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A second-order slip model for arbitrary accomodation at the wall

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Abstract We present a kinetic-theory derivation of second-order slip boundary conditions for a plane isothermal pressure driven gas flowing through a microchannel. In the proposed approach, the distribution function is expanded in terms of orthogonal polynomials and the system of moment equations in the expansion coefficients is analytically solved. The velocity slip coefficients, as well as their Knudsen layer corrections, are obtained by evaluating the solution in the near continuum limit. In comparison with other methods, the present approach is accurate and easy to implement. The results are presented for the Bhatnagar-Gross-Krook-Welander (BGKW) kinetic model equation and Maxwell's boundary conditions, but can be extended to more general collision integral and different scattering kernels.

Keywords: Gas microflows. Half-range Hermite polynomials. Velocity slip coefficients.

1. Introduction

Isothermal gas flows within channels with small scale geometries have received considerable attention in recent years, due to their direct relevance to typical micro-electro-mechanical systems applications (Gad-el-Hak , 1999). Since in these flows, the mean free path of the gas particles, $\tilde{\lambda}$, is of the same order as the characteristic geometric dimension of the channel, $\tilde{L}$, the conventional continuum approach, namely the Navier-Stokes equations with stick boundary conditions, breaks down and one should resort to a description based on kinetic theory (Cercignani , 1988). However, for moderate Knudsen numbers, $\mathrm{Kn} = \tilde{\lambda}/\tilde{L} \lesssim 0.1$, a rigorous asymptotic analysis of the Boltzmann equation shows that the non equilibrium regions are confined to thin regions close to solid surfaces, which are referred to as Knudsen layers. The flow field in the bulk, on the other hand, can be obtained by solving the Navier-Stokes equations albeit subject to boundary conditions which prescribe a slip velocity at the solid walls (Sone , 2002; Hadjiconstantinou , 2006). By assuming that the solid wall is a fixed plate and the gas flow is isothermal, the slip boundary condition takes the form

$$\tilde{\xi}_x = A_1 \tilde{\lambda} \frac{d\tilde{\xi}_x}{d\tilde{y}} - A_2 \tilde{\lambda}^2 \frac{d^2 \tilde{\xi}_x}{d\tilde{y}^2} \qquad (1)$$

where $\tilde{y}$ is the coordinate normal to the plate, $\tilde{\xi}_x$ is the tangential velocity component and A_1, A_2 are the first- and second-order velocity slip coefficients, respectively. It is worth mentioning that a number of slip boundary conditions different from Eq. (1) have been proposed over the years but they will not be considered here since they are either phenomenological in nature (Bahukudumbi and Beskok , 2003) or derived by means of heuristic kinetic theory arguments (Shen et al. , 2007). The first-order velocity slip coefficient, A_1, has been calculated both theoretically and numerically (Cercignani , 1988; Sone , 2002). Determining the second-order velocity slip coefficient, A_2, has proven to be a significantly harder task and most of the results obtained so far are lacking in two respects. First, these are usually determined under the hypothesis of complete accommodation at the solid walls, primarily because most engineering surfaces are expected to satisfy this assumption. However, ascertaining the dependence of the velocity slip coefficients from the accomodation coefficient would be of practical importance in analysing and interpreting experimental measurements (Maurer et al. , 2003). Second, when $\mathrm{Kn} > 0.1$ the slip velocity profiles are not very meaningful since they are expected to be valid outside the Knudsen layers which, for this range of Knudsen numbers, comprise

