phi^2 G^{*'}(\alpha)\left(1+\alpha E(R)-\alpha p\left(E(V)+rE(eV)\right)\right)$$
$$-2\Phi\left(1+\alpha G^{*'}(\alpha)+\alpha(1-G^*(\alpha))E(R)\right)$$

3.6.2 Case 2: No Extended Vacation

Once the vacation ends the server returns to service immediately and there is no extended vacation. We let $r = 0$ and $F^*(m) = 1$ then from the main results of this chapter we obtain

$$S_q(z) = \frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-W^*(m)H^*(m)\right)\right)-aQpG^*(a)\left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{3.57}$$

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(D)-E(R)+pE(V)\right) \tag{3.58}$$

$$N'(1) = Q\Phi\left(\left(1-G^*(\alpha)\right)\left(1+\alpha\left(E(D)+E(R)\right)\right)+\alpha p\, G^*(\alpha)\, E(V)\right) \tag{3.59}$$

$$N''(1) =$$
$$\begin{aligned}&\Phi^2 Q\Big(\alpha\big(1-G^*(\alpha)\big)\big(E(D^2)+E(R^2)+2E(D)E(R)\big)\\&+2G^{*'}(\alpha)\big(1+\alpha(E(D)+E(R))\big)+\alpha pG^*(\alpha)E(V^2)\\&-2pE(V)(\alpha G^{*'}(\alpha)+G^*(\alpha))\Big)\\&+\Lambda Q\Big(\big(1-G^*(\alpha)\big)\big(1+\alpha(E(D)+E(R))\big)+\alpha pG^*(\alpha)E(V)\Big)\\&+2\alpha Q\Phi\big(1-G^*(\alpha)\big)\big(E(D)+E(R)\big)\end{aligned} \tag{3.60}$$

$$D'(1) = -\Phi\Big(\big(1-G^*(\alpha)\big)\big(1+\alpha(E(D)+E(R))\big)+\alpha pG^*(\alpha)E(V)\Big)+\alpha G^*(\alpha) \tag{3.61}$$

$$D''(1) =$$
$$\begin{aligned}&-\Phi^2 G^*(\alpha)\Big(\alpha pE(V^2)-2pE(V)-\alpha\big(E(D^2)+E(R^2)+2E(D)E(R)\big)\Big)\\&-\alpha\Phi^2\big(E(D^2)+E(R^2)+2E(D)E(R)\big)\\&-\Lambda\Big(\big(1-G^*(\alpha)\big)\big(1+\alpha\big(E(D)+E(R)\big)\big)+\alpha pG^*(\alpha)E(V)\Big)\\&-2\Phi^2 G^{*'}(\alpha)\big(1+\alpha\big(E(D)+E(R)\big)-\alpha pE(V)\big)\\&-2\Phi\Big(1+\alpha G^{*'}(\alpha)+\alpha\big(1-G^*(\alpha)\big)\big(E(D)+E(R)\big)\Big)\end{aligned} \tag{3.62}$$

The results obtained in (3.51), to (3.56) agree with the results given in Khalaf, et.al. (2011a).

3.6.3 Case 3: No Delay for Repairs to Start and No Extended Vacation

If there is no delay time we let $E(D)=0$ and $W^*(m)=1$, so we consider there is no extended vacation time we let $r=0$ and $F^*(m)=1$ then from the main results we obtain

$$S_q(z) = \frac{-Q\big(1-G^*(a)\big)\big(m+\alpha z\big(1-H^*(m)\big)\big)-aQpG^*(a)\big(1-B^*(m)\big)}{a\big(z-G^*(a)\big(1-p+pB^*(m)\big)\big)-\alpha z\big(1-G^*(a)\big)H^*(m)} \tag{3.63}$$

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{3.64}$$

$$N'(1) = Q\Phi\Big(\big(1-G^*(\alpha)\big)\big(1+\alpha E(R)\big)+\alpha p\,G^*(\alpha)\,E(V)\Big) \tag{3.65}$$

$$N''(1) = \Phi^2 Q\Big(\alpha(1 - G^*(\alpha))E(R^2) + 2G^{*'}(\alpha)E(R)$$
$$+ \alpha p G^*(\alpha)E(V^2) - 2pE(V)\Big(\alpha G^{*'}(\alpha) + G^*(\alpha)\Big)\Big) \tag{3.66}$$
$$+ \Lambda Q\big((1 - G^*(\alpha))(1 + \alpha E(R)) + \alpha p G^*(\alpha)E(V)\big)$$
$$+ 2\alpha Q\Phi(1 - G^*(\alpha))E(R))$$

$$D'(1) = -\Phi\big((1 - G^*(\alpha))(1 + \alpha E(R)) + \alpha p G^*(\alpha)E(V)\big) + \alpha G^*(\alpha) \tag{3.67}$$

$$D''(1) = -\Phi^2 G^*(\alpha)\big(\alpha p E(V^2) - 2pE(V) - \alpha E(R^2)\big)$$
$$- \alpha\Phi^2 E(R^2) - 2(\lambda E(I))^2 G^{*'}(\alpha)(1 + \alpha E(R) - \alpha p E(V))$$
$$- \Lambda\big((1 - G^*(\alpha))(1 + \alpha E(R)) + \alpha p G^*(\alpha)E(V)\big) \tag{3.68}$$
$$- 2\Phi\Big(1 + \alpha G^{*'}(\alpha) + \alpha(1 - G^*(\alpha))E(R)\Big)$$

The results obtained in (3.63) to (3.68) agree with the results given by Maraghi, et. al. (2.10).

3.7 A Numerical Example

In order to verify the validity of the results of this chapter, we consider the service times, vacation times, delay times, extended vacation times and repair times to be exponentially distributed. All values were chosen arbitrarily in order that the stability conditions are satisfied.

In table 3.1 we choose the following values:

$\mu = 7$, $\beta = 5$, $\gamma = 4$, $\lambda = 2$, $\alpha = 2$, $p = 0.5, r = 0.5$, $E(I) = 1$ and $E(I(I-1)) = 0$, we consider that θ takes the values 5, 6, 7 and 8, while φ takes the values 3, 5, 7 and 9.

Table 3.1: Some queue performance measures values computed when $\mu = 7,\ \beta = 5,\ \gamma = 4,\ \lambda = 2,\ \alpha = 2,\ p = 0.5, r = 0.5$

θ	φ	ρ	Q	L_q	W_q	L	W
5	3	0.919	0.081	12.9653	6.4826	13.8843	6.9422
	5	0.8429	0.1571	5.2961	2.6481	6.139	3.0695
	7	0.8102	0.1898	3.9598	1.9799	4.77	2.385
	9	0.7921	0.2079	3.4142	1.7071	4.2063	2.1031
6	3	0.9024	0.0976	10.5094	5.2547	11.4118	5.7059
	5	0.8262	0.1738	4.6522	2.3261	5.4784	2.7392
	7	0.7935	0.2065	3.5256	1.7628	4.3191	2.1595
	9	0.7754	0.2246	3.0555	1.5278	3.8309	1.9155
7	3	0.8905	0.1095	9.223	4.6126	10.1157	5.0579
	5	0.8143	0.1857	4.2703	2.1352	5.0846	2.5423
	7	0.7816	0.2184	3.2622	1.6311	4.0438	2.0219
	9	0.7635	0.2365	2.8361	1.418	3.5996	1.7998
8	3	0.8815	0.1185	8.4378	4.2189	9.3194	4.6597
	5	0.8054	0.1946	4.0184	2.0092	4.8238	2.4119
	7	0.7727	0.2273	3.0861	1.543	3.8588	1.9294
	9	0.7546	0.2454	2.6885	1.3442	3.443	1.7215

It is clear from table 3.1 that increasing the value of φ or θ decreases the traffic intensity, the average queue length and the average response time, while the server idle time increases. All the trends shown by this table are as expected.

The following graphs show the effect of the new contributions of this chapter (the delay times and the extended vacation times).

In figures 3.1 and 3.2 we consider the first four rows of table 3.1.

In figure 3.1 the horizontal axis represents the delay rate φ and the vertical axis represents the mean response time *W*.

In figure 3.2 the horizontal axis represents the extended vacation rate θ and the vertical axis represents the mean number of customers in the system *L*.

Figure 3.1: the effects of the delay rate on the mean response time

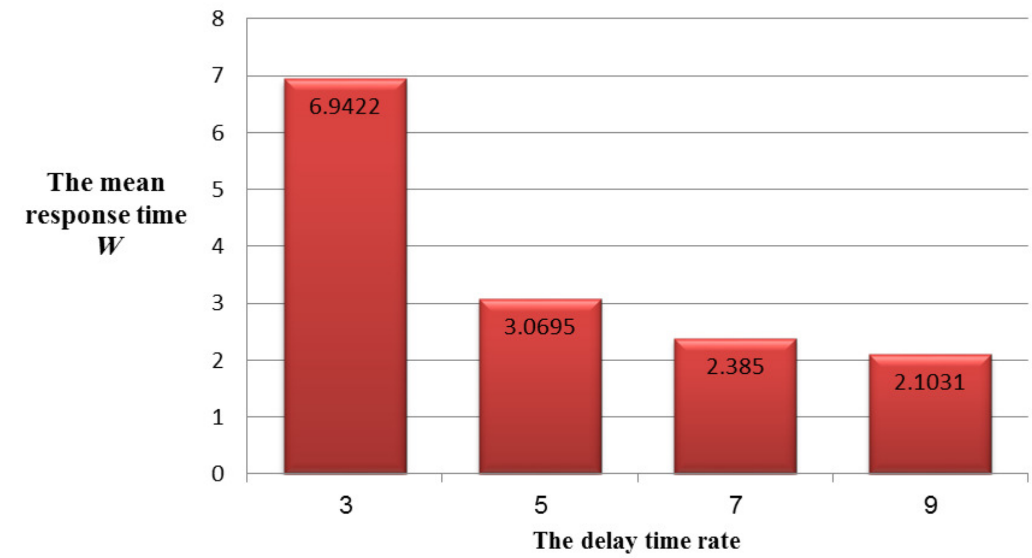
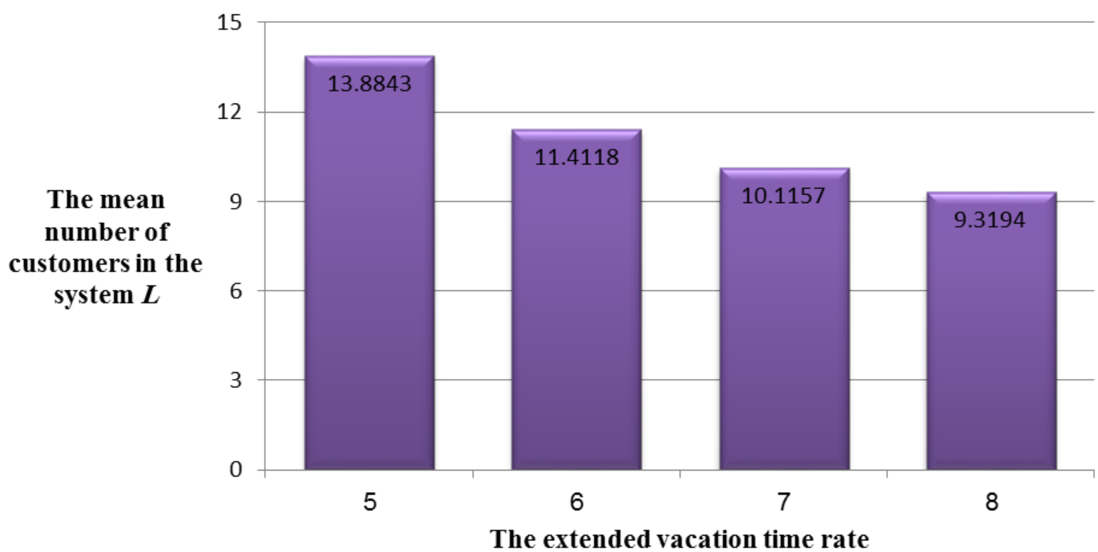


Figure 3.2: the effects of the extended vacation rate on the mean number of customers in the system



➢⮘

Chapter Four: On a Batch Arrival Queuing System Equipped with a Stand-by Server During Vacation Periods and the Repairs Times of the Main Server

4.1 Introduction

In this chapter we study the basic model introduced in chapter two with an additional significant assumption that the system deploys a stand-by server during the vacation period and the repair period of the main server.

Madan (1995) studied the steady state behavior of a queuing system with a stand-by server to serve customers only during the repair period. In that work repair times were assumed to follow an exponential distribution. In this chapter we consider both vacations and breakdowns with the additional assumption of deployment of a stand-by server during the vacation periods and repair periods. Most importantly, we assume that the service times, vacation times, repair times have different general (arbitrary) distributions while the breakdown times and the stand-by service times follow exponential distributions.

This chapter is arranged as follows: section 4.2 gives the mathematical model that we study in this chapter. Equations governing the system and their solutions to find the distribution of the length of the queue at any point of time are given in section 4.3. The mean length of the queue and the mean waiting and response times are given in section 4.4. In section 4.5, we consider three numerical examples to illustrate the application.

4.2 Mathematical Model

We consider the mathematical model in chapter two and add the assumption that a stand-by server starts to serve the customers when the original server is on vacation or under repair. We assume that the stand-by service time distribution follows an exponential distribution with stand-by service rate $\delta > 0$ and mean stand-by service time $1/\delta$.

4.3 Equations Governing the System and the Distribution of the Queue Length at any point of time

Considering that there is a stand-by server to serve the customers for every main server interruption, results in changes to some of the equations governing the basic system in chapter 2. From chapter 2, equations (2.1) and (2.2) and the following equations represent the system described in the previous section

$$V_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\beta(x)\Delta x)(1-\delta\Delta t)V_n(t,x) + \lambda\sum_{i=1}^{n} c_i V_{n-i}(t,x)\Delta x + \delta\Delta t V_{n+1}(t,x) \qquad n \geq 1 \tag{4.1}$$

$$V_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\beta(x)\Delta x)(1-\delta\Delta t)V_0(t,x) \tag{4.2}$$

$$R_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\gamma(x)\Delta x)(1-\delta\Delta t)R_n(t,x) + \lambda\sum_{i=1}^{n} c_i R_{n-i}(t,x)\Delta t + \delta\Delta t R_{n+1}(t,x) \qquad n \geq 1 \tag{4.3}$$

$$R_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\mu(x)\Delta x)(1-\delta\Delta t)R_0(t,x) \tag{4.4}$$

Following the same process in deriving equations (2.9) and (2.10) from (2.1) and (2.2) respectively, from equations (4.1) to (4.4) we get respectively

$$\frac{\partial}{\partial x}V_n(x) = -(\lambda+\beta(x)+\delta)V_n(x) + \lambda\sum_{i=1}^{n} c_i V_{n-i}(x) + \delta V_{n+1}(x), \qquad n \geq 1 \tag{4.5}$$

$$\frac{\partial}{\partial x}V_0(x) = -(\lambda+\beta(x)+\delta)V_0(x) + \delta V_1(x) \tag{4.6}$$

$$\frac{\partial R_n(x)}{\partial x} = -(\lambda+\gamma(x)+\delta)R_n(x) + \lambda\sum_{i=1}^{n} c_i R_{n-i}(x) + \delta R_{n+1}(x) \quad n \geq 1 \tag{4.7}$$

$$\frac{\partial R_0(x)}{\partial x} = -(\lambda+\gamma(x)+\delta)R_0(x) + \delta R_1(x) \tag{4.8}$$

From chapter two the relevant equations for the assumptions of this model are (2.9), (2.10) and (2.16).

