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Dynamic Asset (and Liability) Management under Market and Credit Risk

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DYNAMIC ASSET (AND LIABILITY) MANAGEMENT UNDER MARKET AND CREDIT RISK

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Abstract. We introduce a modelling paradigm which integrates credit risk and market risk in describing the random dynamical behaviour of the underlying fixed income assets. We then consider an asset and liability management (ALM) problem and develop a multistage stochastic programming model which focuses on *optimum risk decisions*. These models exploit the dynamical multiperiod structure of credit risk and provide insight into the corrective recourse decisions whereby issues such as *the timing risk of default* is appropriately taken into consideration. We also present a index tracking model in which risk is measured (and optimised) by the CVaR of the tracking portfolio in relation to the index. Both in- and out-of-sample (backtesting) experiments are undertaken to validate our approach. In this way we are able to demonstrate the feasibility and flexibility of the chosen framework.

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Contents

1. Introduction

1.1. **Credit Markets.** Credit markets have undergone significant changes over the last few years in Europe and in the US. The size of outstanding corporate bonds increased tremendously; this is mainly due to

(i) increased competition in the banking sector,
(ii) increased competition in the corporate world and the explosion in European Mergers & Acquisition,
(iii) institutional investors looking for additional yields due to the low (European) government bond yield environment,
(iv) and rapid increase of cash held by pension and mutual funds.

This growth in the underlying instruments of the credit market led as well to an increase in the credit derivatives[1] market. Traditionally, credit risk exposure was managed by trading the underlying assets, whereas nowadays, credit derivatives are frequently used to transfer, replicate and hedge credit risk.

The BAA Credit Derivatives Report 2002 estimated the size of the global market (excluding asset swaps) over one trillion $ by the end of 2001. They also estimate it to grow to $1952 billion in 2002 and $4.8 trillion by 2004, with London being the dominant center in global credit derivative markets. Clearly, the credit derivative market is becoming increasingly important with the main demand driven by institutional investors, banks and other financial companies. Some of the reasons for this trend can be argued in the following way:

(i) Institutional investors and asset managers can participate in the loan market, whereas direct participation would often be not possible. This offers new diversification possibilities and new asset classes to consider.
(ii) Banks and financial institutions can transfer credit risk off their balance sheet without physically selling the underlying assets. Apart from diversification arguments, it has advantages with respect to lending relationships with important clients.
(iii) Buying default protection leads to reductions in capital requirements ("regulatory capital arbitrage").

This strong increase in demand for credit (derivative) securities lead to an explosion of research efforts both, in academia and industry. Most of this work is devoted to securities valuation and credit risk measurement at a portfolio level.

1.2. **Credit Risk Modelling.** Two alternative approaches have been developed to price credit risky securities; (1) *structural* models and (2) *reduced form* models. Structural models date back to Black and Scholes (1973) and Merton (1974) and focus on determining the default time by an underlying diffusion process describing the value of the firm. Credit

[1]For a detailed introduction to the most important credit derivatives, see O'Kane (2001).

events of a particular firm are determined (or triggered) by the movements of the firm's value relative to some (stochastic or deterministic) threshold or barrier. Given such a setup, credit events are linked to the firms economic fundamentals and capital structure. Equity and debt prices as well as (implied) default probabilities are derived using option pricing theory. Corporate liabilities are modelled as contingent claims on the firms assets.

Reduced-form approaches on the contrary model the time of default as a totally inaccessible stopping time capturing the idea that the timing of default takes the bondholders by surprise. This approach does not define the default event based on the firm's value, but derives instead the default probability as the instantaneous likelihood of default, frequently called the hazard rate. These models were developed by Jarrow and Turnbull (1995), Duffie and Singleton (1999), Lando (1998) and Schönbucher (1998), amongst others. A unified exposition of reduced form models, including intensity and rating based models, can be found in Jobst and Schönbucher (2002). Both, structural and reduced form models are rigorously treated in Bielecki and Rutkowski (2002).

Recently most research is devoted to the modelling of default dependency. Examples are the copula approaches of Li (2000) and Schönbucher and Schuberth (2001), or the contagious default approaches of Davis and Lo (2001) and Jarrow and Yu (2001). These approaches extend portfolio credit risk models such CreditMetrics (RiskMetrics group (1997)) or KMV's approach (KMV-Corporation (1997)) to model the dynamics of default risk at an individual asset level more accurately while capturing also default dependency in a model consistent way.