a large part of the channel. Slip solutions may not yield reasonable results but for a few quantities of engineering interest, such as the average volume flow rate, as long as the effects of the Knudsen layers are properly taken into account (Sone , 2002). Unfortunately the procedure to correct the velocity slip coefficients is approximated since it requires to integrate numerically tabulated functions and, as a consequence, it is often overlooked in engineering analysis (Hadjiconstantinou , 2006).
The main aim of the present paper is to obtain an accurate expression of the first- and second-order velocity slip coefficients. The work is based on the kinetic model equation proposed by Bhatnagar, Gross and Krook and, independently, by Welander (BGKW), with Maxwell's diffuse-specular boundary conditions (Cercignani , 1988). The analysis is carried out for an arbitrary accomodation at the solid walls and includes also the correction of the velocity slip coefficients due to the Knudsen layer structure. The remainder of the paper is organized as follows. In Section 2, we briefly introduce the kinetic theory formulation of the Poiseuille flow problem between two parallel plates. In Section 3, we present details of the moment method of solution. In Section 4, we show that the velocity slip coefficients can be determined by direct inspection of the near continuum solutions. In Section 5, we develop the method for a given expansion and we assess the results through comparison with previously reported values. In Section 6, we summarize our findings and comment on the main results of the paper.

2. Poiseuille flow problem

The situation studied is that of an ideal monatomic rarefied gas confined between two parallel plates at rest located at $\tilde{y} = \pm\tilde{L}/2$. The plates are fixed and kept at equal temperatures $\tilde{T}_0$. A uniform gradient pressure, $-\tilde{K}$, acts in the $\tilde{x}$ direction, and it is assumed small enough that the flow can be considered isothermal and incompressible. The channel is considered to be infinitely long such that a full velocity profile can developed. We choose $\tilde{L}$ and $(\tilde{R}\tilde{T}_0)^{1/2}$, with $\tilde{R}$ the specific gas constant, to define a dimensionless position $\boldsymbol{r} = \tilde{\boldsymbol{r}}/\tilde{L}$ and velocity $\boldsymbol{v} = \tilde{\boldsymbol{v}}/(\tilde{R}\tilde{T}_0)$, respectively. Likewise the dimensionless time is given by $t = (\tilde{R}\tilde{T}_0)^{1/2}/\tilde{L}\tilde{t}$. Since the pressure gradient is taken to be small, that is $K = \tilde{K}\tilde{L}/\tilde{p}_0 \ll 1$ with $\tilde{p}_0 = \tilde{n}_0\tilde{R}\tilde{T}_0$, we are justified to replace the collision integral of the Boltzmann equation by the BGKW kinetic model and write the kinetic equation in its linearized form

$$-v_x + v_y \frac{\partial h}{\partial y} = \frac{1}{\mathrm{Kn}}\sqrt{\frac{\pi}{2}}\,(2v_x\xi_x - h) \qquad (2)$$

where $h(y, \boldsymbol{v})$ is a measure of the perturbation of the molecular distribution function which does not depend on the x and z coordinates because of the symmetry of the problem (Cercignani , 1988). In Eq. (2), the Knudsen number is based on the mean free path $\tilde{\lambda} = \sqrt{\pi/2}\tilde{\mu}_0\sqrt{\tilde{R}\tilde{T}_0}/\tilde{p}_0$, with $\tilde{\mu}_0$ the gas viscosity at the equilibrium, and the x-component of the mean velocity is given by

$$\xi_x = \int_{-\infty}^{+\infty} f_0 h v_x d\boldsymbol{v} \qquad (3)$$

where $f_0(\boldsymbol{v})$ is the unit density equilibrium Maxwellian

$$f_0 = \frac{1}{(2\pi)^{3/2}} e^{-\boldsymbol{v}^2/2} \qquad (4)$$

At the solid walls in $y = \mp 1/2$, we assume the Maxwell's scattering kernel (Cercignani , 1988). Accordingly, the distribution function of atoms emerging from walls has the following expression

$$h(v_x, \pm v_y, v_z) = -\alpha\sqrt{2\pi}\int_{v_y \lesseqgtr 0} f_0 h v_y d\mathbf{v} + (1-\alpha)h(v_x, \mp v_y, v_z), \;\; v_y > 0 \quad (5)$$

where $0 \leq \alpha \leq 1$ is the accommodation coefficient, which gives the fraction of molecules which are diffusely reflected.
For later reference, we report here the volume flow rate