The boundary conditions given in equations (2.17) to (2.20) will be used to solve the above equations by following the same process as for the basic model.

Multiplying equation (4.5) by z^n , summing over n from 1 to ∞ and adding to (4.6) we obtain

$$\frac{\partial}{\partial x}V_q(x,z)+(\lambda-\lambda C(z)+\beta(x)+\delta-\frac{\delta}{z})V_q(x,z)=0 \tag{4.9}$$

Using the same process, from (4.7) and (4.8) we obtain

$$\frac{\partial}{\partial x}R_q(x,z)+(\lambda-\lambda C(z)+\gamma(x)+\delta-\frac{\delta}{z})R_q(x,z)=0 \tag{4.10}$$

Let $\omega=\lambda-\lambda C(z)+\delta-\dfrac{\delta}{z}$ and integrating equation (4.9) from 0 to x, we get

$$V_q(x,z)=V_q(0,z)e^{-\omega x-\int_0^x \beta(t)dt} \tag{4.11}$$

Integrating equation (4.11) by parts with respect to x and using equation (2.31) to substitute for $V_q(0,z)$ we obtain

$$V_q(z)=\frac{pP_q(0,z)G^*(a)\ \left(1-B^*(\omega)\right)}{\omega} \tag{4.12}$$

Multiplying equation (4.11) by $\beta(x)$ and integrating over x we obtain

$$\int_0^\infty V_q(x,z)\beta(x)dx=pP_q(0,z)G^*(a)B^*(\omega) \tag{4.13}$$

Integrating equation (4.10) from 0 to x, we obtain

$$R_q(x,z)=R_q(0,z)e^{-\omega x-\int_0^x \gamma(t)dt} \tag{4.14}$$

Integrating equation (4.14) by parts with regard to x and using equations (2.27) and (2.29) to substitute for $R_q(0,z)$ we obtain

$$R_q(z) = \alpha z P_q(0,z)\left(\frac{\left(1-G^*(a)\right)\left(1-H^*(\omega)\right)}{a\,\omega}\right) \tag{4.15}$$

Multiplying equation (4.14) by $\gamma(x)$ and integrating over x we obtain

$$\int_0^\infty R_q(x,z)\gamma(x)dx = \alpha z P_q(0,z)\left(\frac{1-G^*(a)}{a}\right)H^*(\omega) \tag{4.16}$$

Now using equations (4.13), (4.16) and (2.30) in equation (2.25), to obtain

$$P_q(0,z) = \frac{-amQ}{a\left(z-G^*(a)\left(1-p+pB^*(\omega)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(\omega)} \tag{4.17}$$

From equations (4.17), (2.29), (4.12) and (4.15) we obtain

$$P_q(z) = \frac{-mQ\left(1-G^*(a)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(\omega)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(\omega)} \tag{4.18}$$

$$V_q(z) = \frac{-amQpG^*(a)\ \left(1-B^*(\omega)\right)}{a\omega\left(z-G^*(a)\left(1-p+pB^*(\omega)\right)\right)-\alpha z\omega\left(1-G^*(a)\right)H^*(\omega)} \tag{4.19}$$

$$R_q(z) = \frac{-\alpha zmQ\left(1-G^*(a)\right)\left(1-H^*(\omega)\right)}{a\omega\left(z-G^*(a)\left(1-p+pB^*(\omega)\right)\right)-\alpha z\omega\left(1-G^*(a)\right)F^*(\omega)} \tag{4.20}$$

In this chapter the p.g.f $S_q(z)$ is given by $S_q(z) = P_q(z)+V_q(z)+R_q(z)$.

Then adding equations (4.18), (4.19) and (4.20) we obtain

$$S_q(z) = \frac{-mQ\left(1-G^*(a)\right)\left(\omega+\alpha z\left(1-H^*(\omega)\right)\right)-amQpG^*(a)\ \left(1-B^*(\omega)\right)}{a\omega\left(z-G^*(a)\left(1-p+pB^*(\omega)\right)\right)-\alpha z\omega\left(1-G^*(a)\right)H^*(\omega)} \tag{4.21}$$

The normalization condition $S_q(1)+Q=1$ is used in order to determine Q.

Because of the indeterminate form of $S_q(1)$, L'Hopitals rule is applied twice on equation (4.21), to obtain

$$S_q(1) = \lim_{z \to 1} \frac{N''(z)}{D''(z)} \tag{4.22}$$

where double primes in (4.22) denote the second derivative at $z = 1$. Finding the derivatives at $z = 1$ we have

$$N''(1) = -2Q\lambda E(I)(\lambda E(I) - \delta)\left(\left(1 - G^*(\alpha)\right)\left(1 + \alpha E(R)\right) + \alpha p G^*(\alpha)E(V)\right) \tag{4.23}$$

$$\begin{aligned} D''(1) = 2(\lambda E(I) - \delta)&\left(\left(1 - G^*(\alpha)\right)\left(\lambda E(I) + \alpha\left(1 + \left(\lambda E(I) - \delta\right)E(R)\right)\right)\right. \\ &\left. - \alpha\left(1 - p(\lambda E(I) - \delta)G^*(\alpha)E(V)\right)\right) \end{aligned} \tag{4.24}$$

Therefore,

$$Q = \frac{\alpha\left(1 - p\left(\lambda E(I) - \delta\right)G^*(\alpha)E(V)\right) - \left(1 - G^*(\alpha)\right)\left(\lambda E(I) + \alpha\left(1 + \left(\lambda E(I) - \delta\right)E(R)\right)\right)}{\alpha\left(\delta E(R)\left(1 - G^*(\alpha)\right) + G^*(\alpha)\left(1 + p\delta E(V)\right)\right)} \tag{4.25}$$

From equation (4.25) we can find the utilization factor ρ, where $\rho = 1 - Q$.

As a particular case if we assume that there is no stand by server this means that $\delta = 0$, $m = \alpha = \lambda - \lambda E(I)$ then from the main results we obtain

$$S_q(z) = \frac{-Q\left(1 - G^*(a)\right)\left(m + \alpha z\left(1 - H^*(m)\right)\right) - aQpG^*(a)\left(1 - B^*(m)\right)}{a\left(z - G^*(a)\left(1 - p + pB^*(m)\right)\right) - \alpha z\left(1 - G^*(a)\right)H^*(m)} \tag{4.26}$$

$$Q = 1 - \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} - \frac{1}{\alpha} + \frac{E(R)}{G^*(\alpha)} - E(R) + pE(V)\right) \tag{4.27}$$

This is exactly the case of the basic model in chapter two equations (2.44) and (2.46).

4.4 The Mean Length of the Queue and the Mean Waiting Time

To find L_q , the average queue length in the steady state, where $L_q = \frac{d}{dz} S_q(z)\Big|_{z=1}$, we note that this formula is of the 0/0 form, then using L'Hopital's rule four times we obtain

$$L_q = \lim_{z \to 1} \frac{D''(z)N'''(z) - N''(z)D'''(z)}{3\left(D''(z)\right)^2} \tag{4.28}$$

Where the treble primes denote the third derivative. $N''(1)$ and $D''(1)$ are given in equations (4.23) and (4.24) respectively, and

$$\begin{aligned}
&N'''(1) = \\
&-3Q\Lambda\Psi\left(1 - G^*(\alpha)\right)\left(1 + \alpha E(R)\right) - 6Q\Psi\Phi^2 G^{*'}(\alpha)\left(1 + \alpha E(R)\right) \\
&-3Q\Phi\left(1 - G^*(\alpha)\right)\left(\Omega(1 + \alpha E(R)) + 2\alpha\Psi E(R) + \alpha\Psi^2 E(R^2)\right) \\
&+3pQE(V)\Psi G^*(\alpha)\left(2\Phi^2 - \alpha\Lambda\right) + 6\alpha pQ\Phi^2\Psi E(V) G^{*'}(\alpha) \\
&-3\alpha pQ\Phi G^*(\alpha)\left(\Psi^2 E(V^2) + \Omega E(V)\right)
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&D'''(1) = \\
&3\left(1 - G^*(\alpha) pE(V)\Psi\right)\left(2\Psi\Phi - \alpha\Omega\right) + 6\Phi^2\Psi G^{*'}(\alpha) \\
&+3\left(1 - G^*(\alpha)\right)\left(\Lambda\Psi + \Phi\Omega\right) + 6\alpha\Psi^2\left(1 - G^*(\alpha)\right)E(R) \\
&-3\alpha\Psi p\left(2G^{*'}(\alpha)\Phi\Psi E(V) - G^*(\alpha)\left(E(V^2)\Psi^2 + \Omega E(V)\right)\right) \\
&+3\alpha\left(1 + \Psi E(R)\right)\left(\Omega\left(1 - G^*(\alpha)\right) + 2\Phi\Psi G^{*'}(\alpha)\right) \\
&+3\alpha\Psi\left(1 - G^*(\alpha)\right)\left(\Psi^2 E(R^2) + \Omega E(R)\right)
\end{aligned} \tag{4.30}$$

Where $\Omega = (\lambda E(I(I-1)) + 2\delta)$, $\Psi = \lambda E(I) - \delta$, $\Lambda = \lambda E(I(I-1))$ and $\Phi \lambda \mathrm{E(I)}$.

4.5 A Numerical Example

In order to verify the validity of the results of this chapter, we consider the service times, vacation times, stand-by service times and repair times are all exponentially distributed. We show the effect of the new parameter δ (the stand-by service rate) on the utilization factor, the server idle time, the average length of the queue and the average response time of the customers. All values were arbitrary chosen in order that the stability conditions are satisfied.

4.5.1 Example 1

In this example it is considered that there is no stand by server equipped in the system, i.e. $\delta = 0$. Moreover we consider that $\mu = 5,\ \gamma = 15,\ \lambda = 2,\ \beta = 7$, $E(I) = 1$ and $E(I(I-1)) = 0$, while p takes the values 0.25, 0.5 and 0.75 and α takes the values 1, 2 and 3.

Table 4.1: Some queue performance measures values computed when $(\delta = 0,\ \mu = 5,\ \gamma = 15,\ \lambda = 2,\ \beta = 7)$

α	p	ρ	Q	L_q	W_q	L	W
	0.25	0.4981	0.5019	0.4978	0.2489	0.9959	0.498
1	0.5	0.5695	0.4305	0.6942	0.3471	1.2638	0.6319
	0.75	0.641	0.359	0.9688	0.4844	1.6097	0.8049
	0.25	0.5248	0.4752	0.6118	0.3059	1.1366	0.5683
2	0.5	0.5962	0.4038	0.8413	0.4207	1.4375	0.7188
	0.75	0.6676	0.3324	1.1695	0.5847	1.8371	0.9186
	0.25	0.5514	0.4486	0.7393	0.3697	1.2908	0.6454
3	0.5	0.6229	0.3771	1.0092	0.5046	1.6321	0.816
	0.75	0.6943	0.3057	1.4053	0.7026	2.0995	1.0498

Since we assume that $\delta = 0$ then we have the particular case we mentioned in section 4.3, table 4.1 shows the same results as found in Maraghi, et al. (2010) in table 1.

4.5.2 Example 2

In this example we assume that the arrivals occur in batches of size 3 or 4 with equally likely probability. This means that the probability generating function of the batch size will be given by

$$C(z) = 0.5\left(z^3 + z^4\right)$$

$$C'(z) = 0.5\left(3z^2 + 4z^3\right) \Rightarrow C'(1) = E(I) = 3.5$$
$$C''(z) = 0.5\left(6z + 12z^2\right) \Rightarrow C''(1) = E(I(I-1)) = 0.5(6+12) = 9$$

See Bose (2002) for more details.

Moreover we consider that $\lambda = 2,\ \gamma = 5,\ \alpha = 1,\ \beta = 7,\ p = 0.25$, while μ takes the values 5, 6 and 7 and δ takes the values 3, 4 and 5. In table 4.2 some queue performance measures values are given.

Table 4.2: Some queue performance measures values computed when $\lambda = 2,\ \gamma = 5,\ \alpha = 1,\ \beta = 7,\ p = 0.25$

μ	δ	ρ	Q	L_q	W_q	L	W
	3	0.4494	0.5506	100.2372	50.1186	100.6865	50.3433
5	4	0.4232	0.5768	15.2796	7.6398	15.7029	7.8514
	5	0.4	0.6	1.2752	0.6376	1.6752	0.8376
	3	0.3905	0.6095	81.4748	40.7374	81.8653	40.9327
6	4	0.3695	0.6306	11.5505	5.7753	11.9199	5.96
	5	0.3504	0.6496	0.4615	0.2307	0.8119	0.4059
	3	0.3473	0.6527	70.7303	35.3651	71.0776	35.5388
7	4	0.3295	0.6705	9.4764	4.7382	9.8059	4.903
	5	0.3135	0.6865	0.0692	0.0346	0.3827	0.1914

4.5.3 Example 3

In this example we choose the following values: $\mu = 7,\ \gamma = 3,\ \lambda = 2,\ \alpha = 2,\ p = 0.5,\ E(I) = 1$ and $E(I(I-1)) = 0$, and we consider that β takes the values 6, 7 and 9 while δ takes the values 0, 1 and 3. In table 4.3 some queue performance measures values are given

Table 4.3: Some queue performance measures values computed when $(\mu = 7,\ \gamma = 3,\ \lambda = 2,\ \alpha = 2,\ p = 0.5)$

β	δ	ρ	Q	L_q	W_q	L	W
	0	0.6429	0.3571	1.5587	0.7794	2.2016	1.1008
6	1	0.5455	0.4545	0.8552	0.4276	1.4007	0.7003
	3	0.4186	0.5814	0.4265	0.2132	0.8451	0.4225
	0	0.619	0.381	1.4048	0.7024	2.0238	1.0119
7	1	0.5306	0.4694	0.8064	0.4032	1.3371	0.6685
	3	0.4127	0.5873	0.4142	0.2071	0.8269	0.4134
	0	0.5873	0.4127	1.2357	0.6178	1.823	0.9115
9	1	0.5103	0.4897	0.7474	0.3737	1.2577	0.6288
	3	0.4044	0.5956	0.3972	0.1986	0.8015	0.4008

From tables 4.1, 4.2 and 4.3 we note that increasing the value of α or p increases the traffic intensity, the average queue length and the average response time while the server idle time decreases.

