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A Review of Portfolio Planning: Models and Systems*

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Satchell, S E and Scowcroft A E (Eds.) Butterworth and Heinemann, Oxford

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Abstract

In this chapter, we first provide an overview of a number of portfolio planning models which have been proposed and investigated over the last forty years. We revisit the mean-variance (M-V) model of Markowitz and the construction of the risk-return efficient frontier. A piecewise linear approximation of the problem through a reformulation involving diagonalisation of the quadratic form into a variable separable function is also considered. A few other models, such as, the Mean Absolute Deviation (MAD), the Weighted Goal Programming (WGP) and the Minimax (MM) model which use alternative metrics for risk are also introduced, compared and contrasted. Recently asymmetric measures of risk have gained in importance; we consider a generic representation and a number of alternative symmetric and asymmetric measures of risk which find use in the evaluation of portfolios. There are a number of modelling and computational considerations which have been introduced into practical portfolio planning problems. These include: (a) buy-in thresholds for assets, (b) restriction on the number of assets (cardinality constraints), (c) transaction roundlot restrictions. Practical portfolio models may also include (d) dedication of cashflow streams, and, (e) immunization which involves duration matching and convexity constraints. The modelling issues in respect of these features are discussed. Many of these features lead to discrete restrictions involving zero-one and general integer variables which make the resulting model a quadratic mixed-integer programming model (QMIP). The QMIP is a NP-hard problem; the algorithms and solution methods for this class of problems are also discussed. The issues of preparing the analytic data (financial datamarts) for this family of portfolio planning problems are examined. We finally present computational results which provide some indication of the state-of-the-art in the solution of portfolio optimisation problems.

A Review of Portfolio Planning: Models and Systems

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1. Introduction and overview

The mean-variance (M-V) model of Markowitz is a single period static portfolio planning model and in recent times, it has become the core decision engine of many portfolio analytics and planning systems in the construction of the risk-return efficient frontier.

Markowitz shows that for a rational investor maximizing expected utility, a chosen portfolio is optimal with respect to both expected return and variance of return. He defines such a non-dominated portfolio as *efficient*, that is, it offers the highest level of expected return for a given level of risk and the lowest level of risk for a given level of return. His normative mean-variance rule for investor behaviour both implies and justifies the observable phenomenon of diversification in investment. Determining the efficient set from the investment opportunity set, the set of all possible portfolios, requires the formulation and solution of a parametric quadratic program (QP). Plotted in risk-return space the *efficient* set traces out the *efficient frontier.*

Hanoch and Levy (1969) show that the M-V criterion is a valid efficiency criterion, for any individual's utility function, when the distributions considered are Gaussian normal. A study comparing alternative utility functions appears in Kallberg and Ziemba (1983). They show that portfolios with 'similar' absolute risk-aversion indices have 'similar' optimal compositions, regardless of the functional form and parameters of the utility function. Hence, M-V analysis is justified for any general concave utility function of the Von Neumann-Morgenstern type (Von Neumann and Morgenstern 1944).

The estimation of the underlying parameters (returns, variances and covariances) which are required as the input to M-V analysis is an important modelling step. Small changes in the input can have a large impact on the optimal asset weights. Chopra and Ziemba (1993) found that, for a typical investor's risk-tolerance level, errors in the forecast means are more than ten times as important as errors in the variance and about twenty times as important as errors in covariances. For practical aspects of portfolio analysis see Perold (1984), Hensel and Turner (1998) and Grinold and Kahn

(1995). The modern portfolio theory has developed in tandem with simplifications to the QP required by M-V analysis; these simplifications centre around linearizing the quadratic objective function or reducing the number of parameters to be estimated. Both approaches involve either an approximation or a decomposition of the covariance matrix.

Tobin (1958) developed the separation theorem which states that, in the presence of a risk- free asset, the optimal risky portfolio can be determined without any knowledge of investor preferences. Ziemba *et al.* (1974) show that the solution to the portfolio problem involving a risk-free asset can be obtained by a two-stage process: first solving a deterministic linear complementarity problem and then solving a univariate stochastic program.

Sharpe (1963) proposed that the single-index, or 'market', model was a sufficient model of covariance. Subsequently, Sharpe (1964), Lintner (1965) and Mossin (1966) independently developed the capital asset pricing model. This linear model of equilibrium asset prices explains the covariance of asset returns solely through their covariance with the market. King (1966) presented evidence of the influence of industry factors that the market model did not take into account. Rosenberg (1974) proposed a multifactor model that incorporated industry and other factors. Ross (1976) using factor analysis, developed the arbitrage pricing theory, which is a multi-index equilibrium model.

Since Markowitz's seminal paper (1952), a number of alternative models have been proposed for portfolio planning. The main underlying motivations for these alternative models are (i) such models are easier to process from a computational point of view compared to Markowitz's quadratic programming approach (ii) they take into consideration alternative risk metrics. In section 2 of this chapter we describe a number of alternative models taking into consideration the motivations discussed above. In section 3 we introduce alternative risk measures for financial planning problems. Although, not all of them are used as such in a single period planning model they play an important role in defining measures which can be used in a 'portfolio analytics' tool. In section 4 we present a number of extensions of the original Markowitz model. Some, if not all of these extensions are used in many modern portfolio planning systems. Preparation of asset data in a financial data mart is an important aspect of portfolio systems. The method of preparation of these analytic information is discussed in section 5. Real world portfolio planning problems include various practical restrictions which reflect financial industry realisms in respect of threshold constraints, cardinality of assets held and transaction roundlots. These translate to discrete optimisation problems with a convex quadratic objective function. The resulting problems are NP-hard. In section 6 we discuss solution methods for processing such QP and quadratic mixed integer programming (QMIP) problems. In section 7 we consider computational results based on our experience of a current state-of-the-art portfolio optimisation system. We conclude the chapter with a discussion of the leading issues. In Appendix 1, we set out the method of linearizing and also approximating the QP. In Appendix 2, we provide a comparative analysis of alternative portfolio selection models and their relative performance in respect of a small, yet representative dataset of assets.

2. Alternative computational models

In this section, we present five different portfolio planning models; (1) Markowitz's MV model presented as two quadratic programs (QP1 and QP2), (2) reformulation of QP as diagonal models (DIAG1, DIAG2, DIAG3); the piecewise linear approximations of these models are given in **Appendix 1**, (3) the mean-absolute deviation model (MAD), (4) the weighted goal programming (WGP) model and (5) the minimax (MM) model. These five models are presented within a unified framework. The basic set of notations common to all these models is defined below as Indices, Parameters and Decision variables.

Indices:
Let
i, j=1,…N: denote the different risky assets
t=1,…,T: … the time periods of past historical data

Parameters:
Let
r_{it} : denote the return of asset i at time t
μ_i : … the expected return of asset i
σ_{ij} : … the coefficients of the (N×N) variance-covariance matrix V defined for asset i and asset j
($\sigma_{ii} = \sigma_i$ the diagonal coefficients for the asset i)

ρ : … the desired level of return for the portfolio

Decision variables:
Let
x_i : denote the fraction of the portfolio value invested in asset i $(0 \le x_i \le 1)$
x: … the $N\times 1$ vector of portfolio weights x_i

2.1 The Markowitz mean-variance model and the risk-return frontier

The portfolio selection model of Markowitz (1952, 1959) laid the basis of Modern Portfolio Theory. The Markowitz model put forward in 1952, is a multi-(two) objective optimisation model which is used to balance the expected return and variance of a portfolio. Markowitz (1952) shows how rational investors can construct optimal portfolios under conditions of uncertainty. For an investor, the returns (for a given portfolio) and the stability or its absence (volatility) of the returns are the crucial aspects in the choice of portfolio. Markowitz uses the statistical measurements of expectation and variance of return to describe, respectively, the benefit and risk associated with an investment. The objective is either to minimise the risk of the portfolio for a given level of return, or, to maximise the expected level of return for a given level of risk. The mean-variance (M-V) approach still underpins much of the quantitative analysis of portfolio selection as carried out by the financial industry today.

The classical M-V model (Markowitz (1952, 1959)) and an alternative approach towards computing the Markowitz Efficient Frontier (MEF) are set out below.

QP 1:

$$\text{Min} \quad Z_{QP1} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} \tag{1}$$

Subject to

$$\sum_{i=1}^{N} x_i \mu_i = \rho \tag{2}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{3}$$

$$x_i \geq 0 \qquad i=1,\ldots,N \tag{4}$$

Varying the desired level of return, ρ, in QP1 and repeatedly solving the quadratic program identifies the minimum variance portfolio for each value of ρ. These are the efficient portfolios that compose the efficient set. By plotting the corresponding values of the objective function (the variance) and ρ (the return) respectively, we trace out the MEF in the mean-variance plane. Markowitz (1956) describes a 'critical line' solution algorithm tracing out the efficient frontier by identifying 'corner' portfolios-points at which a stock either enters or leaves the current portfolio. It is normal practice to use standard deviation rather than variance as the risk measure because the σ versus ρ frontier is linear if a risk-free asset exists, see Tobin (1958) and Ziemba *et al.* (1974).

An alternative formulation of QP1 explicitly trades risk against return in the objective function using the Arrow-Pratt absolute risk-aversion index R_A (see Kallberg and Ziemba 1983). R_A is defined as

$$R_A = \frac{u''(w)}{u'(w)} \tag{5}$$

where w is portfolio wealth and u', u'' are the first and second derivatives of a Von Neumann-Morgenstern utility function u.