$$Q = \frac{1}{K}\int_{-1/2}^{1/2} \xi_x dx \qquad (6)$$

3. Half-range polynomial solution

The solution of the linearized BGKW kinetic model equation subjected to Maxwell's diffuse-specular boundary conditions has the form $h(y, \mathbf{v}) = v_x \phi(y, v_y)$ (Cercignani , 1988). Equation (2) can thus be solved by expanding $\phi(y, v_y)$ in a complete set of orthogonal velocity polynomials with space varying coefficients. The boundary conditions, Eq. (5), however, implies a discontinuity of the distribution function, or some of its derivatives, considered as a function of the velocity component v_y. If the distribution function were expanded in continuous orthogonal polynomials, it would be necessary to go to high order to obtain an adequate representation of the low pressure region and of the boundary conditions. We thus consider the following expansion of the perturbed distribution function

$$h = v_x \sum_{i=0}^{N_y-1} \left[h_i^- H_i^- + h_i^+ H_i^+ \right] \qquad (7)$$

where $h_i^{\pm}(y)$ and $H_i^{\pm}(v_y)$ are the unknown expansion coefficients and the half-range Hermite polynomials, respectively (Gross et al. , 1957; Frezzotti et al. , 2009). Expansion (7) allows to exactly satisfy the boundary conditions, Eq. (5), for every value of N_y and it is quickly convergent even in the early transition regime. Substituting Eq. (7) into Eqs. (2), multiplying by $f_0(\mathbf{v}) H_j^{\pm}(v_y)$ and integrating in velocity, we obtain a set of N coupled non homogeneous ordinary differential equations for the N space varying coefficients, with $N = 2N_y$. The system of moment equations written in matrix form reads

$$\boldsymbol{A} \frac{d\boldsymbol{h}}{dy} - \frac{1}{\mathrm{Kn}} \boldsymbol{B} \boldsymbol{h} = \boldsymbol{a} \qquad (8)$$

where $\mathbf{h}$ is the N column vector composed of the coefficients $h_i^{\pm}$, $\boldsymbol{A}$ is the $N \times N$ tridiagonal matrix whose entries are the recurrence coefficients of the half-range Hermite polynomials and $\boldsymbol{B}$ is a $N \times N$ nearly identity matrix except for two extra-diagonal coefficients that couple the moment equations for the negative and positive half-range (Frezzotti et al. , 2009). Likewise, the boundary conditions, Eqs. (5), can be rewritten as

$$\boldsymbol{C} \boldsymbol{h} = 0 \qquad (9)$$

where $\boldsymbol{C}$ is a $N \times N$ sparse matrix whose entries depend on the accommodation coefficient. Since Eq. (8) is a linear system of differential equations with constant coefficients, its solution can be achieved by assuming a particular solution of the non homogeneous equation and then coupling the solution of homogeneous equation. The homogeneous equation is solved by looking for exponential solutions in the form $\boldsymbol{h} = \exp(\lambda y) \boldsymbol{w}$ where λ and $\boldsymbol{w}$ are constants. This leads to the eigenvalue problem

$$\left[\frac{\boldsymbol{A}^{-1} \boldsymbol{B}}{\mathrm{Kn}} - \lambda \boldsymbol{I} \right] \boldsymbol{w} = \boldsymbol{0} \qquad (10)$$

where $\boldsymbol{I}$ is the identity matrix of size $N \times N$. It is possible to prove that Eq. (10) admits two zero eigenvalues and $N/2 - 1$ pair of real eigenvalues with opposite sign (Huang and Giddens , 1967). Therefore the complementary solution exhibits a quadratic form in addition to the $(N-1)/2$ pairs of exponential solutions. Since the non homogeneous term is constant, the particular solution is seeked in polynomial form $\boldsymbol{h}_p = \boldsymbol{u}_1 + \boldsymbol{u}_2 y + \boldsymbol{u}_3 y^2$. The vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_3$ can be determined by solving the three systems of linear equations which are obtained by substituting the particular solution into Eq. (8) and equating to zero the terms which multiply $1, y$ and y^2. Overall, the solution of Eq. (8) may thus be written in the form

$$\boldsymbol{h} = \boldsymbol{u}_1 + \boldsymbol{u}_2 y + \boldsymbol{u}_3 y^2 + b_0 \boldsymbol{p}_0 + c_0 \boldsymbol{m}_0 y + \sum_{i=1}^{N/2-1} \left[b_i \boldsymbol{p}_i \exp\left(\frac{\lambda_i y}{\mathrm{Kn}} \right) + c_i \boldsymbol{m}_i \exp\left(\frac{-\lambda_i y}{\mathrm{Kn}} \right) \right] \qquad (11)$$