Increasing the value of β, μ or δ decreases the traffic intensity, the average queue length and the average response time, while the server idle time increases. These trends are as expected.

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Chapter Five: On an $M^{(X)}/G/1$ Queueing System with Random Breakdowns, Server Vacations, Delay Times and a Stand-by Server

5.1 Introduction

In this chapter we extend the basic model introduced in chapter two with two additional significant assumptions. The first assumption is that the repair process does not start immediately after a breakdown, consequently there is a delay time before starting repairs. The second assumption is that the system deploys a stand-by server during the vacation period.

In chapter three the concept of the delay times was introduced with the concept of extended vacation times, but in this chapter we do not consider extended vacations. The queueing system studied in chapter four introduced a stand by server who works during the repair process and during the vacation period. In this chapter the stand-by server works only during the vacation period.

The service times, vacation times, repair times and delay times are assumed to follow different general arbitrary distributions while the service times of the stand-by server follow an exponential distribution.

This chapter is organized as follows: section 5.2 gives the assumptions underlying the mathematical model under investigation. Equations governing the system and the queue size distribution at a random epoch are formulated in section 5.3. Two special cases are discussed in section 5.4. The average length of the queue and the average waiting time are given in section 5.5. In section 5.6 we consider a numerical example and use MathCAD to illustrate the results of our application.

5.2 Mathematical Model

In this chapter we consider the mathematical model of chapter two, its features were a batch arrival queueing system, Bernoulli scheduled general vacations, general service times, random breakdowns and general repair times. The new contribution in this chapter is that we assume that there is a stand-by server similar to the one in the model of chapter four but this time the stand-by server serves the customers only during the vacation period, and not during the repair process. Moreover, we assume that there is a delay time waiting for repairs to start.

We recall that; the service times are generally distributed with distribution function $G(x)$. The vacation times have a general distribution with distribution function $B(x)$. We assume that breakdowns occur according to a Poisson distribution with mean breakdown rate $\alpha > 0$. $W(x)$ is the distribution function of the delay times which follow a general distribution. The duration of repairs follows a general (arbitrary) distribution with distribution function $H(x)$. The stand-by service time follows an exponential distribution with stand-by service rate $\delta > 0$.

5.3 Equations Governing the System and the Distribution of the Queue Length at any Point of Time

The equations governing the system of this chapter are a combination of equations from chapter two, where we introduced the basic model, from chapter three where there was a delay time waiting for the repair process to start and from chapter four, where the stand-by server is available in the system.

From chapter two the equations appropriate for this system are the equations of the service probabilities (2.9), (2.10) the equations of the repair probabilities (2.13), (2.14) and the equation of the idle server probabilities (2.16). From chapter three the necessary equations are the delay time probabilities (3.6) and (3.7). From chapter four we use equations (4.5) and (4.6) which are related to the stand-by server during the vacation. The set of the equations mentioned above are the differential equations governing the system we study in this chapter.

The boundary conditions used to solve the equations in the previous paragraph are a mixture of the boundary conditions in chapters two and three. These conditions are (2.17), (2.18), (3.14), (3.15) and (3.16).

All the equations mentioned in the previous paragraphs are listed in appendix A, (A.5).

The results of the equations and the boundary conditions we considered from chapter two results in

$$P_q(z) = P_q(0,z)\left(\frac{1-G^*(a)}{a}\right) \tag{5.1}$$

and

$$\int_0^\infty P_q(x,z)\mu(x)dx = P_q(0,z)G^*(a) \tag{5.2}$$

Which are given in equations (2.29) and (2.30) respectively.

From the equations and conditions we take from chapter three we get

$$D_q(z) = \frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)\left(1-W^*(m)\right)}{a\,m} \tag{5.3}$$

$$R_q(z) = \frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)W^*(m)\left(1-H^*(m)\right)}{a\,m} \tag{5.4}$$

$$\int_0^\infty R_q(x,z)\gamma(x)dx = \frac{\alpha z P_q(0,z)\left(1-G^*(a)\right)W^*(m)H^*(m)}{a} \tag{5.5}$$

Which are given in equations (3.27), (3.31) and (3.32) respectively.

As a result of the equations taken from chapter four we get

$$V_q(z) = \frac{pP_q(0,z)G^*(a)\ \left(1-B^*(\omega)\right)}{\omega} \tag{5.6}$$

and

$$\int_0^\infty V_q(x,z)\beta(x)dx = pP_q(0,z)G^*(a)B^*(\omega) \tag{5.7}$$

These are respectively equations (4.12) and (4.13).

To find $P_q(0,z)$ we need to re solve equation (2.25) given by

$$\begin{aligned} zP_q(0,z) &= (1-p)\int_0^\infty P_q(x,z)\mu(x)dx + \int_0^\infty V_q(x,z)\beta(x)dx \\ &+ \int_0^\infty R_q(x,z)\gamma(x)dx + \lambda Q(C(z)-1) \end{aligned} \tag{5.8}$$

Using equations (5.2), (5.5) and (5.7) in equation (5.8) we obtain

$$P_q(0,z) = \frac{-amQ}{a\left(z - G^*(a)\left(1-p+pB^*(\omega)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.9}$$

From equation (5.9) equations (5.1), (5.3), (5.4) and (5.6) become respectively

$$P_q(z) = \frac{-mQ\left(1-G^*(a)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(\omega)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.10}$$

$$D_q(z) = \frac{-Q\alpha z\left(1-G^*(a)\right)\left(1-W^*(m)\right)}{a\left(z - G^*(a)\left(1-p+pB^*(\omega)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.11}$$

$$R_q(z) = \frac{-\alpha z Q(1-G^*(a))W^*(m)(1-H^*(m))}{a\left(z - G^*(a)\left(1-p+pB^*(\omega)\right)\right) - \alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.12}$$

$$V_q(z) = \frac{-amQpG^*(a)\left(1-B^*(\omega)\right)}{a\,\omega\left(z - G^*(a)\left(1-p+pB^*(\omega)\right)\right) - \alpha\,\omega\, z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.13}$$

In this chapter the p.g.f $S_q(z)$ is given by $S_q(z) = P_q(z) + V_q(z) + D_q(z) + R_q(z)$, so adding equations from (5.10) to (5.13) we obtain

$$S_q(z) = \frac{-\omega Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-W^*(m)H^*(m)\right)\right) - amQpG^*(a)\ \left(1-B^*(\omega)\right)}{a\,\omega\left(z-G^*(a)\left(1-p+pB^*(\omega)\right)\right) - \alpha\,\omega\,z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.14}$$

The normalization condition $S_q(1) + Q = 1$ is used in order to determine Q.

As in chapter four, for $z = 1$, $S_q(1)$ is in the indeterminate of 0/0 form. Therefore, we apply L'Hopital's rule twice on equation (5.14), to obtain

$$S_q(1) = \lim_{z\to 1} \frac{N''(z)}{D''(z)} \tag{5.15}$$

Finding the derivatives at $z = 1$ we have

$$\begin{aligned} N''(1) &= \\ & -2Q\lambda E(I)\left(\lambda E(I)-\delta\right)\left(\left(1-G^*(\alpha)\right)\left(1+\alpha\left(E(D)+E(R)\right)\right)\right. \\ & \left.+ p\alpha G^*(\alpha)E(V)\right) \end{aligned} \tag{5.16}$$

$$\begin{aligned} D''(1) = {} & 2\lambda E(I)\left(\lambda E(I)-\delta\right)\left(1-G^*(\alpha)\right)\left(1+\alpha\left(E(D)+E(R)\right)\right) \\ & -2\alpha\left(\lambda E(I)-\delta\right)G^*(\alpha)\left(1-pE(V)\left(\lambda E(I)-\delta\right)\right) \end{aligned} \tag{5.17}$$

Therefore, adding Q to equation (5.15) and equating to 1 and simplifying we get

$$Q = -\frac{\lambda E(I)\left(1-G^*(\alpha)\right)\left(1+\alpha\left(E(D)+E(R)\right)\right)}{\alpha G^*(\alpha)\left(1+\delta\, pE(V)\right)} - \frac{pE(V)\left(\lambda E(I)-\delta\right)-1}{\left(1+\delta\, pE(V)\right)} \tag{5.18}$$

From equation (5.18) we can find the utilization factor, ρ, where $\rho = 1 - Q$.

5.4 Particular Cases

5.4.1 Case1: No Stand-by Server

If we assume there is no stand by server this means that $\delta = 0$, $m = \alpha$, then from the main results we obtain

$$S_q(z) = \frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-W^*(m)H^*(m)\right)\right)-aQpG^*(a)\ \left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(m)H^*(m)} \tag{5.19}$$

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(D)-E(R)+pE(V)\right) \tag{5.20}$$

The results obtained in equations (5.19) and (5.20) agree with the results given in Khalaf, et al. (2011a).

5.4.2 Case 2: No Delay and no Stand-by Server

If we consider that repairs start immediately after the breakdown and there is no stand by server this means that $E(D) = 0$, $W^*(\omega) = 1$, $\delta = 0$, $m = \alpha$ then from the main results we obtain,

$$S_q(z) = \frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-H^*(m)\right)\right)-aQpG^*(a)\ \left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \tag{5.21}$$

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{5.22}$$

The results obtained in equations (5.21) and (5.22) agree with the results by Maraghi, et. al. (2010).

5.5 The Mean Length of the Queue Size and the Mean Waiting Time

To find the average queue size and consequently the mean waiting time, we follow the procedure as set out in chapter four, section 4.4. Finding the required derivatives of equation (5.14) at $z = 1$, we obtain

$$\begin{aligned} N'''(1) = & -3Q\Psi\left(1-G^*(\alpha)\right)\left(\Lambda\left(1+\left(E(D)+E(R)\right)\right)+2\alpha\Phi\left(E(D)+E(R)\right)\right. \\ & \left.+\alpha\left(\Phi^2\left(E(D^2)+E(R^2)+2E(D)E(R)\right)\right)\right) \\ & -3Q\Phi\left(1+\alpha\left(E(D)+E(R)\right)\right)\left(\Omega\left(1-G^*(\alpha)\right)+2\Phi\Psi G^{*'}(\alpha)\right) \\ & +6pQ\Phi^2 E(V)\Psi\left(G^*(\alpha)+\alpha G^{*'}(\alpha)\right) \\ & -3pQ\alpha G^*(\alpha)E(V)\left(\Lambda\Psi+\Phi\Omega\right)-3pQ\alpha\Phi G^*(\alpha)\Psi^2 E(V^2) \end{aligned} \tag{5.23}$$

$$\begin{aligned} & D'''(1) = \\ & 3\left(1+\Phi G^{*'}(\alpha)-p\Psi G^*(\alpha)E(V)\right)\left(2\Phi\Psi-\alpha\Omega\right) \\ & +\left(1-G^*(\alpha)\right)\left(3\Lambda\Psi+3\Phi\Omega\right)+3\alpha\Omega\left(1-G^*(\alpha)\right)+3\alpha\Phi\left(\Lambda+2\Phi\right)G^{*'}(\alpha) \\ & +3\alpha\Psi\left(-2p\Phi\Psi G^{*'}(\alpha)E(V)+pG^*(\alpha)\left(E(V^2)\Psi^2+\Omega E(V)\right)\right) \\ & +3\alpha\Phi\left(\Lambda+2\Phi\right)\left(1-G^*(\alpha)\right)\left(E(D)+E(R)\right) \\ & +6\alpha\Phi^2\Psi G^{*'}(\alpha)\left(E(D)+E(R)\right) \\ & +3\alpha\Psi\left(1-G^*(\alpha)\right)\left(\Phi^2\left(E(D^2)+E(R^2)+2E(D)E(R)\right)+\Lambda\left(E(D)+E(R)\right)\right) \end{aligned} \tag{5.24}$$

Where $\Omega = (\lambda E(I(I-1))+2\delta)$, $\Psi = \lambda E(I)-\delta$, $\Lambda = \lambda E(I(I-1))$ and $\Phi = \lambda \mathrm{E(I)}$.

$N''(1)$ and $D''(1)$ are given in equations (5.16) and (5.17) respectively.

5.6 A Numerical Example

In order to verify the validity of the results of this chapter, we present the following two examples:

5.6.1 Example 1

In this example the second particular case (no stand-by server) will be considered and the delay times are assumed to have a *k*- Erlang distribution then:

$$W^*(m) = \frac{(k\varphi)^k}{(m+k\varphi)^k}, \quad E(D) = \frac{1}{\varphi}, \quad Var(D) = \frac{1}{k\varphi^2} \quad \text{and} \quad E(D^2) = \frac{(k+1)}{k\varphi^2}$$

All the *k*- Erlang distribution equations are taken from Allen (1990).

It is assumed that the service times, vacation times and repair times are all exponentially distributed.

We consider the following values: $\mu = 7,\ \lambda = 2,\ \alpha = 2, \beta = 5\ \gamma = 4, p = 0.5,$ $E(I) = 1$ and $E(I(I-1)) = 0$. The delay rate φ takes the values 2, 4 and 6, while k takes the values 3, 5 and 7.

Table 5.1: Some queue performance measures values computed when the delay times follow the *k*-Erlang distribution and
$\mu = 7,\ \lambda = 2,\ \alpha = 2, \beta = 5\ \gamma = 4, p = 0.5$

k	φ	ρ	Q	L_q	W_q	L	W
	2	0.9143	0.0857	13.7032	6.8516	14.6175	7.3087
3	4	0.7714	0.2286	3.3976	1.6988	4.169	2.0845
	6	0.7238	0.2762	2.4082	1.2041	3.132	1.566
	2	0.9143	0.0857	13.481	6.7405	14.3952	7.1976
5	4	0.7714	0.2286	3.3768	1.6884	4.1482	2.0741
	6	0.7238	0.2762	2.4005	1.2002	3.1243	1.5622
	2	0.9143	0.0857	13.3857	6.6929	14.3	7.15
7	4	0.7714	0.2286	3.3679	1.6839	4.1393	2.0696
	6	0.7238	0.2762	2.3972	1.1986	3.121	1.5605

5.6.2 Example 2

In this example we assume that the arriving batches are of size 2, then

$C(z) = z^2$

$C'(z) = 2z \Rightarrow C'(1) = E(I) = 2$

$C''(z) = 2 \Rightarrow \quad C''(1) = E(I(I-1)) = 2.$

The delay times follow a Hyperexponential distribution (H_k), with two phases ($k = 2$), the probability of the first phase is $p_1 = 0.3$ and the probability of the second phase is $p_2 = 0.7$, the mean of the first phase ($\varphi_1 = 2$), the mean of the second phase ($\varphi_2 = 3$), thus

$$E(D) = \sum_{i=1}^{2} \frac{p_i}{\varphi_i} = \frac{0.3}{2} + \frac{0.7}{3} = \frac{0.9 + 1.4}{6} = 0.383,$$

$$E(D^2) = 2\sum_{i=1}^{2} \frac{p_i}{(\varphi_i)^2} = 2\left(\frac{0.3}{4} + \frac{0.7}{9}\right) = 0.306.$$

The repair times are assumed to have a *k*- Erlang distribution, with three phases $(k = 3)$ then:

$$E(R) = \frac{1}{\gamma}, \quad Var(R) = \frac{1}{k\gamma^2} = \frac{1}{3\gamma^2} \quad \text{and} \quad E(R^2) = \frac{(k+1)}{k\gamma^2} = \frac{4}{3\gamma^2} \ .$$

We consider the service times to be exponentially distributed with service rate $\mu = 7$ and vacation times are exponentially distributed with vacation rate $\beta = 9$.