QP2:

$$\text{Max} \quad Z_{QP2} = \sum_{i=1}^{N} x_i \mu_i - \frac{R_A}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} \tag{6}$$

subject to

$$\sum_{i=1}^{N} x_i = 1 \tag{7}$$

$$x_i \geq 0 \qquad i=1,\ldots,N \tag{8}$$

By increasing R_A from zero and solving the different instances of **QPs**, we trace out the efficient frontier. Empirical results by Kallberg and Ziemba (1983) show that $R_A \geq 6$ leads to very risk-averse portfolios, $2 \leq R_A \leq 4$ represents moderate absolute risk aversion and $R_A \leq 2$ leads to risky portfolios. $R_A = 4$ corresponds approximately to pension fund management (typically, holdings of 60% stocks and 40% bonds). In practice, it is common to model the risk-return trade-off using a parameter λ, $0 \leq \lambda \leq 1$, with the following objective function

$$\text{Min } Z = \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} - (1-\lambda) \sum_{i=1}^{N} x_i \mu_i \tag{9}$$

Setting

$$\frac{R_A}{2} = \frac{\lambda}{(1-\lambda)} \tag{10}$$

shows equivalence with the objective function in QP2. The same efficient frontier generated by QP1 can be traced out by varying the value of λ and repeatedly solving QP2. This is the most frequently used way of generating the efficient frontier, the parameter λ is systematically varied between 0 and 1, which correspond to the maximum return and minimum variance portfolios respectively.

2.2 Models with diagonal quadratic form as objectives

Diagonal models are of interest as the corresponding quadratic forms can then be expressed as variable separable functions which in turn are approximated as piecewise linear functions (see appendix). Since these are convex programming problems, piecewise linear approximations lead to a linear programming (LP) reformulation of the given problem; the solution of the LP guarantees global optimum solution of the given approximated QP. A detailed description of diagonalisation methods (based on Cholesky decomposition, an approach that exploits the decomposition of the covariance matrix (see also Vanderbei and Carpenter (1993)) and diagonal QPs based on index or factor models for describing asset returns) can be found in Horniman et al. (2000).

Diagonal model 1

By applying Cholesky decomposition the given covariance matrix, *V,* can be re-expressed as

$$V = L^T L$$

where *L(N×N)* is a lower triangular matrix. The objective function of model QP1, in matrix form is $Z_{QP1} = \mathbf{x}^T\ V\ \mathbf{x}$ which can be expressed as $Z_{QP1} = \mathrm{x}^T\ L^T L\mathbf{x}$. Defining a new vector **y** *(N×1)* such that **y** = *L***x** with elements

$$y_i = \sum_{j=1}^{i} l_{ij} x_j \qquad i = 1,\ldots,N \qquad (11)$$

leads to the equivalent formulation of the portfolio selection problem, model **DIAG1**. The number of terms in the objective function of model QP1 is reduced from N^2 to N at the cost of N additional constraints (11) and N additional variables (11).

DIAG1:

$$\text{Min} \quad \mathbf{Z}_{DIAG1} = \sum_{i=1}^{N} y_i^2 \qquad (12)$$

Subject to

$$y_i = \sum_{j=1}^{i} l_{ij} x_j \qquad i = 1,\ldots,N \qquad (13)$$

$$\sum_{i=1}^{N} x_i \mu_i = \rho \qquad (14)$$

$$\sum_{i=1}^{N} x_i = 1 \qquad (15)$$

$$\mathrm{x}_i \geq 0 \qquad \mathrm{i=1,\ldots,N} \qquad (16)$$

$$g_i^y \leq y_i \leq h_i^y \qquad \mathrm{i=1,\ldots,N} \qquad (17)$$

For a general quadratic form, y_i would be a free variable ($-\infty < y_i < +\infty$). However, with constraints (14,15, and 16), there are finite upper and lower bounds on x_i for i=1,…,N. As a consequence, there exist finite upper (h_i^y) and lower (g_i^y) bounds on y_i, i=1,…,N (see Brearley et al. (1975)). These bounds are a necessity for the piecewise linear approximations of the quadratic terms.

Diagonal model 2

A similar approach (see Vanderbei and Carpenter (1993)) exploits the composition of the covariance matrix V given that it has been calculated from returns R observed over T periods. Given that the matrix of mean returns is $\overline{R}$, the covariance matrix V is calculated as

$$V = \frac{1}{N-1}(R-\overline{R})^T(R-\overline{R})$$

Defining S ($T \times N$) as

$$S = \frac{1}{\sqrt{N-1}}(R-\overline{R}),$$

The covariance matrix can be expressed as

$$V = S^T S$$

This leads to a model similar to DIAG1 with T (instead of N) new decision variables y_t and T (instead of N) additional constraints compared to model QP1. We refer to it as **DIAG2**.

DIAG2:

$$\text{Min } Z_{DIAG2} = \sum_{t=1}^{T} y_t^2 \qquad (18)$$

subject to

$$y_t = \sum_{i=1}^{N} s_{it} x_i \qquad t=1,\ldots,T \qquad (19)$$

$$\sum_{i=1}^{N} x_i \mu_i = \rho \qquad (20)$$

$$\sum_{i=1}^{N} x_i = 1 \qquad (21)$$

$$x_i \geq 0 \qquad i=1,\ldots,N \qquad (22)$$

$$g_t^y \leq y_t \leq h_t^y \qquad t=1,\ldots,T \qquad (23)$$

In this instance, the number of terms in the objective function (19) is reduced to *T*, with the addition of *T* variables (23) and *T* constraints (19). Again, there are finite upper (h_t^y) and lower (g_t^y) bounds on y_t, $t=1,\ldots,T$.

Diagonal model 3

The use of factor models to describe asset returns can also lead to a diagonal form provided the composition of the covariance matrix is appropriately exploited. Sharpe (1971) introduced this feature for the single index model and the technique can be extended to any number of factors or indices (see Rosenberg (1974) and Perold (1984)). For a model with *K* factors, let f_k denote the level of the k^{th} factor, β_{ik} the sensitivity of asset *i* to factor *k*, α_i the mean return of asset *i* and e_i the random component of return of asset *i*; then asset returns r_i can be expressed as a linear form by:

$$r_i = \alpha_i + \sum_{k=1}^{K} \beta_{ik} f_k + e_i$$

If the factors are constructed (or transformed) so that there is no correlation between the factors and specific returns, and it is further assumed that the specific returns are uncorrelated, the covariance matrix, *V*, can be decomposed as

$$V = B^T QB + D$$

where B is the *K*×*N* matrix of factor sensitivities, *Q* is the *K*×*K* diagonal matrix of factor variances $\sigma_{f_k}^2$, and *D* is the *N*×*N* diagonal matrix of specific variances, $\sigma_{\varepsilon_i}^2$. (If the factors are constructed to be orthonormal, then Q reduces to the *K*×*K* identity matrix).

Having decomposed the covariance matrix in this fashion, model **DIAG3** can be stated as follows.

DIAG3:

$$\text{Min} \quad Z_{DIAG} = \sum_{k=1}^{K} y_{P,k}^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\varepsilon_i}^2 \qquad (24)$$

subject to

$$y_{P,k} = \sum_{i=1}^{N} x_i \beta_{ik} \sigma_{f_k} \qquad k= 1,\ldots,K \qquad (25)$$

$$\sum_{i=1}^{N} x_i \mu_i = \rho \tag{26}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{27}$$

$$x_i \geq 0 \qquad i=1,\ldots,N \tag{28}$$

$$l_k^y \leq \quad y_{P,k} \leq u_k^y \qquad k= 1,\ldots,K \tag{29}$$

In this approach, the objective function of QP1 is reduced to a sum of squares in $N+K$ terms with an additional K variables $y_{P,k}$ (expressed as linear forms of x_i) (25) with finite upper (u_k^y) and lower (l_k^y) bounds (see (29)).

2.3 The mean-absolute deviation (MAD) model

Konno (1988) proposed a portfolio optimisation model using a piecewise linear risk function. The MAD model, a special case of the piecewise linear risk model, has been shown to be equivalent to the Markowitz model under the assumption that returns are multivariate normally distributed (Konno and Yamazaki (1991)). That is, under this assumption, the minimisation of the L_1 measure, (the sum of absolute deviations of portfolio returns about the mean), is equivalent to the minimisation of the L_2 measure, (the variance). Let m_t denote the absolute deviation of the portfolio return (from the mean) at time *t*, then the **MAD** model is stated as:

MAD:

$$\text{Min} \quad Z_{MAD} = \frac{1}{T}\sum_{t=1}^{T} m_t \tag{30}$$

subject to

$$\sum_{i=1}^{N} (r_{it} - \mu_i) x_i \leq m_t \qquad t=1,\ldots,T \tag{31}$$

$$\sum_{i=1}^{N} (r_{it} - \mu_i) x_i \geq -m_t \qquad t=1,\ldots,T \tag{32}$$

$$\sum_{i=1}^{N} x_i \mu_i = \rho \tag{33}$$

$$\sum_{i=1}^{N} x_i = 1 \qquad (34)$$

$$m_t \geq 0 \qquad t=1,\ldots,T \qquad (35)$$

$$x_i \geq 0 \qquad i=1,\ldots,N \qquad (36)$$

The objective function (30) minimises the mean of the absolute deviation calculated using constraints (31) and (32), with m_t restricted to be non-negative (35).

A comparison of the M-V model and the MAD model

Konno and Yamazaki claim that the MAD model credibly replaces the M-V model as it incorporates all its positive features; they present the following three arguments in support of their claim.

a) In the formulation of the MAD model, there is no requirement for the covariance matrix of asset returns,
b) the relative ease with which a linear program can be solved compared to a quadratic one- thus large scale problems can be solved faster and more efficiently,
c) mean absolute deviation portfolios have fewer assets- this fact implies lower transaction costs in portfolio revisions.

Simaan (1997) discusses the advantages and disadvantages of the MAD model. He puts forward a contrary viewpoint and shows that ignoring the covariance matrix results in greater estimation risk that outweighs the benefits. In both models, estimation risk is more severe in small samples (small observations relative to the number of assets) and for investors with high risk tolerance. The M-V model's lower estimation risk is most striking in small samples and for investors with a low risk tolerance.

2.4 The goal programming model

Goal Programming is a branch of multi-objective decision-making and is based on the concept of finding feasible points as close as possible to a number of goals. A set of targets/goals is chosen by the decision maker. Any (unwanted) deviations from these targets are penalised in order to get a satisfactory solution. How these penalties are implemented depends on the type of goal program.

