TOWARDS A SPLITTER THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS

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Abstract. We prove that if M is a 4-connected binary matroid and N is an internally 4-connected proper minor of M with at least 7 elements, then, unless M is a certain 16-element matroid, there is an element e of $E(M)$ such that either $M\backslash e$ or M/e is internally 4-connected having an N-minor. This strengthens a result of Zhou and is a first step towards obtaining a splitter theorem for internally 4-connected binary matroids.

1. Introduction

Our goal in this article is to make progress towards a splitter theorem for internally 4-connected binary matroids. Such a theorem would provide a guarantee that if M and N are internally 4-connected binary matroids, and M has a proper N-minor, then M has a minor M' such that M' is internally 4-connected with an N-minor, and M' can be produced from M by a bounded number of simple operations.

A chain theorem resembles a splitter theorem, except that the requirement that M' has an N-minor is dropped. In a previous article we proved a chain theorem for internally 4-connected binary matroids [1]. In particular, we showed that if M is an internally 4-connected binary matroid, then M has an internally 4-connected minor, M', such that $|E(M)| - |E(M')| \leq 6$. (In almost every case, this bound can be improved to 3.) In this paper, we take a necessary step towards a splitter theorem, by proving that, as long as M is 4-connected, we can produce a proper minor M' of M such that M' has an N-minor and $|E(M)| - |E(M')| \leq 2$. (In almost every case, this bound can be improved to 1.)

We note here that there is no hope of extending our main theorem to the case where M, N, and M' are all required to be 4-connected. This is true even if we relax the bound on $|E(M)| - |E(M')|$ to be any fixed constant. To see this, consider the toroidal grid graph $G_{m\times n}$ with vertex set $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$, where (i, j) and (x, y) are adjacent if and only if $i = x$ and $j - y \equiv \pm 1 \mod n$, or if $j = y$ and $i - x \equiv \pm 1 \mod m$. If m is any positive integer, then $N = M(G_{m\times m})$ is a proper minor

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of $M = M(G_{(m+1)\times m})$, and both matroids are 4-connected. But there is no proper minor M' of M such that N is a proper minor of M', and M' is 4-connected. Further examples demonstrating the limits of possible splitter theorems can be found in [3].

We recall some key definitions before stating our main result. Let M be a matroid on the ground set E. If $X \subseteq E$, then $\lambda_M(X)$ is defined to be

$$r(X) + r^*(X) - |X| = r(X) + r(E - X) - r(M).$$

Note $\lambda_M(X) = \lambda_M(E - X)$. A partition (X, Y) of E is a *k-separation*, for a positive integer k, if $|X|, |Y| \geq k$ and $\lambda_M(X) < k$. If $\lambda_M(X) < k$, then X is said to be *k-separating.* If every k-separation of M satisfies $k \geq n$, for some value n, then M is n-connected. If M is 3-connected, and every 3-separation (X, Y) satisfies $\min\{|X|, |Y|\} = 3$, then M is *internally 4-connected.*

Theorem 1.1. *Let M be a 4-connected binary matroid and N be an internally 4-connected proper minor of M with at least 7 elements. Then, for some e in $E(M)$, either $M\backslash e$ or M/e is internally 4-connected having an N-minor unless $M \cong D_{16}$. In the exceptional case, there are elements $e, f \in E(M)$ such that $M' = M\backslash e/f$ is internally 4-connected with an N-minor.*

In the statement of Theorem 1.1, D_{16} refers to the 16-element rank-8 binary matroid represented over $GF(2)$ by the matrix $[I_8|A]$, where A is the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Evidently D_{16} is isomorphic to its dual. Moreover, D_{16} has two AG$(3, 2)$-minors on disjoint ground sets.

Theorem 1.1 strengthens the following result by Zhou [5, Theorem 3.1], which plays a fundamental role in our proof. A matroid is *weakly 4-connected* if it is 3-connected, and, whenever (X, Y) is a 3-separation, $\min\{|X|, |Y|\} \leq 4$.

Theorem 1.2. *Let M be a 4-connected binary matroid and N be an internally 4-connected proper minor of M with at least 7 elements. Then, for some e in $E(M)$, either $M\backslash e$ or M/e is weakly 4-connected having an N-minor.*

We briefly describe the structure of the proof of Theorem 1.1. We assume that M and N are as in the statement of the theorem, and that there is no element $e \in E(M)$ such that $M\backslash e$ or M/e is internally 4-connected with an N-minor. By duality and Theorem 1.2, there is an element $e \in E(M)$ such that $M\backslash e$ is weakly 4-connected with an N-minor. We deduce that $M\backslash e$ contains a *quad* Q, that is, a 4-element circuit-cocircuit. Lemma 2.3 says that if 1 is an arbitrary element in Q, then either $M\backslash 1$ or $M/1$ is weakly 4-connected with an N-minor. The first case quickly leads to a contradiction, so $M/1$ is weakly 4-connected, and must contain a quad Q_1. In fact, if $Q = \{1, 2, 3, 4\}$, then M/i is weakly 4-connected, and contains a quad Q_i, for every element $i \in Q$. We show that $e \in Q_i$ for each i. Let Q_i be $\{e, x_i, y_i, z_i\}$. We gain additional structure by considering the minors $M\backslash x_1$, $M\backslash y_1$, $M\backslash z_1$, $M\backslash y_2$, $M\backslash z_2$, and $M\backslash x_3$. Each of these is weakly 4-connected with a quad. By repeatedly exploiting the fact that circuits meet cocircuits in an even number of elements in binary matroids, we find that $Q_1 \cup \cdots \cup Q_4 = \{e, x_1, y_1, z_1, y_2, z_2, x_3\}$. The entire ground set consists of these 7 elements together with $\{1, 2, 3, 4\}$ and 5 other elements found in various quads. At this point, we have learned enough about the structure of M to construct a representation for it and deduce that it is isomorphic to D_{16}.

We conclude the paper by showing that it really is necessary to make an exception for D_{16} in the statement of Theorem 1.1; that is, D_{16} really is 4-connected and has an internally 4-connected minor, N, such that no single-element deletion or contraction of D_{16} is internally 4-connected with an N-minor.

2. Some preliminaries

Recall that a *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. An n-connected matroid with at least $2(n-1)$ elements does not contain a circuit or cocircuit with fewer than n elements [2, Proposition 8.2.1]. Hence a 4-connected matroid with at least 6 elements does not contain a triangle or triad.

A circuit and a cocircuit cannot meet in a single element. We refer to this property as *orthogonality.* Let M be a binary matroid. Then a circuit and a cocircuit of M must intersect in an even number of elements [2, Theorem 9.1.2 (ii)]. If C_1 and C_2 are circuits of M, then $C_1 \triangle C_2$, the symmetric difference of C_1 and C_2, is a disjoint union of circuits [2, Theorem 9.1.2 (iv)].

Let (X, Y) be a k-separation of the matroid M. If $y \in Y$ is in $\mathrm{cl}(X)$, then $r(X \cup y) = r(X)$. As $r(Y - y) \leq r(Y)$, it follows that $(X \cup y, Y - y)$ is a k-separation of M (provided $|Y - y| \geq k$). Corollary 8.1.5 of [2] implies that (X, Y) is a k-separation of M if and only if it is a k-separation of M^*. Therefore, if y is in $Y \cap \mathrm{cl}^*(X)$ and $|Y - y| \geq k$, then $(X \cup y, Y - y)$ is a k-separation of M^*, and hence of M.

Lemma 2.1. *Let M be a 3-connected binary matroid and (X, Y) be a 3-separation of M. If $|X| = 5$ and $r(X) = 3$, then X is not a cocircuit of M.*

Proof. Assume that X is a cocircuit. We may view M as a restriction of $\mathrm{PG}(r-1,2)$ where $r = r(M)$. As (X, Y) is a 3-separation of M, the subspaces of $\mathrm{PG}(r-1,2)$ spanned by X and Y meet in a rank-2 flat of $\mathrm{PG}(r-1,2)$. Since X is a cocircuit of M, it follows that $X \cap \mathrm{cl}(Y) = \emptyset$, so this rank-2 flat avoids X. Thus X is a subset of the 4-element set that is obtained from the binary projective plane, $\mathrm{PG}(2,2)$, by deleting a line. As $|X| = 5$, this is impossible. □

Lemma 2.2. *Let Q be a quad of the binary matroid M. If x and y are elements of Q, then $M\backslash x$ is isomorphic to $M\backslash y$.*

Proof. We may as well assume $x \neq y$. Let E be the ground set of M and let $Q = \{x, y, a, b\}$. Let $\phi\colon (E - x) \to (E - y)$ be defined so that $\phi(y) = x$, $\phi(a) = b$, $\phi(b) = a$, and $\phi(e) = e$ for every element $e \in E - Q$.