Moreover we assume that

$\gamma = 2$, $\lambda = 2$, , $\alpha = 2$ and $p = 0.5$, and that δ takes the values 0, 1 and 3, while γ takes the values 2, 3, 4 and 7.

All values were arbitrarily chosen in order that the stability conditions are satisfied.

Table 5.2: Some queue performance measures values computed when the delay times have Hyperexponential distribution, and the repair times have *k*- Erlang distribution

δ	γ	ρ	Q	L_q	W_q	L	W
0	2	0.9014	0.0986	13.9042	6.9521	14.8056	7.4028
	3	0.8062	0.1938	5.5188	2.7594	6.325	3.1625
	4	0.7585	0.2415	3.8724	1.9362	4.631	2.3155
	7	0.6973	0.3027	2.5682	1.2841	3.2655	1.6328
1	2	0.854	0.146	8.9989	4.4995	9.8529	4.9264
	3	0.7637	0.2363	4.3426	2.1713	5.1063	2.5531
	4	0.7186	0.2814	3.1876	1.5938	3.9062	1.9531
	7	0.6606	0.3394	2.1972	1.0986	2.8579	1.4289
3	2	0.7726	0.2274	5.4294	2.7147	6.202	3.101
	3	0.691	0.309	3.1283	1.5642	3.8193	1.9097
	4	0.6502	0.3498	2.4194	1.2097	3.0696	1.5348
	7	0.5977	0.4023	1.7525	0.8762	2.3502	1.1751

Tables 5.1 and 5.2 show that increasing the value of k, φ, δ or γ decreases the traffic intensity, the average queue length and the average response time while the server idle time increases. These trends are as expected.

In the next graphs, graph 5.1 shows the effect of the stand-by service rate δ on the mean number of customers in the queue, where we consider that $\gamma = 2$

Graph 5.2 shows the effect of the repair rate γ on the mean waiting time in the system, where we consider that $\delta = 1$ (the second four rows in table 5.2).

Figure 5.1: the effect of the stand-by service rate on the mean number of customers in the queue

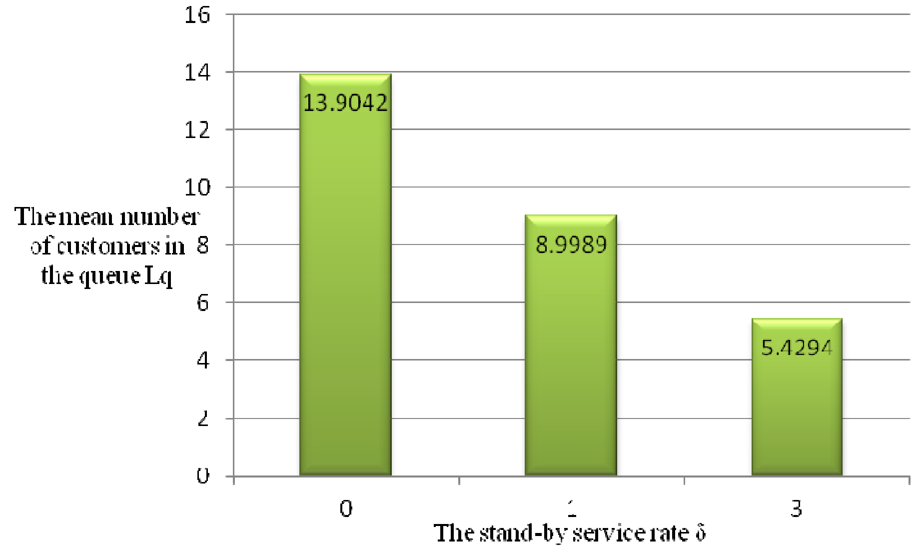
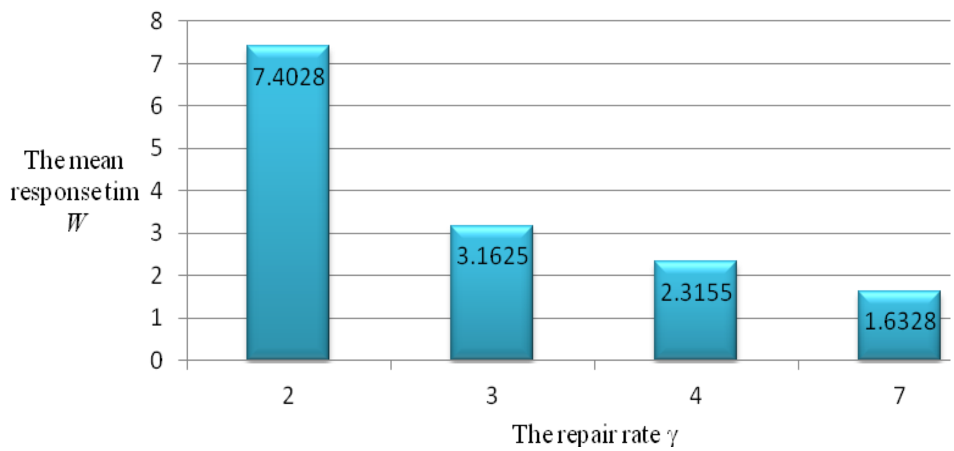


Figure 5.2: the effect of the repair rate on the mean response time W



Chapter Six: On a Batch Arrival Queue with General Vacations Followed by a Further Optional Extended Vacation, Random Breakdowns, and an Exponential Stand-by Server during General Repair Times

6.1 Introduction

In this chapter we study a batch arrival queueing system where after every service completion, the server has the option to go on a vacation of random length with probability p or continue serving with probability $1-p$. The server may go on an extended vacation after the original vacation completion with probability r or rejoins the system to serve the customers directly after the vacation with probability $1-r$. In addition to vacations and extended vacations, the system may suffer from random breakdowns from time to time. A stand-by server is available in addition to the main server**.** The stand-by server provides service to customers only during the repair process. The service times, vacation times, extended vacation times and repair times are assumed to follow general arbitrary distributions while the stand-by service times follow an exponential distribution.

The study of this chapter generalizes the results obtained by Madan (1995).

This chapter is arranged as follows: section 6.2 gives the mathematical model and equations governing the system. In section 6.3 we derive the distribution of the queue length at any point of time. Some special cases have been discussed in section 6.4. The average queue size and the average waiting time are derived in section 6.5. In section 6.6 we consider two numerical examples and use MathCAD to illustrate the results of our applications.

6.2 Mathematical Model and Equations Governing the System

The basic mathematical model of chapter two is considered in this chapter with two more assumptions. The first assumption is that we assume that the server may go on an extended vacation as defined in chapter three. The second added assumption is that we assume that the system is equipped with a stand-by server who works during the repair process. This means that the equations governing the system when the main server serves the customers or the main server is on the normal vacation will be the same equations as for the basic model because in this case the extended vacation time or stand-by service time will not effect the probabilities $P_n(t,x)$ or $V_n(t,x)$, so from the basic model in chapter two we use the differential equations (2.9), (2.10), (2.11) and (2.12).

From chapter three we consider the differential equations related to the extended vacation i.e. equations (3.8), (3.9) and the differential equation of the idle server probabilities (3.10). Finally from chapter four we use the differential equations in the case where the server is under repair and where there is a stand by server to serve the customers, these are equations (4.7) and (4.8).

The boundary equations used to solve the above equations are a combination from chapters two and three. These boundary conditions are (2.18), (2.19), (2.20), (3.11) and (3.13).

All the equations mentioned in the previous paragraphs are listed in appendix A, (A.6).

6.3 The Distribution of the Queue Length at any Point of Time

According to the set of equations and the boundary conditions mentioned in section 6.2, and following the equations taken from chapter two we have

$$P_q(z) = P_q(0,z)\left(\frac{1-G^*(a)}{a}\right) \tag{6.1}$$

$$\int_0^\infty P_q(x,z)\mu(x)dx = P_q(0,z)G^*(a) \tag{6.2}$$

$$V_q(z) = \frac{pP_q(0,z)G^*(a)\ \left(1-B^*(m)\right)}{m} \tag{6.3}$$

$$\int_0^\infty V_q(x,z)\beta(x)dx = pP_q(0,z)G^*(a)B^*(m) \tag{6.4}$$

From the equations taken from chapter three we have

$$E_q(z) = \frac{rpP_q(0,z)G^*(a)B^*(m)\left(1-F^*(m)\right)}{m} \tag{6.5}$$

$$\int_0^\infty E_q(x,z)\varphi(x)dx = rpP_q(0,z)G^*(a)B^*(m)F^*(m) \tag{6.6}$$

And from the equations of chapter four we get

$$R_q(z) = \alpha z P_q(0,z)\left(\frac{\left(1-G^*(a)\right)\left(1-H^*(\omega)\right)}{a\,\omega}\right) \tag{6.7}$$

$$\int_0^\infty R_q(x,z)\gamma(x)dx = \alpha z P_q(0,z)\left(\frac{1-G^*(a)}{a}\right)H^*(\omega) \tag{6.8}$$

Now we need to find $P_q(0,z)$ given in equation (3.19) by using equations (6.2), (6.4), (6.6) and (6.8), we obtain

$$P_q(0,z) = \frac{-amQ}{a\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(\omega)} \tag{6.9}$$

from equation (6.9) equations (6.1), (6.3), (6.5), and (6.7) become respectively

$$P_q(z) = \frac{-mQ\left(1-G^*(a)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(\omega)} \tag{6.10}$$

$$V_q(z) = \frac{-aQpG^*(a)\ \left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(\omega)} \tag{6.11}$$

$$E_q(z) = \frac{-aQrpG^*(a)B^*(m)\left(1-F^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(\omega)} \tag{6.12}$$

$$R_q(z)=\frac{-\alpha zmQ\left(1-G^*(a)\right)\left(1-H^*(\omega)\right)}{a\,\omega\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\,\omega\left(1-G^*(a)\right)H^*(\omega)} \quad (6.13)$$

In this chapter the p.g.f $S_q(z)$ is given by $S_q(z)=P_q(z)+V_q(z)+D_q(z)+R_q(z)$.

Then adding equations (6.10), (6.11), (6.12) and (6.13) we obtain

$$S_q(z)=$$
$$\frac{-mQ\left(1-G^*(a)\right)\left(\omega+\alpha z\left(1-H^*(\omega)\right)\right)-a\omega pQG^*(a)\left(1-B^*(m)\left(1-r\left(1-F^*(m)\right)\right)\right)}{a\,\omega\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\,\omega\left(1-G^*(a)\right)H^*(\omega)} \quad (6.14)$$

The normalization condition $S_q(1)+Q=1$ is used to determine Q.

Because of the indeterminate form of $S_q(1)$, L'Hopital's rule is applied twice on equation (6.14), to obtain

$$S_q(1)=\lim_{z\to 1}\frac{N''(z)}{D''(z)} \quad (6.15)$$

Finding the derivatives at $z=1$ we obtain

$$\begin{aligned}N''(1)=-2Q\lambda E(I)(\lambda E(I)-\delta)\Big(&\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)\\&+p\alpha G^*(\alpha)\left(E(V)+rE(eV)\right)\Big)\end{aligned} \quad (6.16)$$

$$\begin{aligned}D''(1)=\;&2(\lambda E(I)-\delta)\left(1-G^*(\alpha)\right)\left(\lambda E(I)+\alpha+\alpha(\lambda E(I)-\delta)E(R)\right)\\&-2\alpha(\lambda E(I)-\delta)\left(1-p\lambda E(I)G^*(a)\left(E(V)+rE(eV)\right)\right)\end{aligned} \quad (6.17)$$

Therefore, adding Q to equation (6.15) and equating to 1 and simplifying we get

$$\begin{aligned}Q=\;&\frac{-\lambda E(I)\left(1-G^*(\alpha)\right)}{\alpha\left(\delta E(R)\left(1-G^*(\alpha)\right)+G^*(\alpha)\right)}\\&+\frac{\alpha\left(1-p\lambda E(I)G^*(\alpha)\left(E(V)+rE(eV)\right)-\left(1-G^*(\alpha)\right)\left(1+(\lambda E(I)-\delta)E(R)\right)\right)}{\alpha\left(\delta E(R)\left(1-G^*(\alpha)\right)+G^*(\alpha)\right)}\end{aligned} \quad (6.18)$$

From equation (6.18) we can find the utilization factor, ρ, where $\rho=1-Q$.

6.4 Particular Cases

6.4.1 Case 1: No Extended Vacation

Once the server finishes the original vacation it starts to serve the customers immediately and there is no extended vacation time. We let $r = 0$, then from the main results we obtain

$$S_q(z) = \frac{-mQ\left(1-G^*(a)\right)\left(\omega+\alpha z\left(1-H^*(\omega)\right)\right)-a\omega pQG^*(a)\left(1-B^*(m)\right)}{a\,\omega\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\,\omega\left(1-G^*(a)\right)H^*(\omega)} \tag{6.19}$$

$$\begin{aligned} Q = & \frac{-\lambda E(I)\left(1-G^*(\alpha)\right)}{\alpha\left(\delta E(R)\left(1-G^*(\alpha)\right)+G^*(\alpha)\right)} \\ & + \frac{\alpha\left(1-p\lambda E(I)G^*(\alpha)E(V)-\left(1-G^*(\alpha)\right)\left(1+(\lambda E(I)-\delta)E(R)\right)\right)}{\alpha\left(\delta E(R)\left(1-G^*(\alpha)\right)+G^*(\alpha)\right)} \end{aligned} \tag{6.20}$$

6.4.2 Case 2: No Stand-by Server

If we assume there is no stand by server this means that $\delta = 0$, $m = \omega$, then from the main results we obtain

$$\begin{aligned} & S_q(z) = \\ & \frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-H^*(m)\right)\right)-apQG^*(a)\left(1-B^*(m)\left(1-r(1-F^*(m))\right)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\left(1-r+rF^*(m)\right)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)} \end{aligned} \tag{6.21}$$

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}-\frac{1}{\alpha}+\frac{E(R)}{G^*(\alpha)}-E(R)+p\left(E(V)+rE(eV)\right)\right) \tag{6.22}$$

6.4.3 Case 3. No Extended Vacation and no Stand-by Server

If we consider that services start immediately and there is no stand by server this means that, $r = 0$, $\delta = 0$, $m = \omega$ using this in the main results of this chapter, we get,

$$S_q(z) = \frac{-Q\left(1-G^*(a)\right)\left(m+\alpha z\left(1-H^*(m)\right)\right) - apQG^*(a)\left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right) - \alpha z\left(1-G^*(a)\right)H^*(m)} \tag{6.23}$$

$$Q = 1 - \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} - \frac{1}{\alpha} + \frac{E(R)}{G^*(\alpha)} - E(R) + pE(V)\right) \tag{6.24}$$

The results obtained in equations (6.23) and (6.24) agree with the results by Maraghi, et. al. (2010).