In Eq. (11), λ_i are the positive eigenvalues, $\{\boldsymbol{p}_i, \boldsymbol{m}_i\}$ are the eigenvectors associated to the positive and negative eigenvalues, respectively, whereas $\{\boldsymbol{p}_0, \boldsymbol{m}_0\}$ are the generalized eigenvectors corresponding to the zero eigenvalue of multiplicity two. Likewise, $\{b_i, c_i\}$ with $i = 0, \ldots, N/2 - 1$ are N arbitrary constants which can be determined by the boundary conditions, Eq. (9).

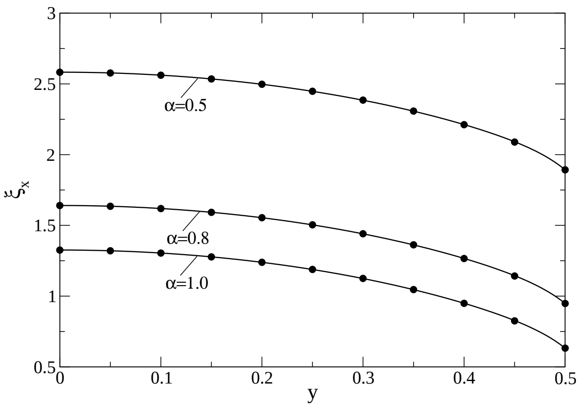

Figure 1: Profile of the mean velocity, ξ_x, through the half-channel for different values of the accomodation coefficient, α. Solid line: closed form solution, Eq. (12). Solid symbols: numerical results reported in (Barichello and Siewert , 1999). $\delta = 2$; $K = 1$.

4. Velocity slip coefficients

The substitution of expansion (7) into Eq. (3), with the expansion coefficients as given by Eq. (11), yields the x-component of the mean velocity

$$\xi_x = K \left\{ \sqrt{\frac{\pi}{8}} \frac{1}{\mathrm{Kn}} y^2 + d_0 + \sum_{i=1}^{N/2-1} d_i \left[\exp\left(\frac{\lambda_i y}{\mathrm{Kn}}\right) + \exp\left(\frac{-\lambda_i y}{\mathrm{Kn}}\right) \right] \right\} \quad (12)$$

Eq. (12) is an even function of y in accordance with the symmetry of the problem. The coefficients $d_i(\mathrm{Kn}, \alpha)$ are rational functions of exponentials which depend on both the Knudsen number and the accommodation coefficient. However, after some algebra, it is possible to show that when $\mathrm{Kn} \to 0$, they asymptotically behave as

$$d_0 \sim q_0 + r_0 \mathrm{Kn} \quad (13)$$

$$d_i \sim [q_i + r_i \mathrm{Kn}] \exp\left(-\frac{\lambda_i}{2\mathrm{Kn}}\right) \quad (14)$$

where $q_k(\alpha)$ and $r_k(\alpha)$, with $k = 0, \ldots, N/2-1$, depend only on the accommodation coefficient, α. By virtue of Eqs. (13) and (14), in the limit of a vanishing Knudsen number, Eq. (12) yields

$$\xi_x \sim K \sqrt{\frac{\pi}{8}} \left[\frac{1}{\mathrm{Kn}} \left(\frac{1}{4} - y^2 \right) + A_1 + 2\mathrm{Kn} A_2 \right] \quad (15)$$