Let C be a circuit of $M\backslash x$. If $C \subseteq E - Q$, then clearly $\phi(C) = C$ is a circuit of $M\backslash y$. Assume that C meets $Q - x$. Since $Q - x$ is a cocircuit of $M\backslash x$, it follows that $|C \cap (Q - x)| = 2$. If $y \notin C$, then $\phi(C) = C$ is a circuit of $M\backslash y$, so we assume $y \in C$. Then $\phi(C) = C \triangle Q$ is a disjoint union of circuits of M. No circuit of M can meet Q in a single element, and no circuit can be properly contained in C. Therefore $\phi(C)$ is a circuit of M that does not contain y. Hence $\phi(C)$ is a circuit of $M\backslash y$. A similar argument shows that if C is a circuit of $M\backslash y$ that meets $Q - y$, then $\phi^{-1}(C)$ is a circuit of $M\backslash x$. Hence ϕ is the desired isomorphism. □

Lemma 2.3. *Let M be a 4-connected binary matroid. Let e be an element such that $M\backslash e$ is weakly 4-connected. Suppose $M\backslash e$ has a quad Q. Let 1 be an element of Q. Then the following statements hold.*

(i) *$M\backslash e\backslash 1$ is 3-connected and $M\backslash 1$ is weakly 4-connected.*
(ii) *$M\backslash e/1$ is 3-connected and $M/1$ is weakly 4-connected.*

Proof. Assume $|E(M)| < 6$. It is trivial to check that there are no 3-connected binary matroids with 4 or 5 elements. Therefore $|E(M)| \leq 3$, which contradicts the fact that $M\backslash e$ has a quad. Therefore $|E(M)| \geq 6$, so M has no triangles or triads.

We first establish (i).

2.3.1. *$M\backslash e\backslash 1$ is 3-connected.*

If not, then $M\backslash e\backslash 1$ has a 2-separation (U, V). Without loss of generality, $|U \cap (Q - 1)| \geq 2$. If $|U \cap (Q - 1)| = 3$, then $1 \in \mathrm{cl}_{M\backslash e}(U)$, so $(U \cup 1, V)$ is a 2-separation of $M\backslash e$; a contradiction. Thus we may assume that $|U \cap (Q-1)| = 2$, so $V \cap (Q-1) = \{g\}$, say. Since $Q - 1$ is a cocircuit of $M\backslash e\backslash 1$, $g \in \mathrm{cl}^*_{M\backslash e\backslash 1}(U)$. Therefore $(U \cup g, V - g)$ is a 2-separation of $M\backslash e\backslash 1$ unless $|V| = 2$. If $(U \cup g, V - g)$ is a 2-separation of $M\backslash e\backslash 1$, then, as $U \cup g \supseteq Q - 1$, we obtain a contradiction as above. Thus we may assume that $|V| = 2$.

Since $M\backslash e\backslash 1$ is certainly 2-connected, it follows from [2, Corollary 8.2.2] that V is a circuit or cocircuit of $M\backslash e\backslash 1$. As $Q - 1$ is a cocircuit meeting V in $\{g\}$, orthogonality implies V is a cocircuit. Since M has no cocircuits with fewer than 4 elements, $V \cup \{e, 1\}$ is a cocircuit of M. Now $Q \cap (V \cup \{e, 1\}) = \{g, 1\}$. As Q is a quad in $M\backslash e$, but not in M, $Q \cup e$ is a cocircuit of M. Therefore $(Q \cup e) \triangle (V \cup \{e, 1\})$ is a disjoint union of cocircuits of M. But the last set has only 3 elements, contradicting the fact that M is 4-connected. We conclude that (2.3.1) holds.

Suppose $M\backslash 1$ is not weakly 4-connected. Then it has a 3-separation (X, Y) with $|X|, |Y| \geq 5$. Without loss of generality, $e \in X$. Since neither $(X \cup 1, Y)$ nor $(X, Y \cup 1)$ is a 3-separation of M, neither $\mathrm{cl}_M(X)$ nor $\mathrm{cl}_M(Y)$ contains 1. Therefore $Q - 1$ is contained in neither X nor Y.

We first assume that $|(Q - 1) \cap X| = 2$ and let $(Q - 1) \cap Y = \{f\}$. Then $f \in \mathrm{cl}^*_{M\backslash 1}(X)$, since $(Q \cup e) - 1$ is a cocircuit of $M\backslash 1$, so $(X \cup f, Y - f)$ is a 3-separation of $M\backslash 1$. However, $1 \in \mathrm{cl}_M(X \cup f)$, so this implies that $(X \cup \{f, 1\}, Y - f)$ is a 3-separation of M, which is impossible.

We deduce that $|(Q-1) \cap Y| = 2$. Let g be the single element in $(Q-1) \cap X$. Now $(X - e, Y)$ is a 3-separation in $M\backslash 1\backslash e$. As $Q - 1$ is a cocircuit of $M\backslash 1\backslash e$, it follows that $g \in \mathrm{cl}^*_{M\backslash 1\backslash e}(Y)$, so $(X - \{e, g\}, Y \cup g)$ is a 3-separation in $M\backslash 1\backslash e$. But $Q \subseteq Y \cup \{g, 1\}$, so $1 \in \mathrm{cl}_{M\backslash e}(Y \cup g)$. Therefore $(X - \{e, g\}, Y \cup \{g, 1\})$ is a 3-separation in $M\backslash e$. As $M\backslash e$ is weakly 4-connected, it follows that $|X - \{e, g\}| \leq 4$, so $|X|$ is 5 or 6.

Now e must be in $\mathrm{cl}_{M\backslash 1}(X - e)$, for otherwise $(X - e, Y)$ is a 2-separation in $M\backslash 1\backslash e$, contradicting (2.3.1). On the other hand, $e \notin \mathrm{cl}_M(X - \{e, g\})$, or else $(X - g, Y \cup \{g, 1\})$ is a 3-separation in M, which contradicts the fact that M is 4-connected. We deduce from this that there is a circuit C contained in X that contains both e and g.

Assume that $|X| = 5$. Then $X - \{e, g\}$ is a 3-element 3-separating set in $M\backslash e$. As M has no triangles, $X - \{e, g\}$ is a triad of $M\backslash e$, so $X - g$ is a cocircuit of M. Furthermore, $|C| > 3$, and $|C \cap (X - g)|$ is even, so C must be equal to X. Therefore $r_{M\backslash 1}(X) = 4$. As

$$\lambda_{M\backslash 1}(X) = r_{M\backslash 1}(X) + r^*_{M\backslash 1}(X) - |X| = 2,$$

it follows that $r^*_{M\backslash 1}(X) = 3$. Now $M^*/1 = (M\backslash 1)^*$ is 3-connected, (X, Y) is a 3-separation in $M^*/1$, $r_{M^*/1}(X) = 3$, and X is a cocircuit in $M^*/1$. This contradiction to Lemma 2.1 shows that $|X| = 6$.

Since $X - \{e, g\}$ is a 4-element 3-separating set in $M\backslash e$ that contains no triangles, it is a quad of $M\backslash e$. Therefore $X - \{e, g\}$ and $X - g$ are a circuit and a cocircuit in M, respectively. Thus $|C \cap (X - g)|$ is even. As $|C| > 3$, this means that $|C \cap (X - g)| = 4$. Now $C \triangle (X - \{e, g\})$ has cardinality 3 and is a disjoint union of circuits. This contradiction completes the proof of statement (i).

To prove (ii), we first show that

2.3.2. *$M\backslash e/1$ is 3-connected.*

Suppose $M\backslash e/1$ has (U, V) as a 2-separation. We can assume $|(Q - 1) \cap U| \geq 2$. Now $Q - 1$ is a circuit of $M\backslash e/1$. If $Q - 1 \subseteq U$, then, as Q is a cocircuit of $M\backslash e$, we deduce that $(U \cup 1, V)$ is a 2-separation of $M\backslash e$; a contradiction. If $|(Q-1)\cap U| = 2$ and $(Q-1)\cap V = \{f\}$, then either $(U\cup f, V-f)$ is a 2-separation of $M\backslash e/1$ with $Q-1\subseteq U\cup f$, or $|V| = 2$. In the former case, we argue as above. In the latter case, V is a circuit or a cocircuit of M, contradicting the fact that M has no triangles and no triads. Hence (2.3.2) holds.