6.5 The Mean Length of the Queue Size and the Mean Waiting Time

To find L_q , the steady state of the average queue length and because the formula $L_q = \frac{d}{dz} S_q(z)\Big|_{z=1}$ is in an indeterminate form, we have to use L'Hopital's rule four times to obtain

$$L_q = \lim_{z \to 1} \frac{D''(z)N'''(z) - N''(z)D'''(z)}{3\left(D''(z)\right)^2} \tag{6.25}$$

Where $N''(1)$ and $D''(1)$ are given in equations (6.16) and (6.17) respectively, and

$$\begin{aligned}
N'''(1) = {}& \\
& -3Q\Psi\left(1+\alpha E(R)\right)\left(\Lambda\left(1-G^*(\alpha)\right) + 2\Phi^2 G^{*\prime}(\alpha)\right) \\
& -3Q\Phi\left(1-G^*(\alpha)\right)\left(\Omega\left(1+E(R)\right) + 2\alpha\Psi E(R) + \alpha\Psi^2 E(R^2)\right) \\
& +3pQ\Phi\left(E(V)+rE(eV)\right)\left(2\Phi\Psi\left(G^*(\alpha)+\alpha G^{*\prime}(\alpha)\right) - \alpha\Omega G^*(\alpha)\right) \\
& -3pQ\alpha\Psi G^*(\alpha)\left(\Phi^2\left(E(V^2)+rE(eV^2)+2rE(V)E(eV)\right) + \Lambda\left(E(V)+rE(eV)\right)\right)
\end{aligned} \tag{6.26}$$

$$
\begin{aligned}
&D'''(1) = \\
&3(2\Phi\Psi - \alpha\Omega)\left(1 + \Phi G^{*'}(\alpha) - p\Phi G^{*}(a)\big(E(V) + rE(eV)\big)\right) \\
&+ (3\Lambda\Psi + 3\Phi\Omega)\big(1 - G^{*}(\alpha)\big) + 3\alpha\Omega\big(1 + \Psi E(R)\big)\big(1 - G^{*}(\alpha)\big) \\
&- 3\alpha\Psi\Big(2p\Phi^{2} G^{*'}(\alpha)\big(E(V) + rE(eV)\big) \\
&- pG^{*}(\alpha)\Big(\Phi^{2}\big(E(V^{2}) + rE(eV^{2}) + 2rE(V)E(eV)\big) + \Lambda\big(E(V) + rE(eV)\big)\Big)\Big) \\
&+ 3\alpha\Omega\Phi G^{*'}(\alpha) + 3\alpha\Psi\big(E(R)(2\Phi + \Lambda) + E(R^{2})\Psi^{2}\big)\big(1 - G^{*}(\alpha)\big) \\
&+ 6\alpha\Phi\Psi G^{*'}(\alpha)\big(1 + E(R)\Psi\big)
\end{aligned}
\tag{6.27}
$$

Where

$\Omega = (\lambda E(I(I-1)) + 2\delta)$, $\Psi = \lambda E(I) - \delta$, $\Lambda = \lambda E(I(I-1))$ and $\Phi = \lambda \mathrm{E(I)}$.

6.6 A Numerical Example

In this section we illustrate some numerical results to show the effect of the new contributions (the extended vacation times and the stand-by service times) on the performance measures of the system. To illustrate the results of this chapter numerically we consider the service times, vacation times, stand-by service times and repair times to be exponentially distributed. All values were chosen arbitrarily in order that all stability conditions are satisfied.

6.6.1 Example 1

In this example it is assumed that the extended vacation times follow a Hyperexponential distribution (Hyper-*k*, H_k), with two phases ($k = 2$), the probability of the first phase is $p_1 = 0.4$ and the probability of the second phase is $p_2 = 0.6$, the mean of the first phase ($\theta_1 = 3$), the mean of the second phase ($\theta_2 = 4$), thus

$$E(eV) = \frac{p_1}{\theta_1} + \frac{p_2}{\theta_2} = \frac{0.4}{3} + \frac{0.6}{4} = 0.2833$$

$$E(eV^2) = 2\left(\frac{p_1}{(\theta_1)^2} + \frac{p_2}{(\theta_2)^2}\right) = 2\left(\frac{0.4}{9} + \frac{0.6}{16}\right) = 0.1638 \ .$$

All the Hyperexponential distribution (H_k) equations are taken from Allen (1990).

Moreover it is assumed that $\lambda = 3$, $\mu = 7$, $\gamma = 4$, $\alpha = 1$, $p = 0.5$, $r = 0.5$, $E(I) = 1$ and $E(I(I-1)) = 0$, we consider that β takes the values 6, 7 and, 8 and δ takes the values 0, 1 and 2.

Table 6.1: Some queue performance measures values computed when $\lambda = 3$, $\mu = 7$, $\gamma = 4$, $\alpha = 1$, $p = 0.5$, $r = 0.5$

β	δ	ρ	Q	L_q	W_q	L	W
	0	0.9982	0.0018	569.2825	189.7608	570.2807	190.0936
6	1	0.9638	0.0362	23.8405	7.9468	24.8043	8.2681
	2	0.9316	0.0684	9.3192	3.1064	10.2509	3.417
	0	0.9625	0.0375	25.7738	8.5913	26.7363	8.9121
7	1	0.9293	0.0707	11.3994	3.7998	12.3287	4.1096
	2	0.8983	0.1017	5.7771	1.9257	6.6754	2.2251
	0	0.9357	0.0643	14.3487	4.7829	15.2844	5.0948
8	1	0.9034	0.0966	7.9323	2.6441	8.8358	2.9453
	2	0.8733	0.1267	4.3683	1.4561	5.2416	1.7472

6.6.2 Example 2

In this example we consider that the extended vacation times follow an exponential distribution. It is considered that $\mu = 7$, $\lambda = 2$, $\alpha = 2$, $\delta = 3$, $\gamma = 5$, $r = 0.5$, $p = 0.5$, $E(I) = 1$ and $E(I(I-1)) = 0$, while θ takes the values 2, 3, and 4 and β takes the values 5, 7, and 9.

Table 6.2: Some queue performance measures values computed when $(\mu = 7,\ \lambda = 2,\ \alpha = 2,\ \delta = 3,\ \gamma = 5,\ r = 0.5,\ p = 0.5)$

β	θ	ρ	Q	L_q	W_q	L	W
5	2	0.7256	0.2744	3.7293	1.8647	4.4549	2.2275
	3	0.6545	0.3455	2.4135	1.2068	3.068	1.534
	4	0.6189	0.3811	1.9865	0.9932	2.6054	1.3027
7	2	0.6768	0.3232	2.8976	1.4488	3.5744	1.7872
	3	0.6057	0.3943	1.9152	0.9576	2.5209	1.2605
	4	0.5701	0.4299	1.5874	0.7937	2.1575	1.0787
9	2	0.6497	0.3503	2.5494	1.2747	3.1991	1.5996
	3	0.5786	0.4214	1.6997	0.8498	2.2783	1.1391
	4	0.543	0.457	1.413	0.7065	1.956	0.978

Tables 6.1 and 6.2 show that increasing the value of δ, β or θ decreases the traffic intensity, the average queue length and the average response time while the server idle time increases. These trends are as expected.

Chapter Seven: Conclusions and Future Work

7.1 Conclusions

In this research we investigated the batch arrival queuing system $M^{[X]}/G/1$, with random breakdowns and Bernoulli scheduled vacations where after completion of the service of a customer the server may take a vacation with probability p or stay in the system to serve customers with probability $1-p$. We investigated this system by extending it in many directions. In each chapter we added more than one new assumption. In this way we developed a different, more advanced queuing system. A number of queuing systems developed by many authors are special cases of our systems. In each chapter our goals were to find the closed form solution of important performance measures including the intensity parameter (the utilization factor), the mean idle time, the mean number of customers in the queue, the mean number of customers in the system, the mean waiting time in the queue and the mean response time.

The supplementary variable technique has been used to solve the system of equations. The elapsed service time, elapsed vacation time, elapsed repair time, elapsed delay time and the elapsed extended vacation time have been introduced as supplementary variables.

In chapter two we described the basic mathematical model. This was introduced to avoid duplicating these fundamental equations in later chapters. For the next four chapters we added new assumptions to the basic model and developed a new queueing system.

In chapter three we considered the basic model with two added assumptions. The first assumption is that after a breakdown occurs the server does not enter the repair process immediately instead it may have to wait for a period of time called the "delay time", until it starts being repaired. The second assumption is that when the server finishes a vacation period it does not enter the system immediately to start serving the customers. The server can wait for an extra period of time for possible required actions before the first service. We call this period of time the “extended vacation time” where we introduced this term for the first time.

The stand-by server available during any interruption is the new contribution added to the basic model and analyzed in chapter four. In this chapter we study the batch arrival queuing system assuming that there is a stand-by server to serve the customers during the vacation time and during the repair process.

In chapter five we developed chapter two by adding a stand-by server to serve the customers during the vacation times. Moreover in this chapter it is assumed that when the server breaks down it does not enter the repair process immediately, where there is a delay time waiting for repairs to start.

Finally in chapter six the basic model was extended by adding two more assumptions, where it is assumed that the server can go on an extended vacation immediately after the original vacation. The system in chapter six is equipped with a stand-by server to serve the customers during the repair process.

Throughout all the four chapters (from chapter three to six), we conclude that increasing the delay rate and extended vacation rate increases the server idle time and decreases the mean waiting time, the mean response time, the mean number of customers in the system and the value of traffic intensity. Also we conclude that increasing the stand-by service rate decreases the server idle time and increases the traffic intensity, mean waiting time, mean response time and mean number of customers in the system.

Although the conclusions are in accordance with what is expected systematically and logically, much work went into the details of establishing the closed form solutions and to determine the critical values of performance measures of each system studied. Consequently we expect that our work will greatly help system designers in their decision regarding the system parameter and that this work provides new knowledge in queueing theory.

7.2 Future Work

The researchers suggest the following queueing systems to be developed according to the results found in this research:

(1) Batch arrival queueing system with random breakdowns, Bernoulli scheduled general vacation times, general service times, general extended vacation times, general delay times, general repair times and general stand-by server works during every main server interruption. In this case all the new systems introduced in this research investigation (from chapter two to chapter six) would be special cases of this system.

(2) Batch arrival queueing system with random breakdowns, server Bernoulli schedule vacation, general service times and general delay times where the server provides two phases of heterogeneous service and the rates of breakdowns are different in every service breakdown.

(3) Batch arrival queueing system with server Bernoulli schedule vacation, general service times, general extended vacation times, random breakdowns, general delay times and two types of general repairs.

(4) Batch arrival queueing systems with random breakdowns and server vacations based on multiple vacation policy or N-policy with general service times, general delay times and general extended vacation times.

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Appendix A

A.1 Explanation of Obtaining $g(x)$

$$\mu(x) = \frac{g(x)}{(1 - G(x))}$$

$$\int_0^x \mu(s)ds = -\ln(1 - G(s))\Big]_0^x$$

$$-\int_0^x \mu(s)ds = \ln\left(\frac{1 - G(x)}{1 - G(0)}\right) \qquad G(0) = 0$$

$$e^{-\int_0^x \mu(s)ds} = (1 - G(x))$$

$$-\mu(x)e^{-\int_0^x \mu(s)ds} = -g(x) \quad \Rightarrow \quad g(x) = \mu(x)e^{-\int_0^s \mu(s)ds}$$

A.2 Details of Integrating Equation (2.28) by Parts

$$\int_0^\infty P_q(x,z)dx = \int_0^\infty P_q(0,z)e^{-ax-\int_0^x \mu(t)dt}dx$$

Let $u = e^{-\int_0^x \mu(t)dt} \qquad \Rightarrow du = -\mu(x)e^{-\int_0^x \mu(t)dt}dx$

Let $dv = e^{-ax}dx \qquad \Rightarrow v = \dfrac{-1}{a}e^{-ax}$

$\int udv = uv - \int vdu$

$$\int_0^\infty e^{-ax-\int_0^x \mu(t)dt}dx = \left[\frac{-e^{-ax}e^{-\int_0^x \mu(t)dt}}{a}\right]_0^\infty - \int_0^\infty \left(\frac{1}{a}e^{-ax}\left(\mu(x)e^{-\int_0^x \mu(t)dt}\right)\right)dx$$

$$= \frac{1}{a} - \frac{1}{a}\int_0^\infty e^{-ax}\mu(x)e^{-\int_0^x \mu(t)dt}dx = \frac{1}{a} - \frac{1}{a}\int_0^\infty e^{-ax}dG(x) = \frac{1}{a}(1 - G^*(a))$$

A.3 Details of Applying L'Hopital's Rule on Equation (2.44) to Obtain Q in Chapter Two