where $A_1(\alpha)$ and $A_2(\alpha)$ are rational functions of the accommodation coefficient. It is worth noticing that the exponential terms in Eq. (12) do not directly contribute to the bulk velocity profile given by Eq. (15). As a matter of fact, they are expected to disappear when $\mathrm{Kn} \to 0$ since they represent the Knudsen layer correction to the continuum solution. In Eq. (15), A_1 and A_2 may be identified with the first- and second-order velocity slip coefficients. In fact, Eq. (15) is also the solution of the linearized Navier-Stokes equation

$$\frac{d^2 \xi_x}{dy^2} = -K \sqrt{\frac{\pi}{2}} \frac{1}{\mathrm{Kn}} \quad (16)$$

subjected to the slip boundary conditions Eq. (1) which, in dimensionless form, reads

$$\xi_x = \mp A_1 \mathrm{Kn} \frac{d\xi_x}{dy} - A_2 \mathrm{Kn}^2 \frac{d^2 \xi_x}{dy^2} \quad (17)$$

at $y = \pm 1/2$. Both Eqs. (16) and (17) have been written using the dimensionless unit introduced in Section 2.
In order to account for the structure of the Knudsen layer, the velocity slip coefficients should be evaluated on the basis of the volume flow rate instead of the velocity profile. By substituting Eq. (12) into Eq. (6) and taking $\mathrm{Kn} \to 0$, we obtain

$$Q \sim \frac{1}{12} \sqrt{\frac{\pi}{2}} K \left(\frac{1}{\mathrm{Kn}} + 6A_1 + 12\mathrm{Kn}\bar{A}_2 - 12\mathrm{Kn}^2 \bar{A}_3 \right) \quad (18)$$

where $\bar{A}_2(\alpha)$ and $\bar{A}_3(\alpha)$ are rational functions of the accommodation coefficient. The first three terms in Eq. (18) can also be obtained by integrating the velocity profile as deduced by the linearized Navier-Stokes equation, Eq. (16), subjected at $y = \pm 1/2$ to the boundary condition given by Eq. (17) with $\bar{A}_2$ in place of A_2. Hence $\bar{A}_2$ may be identified with the second-order velocity slip coefficient corrected to account for the structure

	$\alpha = 0.5$		$\alpha = 0.8$		$\alpha = 1.0$	
δ	Eq. (12)	Ref. [1]	Eq. (12)	Ref. [1]	Eq. (12)	Ref. [1]
0.10	3.22401	3.22187	1.91371	1.91466	1.43194	1.43734
0.30	2.67287	2.67178	1.58928	1.58729	1.20661	1.20383
0.50	2.50618	2.50625	1.48637	1.48653	1.13246	1.13269
0.70	2.43062	2.43080	1.44135	1.44163	1.10216	1.10251
0.90	2.39269	2.39277	1.42065	1.42075	1.09010	1.09022
1.00	2.38166	2.38169	1.41550	1.41554	1.08798	1.08801
2.00	2.38764	2.38760	1.44352	1.44348	1.12777	1.12774
5.00	2.66893	2.66890	1.72411	1.72409	1.40770	1.40769
7.00	2.89075	2.89073	1.94181	1.94179	1.62277	1.62276
9.00	3.11849	3.11848	2.16621	2.16619	1.84503	1.84502

Table 1: Volume flow rate, Q, versus the rarefaction parameter, $\delta = \sqrt{\pi}/(2\mathrm{Kn})$, for different values of the accommodation coefficient, α. Ref [1]: (Loyalka et al. , 1975).

of the Knudsen layer. By virtue of Eqs. (13) and (14), all the exponentials in Eq. (12) contribute terms of order Kn and Kn^2 to the volume flow rate when integrated along the width of the channel. The first-order velocity slip coefficient is thus the same regardless of whether Eq. (15) or (18) is used. It is worth noticing that by adding the term $\bar{A}_3 \mathrm{Kn}^3 d^2\xi_x/dy^2$ to the boundary condition it would be possible to achieve a volume flow rate which is second-order accurate in the Knudsen number. This third-order velocity slip coefficient, $\bar{A}_3$, however, has not a physical meaning, and its only rationale is to provide a term to match the volume flow rate given by Eq. (18) at the second order in the Knudsen number.