Weighted Goal Programming (WGP) attaches weights according to the relative importance of each objective as perceived by the decision maker and minimises the sum of the weights. Zero weights are attached to deviations that do not have to be minimised (for example, positive deviations from the expected portfolio return goal).

Lexicographic Goal Programming (LGP) separates the objectives into a number of priority levels where the satisfaction of goals with higher priority is regarded as infinitely more important than the satisfaction of lower level goals. A practical LGP model, first introduced by Lee (1980), and a WGP formulation of Lee's model can be found in Tamiz (1996).

Tamiz et al. (1996), using a factor model of stock returns, measure the risk of a portfolio as the sum of absolute deviations of the portfolio's factor sensitivities from those of a specified target. Unsystematic risk receives no direct treatment. To force diversification of the stock specific risks, they apply a constraint on the total holdings allowed in each industry sector.

We present a simplified version of such a **WGP** model; the objective of the model is to minimise the risk associated with the portfolio *and* maximise the expected return. We do not specify a particular measure of risk. The only limit on the risk measure is that the portfolio risk is a linear combination of the risks associated with the component stocks.

We also introduce a few additional parameters and variables for this model.

Parameters:

Let

W_1: denote the positive penalty weight associated with shortfalls in portfolio return below the target

W_2: denote the positive penalty weight associated with excess portfolio risk in relation to the target

$Risk_P$: denote risk associated with the portfolio

$Risk_i$: … risk associated with the asset *i*

Decision variables:

Let

n_1: denote the negative deviation from the target level of portfolio return

p_1: … the positive deviation from the target level of portfolio return

n_2: … the negative deviation from the target risk level

p_2: … the positive deviation from the target risk level

WGP:

$$\text{Min} \quad Z_{WGP} = W_1 n_1 + W_2 p_2 \tag{37}$$

subject to

$$\sum_{i=1}^{N} x_i \mu_i + n_1 - p_1 = \rho \tag{38}$$

$$\sum_{i=1}^{N} Risk_i x_i + n_2 - p_2 = Risk_P \quad (39)$$

$$\sum_{i=1}^{N} x_i = 1 \quad (40)$$

$$n_1, n_2, p_1, p_2 \geq 0 \quad (41)$$

$$x_i \geq 0 \qquad i=1,\ldots,N \quad (42)$$

The objective function (37) seeks to minimise risk and maximise return by penalising excess risk and shortfalls in return, relative to the respective targets. Lower levels of risk and higher levels of return are not penalised. The shortfalls in return and excesses in risk are determined by constraints (38) and (39) respectively.

The MAD model can be formulated as a weighted goal program. By replacing inequalities (31) and (32) with the constraints

$$\sum_{i=1}^{N} (r_{it} - \mu_i) x_i = p_t - n_t \quad (43)$$

$$p_t \geq 0 \quad (44)$$

$$n_t \geq 0 \quad (45)$$

and replacing m_t in the objective function by $p_t + n_t$. This results in a weighted goal program that penalises absolute deviations from the portfolio mean. By not penalising deviations above the mean, using a zero penalty weight on p_t, leads to a weighted goal program version of a negative semi-MAD model, such as employed by Speranza (1996).

2.5 The Minimax model (MM)

The principle underlying this model (Young (1988)) can be described as choosing a portfolio based directly on how it would have performed in the past, over the historical observations t=1,…,T. The minimum return that could have occurred in the past is employed as the measure of risk. The model seeks to maximise this value while achieving a specified level of expected return. An alternative, and perhaps more appropriate statement of the minimax portfolio selection rule is the minimisation of the maximum loss that would have occurred over the observation period. The minimax model uses the L_∞ norm to measure risk which implies a strong absolute aversion to downside risk (Gonin and Money ((1989)). The solution can be strongly affected by only one outlying value in the data.

We introduce a variable M_P which represents the minimum return achieved by the portfolio over all observation periods. That is, $M_P = \min_t \sum_{i=1}^{N} x_i r_{it}$.

The Minimax model (**MM**) is then stated as

MM:

$$\text{Max} \quad Z_{MM} = M_P \tag{46}$$

subject to

$$\sum_{i=1}^{N} x_i r_{it} \geq M_P \qquad t=1,\ldots,T \tag{47}$$

$$\sum_{i=1}^{N} x_i \mu_i = \rho \tag{48}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{49}$$

$$l^{M_P} \leq M_P \leq u^{M_P} \tag{50}$$

It is easily seen that finite upper bound (u^{M_P}) and finite lower bound (l^{M_P}) apply to the variable M_P. Young (1998) also suggests an alternative formulation of the model that maximises the expected portfolio return subject to a given lower bound on the portfolio return for every observation period.

3. Symmetric and asymmetric measures of risk

3.1 Sources of risk and choice of appropriate measures: risk dilemmas

The introduction of Markowitz's M-V framework provided financial institutions and portfolio managers a powerful tool that allowed them, for the first time, to utilise the concepts of risk and return in a combined paradigm. Despite the progressive acceptance and wide-spread use of the M-V framework, and its numerous extensions, in practice there has been a considerable debate among academics and practitioners on the validity of variance as a representative measure of risk. The notion of risk has found practical application within the science of *Risk Management*, also known as *Risk Control.* Risk Control deals with limiting or eliminating specific types of risk, in as much as this is possible by taking an active position in one or more types of risk. Deciding which types of risk to mitigate is the *first dilemma* of a financial institution and demands considerable attention, since focusing on one particular risk category may lead to a hedged portfolio for a particular source of risk but may result in exposure to other sources of risk. This issue becomes more challenging when optimisation models are used (see Zenios and Georgiadou (2000)). For instance, optimisation may result in minimisation of the risk (measure) included in the model, but the solution may be sensitive to other sources of risks that were not considered and better measured by another metric.

In general, risk measures can be divided into two groups depending on the perception of risk. The first group contains the so-called dispersion risk measures that quantify risk in terms of probability-weighted dispersion of results around a specific reference point, usually the expected value, and are otherwise classified as *symmetric measures of risk*. Measures in this category penalize negative as well as positive deviations from a pre-specified target. Two of the most well-known and widely applied risk measures, in this group, are Markowitz's (1952, 1959) *variance* or *standard deviation* and the *expected* or *mean absolute deviation (MAD)* of Atkison (1970) and Konno and Yamasaki (1991). The second group comprises measures which quantify risk according to results and probabilities below reference points, selected either subjectively or objectively, and are otherwise classified as *asymmetric measures of risk*. Such risk measures include the *Expected Value of Loss* from Domar and Musgrave (1944), Roy's (1952) *Safety First*, the *Semi-Variance* proposed by Markowitz (1959), *Value at Risk – VaR* – (JP Morgan, 1993) and its extension *Conditional VaR – CVaR* – (Uryasev (1999)), and Fishburn's α-t criterion (1977). The latter not only constitutes the generalized case for the above 'below-target' risk measures, but it is also capable of representing the symmetric risk measures. Set against this background, a financial institution faces a *second dilemma* of deciding which of the two main risk metric categories - symmetric or asymmetric measures of risk – represent its attitude towards risk and, therefore, should be utilised.

The incorporation of risk in the investment decision process should also reflect the benchmark relative to which a financial institution or an individual assesses its portfolio performance. The simplest approach is that of comparing the performance relative to the portfolio's past history. This is achieved by computing the risk measure as a function of the portfolio composition and the random returns of the assets.

Typically, the standard deviation would then reflect the deviation of the asset returns from the expected portfolio return. On the other hand, the portfolio performance can be measured relative to a benchmark index or an alternative investment opportunity. In this case, the risk measure is also a function of a target level of return. The standard deviation in this case would then reflect the deviation of the asset returns from the expected target return (eg. FTSE100). Utilising the two alternative approaches – *portfolio return* and *target return* – implies tackling different planning problems. In particular, the portfolio return approach is mostly suitable for maximum return strategies, whereas the target return framework is suitable for 'index tracking' or 'goal achievement' strategies. Further, the two approaches lead to different portfolio asset mix decisions and, therefore, for financial institutions choosing the appropriate framework becomes the *third dilemma*.

3.2 A generic approach to risk representation and quantification

Bawa (1975) and Fishburn (1977) consolidated the existing research on risk measures up to that time, and developed the *α-t model*, and introduced a general definition of 'below target' risk in the form of lower partial moments (LPM).

Let α be a parameter specifying the moment of the return distribution. In some cases α may be taken as indicating different attitudes towards risk. Let τ be a predefined target level of the investment return, and *F(x)* the cumulative probability distribution function of the investment with return x. The LPM of order α for a given τ defines the *α-t model* and has the following form:

$$F_\alpha(\tau) \equiv LPM_\alpha(\tau; x) = \int_{-\infty}^{\tau} (\tau - x)^\alpha f(x)\,dx = E\{(\max[0, \tau - x])^\alpha\}\ , \alpha > 0 \qquad (51)$$

The introduction of the LPM is a major advance in the field of risk, as it provides the most generic representation of risk. Within this framework both symmetric and asymmetric measures of risk are encapsulated. Alternative formulations of well-known symmetric and asymmetric risk measures are shown below as special cases of the generic approach of LPM.

Symmetric Measures of Risk

The main difference of the symmetric measures of risk, when compared with the asymmetric, is that returns above the pre-specified target are also included. In that case, the returns used to calculate the risk measures can take values between [-∞, +∞]. The two symmetric risk metrics we consider are the Variance and MAD.

(S.1) Variance: the classical representation of variance deals with measuring the spread of the expected returns relative to the average expected portfolio return. Therefore, $\tau = \bar{x}$ and α=2.

$$\sigma^2 \equiv LPM_2(\bar{x}; x) = \int_{-\infty}^{+\infty} (\bar{x} - x)^2 f(x)dx = E\{(\bar{x} - x)^2\} \qquad (52)$$

In the case that the target level of return is not equal to the average expected portfolio return, the representation of the variance from target τ is given by:

$$\sigma^2 \equiv LPM_2(\tau; x) = \int_{-\infty}^{+\infty} (\tau - x)^2 f(x)dx = E\{(\tau - x)^2\} \qquad (53)$$

(S.2) Mean Absolute Deviation: by setting α=1, the MAD measure of risk can be represented as:

$$MAD \equiv LPM_1(\bar{x};x) = \int_{-\infty}^{+\infty} |\bar{x}-x|^1 f(x)dx = E\{(|\bar{x}-x|)^1\} \qquad (54)$$

Asymmetric Measures of Risk

It is easily seen that all asymmetric risk measures for different levels of τ and α are special cases of the *a-t* risk. Adopting the general *a-t* risk measure, we provide the formulations of a set of (interesting) below-target risk measures.