A.3.1 Applying L'Hopital's to Find $\lim_{z\to 1} S_q(z)$

$$S_q(z)=\frac{-Q(1-G^*(a))\{m+\alpha z(1-H^*(m))\}-apQG^*(a)\ \left(1-B^*(m)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)}=\frac{N(z)}{D(z)}$$

$$N(z)=-Q(1-G^*(a))\{m+\alpha z(1-H^*(m))\}-apQG^*(a)\ \left(1-B^*(m)\right)$$

$$N'(z)=-Q\left(-G^{*'}(a)a'\right)\left(m+\alpha z(1-H^*(m))\right)-Q\left(1-G^*(a)\right)\left(m'+\alpha(1-H^*(m))+\alpha z(-H^{*'}(m)m')\right)$$
$$-pQa'G^*(a)\ \left(1-B^*(m)\right)-pQaG^{*'}(a)a'\left(1-B^*(m)\right)-pQaG^*(a)\left(-B^{*'}(m)m'\right)$$

$$N'(1)=-Q\left(1-G^*(a)\right)\left(m'+\alpha(-H^{*'}(m)m')\right)-pQaG^*(a)\left(-B^{*'}(m)m'\right)$$

$$\begin{aligned}N'(1)&=-Q\left(1-G^*(\alpha)\right)\left(-\lambda E(I)-\alpha\lambda E(I)E(R)\right)+pQ\alpha G^*(\alpha)\ \lambda E(I)E(V)\\&=\lambda E(I)Q\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+p\alpha\lambda E(I)QG^*(\alpha)\ E(V)\end{aligned}$$

$$D(z)=a\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)-\alpha z\left(1-G^*(a)\right)H^*(m)$$

$$D'(z)=a'\left(z-G^*(a)\left(1-p+pB^*(m)\right)\right)+a\left(1-G^{*'}(a)a'\left(1-p+pB^*(m)\right)-G^*(a)\left(pB^{*'}(m)m'\right)\right)$$
$$-\alpha\left(1-G^*(a)\right)H^*(m)-\alpha z\left(-G^{*'}(a)a'\right)H^*(m)-\alpha z\left(1-G^*(a)\right)H^{*'}(m)m'$$

$$D'(1)=a'\left(1-G^*(a)\right)+a\left(1-G^{*'}(a)a'-G^*(a)pB^{*'}(m)m'\right)$$
$$-\alpha\left(1-G^*(a)\right)+\alpha G^{*'}(a)a'-\alpha\left(1-G^*(a)\right)H^{*'}(m)m'$$

$$D'(1)=\alpha\left(1-pG^*(\alpha)\lambda E(I)E(V)\right)-\lambda E(I)\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)-\alpha\left(1-G^*(\alpha)\right)$$

$$\lim_{z\to 1}S_q(z)=\frac{\lambda E(I)Q\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)+p\alpha\lambda E(I)QG^*(\alpha)\ E(V)}{\alpha\left(1-pG^*(\alpha)\lambda E(I)E(V)\right)-\lambda E(I)\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)\right)-\alpha\left(1-G^*(\alpha)\right)}$$

A.3.2 Finding Q

$$\lim_{z\to 1} S_q(z) + Q = 1$$

$$\frac{\lambda E(I)Q(1-G^*(\alpha))(1+\alpha E(R)) + p\alpha\lambda E(I)QG^*(\alpha)\ E(V)}{\alpha(1-pG^*(\alpha)\lambda E(I)E(V)) - \lambda E(I)(1-G^*(\alpha))(1+\alpha E(R)) - \alpha(1-G^*(\alpha))} + Q = 1$$

$$\frac{\alpha G^*(\alpha)Q}{\alpha(1-pG^*(\alpha)\lambda E(I)E(V)) - \lambda E(I)(1-G^*(\alpha))(1+\alpha E(R)) - \alpha(1-G^*(\alpha))} = 1$$

$$Q = \frac{\alpha(1-pG^*(\alpha)\lambda E(I)E(V)) - \lambda E(I)(1-G^*(\alpha))(1+\alpha E(R)) - \alpha(1-G^*(\alpha))}{\alpha G^*(\alpha)}$$

$$= \frac{\alpha}{\alpha G^*(\alpha)} - \frac{\alpha p G^*(\alpha)\lambda E(I)E(V)}{\alpha G^*(\alpha)}$$

$$-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{\alpha E(R)}{\alpha G^*(\alpha)} - \frac{G^*(\alpha)}{\alpha G^*(\alpha)} - \frac{\alpha G^*(\alpha)E(R)}{\alpha G^*(\alpha)}\right) + \frac{-\alpha}{\alpha G^*(\alpha)} + \frac{\alpha G^*(\alpha)}{\alpha G^*(\alpha)}$$

$$Q = -p\lambda E(I)E(V) - \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{E(R)}{G^*(\alpha)} - \frac{1}{\alpha} - E(R)\right) + 1$$

$$Q = 1 - \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{E(R)}{G^*(\alpha)} - \frac{1}{\alpha} - E(R) + pE(V)\right)$$

Where

The expression	**Its value when** $z=1$
$C(z)$	1
$C'(z)$	$E(I)$
$a = \lambda - \lambda C(z) + \alpha$	α
$\frac{da}{dz} = a' = -\lambda C'(z)$	$-\lambda E(I)$
$m = \lambda - \lambda C(z)$	0
$\frac{dm}{dz} = m' = -\lambda C'(z)$	$-\lambda E(I)$
$B^*(m)$	1
$\frac{dB^*(m)}{dz} = \frac{dB^*(m)}{dm} \times \frac{dm}{dz}$	$B^{*\prime}(0)(-\lambda C'(0)) = (-E(V))(-\lambda E(I))$ $= \lambda E(I)E(V)$

$H^*(m)$	1
$\frac{dH^*(m)}{dz} = \frac{dH^*(m)}{dm} \times \frac{dm}{dz}$	$H^{*'}(0)\left(-\lambda C'(0)\right) = \lambda E(I)E(R)$

A.4 Details of Obtaining Equation (2.48)

$$L_q = \frac{d}{dz} S_q(z)\Big|_{z=1}$$

$$S_q(z) = \frac{N(z)}{D(z)} \qquad \left(N(1) = 0 \ \& \ D(1) = 0\right)$$

$$\Rightarrow \frac{dS_q(z)}{dz} = \frac{D(z).N'(z) - N(z)D'(z)}{\left(D(z)\right)^2} = \frac{0}{0}$$

By L'Hopital's rule for the first time

$$\frac{d\left(D(z).N'(z) - N(z)D'(z)\right)}{d\left(\left(D(z)\right)^2\right)} = \frac{D(z).N''(z) - N(z)D''(z)}{2D(z).D'(z)} = \frac{0}{0}$$

By L'Hopital's rule for the second time

$$\frac{d\left(D(z).N''(z) - N(z)D''(z)\right)}{d\left(2D(z).D'(z)\right)} = \frac{D(z).N'''(z) + D'(z).N''(z) - N'(z)D''(z) - N(z)D'''(z)}{2D'(z).D'(z) + 2D(z).D''(z)}$$

$$\therefore L_q = \lim_{z \to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2\left(D'(z)\right)^2}$$

A.5 Equations Governing the System Studied in Chapter Five

The set of the differential equations governing chapter five are as follows

$$\frac{\partial}{\partial x} P_n(x) = -(\lambda + \mu(x) + \alpha) P_n(x) + \lambda \sum_{i=1}^{n} c_i P_{n-i}(x) \qquad n \geq 1 \tag{2.9}$$

$$\frac{\partial}{\partial x} P_0(x) = -(\lambda + \mu(x) + \alpha) P_0(x) \tag{2.10}$$

$$\frac{\partial}{\partial x} R_n(x) + (\lambda + \gamma(x)) R_n(x) = \lambda \sum_{i=1}^{n} c_i R_{n-i}(x) \quad n \geq 1 \tag{2.13}$$

$$\frac{\partial}{\partial x} R_0(x) = -(\lambda + \gamma(x)) R_0(x) = 0 \tag{2.14}$$

$$\lambda Q=\int_0^\infty R_0(x)\gamma(x)dx+\int_0^\infty V_0(x)\beta(x)dx+(1-p)\int_0^\infty P_0(x)\mu(x)dx \tag{2.16}$$

$$\frac{\partial}{\partial x}D_n(x)+(\lambda+\varphi(x))D_n(x)=\lambda\sum_{i=1}^{n}c_iD_{n-i}(x) \quad n\ge 1 \tag{3.6}$$

$$\frac{\partial}{\partial x}D_0(x)=0 \tag{3.7}$$

$$\frac{\partial}{\partial x}V_n(x)=-(\lambda+\beta(x)+\delta)V_n(x)+\lambda\sum_{i=1}^{n}c_iV_{n-i}(x)+\delta V_{n+1}(x), \qquad n\ge 1 \tag{4.5}$$

$$\frac{\partial}{\partial x}V_0(x)=-(\lambda+\beta(x)+\delta)V_0(x)+\delta V_1(x) \tag{4.6}$$

The set of the boundary conditions of chapter five are as follows

$$\begin{aligned}P_n(0)=(1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx+\int_0^\infty V_{n+1}(x)\beta(x)dx\\ +\int_0^\infty R_{n+1}(x)\gamma(x)dx+\lambda c_{n+1}Q \qquad n\ge 0\end{aligned} \tag{2.17}$$

$$V_n(0)=p\int_0^\infty P_n(x)\mu(x)dx, \qquad n\ge 0 \tag{2.18}$$

$$D_n(0)=\alpha\int_0^\infty P_{n-1}(x)dx=\alpha P_{n-1} \qquad n\ge 1 \tag{3.14}$$

$$R_n(0)=\int_0^\infty D_n(x)\varphi(x)dx, \qquad n\ge 0 \tag{3.15}$$

$$D_0(0)=R_0(0)=0 \tag{3.16}$$

A.6 Equations Governing the System Studied in Chapter Six

The set of the differential equations of chapter six are as follows

$$\frac{\partial}{\partial x} P_n(x) = -(\lambda + \mu(x) + \alpha) P_n(x) + \lambda \sum_{i=1}^{n} c_i P_{n-i}(x) \qquad n \geq 1 \tag{2.9}$$

$$\frac{\partial}{\partial x} P_0(x) = -(\lambda + \mu(x) + \alpha) P_0(x) \tag{2.10}$$

$$\frac{\partial}{\partial x} V_n(x) = -(\lambda + \beta(x)) V_n(x) + \lambda \sum_{i=1}^{n} c_i V_{n-i}(x) \qquad n \geq 1 \tag{2.11}$$

$$\frac{\partial}{\partial x} V_0(x) = -(\lambda + \beta(x)) V_0(x) \tag{2.12}$$

$$\frac{\partial}{\partial x} E_n(x) + (\lambda + \theta(x)) E_n(x) = \lambda \sum_{i=1}^{n} c_i E_{n-i}(x) \quad n \geq 1 \tag{3.8}$$

$$\frac{\partial}{\partial x} E_0(x) = -(\lambda + \theta(x)) E_0(x) \tag{3.9}$$

$$\begin{aligned} \lambda Q &= \int_0^\infty R_0(x)\gamma(x)dx + (1-p)\int_0^\infty P_0(x)\mu(x)dx \\ &\quad + (1-r)\int_0^\infty V_0(x)\beta(x)dx + \int_0^\infty E_0(x)\theta(x)dx \end{aligned} \tag{3.10}$$

$$\frac{\partial R_n(x)}{\partial x} = -(\lambda + \gamma(x) + \delta) R_n(x) + \lambda \sum_{i=1}^{n} c_i R_{n-i}(x) + \delta R_{n+1}(x) \quad n \geq 1 \tag{4.7}$$

$$\frac{\partial R_0(x)}{\partial x} = -(\lambda + \gamma(x) + \delta) R_0(x) + \delta R_1(x) \tag{4.8}$$

The set of the boundary conditions of chapter six are as follows

$$\begin{aligned} P_n(0) &= (1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx + (1-r)\int_0^\infty V_{n+1}(x)\beta(x)dx \\ &\quad + \int_0^\infty E_{n+1}(x)\theta(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + \lambda c_{n+1} Q \qquad n \geq 0 \end{aligned} \tag{3.11}$$

$$E_n(0) = r\int_0^\infty V_n(x)\beta(x)dx \qquad n \geq 0 \tag{3.13}$$

$$V_n(0) = p\int_0^\infty P_n(x)\mu(x)dx, \qquad n \geq 0 \tag{2.18}$$

$$R_n(0) = \alpha\int_0^\infty P_{n-1}(x)dx = \alpha P_{n-1} \qquad n \geq 1 \tag{2.19}$$

$$R_0(0) = 0 \tag{2.20}$$

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Appendix B

B.1 Full Explanation of the System Equations in Chapter Two

B.1.1 Equation (2.1)

$$P_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\mu(x)\Delta x)(1-\alpha\Delta t)P_n(t,x)$$
$$+\lambda\sum_{i=1}^{n} c_i P_{n-i}(t,x)\Delta t \qquad n \geq 1$$

By connecting the system probabilities at time t (x) with those at time $t+\Delta t$ ($x+\Delta x$) by considering $P_n(t+\Delta t, x+\Delta x)$ which means the probability that at time $t+\Delta t$, there are n ($n \geq 0$) customers in the queue excluding the customer in service and the elapsed service time of this customer is $x+\Delta x$. Then we have the following two mutually exclusive cases:

(1) At time t, there are n customers in the queue excluding the customer in service and the elapsed service time of this customer is x and there is no arrival, no service completion and no breakdown during $(t, t+\Delta t)$. This case has the joint probability $(1-\lambda\Delta t)(1-\mu(x)\Delta t)(1-\alpha\Delta t)P_n(t,x)$.

(2) At time t, there are $n-i$ customers in the queue excluding the customer in service and the elapsed service time of this customer is x and a batch of size i customers arrives at the system during $(t, t+\Delta t)$. This case has the probability $\lambda\sum_{i=1}^{n} c_i P_{n-i}(t,x)\Delta t$.

B.1.2 Equation (2.2)

$$P_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\mu(x)\Delta x)(1-\alpha\Delta t)P_0(t,x)$$

By connecting the system probabilities at time t with those at time $t+\Delta t$ by considering $P_0(t+\Delta t, x+\Delta x)$ which means the probability that at time $t+\Delta t$, there are no customers in the queue excluding the customer in service and the elapsed service time of this customer is $x+\Delta x$. Then we have only the following case:

(1) At time t, there are no customers in the queue excluding the customer in service and the elapsed service time of this customer is x and there is no arrival, no service completion and no breakdown during $(t, t+\Delta t)$. This case has the joint probability $(1-\lambda\Delta t)(1-\mu(x)\Delta x)(1-\alpha\Delta t)P_0(t,x)$.

B.1.3 Equation (2.3)

$$V_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\beta(x)\Delta x)V_n(t,x) + \lambda\sum_{i=1}^{n} c_i V_{n-i}(t,x)\Delta x \qquad n \geq 1$$

By connecting the system probabilities at time t with those at time $t+\Delta t$ by considering $V_n(t+\Delta t, x+\Delta x)$ which means the probability that at time $t+\Delta t$, there are n $(n \geq 0)$ customers in the queue and the server is on vacation with elapsed vacation time $x+\Delta x$. Then we have the following two mutually exclusive cases:

(1) At time t, there are n $(n \geq 0)$ customers in the queue and the server is on vacation with elapsed vacation time x and there is no arrival and no vacation period completion during $(t, t+\Delta t)$. This case has the joint probability $(1-\lambda\Delta t)(1-\beta(x)\Delta x)V_n(t,x)$.

(2) At time t, there are $n-i$ customers in the queue and the server is on vacation with elapsed vacation time x and a batch of size I customers arrives at the system during $(t, t+\Delta t)$. This case has the probability $\lambda\sum_{i=1}^{n} c_i V_{n-i}(t,x)\Delta t$.