1.3. **Credit Risk Management.** Modern credit risk management approaches incorporate many of these features that are important for security valuation. An example is the timing of defaults that lead to a liquidation of positions if default seems unavoidable under a given set of scenarios. Hence, modern credit risk management tools need to go beyond the traditional risk controlling approaches that limit (arbitrarily) the investment in bonds or shares of a specific corporation or the exposure to certain sectors. Quantitative approaches to credit risk management that allow portfolio managers to quantify the overall risk in their positions, and in particular optimisation models (e.g. portfolio optimisation models under credit risk) are still at the early stages of their development. Clearly, the tail of the overall credit loss distribution of a portfolio of obligors is critical. Practical estimation of tail risk measures or downside risk measures is challenging, but becoming increasingly important considering the growth in credit markets and complex credit products. According to Ramaswamy (Spring 2002) diversification of credit risk is much more difficult than in a market risk context, especially due to the risk of overexposure to a particular issuer or industry (concentration risk). To avoid this, one could consider including a large number of issuers in the portfolio. However, such a strategy is not based on any efficient quantitative framework and implies considerable transactions costs.

The literature on portfolio optimisation under credit risk is not extensive and only recently practitioners and academics have started to investigate a number of alternative approaches. Some of these methods build on the Markowitz' idea of mean-variance analysis where the mean and variance refer to the actual loss distribution of the portfolio (e.g.

Kealhofer (March/April 2002), Ramaswamy (Spring 2002), or Dynkin et al. (2001)). However, measuring the risk of downgrading and default by the standard deviation (of losses) does not allow any statement about the worst losses or the severity of losses at a certain percentile of the return/loss distribution. Such a model only reduces the standard deviation, hoping that the chance of catastrophic events is reduced, too. It is well known that standard deviation penalizes positive as well as negative deviations equivalently and is less suitable for asymmetric distributions. These approaches may have an immediate appeal to practitioners who are familiar with the mean-variance framework, however, optimisation methods based on downside risk and tail measures (especially CVaR) are much more adequate given the nature of credit risk.

Zagst et al. (2002) consider an asset and liability management problem in which they maximise the expected final value of the portfolio and risk as measured by lower partial moments (LPM) is introduced as a set of constraints. The model incorporates LPM constraints of order $m = (0, 1, 2)$ which is reformulated as a mixed-integer linear problem. The authors claim that these models can be rather easily solved by commercial optimisation tools.

Andersson et al. (2000) consider a single period (anticipative) model that minimizes the conditional value at risk of a emerging market bond portfolio. For a discrete and finite sample distribution, the CVaR minimization model is formulated as a linear program (see Rockafellar and Uryasev (2000)). Linear programming algorithms are very efficient and hence, large scale real world models can be tackled. Furthermore Andersson et al. (2000) show that given a specific initial portfolio, optimisation leads to reductions in many risk measures, such as CVaR, VaR, expected loss, and standard deviation of the losses.

Jobst and Zenios (2001a) investigate the adequacy of different risk metrics in a credit risk optimisation context. Single period Mean-Absolute-Deviation (MAD) and CVaR models are investigated. In Jobst and Zenios (2001b) we investigate a single period tracking model and report extensive backtesting results based on real-world data. We tackle the problem of tracking bond-indices under market and credit risk and consider the importance of alternative risk factors.

1.4. **Ex Ante and Ex Post decision making: A two-phase modelling paradigm.** The previous two sections highlight the complexity of active and quantitative risk management as applied to the credit risk of fixed income assets.

Our choice of stochastic programming as the underlying approach allows us to combine two important modelling paradigms in the following ways. Multistage stochastic programming with recourse brings together (a) models of dynamical random behaviour with (b) models of optimum decision making and resource allocation under constraints. The random dynamical behaviour is represented by scenarios which are also part of descriptive simulation. The initial (minimal) set of scenarios generated to create a *hedged optimisation* model are often called *in-sample* scenarios and are necessary to create the decision model. These sample scenarios are necessary to obtain a predictive (forward looking) representation of the state of the world, hence it can be viewed as *ex ante* approach to decision making *here and now* responding to possible states of the future world.

The existence of a dependable scenario generator, however, allows us to pursue the descriptive model as an evaluation tool. Thus the optimum hedged decision (or for that matter any other decision) can be validated by creating a large number of *out-of-sample* scenarios for which the performance of any given credit planning decision can be evaluated and validated. Thus *historical backtesting* and *out-of-sample* (simulation) testing are seen to be the *ex poste* approach and form the back bone of our validation procedure.

We also see the very important bridging role of scenario based SP recourse models. Whereas optimum decision models are valuable the decision makers see these as *opaque* black boxes. On the other hand *simulation* models are *transparent* and give clear evaluation of a given plan and gain decision maker's confidence. The simulation models, however, are not designed to provide *optimum decisions.* Our two phase ex ante and ex poste approach provides a very attractive combined framework for this class of problems which require decision making in a dynamic volatile environment and validating the decision to gain the confidence and acceptance of the problem owners.