(A.1) Safety First: The *'Safety First Criterion'* is a special case of the α-t risk when α → *0*.

$$SF \equiv LPM_{\alpha\to 0}(\tau;x) = F_{\alpha\to 0}(\tau) = \int_{-\infty}^{\tau} (\tau-x)^{\theta\to 0} f(x)dx \;= E\{(\max[0,\tau-x])^{\alpha\to 0}\} \qquad (55)$$

(A.2) Expected Downside Risk: When α=1 the *a-t* model equals the expected downside risk.

$$\overline{D} \equiv LPM_1(\tau;x) = F_1(\tau) = \int_{-\infty}^{\tau} (\tau-x)^1 f(x)dx = E\{(\max[0,\tau-x])^1\} \qquad (56)$$

If the target is set equal to the expected portfolio return then the measure can be viewed as a special case of the MAD risk measure where only the negative deviations from the target are considered, thus leading to the Semi-MAD measure:

$$MAD^- \equiv LPM_1(\bar{x};x) = F_1(\bar{x}) = \int_{-\infty}^{\bar{x}} (\bar{x}-x)^1 f(x)dx = E\{(\max[0,\bar{x}-x])^1\} \qquad (57)$$

(A.3) Semi-Variance: as shown by Fishburn in his seminal paper, the semi-variance is a special case of the *a-t* model, for α=*2*.

$$\sigma^{-2} \equiv LPM_2(\tau;x) = F_2(\tau) = \int_{-\infty}^{\tau} (\tau-x)^2 f(x)dx = E\{(\max[0,\tau-x])^2\} \qquad (58)$$

(A.4) Worst Case Scenario: For α→+∞ the *a-t* model defines the worst-case scenario as considered by Boudoukh, Matthew & Richardson (1995).

$$WCS \equiv F_{\alpha\to +\infty}(\tau) = LPM_{\alpha\to +\infty}(\tau;x) = \int_{-\infty}^{\tau} (\tau-x)^{\alpha\to +\infty} f(x)\,dx \;= E\{(\max[0,\tau-x])^{\alpha\to +\infty}\} \qquad (59)$$

(A.5) Value-at-Risk (VaR): the VaR of a portfolio at the β probability level is the left quantile of the losses of the portfolio, i.e, the lowest possible value such that the probability of losses less than VaR exceeds $\beta \times 100\%$. The VaR is given as

$$VaR(x,\beta) = \theta \qquad (60)$$

where the corresponding LPM is

$$LPM_0(\theta;x) = F_0(\theta) = \int_{-\infty}^{\theta} (\theta-x)^0 f(x)dx = 1-\beta$$

4. Computational models in practice

The M-V model as described in section 2.1 and the alternative models described in the rest of section 2 provide adequate mathematical description of the investment decision problem in its general form. In real life situations, to apply such models it is necessary to consider the trading requirements and other aspects of portfolio performance. For instance, it is meaningful: (a) not to have very small holdings, (b) to restrict the total number of holdings and (c) to take into consideration the roundlot of assets that can be bought or sold in a bunch. These requirements can be modelled as threshold constraints (section 4.1), cardinality constraints (section 4.2) and roundlot constraints (section 4.3); in general they all lead to sets of discrete variables and constraints.

The original perspective (which is also a restrictive and narrow view) of portfolio planning is that of asset management namely buying, selling and rebalancing of assets. In this approach no explicit attention is paid to the investor's liabilities. Yet if the assets are bonds/ fixed income securities then coupon payments, reinvestment of cash and the fund's liabilities immediately call for cash flow matching. This is formally known as portfolio dedication and is discussed in section 4.4. The prices of fixed income securities are dependent on the term structure of interest rates and hence exposed to interest rate risk. Thus, measurement and modelling of such risks using duration and convexity and the corresponding restrictions also known as immunization are described in section 4.5.

4.1 Buy-in thresholds for assets

Buy-in thresholds and cardinality constraints are formulated using a discrete programming modelling structure which is well known as variable upper and lower bounds or semi-continuous variables (Beale and Forrest (1976)). For discrete programming solution systems which do not support 'semi-continuous' variables, such threshold restrictions may be specified using a binary variable and a pair of bounding restrictions. Using finite upper and lower bounds l_i, u_i for the stock weight x_i and the binary variable δ_i, the corresponding threshold restriction is represented by the constraint pair

$$l_i\delta_i \le x_i \le u_i\delta_i \qquad \text{and} \qquad \delta_i = 0,1 \qquad I = 1,\ldots,N.$$

The introduction of the binary variables transforms the QP to a quadratic mixed-integer program (QMIP) which becomes larger in size and computationally more complex. These constraints and the binary variables δ_i are also used to represent cardinality constraints which specify the number of stocks in a given portfolio. Imposing constraints that restrict stock holdings to be integer multiples of specified roundlots increases the complexity of the model yet further.

The reformulation of model QP1 with buy-in thresholds is set out below.

BUY-IN:

$$\text{Min} \quad Z_{BUY-IN} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} \tag{61}$$

subject to

$$\sum_{i=1}^{N} x_i \mu_i = \rho \tag{62}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{63}$$

$$l_i \delta_i \le x_i \le u_i \delta_i \qquad i = 1,\ldots,N \tag{64}$$

$$\delta_i = 0 \text{ or } 1 \qquad i = 1,\ldots,N \tag{65}$$

$$\mathrm{x}_i \ge 0 \qquad i=1,\ldots,N \tag{66}$$

Constraints (64) and (65) ensure that if $\delta_i = 1$, then $l_i \le x_i \le u_i$ otherwise $\delta_i = 0$ which imposes x_i=0.

4.2 Cardinality constraints

In order to control transaction costs or for other monitoring and control issues, some investors may wish to limit the number of assets held in their portfolios. By counting the binary variables introduced in model BUY-IN we can construct the cardinality constraint which limits the portfolio to a fixed number of assets *k*. Thus, by adding the restriction

$$\sum_{i=1}^{N} \delta_i = k, \tag{67}$$

to the model BUY-IN above we extend it to model CARD.

It may be worthwhile to point out that buy-in thresholds and cardinality constraints are implicitly linked. For example, a buy-in threshold of 10% of the value of a portfolio implies that up to 10 stocks can be bought.

4.3 Roundlot transactions

In the transaction roundlot model, we introduce the requirement that we can purchase stocks in set 'blocks'. Each block, or roundlot, can be described as a cash value or a number of stocks. For each asset *i*, a block is defined as a fraction f_i of the total portfolio wealth. Introducing integer number of blocks y_i, we re-express x_i as

$$x_i = y_i f_i, \qquad i = 1,\ldots,N$$

which is the fraction of portfolio wealth to be invested in stock *i*. The roundlot model can be stated as follows.

LOT:

$$\text{Min} \quad Z_{LOT} = \sum_{i=1}^{N} \sum_{j=1}^{N} y_i f_i y_j f_j \sigma_{ij} + \gamma\varepsilon^- + \gamma\varepsilon^+ \qquad (68)$$

subject to

$$\sum_{i=1}^{N} y_i f_i \mu_i = \rho \qquad (69)$$

$$\sum_{i=1}^{N} y_i f_i + \varepsilon^- - \varepsilon^+ = 1 \qquad (70)$$

$$l_i \leq y_i f_i \leq u_i \qquad i = 1,\ldots,N \qquad (71)$$

$$y_i \text{ integer} \qquad i = 1,\ldots,N \qquad (72)$$

$$\varepsilon^-, \varepsilon^+ \geq 0 \qquad (73)$$

Using discrete lot sizes of share purchases, it may not be possible to satisfy exactly the requirement $\sum_{i=1}^{N} x_i = 1$. Hence, this restriction is made ‘elastic’ as in goal programming. Thus (70) includes undershoot and overshoot variables $\varepsilon^-, \varepsilon^+$ respectively which are in turn penalised in the objective function with a high cost γ. As a consequence in an optimum solution $\varepsilon^-, \varepsilon^+$ are made as small as possible and the fractional stock holdings x_i sum to a value ‘ as close as possible’ to 1.

4.4 Portfolio dedication

Given that the investment process is in general dynamic and that there are liabilities or obligations to be taken into account, the fund managers need to:

i) match cash flows for known obligations arising out of, say, general investment contracts (GICs), and,

ii) plan borrowing of shortfall and reinvestment of surplus,

both considered over future time periods.

Let

F_{it} denote the positive cash flows from asset *i* in time period t,

L_t … liability in time period t,

ρ_t … a reinvestment rate,

$\rho_t + \Delta$ … the borrowing rate with Δ as the difference between this rate and the reinvestment rate.

We introduce two variables, v_t^+, v_t^- as cash surplus and shortfall respectively in time period t.

Then the restrictions set out below

$$\sum_{i=1}^{N} F_{i0} x_i + v_0 + v_0^- = v_0^+ ,$$

$$\sum_{i=1}^{N} F_{it} x_i + (1+\rho_t) v_{t-1}^+ + v_t^- = L_t + v_t^+ + (1+\rho_t + \Delta) v_{t-1}^- , \quad \text{for all } t=1,\ldots,T.$$

capture portfolio dedication as cashflows matching with borrowing and reinvestment. For a detailed discussion of this and related topics, see Zenios (2002).