The same explanation can be considered to explain equations (2.5), (3.1) and (3.3), bearing in mind the state of the server in each equation.

B.1.4 Equation (2.4)

$$V_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\beta(x)\Delta x)V_0(t,x)$$

By connecting the system probabilities at time t with those at time $t+\Delta t$ by considering $V_0(t+\Delta t, x+\Delta x)$ which means the probability that at time $t+\Delta t$, there are no customers in the queue and the server is on vacation with elapsed vacation time $x+\Delta x$. Then we have the following case:

(1) At time *t*, there are no customers in the queue and the server is on vacation with elapsed vacation time x and there is no arrival, no vacation period completion during $(t, t+\Delta t)$. This case has the joint probability $(1-\lambda\Delta t)(1-\beta(x)\Delta x)V_0(t,x)$.

The same explanation can be considered to explain equations (2.6) and (3.4), bearing in mind the state of the server in each equation.

B.1.5 Equation (2.7)

$$Q(t+\Delta t) = (1-\lambda\Delta t)Q(t) + (1-p)\int_0^\infty P_0(t,x)\mu(x)\Delta t dx$$
$$+\int_0^\infty R_0(t,x)\gamma(x)\Delta t dx + \int_0^\infty V_0(t,x)\beta(x)\Delta t dx$$

By connecting the system probabilities at time t with those at time $t+\Delta t$ by considering $Q(t+\Delta t)$ which means the probability that at time $t+\Delta t$, there are no customers in the system and the server is idle but available in the system. Then we have the following four mutually exclusive cases:

(1) At time *t*, there are no customers in the system and the server is idle but available in the system and there is no arrival during $(t, t+\Delta t)$. This case has the probability $(1-\lambda\Delta t)Q(t)$.

(2) At time *t*, there are no customers in the queue excluding the customer in the service and the server completes the service of this customer and decides to stay in the system and not to go on vacation during $(t, t+\Delta t)$. This case has the probability $(1-p)\int_0^\infty P_0(t,x)\mu(x)\Delta t dx$.

(3) At time t, there are no customers in the system and the server is broken down under repairs and the repair process completed during $(t, t+\Delta t)$. This case has the probability $\int_0^\infty R_0(t,x)\gamma(x)\Delta t dx$.

(4) At time t, there are no customers in the system and the server is on vacation and the vacation period completed during $(t, t+\Delta t)$. This case has the probability $\int_0^\infty V_0(t,x)\beta(x)\Delta t dx$.

For the $M^{[X]}/G/1$ queueing system we have:

$P(\text{no arrival during } (t,t+\Delta t)) = 1-\lambda\Delta t + o(\Delta t) = 1-\lambda\Delta t$

$P(\text{a batch of size } i \text{ arrives during } (t,t+\Delta t)) = \lambda c_i \Delta t + o(\Delta t) = \lambda c_i \Delta t$

$P(\text{more than one batch arrives during } (t,t+\Delta t)) = o(\Delta t)$

$P(\text{no arrival completes his service during } (t,t+\Delta t)) = 1-\mu(x) + o(\Delta t) = 1-\mu(x)\Delta t$

$P(\text{one arrival completes his service during } (t,t+\Delta t)) = \mu(x) + o(\Delta t) = \mu(x)\Delta t$

$P(\text{more than one arrival complete his service during} (t,t+\Delta t)) = o(\Delta t)$

For more details about the above probabilities see Kashyap & Chaudhry (1988).

B.2 Full Explanation of Boundary Conditions in Chapter Two

B.2.1 Boundary Condition in Equation (2.17)

$$P_n(0) = (1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx + \int_0^\infty V_{n+1}(x)\beta(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + \lambda c_{n+1}Q \qquad n \geq 0$$

$P_n(0) = \lim_{t\to\infty} P_n(t,0)$ is the probability that at time t there are n customers in the queue excluding the customer in the service given that the elapsed service time of this customer is 0 (the service just started). Then we have the following four mutually exclusive cases:

(1) At time t there are $(n+1)$ customers in the queue excluding the customer being served given that the elapsed service time of this customer is x, and the

server completes serving this customer, does not go on vacation and starts serving the next customer in the queue. This case has the probability $(1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx$.

(2) At time t there are $(n+1)$ customers in the queue and the server is on vacation given that the elapsed vacation time is x. The vacation is just completed. This case has the probability $\int_0^\infty V_{n+1}(x)\beta(x)dx$.

(3) At time t there are $(n+1)$ customers in the queue and the server is broken down and under repairs given that the elapsed repair time is x. The repair is just completed. This case has the probability $\int_0^\infty R_{n+1}(x)\gamma(x)dx$.

(4) At time t there are no customers in the system, the server is idle, available in the system and a batch of size $(n+1)$ customers arrive at the system. This case has the probability $\lambda c_{n+1}Q$.

B.2.2 Boundary Condition in Equation (2.18)

$$V_n(0) = p\int_0^\infty P_n(x)\mu(x)dx, \qquad n \geq 0$$

$V_n(0) = \lim_{t\to\infty} V_n(t,0)$ is the probability that at time t, there are n $(n \geq 0)$ customers in the system and the server is on vacation given that the elapsed vacation time is 0, this means that the vacation has just started. Then we have only the following case:

(1) At time t there are n customers in the queue excluding the customer being served given that the elapsed service time of this customer is x and the server completes serving this customer and goes on vacation. This case has the probability $p\int_0^\infty P_n(x)\mu(x)dx$.

B.2.3 Boundary Condition in Equation (2.19)

$$R_n(0) = \alpha\int_0^{\infty} P_{n-1}(x)dx = \alpha P_{n-1} \qquad n \ge 1$$

$R_n(0) = \lim_{t\to\infty} R_n(t,0)$ is the probability that at time *t*, there are n $(n \ge 0)$ customers in the system and the server is down and under repair given that the elapsed repair time is 0, this means that the repair has just started. Then we have only the following case:

(1) At time t there are $(n-1)$ customers in the queue excluding the customer being served given that the elapsed service time of this customer is x at the moment the server breaks down i.e. during the service of this customer. This case has the probability $\alpha\int_0^{\infty} P_{n-1}(x)dx = \alpha P_{n-1}$.

The same explanation can be considered in equation (3.14), bearing in mind the state of the server in each equation.

B.2.4 Boundary Condition in Equation (2.20)

$$R_0(0) = 0$$

$R_0(0)$ is the probability that at time *t*, there are no customers in the system and the server is broken down and under repair given that the elapsed repair time is 0. This cannot happen because we assume that the server breaks down when it is providing service. This means that this probability is 0.

B.3 Full Explanation of the System Equations (3.1) to (3.5)

B.3.1 Equations (3.1) and (3.3)

$$D_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\varphi(x)\Delta x)D_n(t,x) + \lambda\sum_{i=1}^{n} c_i D_{n-i}(t,x)\Delta t \qquad n \ge 1$$

$$E_n(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\theta(x)\Delta x)E_n(t,x) + \lambda\sum_{i=1}^{n} c_i E_{n-i}(t,x)\Delta t \qquad n \ge 1$$

See section B.1.3.

B.3.2 Equation (3.2)

$$D_0(t+\Delta t, x+\Delta x) = 0$$

$D_0(t+\Delta t, x+\Delta x)$ is the probability that at time $t+\Delta t$, there are no customers in the system and the server is down and waiting for repair to start(on delay) given that the elapsed delay time is $x+\Delta x$. This cannot happen because we assume that the server breaks down when it is providing service(at least one customer in the system). This means that this probability is 0.

B.3.3 Equation (3.4)

$$E_0(t+\Delta t, x+\Delta x) = (1-\lambda\Delta t)(1-\theta(x)\Delta x)E_0(t,x)$$

See section B.1.4.

B.3.4 Equation (3.5)

$$Q(t+\Delta t) = (1-\lambda\Delta t)Q(t) + (1-p)\int_0^\infty P_0(t,x)\mu(x)\Delta t dx + \int_0^\infty R_0(t,x)\gamma(x)\Delta t dx$$
$$+ (1-r)\int_0^\infty V_0(t,x)\beta(x)\Delta t dx + \int_0^\infty E_0(t,x)\theta(x)\Delta t dx$$

The explanations of the first three terms $(1-\lambda\Delta t)Q(t)$, $(1-p)\int_0^\infty P_0(t,x)\mu(x)\Delta t dx$ and $\int_0^\infty R_0(t,x)\gamma(x)\Delta t dx$ are given in B.1.5.

The explanations of the last two terms in equation (3.5) are respectively as following

(1) At time *t*, there are no customers in the system, the server is on vacation and the vacation period completes and the server does not go on an extended vacation during $(t, t+\Delta t)$. This case has the probability $(1-r)\int_0^\infty V_0(t,x)\beta(x)\Delta t dx$.

(2) At time t, there are no customers in the system, the server is on extended vacation and the extended vacation period completes during $(t, t+\Delta t)$. This case has the probability $\int_0^\infty E_0(t,x)\theta(x)\Delta t dx$.

B.4 Full Explanation of the System Boundary Conditions in Chapter Three

B.4.1 Equation (3.11)

$$P_n(0) = (1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx + (1-r)\int_0^\infty V_{n+1}(x)\beta(x)dx$$
$$+\int_0^\infty E_{n+1}(x)\theta(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + \lambda c_{n+1}Q \qquad n \geq 0$$

The explanations of the terms $(1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx$, $\int_0^\infty R_{n+1}(x)\gamma(x)dx$ and $\lambda c_{n+1}Q$ are given in section B.2.1. The explanations of the other two terms are as the following

(1) At time t there are $(n+1)$ customers in the queue, the server just finished the original vacation and does not go on an extended vacation. This case has the probability $(1-r)\int_0^\infty V_{n+1}(x)\beta(x)dx$.

(2) At time t there are $(n+1)$ customers in the queue and the server is on an extended vacation given that the elapsed extended vacation time is x. The extended vacation is just completed. This case has the probability $\int_0^\infty E_{n+1}(x)\theta(x)dx$.

B.4.2 Equation (3.12)

$$V_n(0) = p\int_0^\infty P_n(x)\mu(x)dx, \qquad n \geq 0$$

See B.2.2.

B.4.3 Equation (3.13)

$$E_n(0) = r\int_0^{\infty} V_n(x)\beta(x)dx \qquad n \geq 0$$

$E_n(0)$ is the probability that at time *t*, there are n $(n \geq 0)$ customers in the system and the server is on extended vacation given that the elapsed extended vacation time is 0, this means that the extended vacation just started. Then we have only the following case:

(1) At time *t* there are *n* customers in the system the server just finished the original vacation and goes on an extended vacation. This case has the probability $r\int_0^{\infty} V_n(x)\beta(x)dx$

B.4.4 Equation (3.14)

See B.2.3

B.4.5 Equation (3.15)

$$R_n(0) = \int_0^{\infty} D_n(x)\varphi(x)dx, \qquad n \geq 0$$

$R_n(0)$ is the probability that at time *t*, there are n $(n \geq 0)$ customers in the system and the server is broken down and under repair given that the elapsed delay time is 0 this means that the repairs just started. Then we have only the following case:

(1) At time *t* there are *n* customers in the system, the server is broken down waiting for repairs to start (on delay) and the delay period just finished to start the repairs. This case has the probability $\int_0^{\infty} D_n(x)\varphi(x)dx$.

B.4.6 Equation (3.16)

$$D_0(0) = R_0(0) = 0$$

See B.2.4.

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Appendix C

Professional MATHCAD 2001 Sheets

Mathcad is computer software that simplifies calculations by combining equations in a presentable format. It can be used as an intelligent calculator making it easy to keep track of the most complex calculations for verification and validation. All the Mathcad templates used to find the numerical answers in this dissertation are listed in this appendix.

C.1 Chapter Three MATHCAD Work

$\lambda := 2 \quad \theta := 5 \quad \beta := 5 \quad \gamma := 4 \quad \alpha := 2 \quad p := 0.5 \quad \phi := 3 \quad r := 0.5 \qquad E(V) = X \qquad E\left(V^2\right) = Y \qquad E(eV) = Z \qquad E\left(eV^2\right) = W$

$G(\alpha) := \frac{7}{\alpha + 7} \qquad X := \frac{1}{\beta} \qquad Y := \frac{2}{\beta^2} \qquad Z := \frac{1}{\theta} \qquad W := \frac{2}{\theta^2}$

$E(R) = R \qquad E\left(R^2\right) = M \qquad E(D) = D \qquad E\left(D^2\right) = K$

$g := \frac{d}{d\alpha} G(\alpha) \qquad R := \frac{1}{\gamma} \qquad M := \frac{2}{\gamma^2} \qquad D := \frac{1}{\phi} \qquad K := \frac{2}{\phi^2}$

$$Q := 1 - \lambda \cdot \left[\frac{1}{\alpha \cdot G(\alpha)} + \frac{D}{G(\alpha)} + \frac{R}{G(\alpha)} - \frac{1}{\alpha} - D - R + p \cdot (X + r \cdot Z)\right]$$

$$n := Q \cdot \lambda \cdot [\,(1 - G(\alpha)) \cdot [\,1 + \alpha \cdot (D + R)\,] + \alpha \cdot p \cdot G(\alpha) \cdot (X + r \cdot Z)\,]$$

$$N1 := Q \cdot \lambda^2 \cdot [\,\alpha \cdot (1 - G(\alpha)) \cdot (K + M + 2 \cdot D \cdot R) + 2 \cdot g \cdot [\,1 + \alpha \cdot (D + R)\,] - 2 \cdot p \cdot (X + r \cdot Z) \cdot (G(\alpha) + \alpha \cdot g) + \alpha \cdot p \cdot G(\alpha) \cdot (2 \cdot r \cdot Z \cdot X + Y + r \cdot W)\,]$$

$$N2 := 2 \cdot Q \cdot \lambda \cdot \alpha \cdot (D + R) \cdot (1 - G(\alpha))$$

$$d := -\lambda \cdot [\,\alpha \cdot p \cdot G(\alpha) \cdot (X + r \cdot Z) + (1 - G(\alpha)) \cdot [\,1 + \alpha \cdot (D + R)\,]\,] + \alpha \cdot G(\alpha)$$

$$D1 := -\lambda^2 \cdot G(\alpha) \cdot [\,\alpha \cdot p \cdot (Y + r \cdot W + 2 \cdot r \cdot X \cdot Z) - 2 \cdot p \cdot (X + r \cdot Z) - \alpha \cdot (K + M + 2 \cdot D \cdot R)\,] - \alpha \cdot \lambda^2 \cdot (K + M + 2 \cdot D \cdot R)$$

$$D2 := -2 \cdot \lambda^2 \cdot g \cdot [\,1 + \alpha \cdot (D + R) - \alpha \cdot p \cdot (X + r \cdot Z)\,] - 2 \cdot \lambda \cdot [1 + \alpha \cdot (D + R) \cdot (1 - G(\alpha)) + \alpha \cdot g]$$