Suppose $M/1$ is not weakly 4-connected. Then it has a 3-separation (X, Y) with $|X|, |Y| \geq 5$. Without loss of generality, $e \in X$. Therefore $(X-e, Y)$ is a 3-separation of $M/1\backslash e$. Suppose $Q-1\subseteq X$. Then $1 \in \mathrm{cl}^*_{M\backslash e}(X)$, as Q is a cocircuit of $M\backslash e$. Hence $((X-e)\cup 1, Y)$ is a 3-separation of $M\backslash e$. This contradicts the fact that this matroid is weakly 4-connected.

Next suppose $Q-1\subseteq Y$. Then $(X-e, Y\cup 1)$ is a 3-separation of $M\backslash e$. Thus $|X-e|\leq 4$, and $X-e$ is a quad of $M\backslash e$, since otherwise $X-e$ contains a triangle of $M\backslash e$, and hence of M. Therefore X is a cocircuit of M, and of $M/1$. Hence $r^*_{M/1}(X) = 4$, and it follows that $r_{M/1}(X) = 3$. Thus we have a contradiction to Lemma 2.1.

Suppose next that $|(Q-1)\cap X| = 2$ and let $(Q-1)\cap Y = \{f\}$. Then $((X-e)\cup f, Y-f)$ is a 3-separation of $M/1\backslash e$, so $((X-e)\cup\{f,1\}, Y-f)$ is a 3-separation of $M\backslash e$. But $e\in \mathrm{cl}_{M/1}(X-e)$, for otherwise $(X-e, Y)$ is a 2-separation of $M/1\backslash e$, contradicting (2.3.2). Therefore $e \in \mathrm{cl}_M((X-e)\cup 1)$, and it follows that $(X\cup\{f,1\}, Y-f)$ is a 3-separation of M. As M is 4-connected, this is a contradiction.

Finally, suppose $|(Q-1)\cap Y| = 2$ and let $(Q-1)\cap X = \{g\}$. As $Q-1$ is a circuit of $M/1$, it follows that $(X-g, Y\cup g)$ is a 3-separation of $M/1$ with $Q-1\subseteq Y\cup g$. If $|X-g|\geq 5$, then we have reduced to an earlier case. Thus we assume that $|X| = 5$. Then $(X-\{g,e\}, Y\cup g)$ is a 3-separation of $M/1\backslash e$ and $Q-1\subseteq Y\cup g$. Hence $(X-\{g,e\}, Y\cup\{g,1\})$ is a 3-separation of $M\backslash e$. Thus $X-\{g,e\}$ is a triad of $M\backslash e$, so $X-g$ is a cocircuit of M and hence of $M/1$.

We have $r_{M/1}(X)+r^*_{M/1}(X) = 7$. Suppose $r_{M/1}(X) = 3$. Then, as $X-g$ is a cocircuit of $M/1$, we deduce that $(M/1)|X$ is the union of two triangles, T_1 and T_2, that meet in g. Thus $T_1\cup 1$ and $T_2\cup 1$ are circuits of M, so $T_1 \triangle T_2 = X-g$ is a circuit of M. Since it is also a cocircuit, M has a quad, which is impossible.

We may now assume that $r^*_{M/1}(X) = r^*_M(X) = 3$. As X is a 5-element rank-3 set in M^*, it contains a triangle of M^*, and hence M contains a triad. This contradiction completes the proof of (ii). □

3. The main result

Proof of Theorem 1.1. First assume that $|E(N)| = 7$. By duality, we can assume that $r(N)\leq 3$. Then N is a 3-connected binary matroid with rank 3 and 7 elements. Since $\mathrm{PG}(2,2)$ contains only 7 elements, this shows that $N \cong F_7$ or F_7^*. Since $M \neq N$, a result by Zhou [4, Corollary 1.2], shows that M has an N_1-minor, where N_1 is one of 5 possible 10- or 11-element matroids. It is easily confirmed that N_1 is non-regular, and internally 4-connected, but not 4-connected. Thus N_1 has an N-minor and $N_1 \neq M$. By relabeling N_1 as N, we can assume that $|E(N)| \geq 8$.

We will assume that M has no element e such that $M\backslash e$ or M/e is internally 4-connected having an N-minor. This implies the following fact.

1.1.1. *Let x be an element of M.*

(i) *If $M\backslash x$ is weakly 4-connected, and has an N-minor, then $M\backslash x$ has a quad.*
(ii) *If M/x is weakly 4-connected, and has an N-minor, then M/x has a quad.*

To prove (1.1.1), we assume that $M\backslash x$ has an N-minor, and is weakly 4-connected. Our assumption means that $M\backslash x$ is not internally 4-connected. Therefore $M\backslash x$ has a 3-separation (X, Y) such that $|X| = 4$ or $|Y| = 4$. We will assume the former, without loss of generality. If X is not a quad, then it contains both a triangle and a triad. Therefore M contains a triangle, which is impossible. Thus $M\backslash x$ contains a quad. The proof of the second statement is identical.

By Theorem 1.2 and duality, for some e in $E(M)$, the matroid $M\backslash e$ is weakly 4-connected and has an N-minor. Then (1.1.1) implies $M\backslash e$ has a quad $Q = \{1, 2, 3, 4\}$. If $Q \subseteq E(N)$, then Q is a 4-element 3-separating set in N. Since $|E(N)| \geq 8$, this contradicts the fact that N is internally 4-connected. Thus, we can assume that the element $1 \in Q$ is not in $E(N)$, and that therefore N is a minor of $M\backslash e\backslash 1$ or of $M\backslash e/1$. Then, by Lemma 2.3, either

(i) $M\backslash e\backslash 1$ has an N-minor and $M\backslash 1$ is weakly 4-connected; or
(ii) $M\backslash e/1$ has an N-minor and $M/1$ is weakly 4-connected.

For all i in Q, the matroid $M\backslash e\backslash i$ is isomorphic to $M\backslash e\backslash 1$ by Lemma 2.2. Therefore, if (i) holds, then $M\backslash e\backslash i$ has an N-minor and is weakly 4-connected, for all $i \in Q$. By duality and Lemma 2.2, $M\backslash e/i$ is isomorphic to $M\backslash e/1$ for all i in Q. Therefore, if (ii) holds, then $M\backslash e/i$ has an N-minor and is weakly 4-connected for all $i \in Q$.

Suppose first that (i) holds. As $M\backslash 1$ is weakly 4-connected, it has a quad Q_1 by (1.1.1). Now Q and Q_1 are circuits of M, while $Q \cup e$ and $Q_1 \cup 1$ are cocircuits. Since $1 \in Q$, it follows that $|Q_1 \cap (Q - 1)|$ is odd. As $|Q_1 \cap ((Q - 1) \cup e)|$ is even, we deduce that $e \in Q_1$. If

$$|Q_1 \cap (Q - 1)| = 3,$$

then $|Q_1 \triangle Q| = 2$, meaning that M has a circuit of size at most 2. This is impossible, so $|(Q_1 - e) \cap (Q - 1)| = 1$. We may assume that $Q_1 = \{e, 2, x_1, y_1\}$ where $|\{1, 2, 3, 4, e, x_1, y_1\}| = 7$. By symmetry, $M\backslash 2$ has a quad Q_2 and $e \in Q_2$. Thus Q_2 is a circuit of M and $Q_2 \cup 2$ is a cocircuit of M. As above, $|(Q_2 - e) \cap (Q - 2)| = 1$. Note that $M\backslash e\backslash 1 = M\backslash 1\backslash e \cong M\backslash 1\backslash 2$ by Lemma 2.2, because $\{2, e\} \subseteq Q_1$. Thus, by symmetry, $1 \in Q_2$ and $|(Q_2 - 1) \cap (Q_1 - 2)| = 1$. Hence $Q_2 = \{e, 1, x_2, y_2\}$ where $|\{1, 2, 3, 4, e, x_1, y_1, x_2, y_2\}| = 9$.

By symmetry again, $M\backslash 3$ has a quad Q_3 and $e \in Q_3$. Moreover, $|Q_3 \cap (Q-3)| = 1$. Assume that $2 \in Q_3$. Then the cocircuit $Q_3 \cup 3$ meets the circuit Q_2 in at least one element, e. It follows that $|Q_3 \cap Q_2| = 2$. But as $2 \in Q_3$, this means that the circuit Q_3 meets the cocircuit $Q_2 \cup 2$ in 3 elements, which is impossible. Therefore either $4 \in Q_3$ or $1 \in Q_3$.