4.5 Portfolio Immunization

Bond prices are affected by yields which in turn depend on market interest rates; also short bonds and long bonds are affected non-uniformly by the interest rate movement. The interest rate sensitivity or risk is traditionally measured by "bond duration". Duration of a bond is generally defined as the weighted average of the present values of the cash flows (the coupon payments). There are alternative definitions of duration (see Douglas (1990) and Luenberger (1998) but in general duration is a first order condition and provides a measure of the interest rate sensitivity or risk of a given fixed income security. A portfolio which is made up of only bonds can also have a duration measure.

Let D_i denote the duration of the i^{th} bond

Then the duration of the portfolio is computed as

$$D_P = x_1 D_1 + x_2 D_2 + ... + x_N D_N$$

If we also compute the duration of all the liabilities, then by balancing the portfolio duration and liability duration

$$D_P = x_i D_1 + ... + x_N D_N = D_L$$

we immunize the portfolio against interest rate risk.

Convexity restrictions

The price-yield relationship of a fixed-income security is a non-linear function for which the second-order condition (differential) is called convexity. Whereas duration matching ensures that for small changes in term structure of interest rates, asset and liabilities move together, it is necessary to also put a restriction on convexity in order to have comparable shape for larger changes.

Let

Q_i denote the first derivative of duration (with respect to the interest rate) for the asset *i*, *i*=1,…,*N*; then Q_i is defined as the convexity of asset *i*; and let,

Q_L denote the convexity of the liabilities.

Then

$$Q_1 x_1 + Q_2 x_2 + ... + Q_N x_N \geq Q_L ,$$

is a constraint which in some sense restricts the sensitivity to the shape of the term structure or the 'shape risk' of the portfolio.

Factor immunization

Factor models are well established in most modern portfolio systems since they play an important role:

i) in analysing and discovering information within the market data, and,
ii) in defining the quadratic objective function of the risk.

By using a linear factor model (typically principal component analysis) one may choose to include *k* factors to represent return variability. The first order and second order measures can be now redefined in this light as:

a) factor modified duration, and,
b) factor modified convexity.

Further immunization restrictions can be written in terms of these parameters and the corresponding model then includes factor immunization conditions. For a fuller treatment of this topic the readers are referred to Zenios (2002).

5. Preparation of data: financial data marts

Deciding on the portfolio asset mix for a given planning horizon is a core task in the operations of a financial institution. The adoption of portfolio models underpins such a task, and in particular these models are used to make robust hedged decisions. Yet the effective use of the portfolio planning models, described above, in practice requires their inclusion in an integrated decision support framework. In this framework it is necessary to consider the roles of data, information and decision models (see Figure 1). This integrated framework is also underpinned by the concept of translating transactional data into analytical data and the integration of information analysis models together with portfolio optimisation models through the combined use of a common data mart.
Within information systems methodology, there is a clear awareness in respect of data stored in transactional/production databases and information stored in *analytical databases*. Transactional data refer to historical market data and internal (institution specific) data; existing portfolio positions, client orders, cash flows. Information analysis models filter transactional data and synthesise them into information that is then stored into the analytical database. The information is subsequently used to instantiate decision models and in turn the optimal solutions are stored in the *decision database*. The integration of the analytical database and the decision database is better known as *data mart* (Koutsoukis *et al*. (1999)). For industry standard portfolio analysis systems such as Northfield Systems and UBS Warburg PAS (2001, 2002), the use of analytical databases is pivotal, and the underlying information model is illustrated in Figure 1.

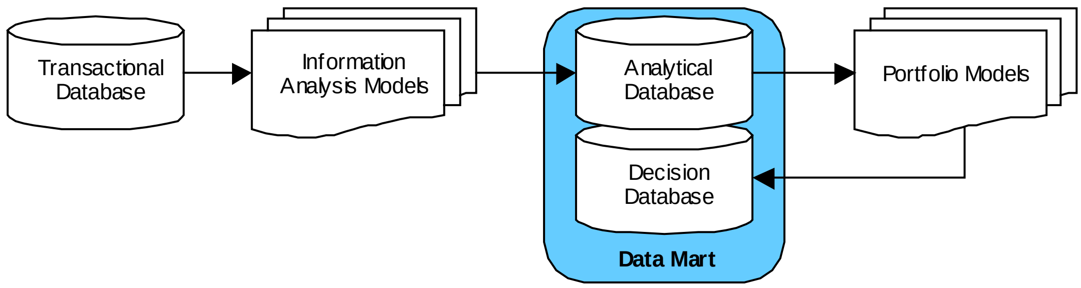

Figure 1: Data information and decision models

In respect of our portfolio applications, the information analysis models themselves can be broken down further into sub-categories taking into consideration the analysis stage in which they are utilised (see Table 1).

Information Analysis Models			
Pre-analysis	**Model Data Parameters**	**Solution Analysis**	**Post-analysis**
Performance Indicators Style Analysis Financial Ratios CAPM APT Simulation Models Internal Company Models	Historical data Weighted Moving Average Factor Models Time Series Models ARCH, GARCH,… Neural Networks Genetic Algorithms Kalman Filters Chaos Internal Company Models	What if Analysis Scenario Analysis Simulation Backtesting Internal Company Models	Performance Indicators Risk Statistics and Indices Financial Ratios CAPM APT Simulation Models Risk Metrics Internal Company Models

Table 1: A Breakdown of Information Analysis Models

Our overall view of the transactional data, information models, analytic database and decision models is set out in Figure 2. This view can be explained in the following way.

The transactional data are collected on a day-to-day basis and stored in the production database which the information analysis models filter into information and generate an analytical database. We refer to this as the *pre-analytical database* because the information is generated before any optimisation takes place. The pre-analytical database comprises:

i) **Pre-analysis data:** information that provides insight on the portfolio performance to date and assist the decision maker to identify market trends to select an appropriate investment style and asset universe. The pre-analysis data includes styles, financial ratios, asset and portfolio statistics, and performance comparisons.

ii) **Model data parameters:** the data input for the portfolio planning model. The model data parameters typically include the asset universe, the expected return of the assets over the given planning horizon, and the expected risk of the assets (variance covariance matrix). The remaining data parameters are application specific and depend on the constraints that the organisation wishes to satisfy. The quality of the data parameters is essential for the quality of the solution that the optimisation model provides and therefore information models for generation of the model parameters can be highly sophisticated (for a review see Grinold and Kahn (1995)) .

The optimisation decision engine processes the portfolio optimisation model which is instantiated by data/information taken from the pre-analytical database. Subsequently the optimisation results (optimum solution) are stored in the *decision database*. The output is model specific and mainly comprises the optimum asset mix, the expected portfolio return(s), and the expected portfolio risk(s). The information within the decision database can be further filtered to obtain additional information utilising once again the information analysis models. The contents and the processing leading to the *post-analytical database* is described below:

i) **Results analysis data:** information on the efficacy and the robustness of the optimal solution. The analyst may carry out *'what-if" analyses*, where the decision-maker changes the input values, that is, using different model data instances. This technique examines the changes of the optimal solution and the optimal objective value with respect to variations of some parameters that are considered to be important. It is usually done by varying one parameter at a time. Another technique that varies uncertain parameters is *scenario analysis*. In this approach different scenarios, that is certain combination of possible values of the uncertain parameters, are considered. Thereafter, the problem is solved for each of these scenarios. Thus, by solving the problem repeatedly for different scenarios and studying the solutions obtained, the decision-maker observes the sensitivities and decides on an appropriate solution by following a heuristic approach.

ii) **Post-analysis data:** information that provides insight on the expected performance of the optimum portfolios. The decision maker can calculate the risk exposure in the form of VaR or expected shortfall of the portfolio and compare its expected return with that of a benchmark index or a chosen portfolio.

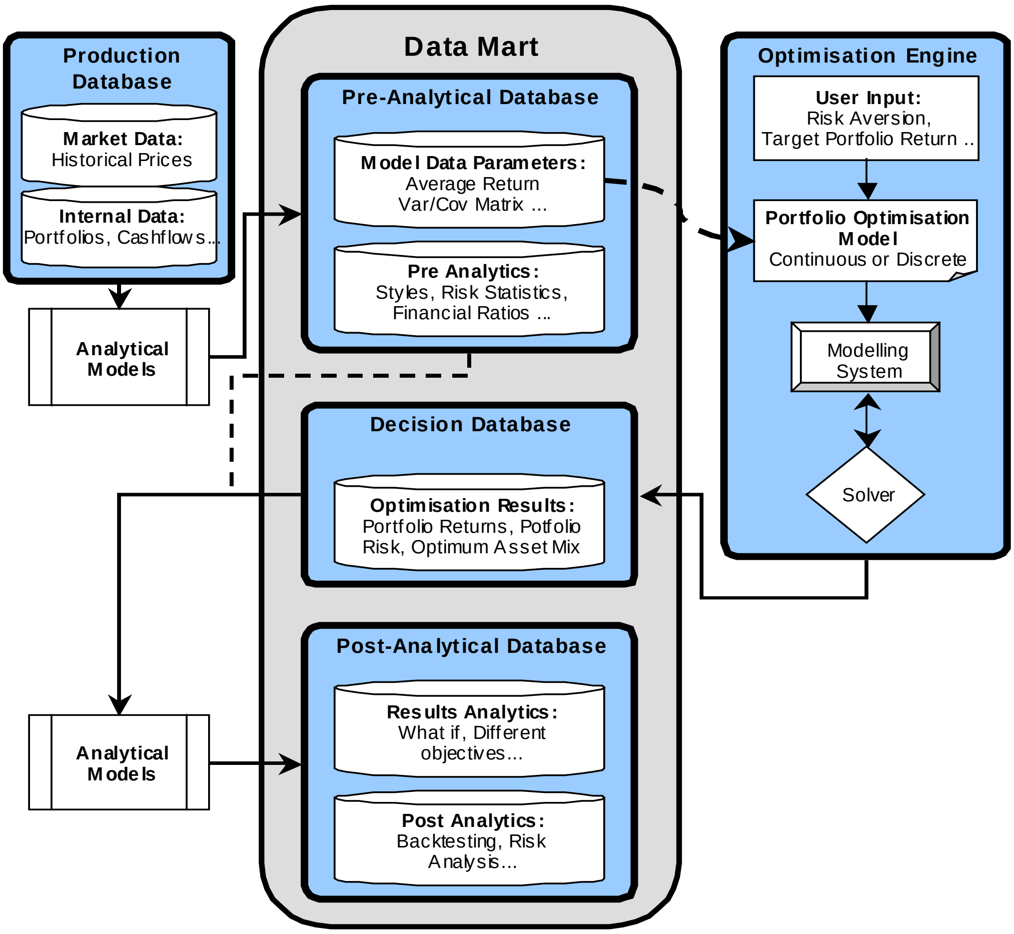

Figure 2: Integrated Decision Support Framework

6. Solution methods

Whereas quadratic programs (QPs) can be solved rapidly using solution algorithms with a low order polynomial complexity, the solution of quadratic mixed integer programs are difficult (NP-hard) and challenging. For instance consider the problem of accurately computing the DCEF. Each point of the DCEF curve represents the global optimum solution of a 'discrete non-convex' optimisation problem. Given that the quadratic form for the minimization problem is positive semidefinite, relaxing the discreteness restriction on the variables leads to a convex programming problem. This continuous variable QP relaxation of the problem provides a lower bound and is easily embedded (see Mitra (1976) and Lawler and Wood (1966)) in a branch-and-bound tree search paradigm.