5. Results and discussion

The number of expansion polynomials in Eq. (7) has to be selected to reflect accuracy needs. Unless otherwise stated, we have set $N_y = 8$.
We first assessed the accuracy of the velocity profile given by Eq. (12). Figure 1 shows a comparison of the mean velocity profile, ξ_x, through the half-channel for different values of the accommodation coefficient and $\delta = 2$, being $\delta = \sqrt{\pi}/(2\mathrm{Kn})$ the rarefaction parameter. Solid lines are the closed form solutions given by Eq. (12) whereas solid symbols are the results obtained by numerically solving the linearized BGKW kinetic model equation with a discrete ordinate method (Barichello and Siewert , 1999). The agreement is fairly good even close to the solid wall where the polynomial solution is expected to converge slowly because of the presence of the Knudsen layer. In order to proceed with a more detailed comparison, in Table 1 we show the volume flow rate obtained by using Eq. (12) in Eq. (6) and the numerical predictions reported in (Barichello and Siewert , 1999). Here and in the following, all the results have been rescaled in order to use the same definition of mean free path and units as in the present work. The agreement at best is five significant figures, but, for the higher Knudsen numbers, we have only three figures of agreement. It is clear that results can be improved to any desired degree of accuracy by increasing the number of expansion polynomials, N_y. However, since we are interested in evaluating the velocity profiles in the near continuum limit, i.e., $\mathrm{Kn} \ll 1$, the solutions obtained with $N_y = 8$ may be considered sufficiently accurate. In any case, it has been verified that the velocity slip coefficients vary only a few percent upon increasing the number of expansion polynomials.
The first- and second-order velocity slip coefficients, as given by Eq. (15), are reported in Table 2 for several values of the accommodation coefficient, α. In order to assess the reliability of the proposed approach, we also list the values of A_1 and A_2 as deduced by the numerical, (Loyalka et al. , 1975), and

	A_1			A_2	
α	Eq. (15)	Ref. [1]	Ref. [2]	Eq. (15)	Ref. [2]
0.1	19.2989	19.2988	19.2596	0.67273	0.66845
0.2	9.28085	9.28081	9.24592	0.70835	0.70028
0.3	5.92979	5.92976	5.89892	0.74348	0.73211
0.4	4.24569	4.24566	4.21859	0.77812	0.76394
0.5	3.22853	3.22851	3.20493	0.81228	0.79577
0.6	2.54498	2.54496	2.52460	0.84594	0.82761
0.7	2.05216	2.05215	2.03475	0.87912	0.85944
0.8	1.67865	1.67864	1.66394	0.91181	0.89127
0.9	1.38476	1.38474	1.37250	0.94401	0.92310
1.0	1.14666	1.14665	1.13662	0.97572	0.95493

Table 2: First- and second-order velocity slip coefficient versus the accommodation coefficient, α. Ref. [1]: (Loyalka et al. , 1975); Ref. [2]: (Loyalka and Hickey , 1989).

variational, (Loyalka and Hickey , 1989), solutions of the linearized BGKW kinetic model equation, which we refer to as Ref. [1] and Ref. [2], respectively. As revealed by the table, the first- and second-order velocity slip coefficients provided by the different approaches compare quite well. In particular, the first-order velocity slip coefficient given by Eq. (15) provides the best agreement with the numerical predictions reported in (Loyalka et al. , 1975). The results for the second-order velocity slip coefficient show larger deviations. However, we point out that, in the complete accommodation case, our prediction of A_2 agrees up to four significant figures with its accurate numerical estimate (Sone , 2002). Low order Padé approximants of A_1, A_2 and $\bar{A}_2, \bar{A}_3$ are given by

$$A_1 = \bar{A}_1 \simeq \frac{2-\alpha}{\alpha}\left(1-\alpha+1.1466\alpha\right) \qquad (19)$$