Assume that $4 \in Q_3$, so $Q_3 = \{e, 4, x_3, y_3\}$. We also know that $M\backslash 4$ has a quad Q_4 and $e \in Q_4$. By symmetry with the previous arguments, $Q_4 = \{e, 3, x_4, y_4\}$ and $|\{1, 2, 3, 4, e, x_3, y_3, x_4, y_4\}| = 9$. Since M is binary, $|(Q_4 - e) \cap (Q_1 - e)| = 1$ and $|(Q_4 - e) \cap (Q_2 - e)| = 1$ so, without loss of generality, $x_4 = x_1$ and $y_4 = y_2$. By symmetry, $x_3 = y_1$ and $y_3 = x_2$. Now let $Z = \{1, 2, 3, 4, e, x_1, y_1, x_2, y_2\}$. Then Z is spanned by $\{1, 2, 3, x_1, y_1\}$ in M. Since $\{1, 2, 3, 4, e\}$ and $\{1, 2, x_1, y_1, e\}$ are cocircuits of M, so is $\{3, 4, x_1, y_1\}$. Hence Z is spanned by $\{1, 2, 3, x_1, e\}$ in M^*. Thus $r(Z) + r^*(Z) - |Z| \leq 1$. Since M is 4-connected, we deduce that $|E(M) - Z| \leq 1$. Hence we obtain a contradiction unless $|E(M)| \in \{9, 10\}$. In the exceptional case, as $M\backslash e$ has a quad and an N-minor, and $|E(N)| \geq 8$, we have $|E(M)| = 10$. Recall that $M\backslash e\backslash 1$ has an N-minor. But $(M\backslash e\backslash 1)^*$ has $\{2, x_1, y_1\}$ and $\{2, 3, 4\}$ as circuits. Now let $E(M) - Z = \{f\}$. Then, as $r(Z) = 5 = r^*(Z)$ and $\{1, 2, 3, 4, e\}$ is a cocircuit of M, we deduce that $r(\{x_1, y_1, x_2, y_2, f\}) = 4$. Thus this set contains a circuit C, and C contains at least 4 elements. Note that $\{1, 2, x_1, y_1, x_2, y_2\}$ is the symmetric difference of Q, Q_3, and Q_4. Since M has no circuits with fewer than 4 elements, it follows that $\{1, 2, x_1, y_1, x_2, y_2\}$ is a circuit. Therefore $C \neq \{x_1, y_1, x_2, y_2\}$. But, by orthogonality with each of the sets $Q_i \cup i$, we deduce that C contains $\{x_1, y_1, x_2, y_2\}$. Hence $C = \{x_1, y_1, x_2, y_2, f\}$. But the symmetric difference of this with $\{1, 2, x_1, y_1, x_2, y_2\}$ is $\{1, 2, f\}$; which contradicts the fact that M has no triangles. We conclude that $4 \notin Q_3$.

We now know that $1 \in Q_3$. Then $Q_3 = \{e, 1, x_3, y_3\}$ for some x_3 and y_3. Thus $\{3, e, 1, x_3, y_3\}$ is a cocircuit. But $\{e, 2, x_1, y_1\}$ is a circuit so $|\{x_1, y_1\} \cap \{x_3, y_3\}|$ is odd. On the other hand, $\{1, 2, x_1, y_1, e\}$ is a cocircuit and $\{e, 1, x_3, y_3\}$ is a circuit, so $|\{x_1, y_1\} \cap \{x_3, y_3\}|$ is even. This contradiction completes the proof that $M\backslash e\backslash 1$ does not have an N-minor.

We now assume that case (ii) holds, so that $M\backslash e/1$ has an N-minor and is weakly 4-connected. Then, by Lemma 2.2 and (1.1.1), for all i in Q, the matroid M/i has an N-minor and is weakly 4-connected having a quad Q_i. Moreover, for any i and f in Q_i, it follows that $M/i/f$ or $M/i\backslash f$ has an N-minor. The first case is dual to the case above, which was eliminated. Thus we may assume that $M/i\backslash f$ has an N-minor. By the dual of Lemma 2.3 (ii), $M\backslash f$ is weakly 4-connected, thus each $M\backslash f$ has a quad by (1.1.1).

Since $Q \cup e$ is cocircuit in M, and $Q \cup i$ is a circuit, for each i in $\{1, 2, 3, 4\}$, the intersection $(Q \cup e) \cap (Q_i \cup i)$ has even cardinality. Therefore $|(Q \cup e) \cap Q_i|$ is odd. Since Q is a circuit and Q_i is a cocircuit, $|Q \cap Q_i|$ is even, so we conclude that $e \in Q_i$ and we let $Q_i = \{e, x_i, y_i, z_i\}$.

1.1.2. $(Q_i - e) \cap Q = \emptyset$ *for all* i *in* $\{1, 2, 3, 4\}$.

As $Q_i \cup i$ is a circuit and $Q \cup e$ is a cocircuit, $|(Q_i - e) \cap (Q - i)|$ is even. Assume $|(Q_i - e) \cap (Q - i)| = 2$. Then, as Q is a circuit, $(Q_i \cup i) \triangle Q$ is a disjoint union of circuits. But $|(Q_i \cup i) \cap Q| = 3$, so $|(Q_i \cup i) \triangle Q| = 3$. This contradicts the fact that M is 4-connected.

We may assume that

1.1.3. $x_1 = x_2$ *and* $\{x_1, y_1, z_1\} \cap \{y_2, z_2\} = \emptyset$.

To see this, observe that $\{e, x_1, y_1, z_1, 1\}$ is a circuit and $\{e, x_2, y_2, z_2\}$ is a cocircuit. Hence $|\{x_1, y_1, z_1\} \cap \{x_2, y_2, z_2\}| = 1$ by (1.1.2), and (1.1.3) holds.

Let $\{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ be a quad of $M \backslash x_1$. The circuit $\{e, x_1, y_1, z_1, 1\}$ and the cocircuit $\{\alpha_1, \beta_1, \gamma_1, \delta_1, x_1\}$ imply that $|\{e, y_1, z_1, 1\} \cap \{\alpha_1, \beta_1, \gamma_1, \delta_1\}|$ is odd. The circuit $\{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ and cocircuit $\{e, x_1, y_1, z_1\}$ imply that $|\{e, y_1, z_1\} \cap \{\alpha_1, \beta_1, \gamma_1, \delta_1\}|$ is even. Thus $1 \in \{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ so, without loss of generality,

1.1.4. $1 = \alpha_1$.

1.1.5. *We may assume that* $2 = \beta_1$ *and*

$$\{\gamma_1, \delta_1\} \cap \{e, x_1, y_1, z_1, y_2, z_2, 1, 2, 3, 4\} = \emptyset.$$

Since $x_2 = x_1$, the set $\{e, x_1, y_2, z_2, 2\}$ is a circuit of M and $\{1, \beta_1, \gamma_1, \delta_1, x_1\}$ is a cocircuit of M by (1.1.4). Thus $|\{e, y_2, z_2, 2\} \cap \{1, \beta_1, \gamma_1, \delta_1\}|$ is odd. In addition, $\{e, x_1, y_2, z_2\}$ is a cocircuit of M by (1.1.3), and $\{1, \beta_1, \gamma_1, \delta_1\}$ is a circuit, so $|\{e, y_2, z_2\} \cap \{1, \beta_1, \gamma_1, \delta_1\}|$ is even. Hence $2 \in \{\beta_1, \gamma_1, \delta_1\}$ and we may assume that $2 = \beta_1$. Then $\{1, 2, \gamma_1, \delta_1, x_1\}$ and $\{e, x_1, y_2, z_2\}$ are cocircuits. If $|\{e, y_2, z_2\} \cap \{1, 2, \gamma_1, \delta_1\}| = 2$, then $|\{1, 2, \gamma_1, \delta_1, x_1\} \triangle \{e, x_1, y_2, z_2\}| = 3$, and this leads to a contradiction. Thus $|\{e, y_2, z_2\} \cap \{1, 2, \gamma_1, \delta_1\}| = 0$. Similarly, $|\{e, y_1, z_1\} \cap \{1, 2, \gamma_1, \delta_1\}| = 0$. Finally, it is clear that $\{1, 2\} \cap \{\gamma_1, \delta_1\} = \emptyset$. If $\{3, 4\} \cap \{\gamma_1, \delta_1\} \neq \emptyset$, then we must have $\{1, 2, 3, 4\} = \{1, 2, \gamma_1, \delta_1\}$ so $\{1, 2, 3, 4, x_1\}$ and $\{1, 2, 3, 4, e\}$ are cocircuits of M, and $e = x_1$; a contradiction. We conclude that (1.1.5) holds.