The FortQP system implemented within the FortMP solver (Ellison *et al* 1999) has both interior point method (IPM) and sparse simplex (SSX) solution capabilities. The system is extensively tested using QLIB test data (Maros and Meszaros 1997) and models from the Finance industry. For the given family of QMIP problems at hand the branch-and-bound algorithm has been specially constructed taking into consideration the following design issues:

SSX versus IPM. In medium-to-large test problems IPM performs better than SSX. Yet as an embedded solver of subproblems within branch-and-bound, IPM is not well suited since the 'warm start' property is relatively poor. We have therefore chosen SSX as our embedded 'optimization engine' for solving subproblems. The dual algorithm is used to solve these subproblems efficiently.

Information sharing and algorithm choice. In solving the subproblems in the child node we share (reuse) the optimum basis information (basis list and the basis factors) of the parent node. We also apply the dual algorithm which reduces the total number of pivotal steps for reoptimization. These features also justify our choice of algorithm and vindicate the useful 'warm start' properties of the SSX.

Integer restart heuristic. In the construction of the DCEF involving, say, 500 points we are unlikely to solve all these models to QMIP optimality. As a consequence, we are likely to lose the 'pareto efficient' property of the frontier and our experiments confirm this. We do, however, adopt a scheme of computing the DCEF from the highest return, and its corresponding risk, to lower return and reduced risk. We use the previous integer solution in this sequence as the 'first feasible and upper bounding QP value' for the next point (problem). Given the previous solution is feasible (or optimal), this solution is automatically a feasible solution for the current optimisation problem, as we decrease the desired level of return from its highest value to the smallest and hence relax the constraint. This has the effect that we obtain an 'efficient' DCEF which is optimal (if all problems are solved to optimality) or sub-optimal (if the algorithm is terminated at a feasible solution). However, the frontier we generate cannot contain inefficient points as we either stay at the previous solution or we improve on it. We believe, and our experimental results vindicate that this approach is preferable to applying modern heuristics to this discrete non-convex programming problem.

7. Computational experience

In this section, we first describe the software system architecture and the computational platform that we use for the investigation of this class of portfolio problems. We also describe our computational experience in respect of the discrete constraint efficient frontier (DCEF) model with threshold (BUY-IN) and cardinality (CARD) constraints using five data sets drawn from the Hang Seng, DAX, FTSE, S&P and Nikkei indices with 31, 85, 89, 98 and 225 stocks respectively (see Beasley 1999). Recently we have further enhanced our discrete QMIP solver to process a range of models supplied by the UBS Warburg PAS system; these computational results are also discussed in this section.

7.1 Modelling and the solution tools

We have adopted a modular component based approach whereby we are able to mix and match modelling and solver tools to process different portfolio problems. The overall computational platform is shown in Figure 3. Data from the datamart (see section 5) is stored and transmitted through EXCEL datasheets. Using the MPL or AMPL algebraic modelling systems (see MAXIMAL, LUCENT), the QP or QMIP as appropriate is generated. The model is then processed by FortMP (QP) or FortMP (QMIP) and the results/ solution files are again stored in the decision database. The system runs under Windows NT and Windows 2000. In the experiments reported in 7.2 and 7.3, we have used a Pentium III, 500 MHZ processor with 128 MB of RAM. This system is also available as a web application; see (OSP-CARFT, 2001).

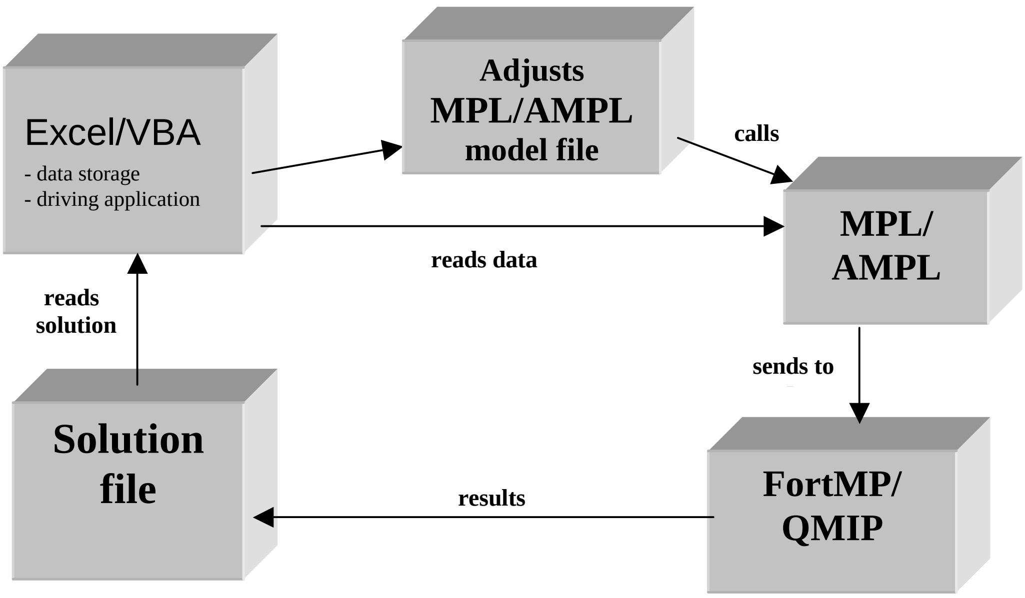

Figure 3: Data, modelling and solver architecture

7.2 DCEF study for five stock indices

We have developed an integer restart heuristic which allows us to rapidly compute points on the DCEF. We investigate our heuristic approach using model CARD for the 5 data sets drawn from the Hang Seng, DAX, FTSE, S&P and Nikkei indices. We set $l_i = 0.01$, i= 1,…, N and use the cardinality constraint $k=10$. To analyse the experimental results we follow the metric used in Chang *et al* (1999). The deviations of the points on the heuristically obtained DCEF are measured as the minimum absolute distance (vertical or horizontal) from the MEF. Since they do not calculate the *exact* DCEF but need to measure the usefulness of the heuristically computed frontier points, this deviation measure which they call 'error' provides a reasonable metric for comparison. These reported 'errors' mainly reflect the systematic deviations due to the discrete constraints. Using the same metric allows a comparison with the modern heuristic results of Chang *et al* (1999). For each data set and solution method, we generate the frontiers by solving 500 optimization problems. This number is chosen arbitrarily and the points are equally spaced with respect to the decrease in the desired level of return, ρ.

The QMIP problems are solved to the second, improving, feasible integer solution subject to a limit of 500 nodes in the branch-and-bound algorithm. Table 2 presents the results for the integer restart method applied to the five data sets. The table includes the mean and median percentage errors, the total number of DCEF points computed, the number of integer optimal points and the total solution time in seconds. The number of optimal points obtained does not appear to influence the size of the errors observed, suggesting that when optimality is not reached, the second integer solution is a good approximation of the optimal solution.

For each data set the mean error is below 0.02% with the median error below 0.015%. In all instances, the mean is greater than the median indicating positively skewed error distributions. The size of the errors reported indicate that the DCEFs obtained are very close to the corresponding MEFs. This is borne out by a mean error of 0.008% (median error 0.006%) for the DCEF solved to optimality (3000) points for the Hang Seng.

Index	Number of stocks	Total number of DCEF points	Number of integer optimal points	Solution time	Mean error	Median error
Hang Seng	31	500	492	57.55	0.014 15	0.009 97
DAX	85	500	228	8 405.33	0.013 99	0.011 59
FTSE	89	500	244	10 978.12	0.011 41	0.008 60
S & P	98	500	192	15 831.97	0.015 86	0.013 25
Nikkei	225	500	486	18 345.56	0.006 18	0.002 52
Hang Seng	31	3000	3000	382.21	0.008 26	0.006 28

Table 2: Results for the integer restart heuristic

In order to establish the computational advantage of the integer restart heuristic, we also calculate the DCEF without starting with the previous solution vector. The integer restart heuristic finds more non-dominated points and more optimal points with a smaller mean deviation in less time. To achieve similar error and optimality results, the number of nodes to be searched in the B&B algorithm needs to be increased. For example, for the S & P data set, the number of nodes has to be increased from 500 to 2500 but the solution time also increases five fold.

Comparison with modern heuristic methods

The integer restart and reoptimization heuristics outperform the modern heuristic methods of Chang *et al* (1999) who report average mean and median deviations in excess of 1% (see table 3). Clearly this makes both of our heuristic schemes very attractive, from the point of view of the quality of the discrete solution. The computational times are difficult to compare. Unfortunately, it is not possible to further compare the results since their full DCEFs are not available (Beasley 2000).