$$A_2 \simeq -0.02442\alpha^2 + 0.36352\alpha + 0.63662 \qquad (20)$$

$$\bar{A}_2 \simeq \alpha\left(-0.04892\alpha + 0.72716\right) \qquad (21)$$

$$\bar{A}_3 \simeq \alpha\left(-0.08645\alpha + 1.6204\right) \qquad (22)$$

Although approximate, Eqs. (19)-(22) are of interest in that they readily provide results which deviate less than 0.1% from the analytical solutions. Figure 2 shows a comparison of the volume flow rate versus the Knudsen number for two different values of the accommodation coefficient, $\alpha = 0.1$ (a) and $\alpha = 1.0$ (b). Solid line is the closed form solutions obtained by using Eq. (12) in Eq. (6) whereas dotted and dashed-dotted lines are the volume flow rate given by Eq. (18) without and with the term containing $\bar{A}_3$. As it is clearly shown, slip solutions match the volume flow rate up to about Kn=0.2, whereas, for higher Knudsen numbers, they quickly deviate.

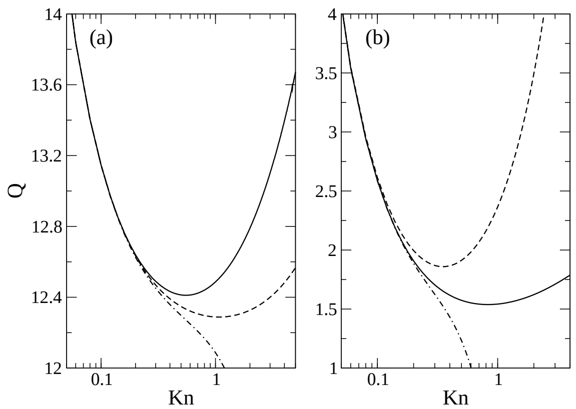

Figure 2: Volume flow rate, Q, versus the Knudsen number, Kn, for $\alpha = 0.1$ (a) and $\alpha = 1.0$ (b). Solid lines: closed form solutions obtained by using Eq. (12) in Eq. (6). Dashed line: Eq. (18) with $\bar{A}_3 = 0$. Dot-dashed line: Eq. (18).

6. Conclusion

First- and second-order velocity slip coefficients, A_1 and A_2, as well as the Knudsen layer correction of the latter, $\bar{A}_2$, have been obtained by solving the plane isothermal Poiseuille flow problem on the basis of the linearized BGKW kinetic model equation, Eq. (2), and the Maxwell's scattering kernel with an arbitrary accomodation at the wall's surfaces, Eq. (5). The velocity slip coefficients have been obtained by firstly expanding the distribution function in terms of half-range Hermite polynomials in order to explicitly account for the discontinuity induced by the solid walls, Eq. (7). By taking weighted averages over the particles velocity, the linearized BGKW kinetic model equation is then reduced to a system of linear ordinary differential equations in the expansion coefficients, Eq. (8), which is solved by finding the eigenvalues and eigenvectors of the coefficient matrix. Finally, the velocity profile and the volume flow rate are evaluated in the near continuum limit, Eqs. (15) and (18). The velocity slip coefficients are obtained by direct inspection of these asymptotic solutions. In particular, Eq. (18) has suggested to introduce beside the first- and second-order velocity slip coefficients, a third coefficient, A_3, by means of which it is possible to achieve a volume flow rate which is accurate to the second order in the Knudsen number. Low order Padè approximants of the dependence of the velocity slip coefficients from the accommodation coefficient are then determined. The main results are summarized by Eqs. (19)-(22). In comparison with previous studies on this subject, the current approach provides the velocity slip coefficients with a good accuracy in the entire range of the accommodation coefficient. The difficulties encountered by umerical studies in evaluating the second derivative of the tangential velocity in the normal direction to the surface are avoided as well as the need to integrate numerically tabulated functions to include the contribution of the Knudsen layer. Moreover, it is not necessary to determine suitable test functions as requested by variational methods. The results obtained encourage to further developments. In particular the extension of the present approach to the Boltzmann equation and more general gas-surface scattering kernels is currently under investigation.

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