1.1.6. $x_1 \notin \{x_3, y_3, z_3\}$.

Recall that $\{1, 2, \gamma_1, \delta_1\}$ is a circuit and $\{e, x_3, y_3, z_3\}$ is a cocircuit, hence $|\{1, 2, \gamma_1, \delta_1\} \cap \{x_3, y_3, z_3\}|$ is even. As $|\{1, 2, \gamma_1, \delta_1, x_1\} \cap \{e, x_3, y_3, z_3, 3\}|$ is even and $3 \notin \{1, 2, \gamma_1, \delta_1, x_1\}$, by (1.1.2) and (1.1.5), it follows that $|\{1, 2, \gamma_1, \delta_1, x_1\} \cap \{e, x_3, y_3, z_3\}|$ is even. Since $e \in \{\gamma_1, \delta_1\}$ by (1.1.5) and $e \notin Q$, we conclude that $e \in \{1, 2, \gamma_1, \delta_1\}$ and therefore (1.1.6) holds.

1.1.7. *We may assume that* $Q_3 = \{e, x_3, y_1, z_2\}$. *Moreover,* $x_3 \notin \{\gamma_1, \delta_1\}$.

To see this, note that the cocircuits $\{e, x_1, y_1, z_1\}$ and $\{e, x_1, y_2, z_2\}$ and the circuit $\{e, x_3, y_3, z_3, 3\}$ of M imply using (1.1.2) and (1.1.6) that each of $\{y_1, z_1\}$ and $\{y_2, z_2\}$ meets $\{x_3, y_3, z_3\}$ in a single element. By (1.1.3), $\{y_1, z_1\} \cap \{y_2, z_2\} = \emptyset$, and the first part of (1.1.7) follows. If $x_3 \in \{\gamma_1, \delta_1\}$, then it follows from (1.1.2) and (1.1.5) that the circuit $\{1, 2, \gamma_1, \delta_1\}$ meets the cocircuit $\{e, x_3, y_1, z_2\}$ in a single element; a contradiction.

Next we consider Q_4. The arguments of (1.1.6) also show that $x_1 \notin \{x_4, y_4, z_4\}$. Since $\{e, x_4, y_4, z_4, 4\}$ is a circuit, and $\{e, x_1, x_2, x_3\}$, $\{e, x_1, y_2, z_2\}$, and $\{e, x_3, y_1, z_2\}$ are cocircuits, it follows that $\{x_4, y_4, z_4\}$ meets each of $\{x_3, y_1, z_2\}$, $\{y_1, z_1\}$, and $\{y_2, z_2\}$ in a single element.

1.1.8. $\{x_3, y_1\} \cap \{x_1, y_2\} = \emptyset = \{x_3, z_2\} \cap \{x_1, z_1\}$.

This follows by considering the intersection of the circuit $\{e, x_3, y_1, z_2, 3\}$ with the cocircuits $\{e, x_1, y_2, z_2\}$ and $\{e, x_1, y_1, z_1\}$.

By using (1.1.3) and the fact that $x_1 \notin \{x_4, y_4, z_4\}$, we deduce that there are the following three possibilities for $\{x_4, y_4, z_4\}$:

(A) $\{y_1, y_2, y'\}$ for some $y' \notin \{y_1, y_2, z_1, z_2, x_1, x_3\}$;
(B) $\{z_1, y_2, x_3\}$;
(C) $\{z_1, z_2, z'\}$ for some $z' \notin \{y_1, y_2, z_1, z_2, x_1, x_3\}$.

Cases (A) and (C) are symmetric, so we may assume that (A) or (B) holds.

Now $M\backslash y_1$ has a quad. By (1.1.4) and symmetry, this quad is $\{1, \beta_2, \gamma_2, \delta_2\}$. Thus $\{1, \beta_2, \gamma_2, \delta_2, y_1\}$ is a cocircuit of M. As $\{e, x_3, y_1, z_2, 3\}$ is a circuit, we deduce that $|\{1, \beta_2, \gamma_2, \delta_2\} \cap \{e, x_3, z_2, 3\}|$ is odd. Also, since $\{1, \beta_2, \gamma_2, \delta_2\}$ is a circuit and $\{e, x_3, y_1, z_2\}$ is a cocircuit, $|\{1, \beta_2, \gamma_2, \delta_2\} \cap \{e, x_3, z_2\}|$ is even. Thus, without loss of generality, and arguing as for (1.1.5), we get that

1.1.9. $3 = \beta_2$ *and* $\{\gamma_2, \delta_2\} \cap \{e, x_3, z_2\} = \emptyset$.

We now have that $\{1, 3, \gamma_2, \delta_2, y_1\}$ is a cocircuit and $\{1, 3, \gamma_2, \delta_2\}$ is a circuit of M. Assume that (A) holds. Then $\{e, y_1, y_2, y', 4\}$ is a circuit of M. Since $|\{1, 3, \gamma_2, \delta_2\} \cap \{e, y_2, y', 4\}|$ is odd and $|\{1, 3, \gamma_2, \delta_2\} \cap \{e, y_2, y'\}|$ is even, it follows that $4 \in \{\gamma_2, \delta_2\}$. Hence $\{1, 3, \gamma_2, \delta_2\} = \{1, 3, 4, 2\}$. But this means that $\{1, 3, 4, 2, y_1\}$ and $\{1, 2, 3, 4, e\}$ are cocircuits of M, so $y_1 = e$; a contradiction. We conclude that (A) does not hold. Thus (B) holds and

1.1.10. *M has $\{e, x_3, y_2, z_1, 4\}$ as a circuit and has $\{e, x_3, y_2, z_1\}$ as a cocircuit.*

The matroid $M\backslash z_1$ has a quad and it must contain 1, by the same argument as (1.1.4). Let $\{1, \beta_3, \gamma_3, \delta_3\}$ be this quad. Then $|\{1, \beta_3, \gamma_3, \delta_3\} \cap \{e, x_3, y_2, z_1\}|$ and $|\{1, \beta_3, \gamma_3, \delta_3, z_1\} \cap \{e, x_3, y_2, z_1, 4\}|$ are both even. Therefore $|\{1, \beta_3, \gamma_3, \delta_3\} \cap \{e, x_3, y_2, 4\}|$ is odd. It follows that $4 \in \{1, \beta_3, \gamma_3, \delta_3\}$. Without loss of generality we assume that $\beta_3 = 4$. Thus we have the following, where the assertion in the last sentence follows by a similar argument used for (1.1.5).

1.1.11. *M has $\{1, 4, \gamma_3, \delta_3\}$ as a circuit and has $\{1, 4, \gamma_3, \delta_3, z_1\}$ as a cocircuit. Moreover,* $\{\gamma_3, \delta_3\} \cap \{e, x_3, y_2\} = \emptyset$.

From (1.1.5) we see that $4 \notin \{\gamma_1, \delta_1\}$. Assume that $2 \in \{\gamma_3, \delta_3\}$. Then $\{1, 2, 3, 4\}$ and $\{1, 4, \gamma_3, \delta_3\}$ are circuits of M intersecting in 3 elements, so $\{1, 2, 3, 4\} = \{1, 4, \gamma_3, \delta_3\}$. Then $\{1, 2, 3, 4, e\}$ and $\{1, 2, 3, 4, z_1\}$ are cocircuits, and this leads to a contradiction. Therefore $2 \notin \{\gamma_3, \delta_3\}$.

Since $\{1,2,\gamma_1,\delta_1\}$ is a circuit and $\{1,2,\gamma_1,\delta_1,x_1\}$ is a cocircuit, $|\{\gamma_3,\delta_3\}\cap\{\gamma_1,\delta_1,x_1\}|$ and $|\{\gamma_1,\delta_1\}\cap\{\gamma_3,\delta_3,z_1\}|$ are both odd. Thus $x_1\in\{\gamma_3,\delta_3\}$ if and only if $z_1\in\{\gamma_1,\delta_1\}$.