Index	Number of stocks	Solution method	Number of efficient points	Mean error	Median error
Hang seng	31	Integer restart heuristic	500	0.014 15	0.009 97
			3000	0.008 26	0.006 28
		GA heuristic	1317	0.945 70	1.181 90
		TS heuristic	1268	0.990 80	1.199 20
		SA heuristic	1003	0.989 20	1.208 20
		Pooled (GA, TS, SA)	2491	0.933 20	1.189 90
DAX	85	Integer restart heuristic	500	0.013 99	0.011 59
		GA heuristic	1270	1.951 50	2.126 20
		TS heuristic	1467	3.063 50	2.538 30
		SA heuristic	1135	2.429 90	2.467 50
		Pooled (GA, TS, SA)	2703	2.192 70	2.462 60
FTSE	89	Integer restart heuristic	500	0.011 41	0.008 60
		GA heuristic	1482	0.878 40	0.596 00
		TS heuristic	1301	1.390 80	0.713 70
		SA heuristic	1183	1.134 10	0.636 10
		Pooled (GA, TS, SA)	2538	0.779 00	0.593 80
S & P	98	Integer restart heuristic	500	0.015 86	0.013 25
		GA heuristic	1560	1.715 70	1.144 70
		TS heuristic	1587	3.167 89	1.148 70
		SA heuristic	1284	2.697 00	1.128 80
		Pooled (GA, TS, SA)	2759	1.310 60	1.068 60
Nikkei	225	Integer restart heuristic	500	0.006 18	0.002 52
		GA heuristic	1823	0.643 1	0.606 2
		TS heuristic	1701	0.989 1	0.591 4
		SA heuristic	1655	0.637	0.629 2
		Pooled (GA, TS, SA)	3648	0.569	0.584 4

Table 3: Comparison with modern heuristic approaches

[GA: Genetic Algorithm**; SA:** Simulated Annealing**; TS:** Tabu Search**]**

7.3 Experience with UBSW-PAS models

The 'optimisation' requirements of the UBSW-PAS system in respect of the average as well as the largest instance of their application pose even greater challenge in respect of processing such portfolio planning applications. Typically the total universe of assets can be as large as 8000 and cardinality constraints (CARD) may have values k=100; we have tested the system for cardinality of k=500 to k=800. Since the solver must be part of a portfolio analytics and solution tool, a good discrete feasible solution must be obtained within a 'reasonable' time frame.

To process these models we have introduced an "enhanced" depth first tree search heuristic to include multiple variable fixing. The heuristic operates in two stages. In the first stage multiple number of discrete variables are fixed in one step; some 'down' ($\delta_i = 0$) and others 'up' ($\delta_i = 1$) fixes are carried out (the number is controlled by a parameter). As a result

i) a number of assets are excluded completely ('down' fixes) , and,

ii) a number of assets are brought into the portfolio ('up' fixes).

This sequence is followed through a number of depths in the tree search until the criteria for invoking stage 2 is realised. In stage 2 only 'up' fixes are undertaken one by one until a full discrete optimum solution is reached. Sub-problems in stage 1 and stage 2 are always solved using the dual algorithm.

Computational results for a set of five models (see Table 4) are summarised in Table 5. These were portfolio rebalancing problems in which portfolios with a given cardinality of holdings were moved to that with an improved new maximum number of holdings.

	Model 1	Model 2	Model 3	Model 4	Model 5
Stock Universe	757	1,304	1,305	1,305	1,305
Initial Portfolio Size	332	251	251	251	251
Target for Maximum Assets	400	250	250	250	250
Risk Acceptance Parameter	0.6	0.6	0.6	0.6	0.6

Table 4: Parameter of a typical UBSW-PAS example

The processing of these models using the built-in QMIP search and the enhanced two-stage heuristic is shown in Table 5 which also includes the objective value of the quadratic function indicating the quality of these discrete solutions. Since this two-stage heuristic is parameter-dependent, we have supplied the average values in respect of nine runs carried out for each model.

Model 1		
Relaxed QP	Objective Value	0.18882922E-13
	Time to optimum (secs)	32.42
FortMP (QMIP)	IP Nodes	400
	IP processing time	1,903.94
	IP Objective	0.28018533E-07
Two-stage Heuristics	IP Nodes	129
	Time (secs)	121.48
	Objective function	0.28437663E-07
Model 2		
Relaxed QP	Objective Value	0.41097040E-01
	Time to optimum (secs)	10.92
FortMP (QMIP)	IP Nodes	250
	IP processing time	163.98
	IP Objective	0.41098065E-01
Two-stage Heuristics	IP Nodes	84
	Time (secs)	62.86
	Objective function	0.41098065E-01
Model 3		
Relaxed QP	Objective Value	0.32291911E-15
	Time to optimum (secs)	172.32
FortMP (QMIP)	IP Nodes	250
	IP processing time	2,943.12
	IP Objective	0.17839276E-05
Two-stage Heuristics	IP Nodes	84
	Time (secs)	235.45
	Objective function	0.15851747E-05
Model 4		
Relaxed QP	Objective Value	0.24351583E-16
	Time to optimum (secs)	130.89
FortMP (QMIP)	IP Nodes	250
	IP processing time	2,992.06
	IP Objective	0.17895076E-5
Two-stage Heuristics	IP Nodes	85
	Time (secs)	228.00
	Objective function	0.159762557E-5
Model 5		
Relaxed QP	Objective Value	0.24748111E-17
	Time to optimum (secs)	121.89
FortMP (QMIP)	IP Nodes	250
	IP processing time	2,936.82
	IP Objective	0.17780700E-5
Two-stage Heuristics	IP Nodes	85
	Time (secs)	235.05
	Objective function	0.15851747E-05

Table 5: Test results for a typical UBSW-PAS example

It is easily seen that the ‘two-stage heuristic’ performs extremely well and reduces the processing time substantially; the quality of the solution is sometimes marginally worse but more often it is better than the straight branch and bound approach labelled as FortMP (QMIP).

8. Discussions and conclusions

Over the last twenty years there have been considerable acceptance and deployment of analytical/ quantitative models for portfolio planning, asset management and asset and liability management. The evolving Basle accord (BIS 1988, 2000) and its impact on the finance industry with respect to measurement and control of risk is already considerable. These regulatory requirements of risk also continue to determine the finance industry's need for models and software systems. Set against this growing recognition and requirements of such tools, we have reviewed and presented in a consolidated form major developments in this field. In conclusion we would like to observe how development of portfolio planning and asset/ liability management systems require a convergence of different skill sets. Thus in addition to:

i) financial engineering and quantitative modelling,

it is necessary to introduce,

ii) information engineering to create analytical databases.

Finally these models must be processed efficiently which require,

iii) algorithmic and software engineering skills;

Only by bringing together all these skill sets, it is possible to create a new generation of financial planning systems.

9. Acknowledgments

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Appendix 1: Piecewise linear approximation of the quadratic form

The advantage of transforming the original quadratic form into a diagonal form which is a variable separable function is that the quadratic objective function can be approximated by a piecewise linear function of line segments. In practice, the choice of the number of line segments is critical if accurate function values are to be computed. Increasing the discrete points by which the function is approximated not only increases the accuracy of the approximation but also increases the model size. An alternative way of increasing the quality of the approximation is to apply standard bound analysis to the linear forms in order to derive a lower and upper bound on each variable appearing in the quadratic function and discretise the function only within this range. Hence, for a given number P, the density of discretising points might now be increased as only the area of interest is taken into consideration. More details about piecewise linear polynomial approximations can be found in Darby-Dowman et al. (1988).

For a set of P points on the function $f(y_i) = y_i^2$, express these as $y_i = a_{ip}, y_i^2 = b_{ip}, p = 1,..., P.$ It is easily seen that $a_{i1} = g_i^y, a_{iP} = h_i^y$.

Model **LA**, the linear approximation to QP1, based on the diagonalisation DIAG1 is now presented.

LA:

$$\text{Min } Z_{LA} = \sum_{i=1}^{N} \sum_{p=1}^{P} \lambda_{ip} b_{ip} \tag{74}$$

subject to

$$\sum_{p=1}^{P} a_{ip}\lambda_{ip} = \sum_{j=1}^{N} l_{ij} x_j \qquad i=1,\dots,N \tag{75}$$

$$\sum_{p=1}^{P} \lambda_{ip} = 1 \qquad i=1,\dots,N \tag{76}$$

$$\sum_{i=1}^{N} x_i \mu_i = \rho \tag{77}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{78}$$

$$\mathrm{x}_i \geq 0 \qquad \mathrm{i}=1,\dots,\mathrm{N} \tag{79}$$

$$0 \leq \lambda_{ip} \leq 1 \qquad i=1,\dots,N;\ \ p=1,\dots,P \tag{80}$$

This linear programming problem is easier to solve than the associated quadratic program. As a result, additional discrete constraints (such as described in the introduction) can be imposed on the model more easily. For LA to be a valid approximation of DIAG1, it is necessary that either only adjacent λ_{ip} 's for a given i are positive or any one λ_{ip} is positive or taking the value unity. These restrictions are known as special ordered set of type 2 (SOS2) restrictions and they are automatically satisfied in a convex programming problem. Hence LA is a valid approximation of DIAG1.

Appendix 2: Comparative computational views of the alternative models

In this appendix we consider a few alternative models; mean absolute deviation (MAD), minimax (MM) and the discrete constraint efficient frontier (DCEF) and study their computational results after applying them to a small illustrative dataset of stocks (equity assets). The respective efficient frontiers of these models are juxtaposed with the M-V model and its frontier; the role of the latter is that of a benchmark (taking standard deviation as the accepted risk measure) against which the performance of the other models are evaluated.

Dataset:

The historical prices of a set of thirty (30) stocks chosen out of the FTSE 100 shares are considered. The four-year price history of these 30 stocks are first downloaded from the DATASTREAM's feed as a table of 208 weekly prices. In order to create the financial datamart for this small universe of 30 stocks the returns are first analysed and filtered against historical facts (typically no extraordinary events, new issues, or administration have occurred). The return on stocks are computed on a logarithmic scale and the 208 price values per stock are used as historical observations (these make up columns of the observation matrix) and are used in turn to calculate:

i) the estimate (average) and,

ii) the variance and covariance,

of return.