$D := D1 + D2 \qquad N := N1 + N2$

$\rho := 1 - Q \qquad Lq := \frac{d \cdot N - n \cdot D}{2d^2}$

$L := Lq + \rho \qquad W := \frac{L}{\lambda} \qquad Wq := \frac{Lq}{\lambda}$

$\rho = 0.919 \qquad Q = 0.081 \qquad Lq = 12.9653 \qquad Wq = 6.4826 \qquad L = 13.8843 \qquad W = 6.9422$

C.2 Chapter Four MATHCAD Work

$\lambda := 2$ $\quad \beta := 7$ $\quad \delta := 3$ $\quad \gamma := 5$ $\quad \alpha := 1$ $\quad p := 0.25$ $\quad E(V) = X$ $\quad E(V^2) = Y$ $\quad E(R) = R$ $\quad E(R^2) = M$

$E(I) = A$ $\quad E(I(I-1)) = T$

$G(\alpha) := \dfrac{5}{\alpha + 5}$ $\quad X := \dfrac{1}{\beta}$ $\quad R := \dfrac{1}{\gamma}$

$A := 3.5$ $\quad T := 9$

$g := \dfrac{d}{d\alpha} G(\alpha)$ $\quad Y := \dfrac{2}{\beta^2}$ $\quad M := \dfrac{2}{\gamma^2}$ $\quad \Psi := \lambda \cdot A - \delta$ $\quad \Omega := \lambda \cdot T + 2 \cdot \delta$ $\quad \Lambda := \lambda \cdot T$ $\quad \Phi := \lambda \cdot A$

$$Q := \frac{\alpha \cdot [1 - p \cdot (\lambda - \delta) \cdot G(\alpha) \cdot X] - (1 - G(\alpha)) \cdot [\lambda + \alpha \cdot [1 + (\lambda - \delta) \cdot R]]}{\alpha \cdot [\delta \cdot R \cdot (1 - G(\alpha)) + G(\alpha) \cdot (1 + p \cdot \delta \cdot X)]}$$

$$n := -2 \cdot Q \cdot \Phi \cdot (\Phi - \delta) \cdot [(1 - G(\alpha)) \cdot (1 + \alpha \cdot R) + \alpha \cdot p \cdot G(\alpha) \cdot X]$$

$$N1 := -3 \cdot Q \cdot \Lambda \cdot \Psi \cdot (1 - G(\alpha)) \cdot (1 + \alpha \cdot R) - 6 \cdot Q \cdot \Psi \cdot \Phi^2 \cdot g \cdot (1 + \alpha \cdot R) - 3 \cdot Q \cdot \Phi \cdot (1 - G(\alpha)) \cdot \left[\Omega \cdot (1 + \alpha \cdot R) + 2 \cdot \alpha \cdot \Psi \cdot R + \alpha \cdot \Psi^2 \cdot M\right]$$

$$N2 := 3 \cdot p \cdot Q \cdot X \cdot \Psi \cdot G(\alpha) \cdot \left(2 \cdot \Phi^2 - \alpha \cdot \Lambda\right) + 6 \cdot \alpha \cdot p \cdot Q \cdot \Phi^2 \cdot \Psi \cdot X \cdot g - 3 \cdot \alpha \cdot p \cdot Q \cdot \Phi \cdot G(\alpha) \cdot \left(\Psi^2 \cdot Y + \Omega \cdot X\right)$$

$$d := 2 \cdot (\lambda - \delta) \cdot [(1 - G(\alpha)) \cdot [\lambda + \alpha \cdot [1 + (\lambda - \delta) \cdot R]] - \alpha \cdot [1 - p \cdot (\lambda - \delta) \cdot G(\alpha) \cdot X]]$$

$$D1 := 3 \cdot (1 - G(\alpha) \cdot p \cdot X \cdot \Psi) \cdot (2 \cdot \Phi \cdot \Psi - \alpha \cdot \Omega) + 6 \cdot \Phi^2 \cdot \Psi \cdot g + 3 \cdot (1 - G(\alpha)) \cdot (\Lambda \cdot \Psi + \Phi \cdot \Omega) + 6 \cdot \alpha \cdot \Psi^2 \cdot (1 - G(\alpha)) \cdot R$$

$$D2 := -3 \cdot \alpha \cdot \Psi \cdot p \cdot \left[2 \cdot g \cdot \Phi \cdot \Psi \cdot X - G(\alpha) \cdot \left(Y \cdot \Psi^2 + \Omega \cdot X\right)\right] + 3 \cdot \alpha \cdot (1 + \Psi \cdot R) \cdot [\Omega \cdot (1 - G(\alpha)) + 2 \cdot \Phi \cdot \Psi \cdot g] + 3 \cdot \alpha \cdot \Psi \cdot (1 - G(\alpha)) \cdot \left(\Psi^2 \cdot M + \Omega \cdot R\right)$$

$D := D1 + D2$ $\quad N := N1 + N2$ $\quad Lq := \dfrac{d \cdot N - n \cdot D}{3d^2}$

$\rho := 1 - Q$ $\quad L := Lq + \rho$ $\quad Wq := \dfrac{Lq}{\lambda}$ $\quad W := \dfrac{L}{\lambda}$

$\rho = 0.4494$ $\quad Q = 0.5506$ $\quad Lq = 100.2372$ $\quad Wq = 50.1186$ $\quad L = 100.6865$ $\quad W = 50.3433$

C.3 Chapter Five MATHCAD Work

$\lambda := 2$ $\beta := 9$ $\gamma := 2$ $\alpha := 2$ $p := 0.5$ $\delta := 0$ $E(V) = X$ $E(V^2) = Y$ $E(D) = D$ $E(D^2) = Z$ $E(R) = R$ $E(R^2) = M$

$G(\alpha) := \frac{7}{\alpha + 7}$ $X := \frac{1}{\beta}$ $D := 0.383$ $R := \frac{1}{\gamma}$ $A := 1$ $T := 0$

$g := \frac{d}{d\alpha}G(\alpha)$ $Y := \frac{2}{\beta^2}$ $Z := 0.306$ $M := \frac{4}{3\gamma^2}$ $\Psi := \lambda \cdot A - \delta$ $\Omega := \lambda \cdot T + 2\cdot\delta$ $\Lambda := \lambda \cdot T$ $\Phi := \lambda \cdot A$

$$Q := \frac{-\Phi\cdot(1 - G(\alpha))\cdot[1 + \alpha\cdot(D + R)]}{\alpha\cdot G(\alpha)\cdot(1 + \delta\cdot p\cdot X)} - \frac{p\cdot X\cdot\Psi - 1}{1 + \delta\cdot p\cdot X}$$

$$n := -2\cdot Q\cdot\Phi\cdot\Psi\cdot[(1 - G(\alpha))\cdot[1 + \alpha\cdot(D + R)] + p\cdot\alpha\cdot G(\alpha)\cdot X]$$

$n = -0.553$

$$N1 := -3\cdot Q\cdot\Psi\cdot(1 - G(\alpha))\cdot\left[\Lambda\cdot(1 + D + R) + 2\cdot\alpha\cdot\Phi\cdot(D + R) + \alpha\cdot\left[\Phi^2\cdot(Z + M + 2\cdot D\cdot R)\right]\right] - 3\cdot Q\cdot\Phi\cdot[1 + \alpha\cdot(D + R)]\cdot[\Omega\cdot(1 - G(\alpha)) + 2\cdot\Phi\cdot\Psi\cdot g]$$

$$N2 := 6\cdot p\cdot Q\cdot\Phi^2\cdot X\cdot\Psi\cdot(G(\alpha) + \alpha\cdot g) - 3\cdot p\cdot Q\cdot\alpha\cdot G(\alpha)\cdot X\cdot(\Lambda\cdot\Psi + \Phi\cdot\Omega) - 3\cdot p\cdot Q\cdot\alpha\cdot\Phi\cdot G(\alpha)\cdot\Psi^2\cdot Y$$

$$d := 2\cdot\Phi\cdot\Psi\cdot(1 - G(\alpha))\cdot[1 + \alpha\cdot(D + R)] - 2\alpha\cdot\Psi\cdot G(\alpha)\cdot(1 - p\cdot X\cdot\Psi)$$

$d = -0.6135$

$$D1 := 3\cdot(1 + \Phi\cdot g - p\cdot\Psi\cdot G(\alpha)\cdot X)\cdot(2\cdot\Phi\cdot\Psi - \alpha\cdot\Omega) + (1 - G(\alpha))\cdot(3\cdot\Lambda\cdot\Psi + 3\cdot\Phi\cdot\Omega) + 3\cdot\alpha\cdot\Omega\cdot(1 - G(\alpha)) + 3\cdot\alpha\cdot\Phi\cdot(\Lambda + 2\cdot\Phi)\cdot g$$

$$D2 := 3\cdot\alpha\cdot\Psi\left[-2\cdot p\cdot\Phi\cdot\Psi\cdot g\cdot X + p\cdot G(\alpha)\cdot\left(Y\cdot\Psi^2 + \Omega\cdot X\right)\right] + 3\cdot\alpha\cdot\Phi\cdot(\Lambda + 2\cdot\Phi)(1 - G(\alpha))\cdot(D + R) + 6\cdot\alpha\cdot\Phi^2\cdot\Psi\cdot g\cdot(D + R)$$

$$D3 := 3\cdot\alpha\cdot\Psi\cdot(1 - G(\alpha))\cdot\left[\Phi^2\cdot(Z + M + 2\cdot D\cdot R) + \Lambda\cdot(D + R)\right]$$

$D := D1 + D2 + D3$ $N := N1 + N2$ $Lq := \frac{d\cdot N - n\cdot D}{3d^2}$

$N = -0.759$

$\rho := 1 - Q$ $D = 27.5494$

$L := Lq + \rho$ $W := \frac{L}{\lambda}$ $Wq := \frac{Lq}{\lambda}$

$\rho = 0.9014$ $Q = 0.0986$ $Lq = 13.9042$ $Wq = 6.9521$ $L = 14.8056$ $W = 7.4028$

C.4 Chapter Six MATHCAD Work

$\lambda := 3$ $\beta := 6$ $\gamma := 4$ $\alpha := 1$ $\delta := 0$ $p := 0.5$ $r := 0.5$ $E(V) = X$ $E(V^2) = Y$ $E(eV) = Z$ $E(eV^2) = W$

$E(R) = R$ $E(R^2) = M$

$G(\alpha) := \dfrac{7}{\alpha + 7}$ $X := \dfrac{1}{\beta}$ $Z := 0.2833$ $R := \dfrac{1}{\gamma}$

$g := \dfrac{d}{d\alpha}G(\alpha)$ $Y := \dfrac{2}{\beta^2}$ $W := 0.1638$ $M := \dfrac{2}{\gamma^2}$

$$Q := \frac{-\lambda\cdot(1 - G(\alpha))}{\alpha\cdot[\delta\cdot R\cdot(1 - G(\alpha)) + G(\alpha)]} + \frac{1 - p\cdot\lambda\cdot G(\alpha)\cdot(X + r\cdot Z) - (1 - G(\alpha))\cdot[1 + (\lambda - \delta)\cdot R]}{\delta\cdot R\cdot(1 - G(\alpha)) + G(\alpha)}$$

$$n := -2\cdot Q\cdot\lambda\cdot(\lambda - \delta)\cdot[(1 - G(\alpha))\cdot(1 + \alpha\cdot R) + p\cdot\alpha\cdot G(\alpha)\cdot(X + r\cdot Z)]$$

$$N1 := -3\cdot Q\cdot(\lambda - \delta)\cdot(1 + \alpha\cdot R)\cdot\left(2\cdot\lambda^2\cdot g\right) - 3\cdot Q\cdot\lambda\cdot(1 - G(\alpha))\cdot\left[2\cdot\delta\cdot(1 + R) + 2\cdot\alpha\cdot(\lambda - \delta)\cdot R + \alpha\cdot(\lambda - \delta)^2\cdot M\right]$$

$$N2 := 3\cdot p\cdot Q\cdot\lambda\cdot(X + r\cdot Z)\cdot[2\cdot\lambda\cdot(\lambda - \delta)\cdot(G(\alpha) + \alpha\cdot g) - 2\cdot\alpha\cdot\delta\cdot G(\alpha)] - 3\cdot p\cdot Q\cdot\alpha\cdot(\lambda - \delta)\cdot G(\alpha)\cdot\left[\lambda^2\cdot(Y + r\cdot W + 2\cdot r\cdot X\cdot Z)\right]$$

$$d := 2\cdot\lambda\cdot(\lambda - \delta)\cdot(1 - G(\alpha)) - 2\cdot\alpha\cdot(\lambda - \delta)\cdot[1 - p\cdot\lambda\cdot G(\alpha)\cdot(X + r\cdot Z)] + 2\cdot\alpha\cdot(\lambda - \delta)\cdot(1 - G(\alpha)) + 2\cdot\alpha\cdot(\lambda - \delta)^2\cdot(1 - G(\alpha))\cdot R$$

$$D1 := 6\cdot\lambda\cdot\delta\cdot(1 - G(\alpha)) + 6\cdot[\lambda\cdot(\lambda - \delta) - \alpha\cdot\delta]\cdot[1 + \lambda\cdot g - p\cdot\lambda\cdot G(\alpha)\cdot(X + r\cdot Z)] + 3\cdot\alpha\cdot(\lambda - \delta)\cdot\left[-2\cdot p\cdot\lambda^2\cdot g\cdot(X + r\cdot Z) + p\cdot G(\alpha)\cdot\left[\lambda^2\cdot(Y + r\cdot W + 2\cdot r\cdot X\cdot Z)\right]\right]$$

$$D2 := 6\cdot\alpha\cdot\delta\cdot[1 + (\lambda - \delta)\cdot R]\cdot R\cdot(1 - G(\alpha)) + 6\cdot\alpha\cdot\delta\cdot\lambda\cdot g + 3\cdot\alpha\cdot(\lambda - \delta)\cdot\left[2\cdot\lambda\cdot R + M\cdot(\lambda - \delta)^2\right]\cdot(1 - G(\alpha)) + 6\cdot\alpha\cdot\lambda\cdot(\lambda - \delta)\cdot g\cdot[1 + R\cdot(\lambda - \delta)]$$

$D := D1 + D2$ $N := N1 + N2$ $Lq := \dfrac{d\cdot N - n\cdot D}{3d^2}$

$\rho := 1 - Q$

$L := Lq + \rho$ $W := \dfrac{L}{\lambda}$ $Wq := \dfrac{Lq}{\lambda}$

$\rho = 0.9982$ $Q = 1.8107 \times 10^{-3}$ $Lq = 569.2825$ $Wq = 189.7608$ $L = 570.2807$ $W = 190.0936$

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