Suppose $x_1\in\{\gamma_3,\delta_3\}$, say $x_1=\gamma_3$. Then $z_1=\gamma_1$, without loss of generality. Thus $\{1,2,z_1,\delta_1\}$ is a circuit and $\{e,x_1,y_1,z_1\}$ is a cocircuit, so $|\{1,2,\delta_1\}\cap\{e,x_1,y_1\}|=1$. By (1.1.2), neither 1 nor 2 is in $\{e,x_1,y_1,z_1\}$, so $\delta_1\in\{e,x_1,y_1\}$. But $\delta_1\neq x_1$ by (1.1.5). If $\delta_1=e$, then $\{1,2,e,z_1\}$ is a circuit and $\{e,x_1,y_2,z_2\}$ is a cocircuit. Note $z_1\neq y_2$ by (1.1.10) and $z_1\neq z_1$ by (1.1.8). Hence $1\in\{y_2,z_2\}$. But $\{x_1,y_2,z_2\}\cap\{1,2,3,4\}=\emptyset$ by (1.1.2). Hence $\delta_1\neq e$. Thus $\delta_1=y_1$. Then $\{1,2,z_1,y_1,x_1\}$ and $\{e,x_1,y_1,z_1\}$ are both cocircuits. Their symmetric difference has exactly 3 elements; a contradiction. We deduce that $x_1\notin\{\gamma_3,\delta_3\}$ and $z_1\notin\{\gamma_1,\delta_1\}$ so

1.1.12. $|\{\gamma_1,\delta_1\}\cap\{\gamma_3,\delta_3\}|=1.$

Now $M\backslash y_2$ has a quad Y_2, so $Y_2\cup y_2$ is a cocircuit of M. By considering the circuit $\{e,x_1,y_2,z_2,2\}$ and the cocircuit $\{e,x_1,y_2,z_2\}$, we deduce that $|Y_2\cap\{e,x_1,z_2\}|$ is even and $|Y_2\cap\{e,x_1,z_2,2\}|$ is odd, so $2\in Y_2$. Similarly, using the circuit $\{e,x_3,y_2,z_1,4\}$ and the cocircuit $\{e,x_3,y_2,z_1\}$, we deduce that $4\in Y_2$. Thus $Y_2=\{2,4,\gamma_5,\delta_5\}$, say.

The matroid $M\backslash z_2$ has a quad Z_2. Since $M/2$ and $M/3$ have $\{e,x_1,y_2,z_2\}$ and $\{e,x_3,y_1,z_2\}$ as quads, it follows that $\{2,3\}\subseteq Z_2$. Thus $Z_2=\{2,3,\gamma_4,\delta_4\}$, say. Similarly, $M\backslash x_3$ has a quad X_3 and $X_3=\{3,4,\gamma_6,\delta_6\}$.

To keep track of the argument to follow, we list in Table 1 the circuits and cocircuits that have arisen from the various quads we have identified. In each of the circuits and cocircuits listed, the elements are distinct.

circuits	cocircuits
$\{1,2,3,4\}$	$\{1,2,3,4,e\}$
$\{e,x_1,y_1,z_1,1\}$	$\{e,x_1,y_1,z_1\}$
$\{e,x_1,y_2,z_2,2\}$	$\{e,x_1,y_2,z_2\}$
$\{e,x_3,y_1,z_2,3\}$	$\{e,x_3,y_1,z_2\}$
$\{e,x_3,y_2,z_1,4\}$	$\{e,x_3,y_2,z_1\}$
$\{1,2,\gamma_1,\delta_1\}$	$\{1,2,\gamma_1,\delta_1,x_1\}$
$\{1,3,\gamma_2,\delta_2\}$	$\{1,3,\gamma_2,\delta_2,y_1\}$
$\{1,4,\gamma_3,\delta_3\}$	$\{1,4,\gamma_3,\delta_3,z_1\}$
$\{2,3,\gamma_4,\delta_4\}$	$\{2,3,\gamma_4,\delta_4,z_2\}$
$\{2,4,\gamma_5,\delta_5\}$	$\{2,4,\gamma_5,\delta_5,y_2\}$
$\{3,4,\gamma_6,\delta_6\}$	$\{3,4,\gamma_6,\delta_6,x_3\}$

Table 1. Some known circuits and cocircuits

Next we prove the following sublemma.

1.1.13. *Suppose that $1\leq i<j\leq 6$. Then $\{\gamma_i,\delta_i\}\neq\{\gamma_j,\delta_j\}$. Moreover, if $\{i,j\}$ is $\{1,6\}$, $\{2,5\}$, or $\{3,4\}$, then $\{\gamma_i,\delta_i\}\cap\{\gamma_j,\delta_j\}=\emptyset$.*

To prove this, we may assume that $i = 1$, as the other cases follow by an identical argument. If $j \in \{2, 3, 4, 5\}$, then $\{\gamma_i, \delta_i\}$ cannot be equal to $\{\gamma_j, \delta_j\}$, for otherwise we can take the symmetric difference of two of the circuits in Table 1 and find a circuit of size at most 2. If $j = 6$ and $\{\gamma_i, \delta_i\} \cap \{\gamma_j, \delta_j\}$ is non-empty, then $\{\gamma_i, \delta_i\}$ and $\{\gamma_j, \delta_j\}$ must be equal, for otherwise the symmetric difference of $\{1, 2, 3, 4\}$, $\{1, 2, \gamma_1, \delta_1\}$, and $\{3, 4, \gamma_6, \delta_6\}$ contains a circuit of size at most 2. Now taking the symmetric difference of $\{1, 2, \gamma_1, \delta_1, x_1\}$ and $\{3, 4, \gamma_6, \delta_6, x_3\}$ shows that $\{1, 2, 3, 4, x_1, x_3\}$ is a cocircuit of M. This is a contradiction, as the cocircuit $\{1, 2, 3, 4, e\}$ leads to a cocircuit of size at most 3. Thus (1.1.13) holds.

We now consider the 6 elements $x_1, y_1, z_1, y_2, z_2, x_3$. From (1.1.3), (1.1.6), and Table 1, these elements are distinct. The 3-element subsets of this set that lie in a known 4-cocircuit with e match up with the 3-point lines in a copy of $M(K_4)$. Moreover, for each 2-element subset $\{i, j\}$ of $\{1, 2, 3, 4\}$, the listed 5-cocircuit containing $\{i, j\}$ contains the unique element of $\{x_1, y_1, z_1, y_2, z_2, x_3\}$ that is common to the indicated 5-circuits containing $\{e, i\}$ and $\{e, j\}$. This reveals more symmetry than may have been immediately apparent.

For example, by repeating the arguments of (1.1.5) with the circuit $\{1, 3, \gamma_2, \delta_2\}$ and the two cocircuits of the form Q_i containing y_1, namely $\{e, x_1, y_1, z_1\}$ and $\{e, x_3, y_1, z_2\}$, we show that $\{\gamma_2, \delta_2\} \cap \{e, x_1, y_1, z_1, z_2, x_3\} = \emptyset$. The orthogonality of the circuit $\{1, 3, \gamma_2, \delta_2\}$ and the cocircuit $\{e, x_1, y_2, z_2\}$ implies that $y_2 \notin \{\gamma_2, \delta_2\}$. Moreover, if $2 \in \{\gamma_2, \delta_2\}$, then $\{1, 2, 3, 4\}$ and $\{1, 3, \gamma_2, \delta_2\}$ must be equal, implying that $\{1, 2, 3, 4, e\}$ and $\{1, 2, 3, 4, y_1\}$ are both cocircuits, which is impossible. Similarly, $4 \notin \{\gamma_2, \delta_2\}$. By applying these arguments in the other symmetric cases we arrive at the following conclusion.

1.1.14. $\{e, x_1, y_1, z_1, y_2, z_2, x_3, 1, 2, 3, 4\}$ *avoids* $\{\gamma_i, \delta_i : 1 \leq i \leq 6\}$.

Moreover, by (1.1.2):

1.1.15. $\{e, x_1, y_1, z_1, y_2, z_2, x_3\}$ *avoids* $\{1, 2, 3, 4\}$.

By using (1.1.14) and comparing circuits and cocircuits in Table 1, we see that $\{\gamma_1, \delta_1\}$ meets each of $\{\gamma_2, \delta_2\}$, $\{\gamma_3, \delta_3\}$, $\{\gamma_4, \delta_4\}$, and $\{\gamma_5, \delta_5\}$ in a single element. From (1.1.13) we know that $\{\gamma_1, \delta_1\}$ avoids $\{\gamma_6, \delta_6\}$.

Without loss of generality, we may assume that $\gamma_1 = \gamma_2$. Then one of the following two cases occurs.