All these calculations are carried out in EXCEL.

The Model Results

We first compute for the model QP1, that is the M-V model, the entire risk-return frontier without imposing any other restrictions. The software system outlined in section 7 is used and by varying the return ρ discretely over a range of $\rho_{\min}$ return corresponding to min value of risk (variance of the portfolio) and $\rho_{\max}$ the max value of return (solved as an LP). In this range, $j = 1,..., P$; $P = 100$ points were used corresponding to returns. $\rho_1 = \rho_{\max}$, $\rho_2 = \rho_1 - \Delta, ...\rho_{100} = \rho_{\min}$. It is easily seen that $\Delta = \dfrac{\rho_{\max} - \rho_{\min}}{P - 1}$.

QP1 and QP2:

We first use the model QP2 equations (9) and (10) and solve it (a) for $\lambda = 1$ which gives us $\rho_{\min}$ and then solve (b) for $\lambda = 0$ which gives us $\rho_{\max}$. We then compute the

M-V efficient frontier for a discretization of *P*=100 points. The frontier is illustrated in Figure 4.

MAD:

In this model we vary the rhs ρ of equation (33) over the same range of values and points $\{(\rho_{\max}, \ \rho_{\min}), \ P{=}100\}$. For each of these expected returns, the standard deviations of the portfolios (of assets) are computed. The corresponding frontier with the same range of return $\rho_{\min} \leq \rho \leq \rho_{\max}$ but the risk recomputed as standard deviation is illustrated in Figure 5.

According to Konno and Yamazaki (1991), the fact that the standard deviation efficient frontier of the MAD model does not coincide with the MEF is largely attributable to the non-normality of the returns data.

MM:

The results of the minimax model are obtained and the corresponding risk figures are recomputed as standard deviation; in this we follow a procedure which is analogous to MAD procedure discussed above. The corresponding efficient frontier is displayed in Figure 6.

The comparison of the MM frontier with the MEF (Figure 5) is not especially meaningful since the minimax rule is not directly related to the quadratic risk term.

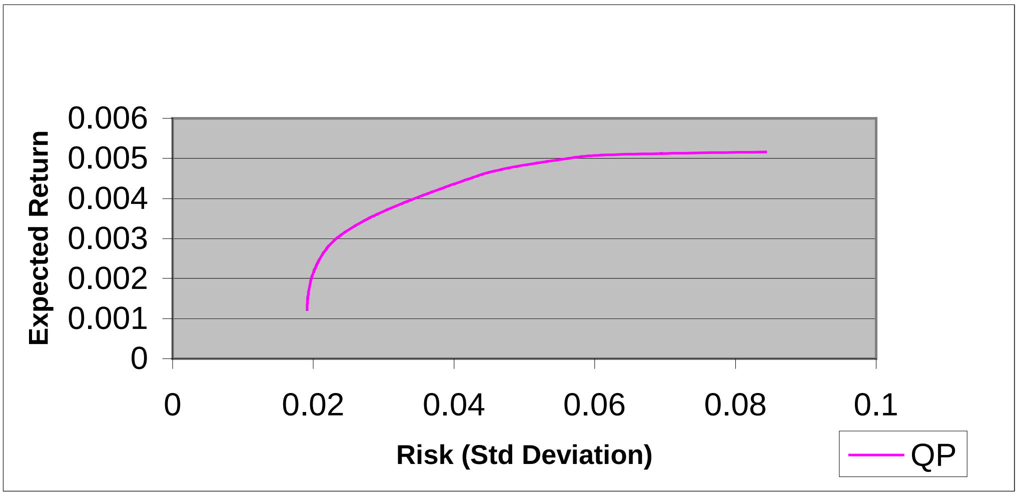

Figure 4: Quadratic Programming model

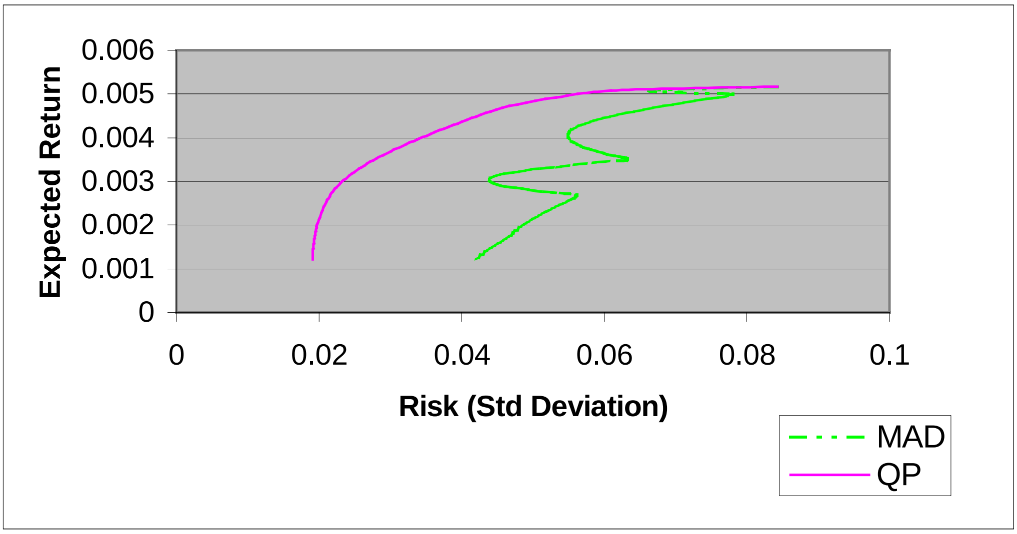

Figure 5: Quadratic Programming and MAD model

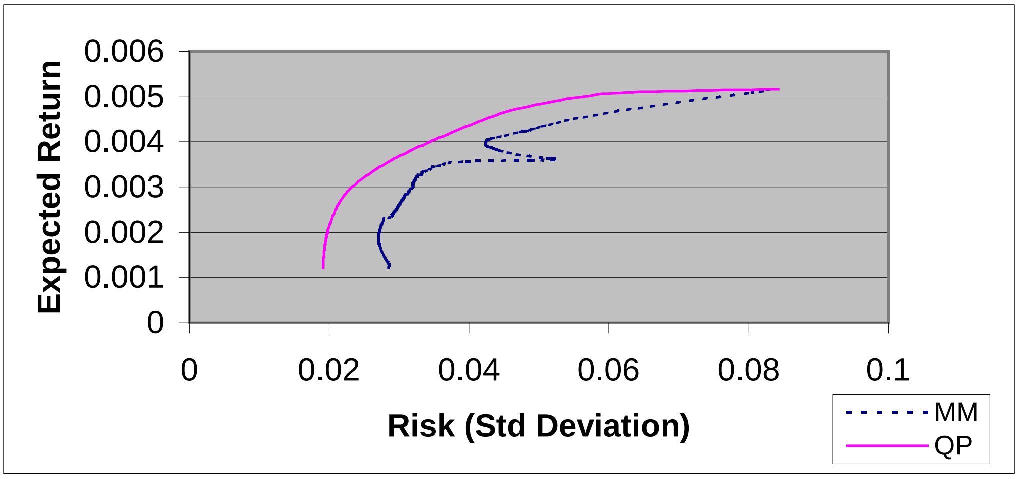

Figure 6: Quadratic Programming and MM model

Discrete constraints efficient frontier (DCEF)

Discrete constraints (see sections 4.1, 4.2) represent practical trading requirements and introduce discontinuities in the otherwise continuous efficient frontier. To illustrate the relationship of the DCEF in respect of the original efficient frontier, we consider the given dataset of the same 30 stocks and introduce a threshold of 0.1 and a cardinality constraint of $k=2$ and $k=4$ (thus only 2 and 4 stocks at a level of 0.1 or more may be included in the portfolio). Figure 7 displays the discrete efficient frontiers for model CARD. The two discrete frontiers were constructed by solving 100 optimisation problems with varying levels of return ρ and in each instance the optimal solution was found. Each of the two DCEFs contain discontinuities; also these discrete frontiers are completely dominated by the continuous M-V efficient frontier.

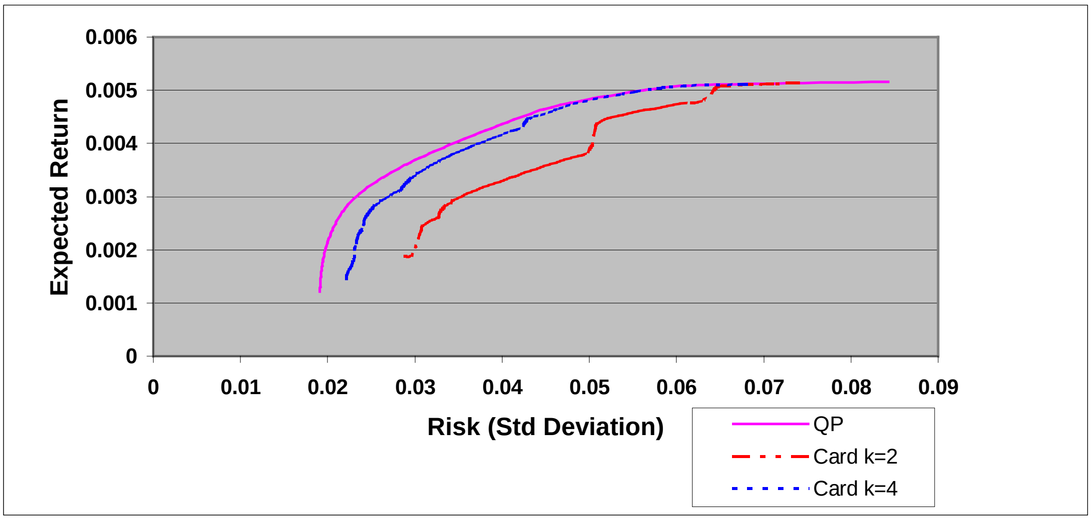

Figure 7: Quadratic Programming model and DCEF

In Jobst et al (2001), we also discuss the missing portion of the DCEF and provide a fuller discussion of these and related issues.