1.1.16. $\{(\gamma_1, \delta_1), (\gamma_2, \delta_2), (\gamma_3, \delta_3), (\gamma_4, \delta_4), (\gamma_5, \delta_5), (\gamma_6, \delta_6)\}$ *is*

(I) $\{(\gamma_1, \delta_1), (\gamma_1, \delta_2), (\delta_1, \delta_2), (\gamma_1, \delta_4), (\delta_1, \delta_4), (\delta_2, \delta_4)\}$; *or*
(II) $\{(\gamma_1, \delta_1), (\gamma_1, \delta_2), (\gamma_1, \delta_3), (\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_2, \delta_3)\}$.

To see that this is true, we consider whether or not γ_1 is in $\{\gamma_3, \delta_3\}$. First assume that it is. Then by relabeling we can assume that $\gamma_3 = \gamma_1$. From

(1.1.13) we see that $\delta_2 \notin \{\gamma_1, \delta_1\}$ and $\delta_3 \notin \{\gamma_1, \delta_1, \delta_2\}$. By orthogonality between $\{2, 3, \gamma_4, \delta_4\}$ and $\{1, 2, \gamma_1, \delta_1, x_1\}$, and between $\{2, 3, \gamma_4, \delta_4\}$ and $\{1, 3, \gamma_1, \delta_2, y_1\}$, we see that

$$|\{\gamma_4, \delta_4\} \cap \{\gamma_1, \delta_1\}| = |\{\gamma_4, \delta_4\} \cap \{\gamma_1, \delta_2\}| = 1.$$

But neither γ_4 nor δ_4 can be be equal to γ_1, for then $\{\gamma_4, \delta_4\}$ and $\{\gamma_3, \delta_3\}$ would not be disjoint, as is demanded by (1.1.13). Thus $\{\gamma_4, \delta_4\} = \{\delta_1, \delta_2\}$. We can assume that $(\gamma_4, \delta_4) = (\delta_1, \delta_2)$. Orthogonality between $\{2, 4, \gamma_5, \delta_5\}$ and the cocircuits $\{1, 2, \gamma_1, \delta_1, x_1\}$ and $\{1, 4, \gamma_1, \delta_3, z_1\}$ shows that

$$|\{\gamma_5, \delta_5\} \cap \{\gamma_1, \delta_1\}| = |\{\gamma_5, \delta_5\} \cap \{\gamma_1, \delta_3\}| = 1.$$

By using (1.1.13), we can assume that $(\gamma_5, \delta_5) = (\delta_1, \delta_3)$. A similar argument shows that we can assume that $(\gamma_6, \delta_6) = (\delta_2, \delta_3)$. Thus we have verified that (II) holds, assuming that $\gamma_1 \in \{\gamma_3, \delta_3\}$.

Next we assume that $\gamma_1 \notin \{\gamma_3, \delta_3\}$. Then $\delta_1 \in \{\gamma_3, \delta_3\}$. Note that $\delta_2 \notin \{\gamma_1, \delta_1\}$. Orthogonality between $\{1, 4, \gamma_3, \delta_3\}$ and $\{1, 3, \gamma_1, \delta_2, y_1\}$ shows that $\delta_2 \in \{\gamma_3, \delta_3\}$, so we may assume that $(\gamma_3, \delta_3) = (\delta_1, \delta_2)$. We know that $|\{\gamma_4, \delta_4\} \cap \{\gamma_1, \delta_1\}| = 1$. But $\delta_1 \notin \{\gamma_4, \delta_4\}$, for $\{\gamma_4, \delta_4\}$ is disjoint with $\{\gamma_3, \delta_3\}$. Thus $\gamma_1 \in \{\gamma_4, \delta_4\}$. We can assume that $\gamma_4 = \gamma_1$. We deduce from (1.1.13) that $\delta_4 \notin \{\gamma_1, \delta_1, \delta_2\}$. By (1.1.13) and orthogonality between $\{2, 4, \gamma_5, \delta_5\}$ and $\{1, 2, \gamma_1, \delta_1, x_1\}$, we see that $\delta_1 \in \{\gamma_5, \delta_5\}$. Applying the same argument to the cocircuit $\{2, 3, \gamma_1, \delta_4, z_2\}$ shows that $\delta_4 \in \{\gamma_5, \delta_5\}$. A similar argument shows that $\{\gamma_6, \delta_6\} = \{\delta_2, \delta_4\}$, so we have completed the proof of (1.1.16).

Now $\{1, 2, \gamma_1, \delta_1\}$ is a quad of $M\backslash x_1$, and $M\backslash x_1/1$ has an N-minor. Thus $M\backslash x_1/\gamma_1$ has an N-minor by Lemma 2.2. Since $M\backslash e$ is weakly 4-connected, Lemma 2.3 implies that $M/1$ is weakly 4-connected. As $\{e, x_1, y_1, z_1\}$ is a quad of $M/1$, this in turn implies that $M\backslash x_1$ is weakly 4-connected, and hence, so is M/γ_1. Thus M/γ_1 has a quad G by (1.1.1). Then G is a cocircuit of M and $G \cup \gamma_1$ is a circuit of M. Since $|G \cap \{1, 2, \delta_1, x_1\}|$ is odd and $|G \cap \{1, 2, \delta_1\}|$ is even, it follows that $x_1 \in G$. Similarly, $\{1, 3, \gamma_2, \delta_2, y_1\} = \{1, 3, \gamma_1, \delta_2, y_1\}$ is a cocircuit, and $|G \cap \{1, 3, \delta_2, y_1\}|$ is odd while $|G \cap \{1, 3, \delta_2\}|$ is even. Hence $y_1 \in G$.

In case (II), $\{1, 4, \gamma_1, \delta_3, z_1\}$ is a cocircuit, and we can argue that z_1 is in G. As $\{x_1, y_1, z_1\} \subseteq G$, and both G and $\{e, x_1, y_1, z_1\}$ are cocircuits, it follows that $G = \{e, x_1, y_1, z_1\}$. Thus $\{e, x_1, y_1, z_1, 1\}$ and $\{e, x_1, y_1, z_1, \gamma_1\}$ are circuits, which leads to a contradiction.

Therefore case (I) holds. Since $\{2, 3, \gamma_1, \delta_4, z_2\}$ is a cocircuit, we can deduce that $z_2 \in G$. Let t be the element of $G - \{x_1, y_1, z_2\}$. By orthogonality, $\{t\}$ is disjoint from the set $J' = \{e, 1, 2, 3, 4, x_1, y_1, z_1, y_2, z_2, x_3, \gamma_1, \delta_1, \delta_2, \delta_4\}$. Let $J = J' \cup t$. Then J is spanned by $\{e, 1, 2, 3, x_1, y_1, y_2, \gamma_1\}$ in M and in M^*. Thus

$$\lambda(J) = r(J) + r^*(J) - |J| \leq 8 + 8 - 16 = 0.$$

Hence $E(M) = J$.

It is easy to show that $\{e, 1, 2, 3, x_1, y_1, y_2, \gamma_1\}$ must be both a basis and cobasis of M, and it is then straightforward to check that M is represented by the matrix $[I_8|A]$, where A is shown in Table 2. Thus $M \cong D_{16}$.

$$\begin{array}{c c} & \begin{array}{cccccccc} x_3 & z_1 & t & e & \delta_1 & 3 & 1 & 4 \end{array} \\ \begin{array}{c} \delta_4 \\ \gamma_1 \\ \delta_2 \\ 2 \\ y_1 \\ y_2 \\ x_1 \\ z_2 \end{array} & \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

TABLE 2. A representation of D_{16}.

As $M/2\backslash e$ has an N-minor, we can complete the proof of Theorem 1.1 by proving the following sublemma.

1.1.17. *$M/2\backslash e$ is internally 4-connected.*

Certainly $M/2\backslash e$ is 3-connected by Lemma 2.3. Assume it is not internally 4-connected and let (X, Y) be a 3-separation of it with $|X|, |Y| \geq 4$. Let $S = \{1, 3, 4, \gamma_1, \delta_1, \delta_2, \delta_4\}$ and $T = \{t, x_1, y_1, z_1, y_2, z_2, x_3\}$. Then (S, T) is a 4-separation of $M/2\backslash e$. Evidently every 4-element subset of S spans S in $M/2\backslash e$. By duality, every 4-element subset of T spans T in $(M/2\backslash e)^*$. Clearly $|S \cap X| \geq 4$ or $|S \cap Y| \geq 4$. Assume the former. If $|Y \cap T| \geq 4$, then, via closure, we can move the elements of $Y \cap S$ into X and, via coclosure, we can move the elements of $X \cap T$ into Y, where each of these moves maintains a 3-separation. It follows that (S, T) is a 3-separation of $M/2\backslash e$; a contradiction. Thus $|Y \cap T| \leq 3$. Now if $|Y| > 4$, we can move elements of $Y \cap S$ into X via closure one at a time until we have a 3-separation (X', Y') with $|Y'| = 4$ and $|Y' \cap T| \leq 3$. If x is an element in $Y' \cap S$, then both Y' and $Y' - x$ are 3-separating. Thus Y' is a 4-element fan of $M/2\backslash e$ so at most one element of Y' is in the closure of X' and at most one element of Y' is in the coclosure of X'. Thus each of $Y' \cap S$ and $Y' \cap T$ has at most one element; a contradiction. We deduce that (1.1.17) holds, and this completes the proof of Theorem 1.1. $\square$

We conclude by demonstrating that it really is necessary to make an exception for D_{16} in the statement of Theorem 1.1. Let $M = [I|A]$, where A is the labeled matrix in Table 2.

We start by showing that M is 4-connected. Assume that this is not the case. When we constructed A during the proof of Theorem 1.1, the element e was chosen so that $M\backslash e$ is weakly 4-connected. Thus $M\backslash e$ is 3-connected, and clearly so is M. Therefore there is a 3-separation (X, Y) of M. It is very easy to confirm that M does not contain any triangles, nor any triads (since it is self-dual). Therefore $|X|, |Y| \geq 4$.

Assume that $|X|, |Y| \geq 5$. Then $(X - \{2, e\}, Y - \{2, e\})$ is a 3-separation of $M/2\backslash e$. Since this matroid is internally 4-connected, by (1.1.17), we can assume that $2, e \in Y$, and that $|Y| = 5$. But it is routine to verify that any 5-element 3-separating set in a 3-connected binary matroid contains a triangle or a triad, so this is impossible. Therefore we can assume that $|Y| = 4$. Moreover, Y is a quad, since otherwise it would contain a triangle or triad.

Let $S_1 = \{\delta_4, \gamma_1, \delta_2, 2\}$, and let $S_2 = \{\delta_1, 3, 1, 4\}$. Moreover, let $T_1 = \{y_1, y_2, x_1, z_2\}$ and let $T_2 = \{x_3, z_1, t, e\}$. Then $M/S_1\backslash S_2$ and $M/T_1\backslash T_2$ are both isomorphic to AG$(3, 2)$. Assume that $Y \subseteq S_1 \cup S_2$. Since AG$(3, 2)$ has no circuits or cocircuits with fewer than 4 elements, Y is one of the 14 quads in $M/T_1\backslash T_2$. But it is easy to verify that none of these is a quad of M. For example, $\{\delta_4, \gamma_1, \delta_2, \delta_1\}$ is a quad in $M/T_1\backslash T_2$. If it were a cocircuit in M, then the rows δ_4, γ_1, δ_2 would sum to the row that is everywhere zero, except in the column labeled δ_1. This is not the case, so $\{\delta_4, \gamma_1, \delta_2, \delta_1\}$ is not a quad of M. In this way we verify that no quad of $M/T_1\backslash T_2$ is a quad of M, and therefore $Y \nsubseteq S_1 \cup S_2$. An identical argument shows that $Y \nsubseteq T_1 \cup T_2$.

It is easy to see that $S_1 \cup S_2$ and $T_1 \cup T_2$ are flats of M, so $|Y \cap (S_1 \cup S_2)| = |Y \cap (T_1 \cup T_2)| = 2$. If $|Y \cap S_1| = 2$ or $|Y \cap S_2| = 2$, then $M/S_1\backslash S_2$ contains a circuit or cocircuit of size 2. Therefore $|Y \cap S_1| = |Y \cap S_2| = 1$. The same argument shows that $|Y \cap T_1| = |Y \cap T_2| = 1$. But it is obvious that no 4-element circuit of M meets S_1, S_2, T_1, and T_2 in a single element each. This contradiction completes the demonstration that M is 4-connected.

By considering the row and column labels of the matrix in Table 2, we see that the permutation that swaps the following pairs is an isomorphism, ϕ, from M to M^*.

$$\{\delta_4, x_3\},\ \{\gamma_1, z_1\},\ \{\delta_2, t\},\ \{2, e\},\ \{y_1, \delta_1\},\ \{y_2, 3\},\ \{x_1, 1\},\ \{z_2, 4\}.$$

Let $N = M/2\backslash e$. Then N is an internally 4-connected minor of M by (1.1.17). We will now show that no single-element deletion or contraction of M is internally 4-connected with an N-minor.

The matrix produced from A by:

(i) pivoting on the entry in the δ_4 row and the δ_1 column;
(ii) swapping the 1 column and the 3 column;
(iii) swapping the x_3 column and the z_1 column;
(iv) swapping the x_1 row and the z_2 row

is identical to A. This shows that there is an automorphism Ω_1 of M swapping the pairs

$$\{\delta_4, \delta_1\},\ \{1, 3\},\ \{x_3, z_1\},\ \{x_1, z_2\}$$

and acting as the identity on the rest of the matroid. Similarly, if we act on A by:

(i) pivoting on the entry in the γ_1 row and the 3 column;
(ii) pivoting on the entry in the x_1 row and the e column;
(iii) swapping the δ_1 column and the 4 column;
(iv) swapping the t column and the x_3 column

then we produce an identical copy of A. Thus there is an automorphism Ω_2 of M that swaps

$$\{\gamma_1, 3\},\ \{x_1, e\},\ \{\delta_1, 4\},\ \{t, x_3\}$$

and acts as the identity on other elements.

Since Ω_1 and Ω_2 are also automorphisms of M^*, we see that $\phi^{-1} \circ \Omega_1 \circ \phi$ and $\phi^{-1} \circ \Omega_2 \circ \phi$ are automorphisms of M that swap, respectively, the pairs

$$\{\delta_4, \gamma_1\},\ \{1, 4\},\ \{x_3, y_1\},\ \{x_1, y_2\} \text{ and}$$
$$\{\delta_4, \delta_2\},\ \{1, 2\},\ \{z_2, y_1\},\ \{z_1, y_2\}$$

while leaving all other elements unchanged. By studying these four automorphisms, we see that

$$O_1 = \{e, t, x_1, y_1, z_1, y_2, z_2, x_3\} \text{ and } O_2 = \{1, 2, 3, 4, \gamma_1, \delta_1, \delta_2, \delta_4\}$$

are contained in orbits of the automorphism group of M.

Consider M/e. It is represented by the matrix $[I_7|A']$ where A' is

$$\begin{array}{c c}
 & \begin{array}{cccccccc} x_3 & z_1 & t & 2 & \delta_1 & 3 & 1 & 4 \end{array} \\
\begin{array}{c} \delta_4 \\ \gamma_1 \\ \delta_2 \\ y_1 \\ y_2 \\ x_1 \\ z_2 \end{array} &
\left[\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{array}\right]
\end{array}.$$

It is easily checked that M/e has no triangles. Since $\{1, 3, 4\}$ is a triangle of N, we deduce that M/e cannot have an N-minor. (This also shows that O_1 and O_2 are in fact orbits.) Certainly $M\backslash e$ is not internally 4-connected, since it contains the quad $\{1, 2, 3, 4\}$. Consequently, we cannot delete or contract an element from O_1 to produces an internally 4-connected matroid with an N-minor.

Since $\phi(e) = 2$, we see that $M^*/2$ does not have an N-minor. As N is self-dual, this means that $M^*/2$ does not have an N^*-minor, so $M\backslash 2$ does not have an N-minor. Moreover, $M/2$ has a quad, so it is not internally 4-connected. Thus we cannot delete or contract any element from O_2 to produce an internally 4-connected matroid with an N-minor, and we have completed the proof of our claim.

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References

[1] Chun, C., Mayhew, D. and Oxley, J., A chain theorem for internally 4-connected binary matroids, *J. Combin. Theory Ser. B* **101** (2011), 141–189.
[2] Oxley, J. G., *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.
[3] Rajan, A., Algorithmic applications of connectivity and related topics in matroid theory, Ph.D. thesis, Northwestern University, 1987.
[4] Zhou, X., On internally 4-connected non-regular binary matroids, *J. Combin. Theory Ser. B* **91** (2004), 327–343.
[5] Zhou, X., Generating an internally 4-connected binary matroid from another, submitted.

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