1.5. **Guided tour.** In the next two sections we discuss the requirements on a quantitative risk management framework for portfolios of credit risky fixed-income assets. In section 2 we briefly investigate the structure of credit risk and provide insight into the importance of incorporating multiple risk factors in a simulation paradigm. In section 3 we provide further insight into the desirable features of a credit risk optimisation framework. That is, optimisation models can be only applied successfully if adequate simulation or scenario generation methods are introduced and combined with an adequate metric that captures the nature of credit risk. In section 4 we develop dynamic stochastic optimisation models for asset (and liability) management. These models extend the (single period) anticipative models to introduce recourse actions at future points in time (stochastic programming recourse models) and therefore specifically address the dynamic, multiperiod structure of credit risk. In section 5, we present in-sample and out-of-sample backtesting results that illustrate the feasibility of the chosen framework and conclusions are set out in section 6.

2. Multiple Risk Factors and Risk Measurement

Modern risk measurement systems are designed to supply portfolio managers and traders with adequate *risk numbers* and, ultimately, assist within hedging or portfolio (re-)structuring processes.

The adequacy of a risk measure depends on many factors, in particular the class of financial instruments and the time horizon of risk exposure under consideration. For short horizons (e.g. a few days), traders and risk managers frequently measure the sensitivity of a financial instrument or derivative to small changes in the price of the underlying asset (the *delta* of a financial instrument). These risk measures (first, second order sensitivity measures) based on small market movements and short time horizons prove very useful in practice; see Zagst (2002) (chapter 6) for an excellent overview. However, managing a portfolio or trading book by using first and second order sensitivity measures means that one has to continuously rebalance the portfolio, or accept small risks and rebalance the portfolio only if the measures exceed certain limits.

Most portfolio managers are also interested in the risk due to large movements over longer time horizons. Especially the latter is true in the presence of credit risk as it may take a considerable amount of time to liquidate certain positions (Jarrow and Turnbull (2000)). The usual time horizons may be a few months or even years. Frequently we are also concerned about the performance of a portfolio compared to given benchmark, hence the risk of the portfolio return falling below the benchmark, known as downside risk is also incorporated in such models.

The complexities of modeling credit risk, in particular when integrating market and credit risk in a unified framework, prevent the use of simple analytic approximations and we therefore have to employ Monte-Carlo simulation methods and scenario analysis to calculate portfolio profit and loss statistics over different risk horizons. These simulation models have to be tailored according to the application at hand. Because of the diverse context no single model has been developed that can be employed across the whole spectrum of credit applications.

In Jobst and Zenios (**?**) we discuss the requirements on an integrated market and credit risk simulation framework and investigate the nature of credit risk in the simulation setting of Figure 1 (further details are presented in the case study of section 5). This way of integrating default, recovery, migration, spread and interest rate risk allows the investigation of these factors to different credit quality instruments and portfolios of debt. Figure 2 (left panel) presents results on the increase of CVaR (at a 99% confidence level)

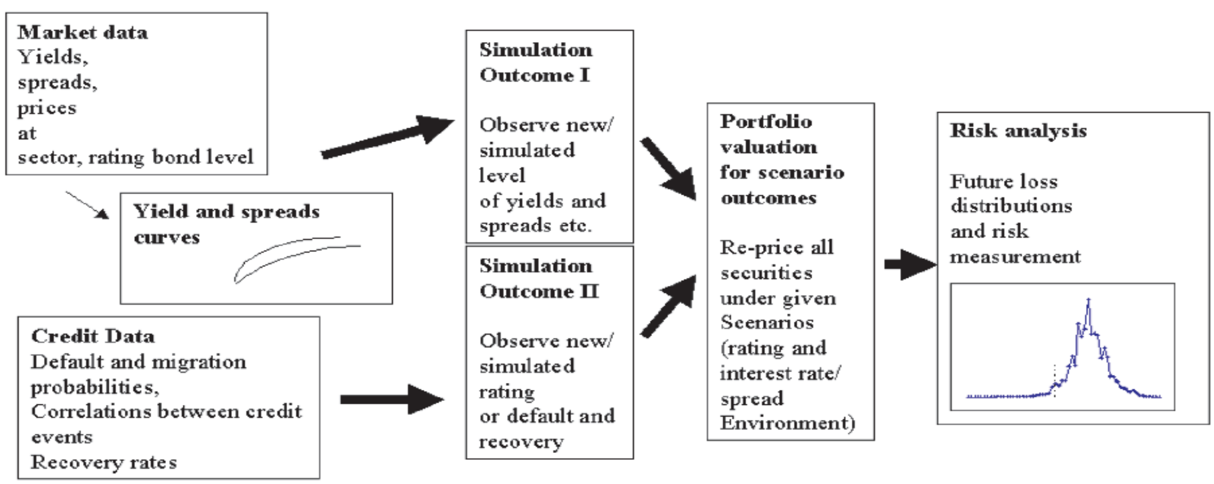


Figure 1. Risk analysis: A descriptive (simulation) modelling paradigm.

for portfolios of different credit quality when considering rating migrations and defaults (RD), RD and spread uncertainty (RD+S) and RD+S and interest rate risk (RD+S+I) in the simulations. We can observe that for portfolios (200 exposures, equally weighted) of high quality corporate bonds, spread and interest rate uncertainty increase the overall

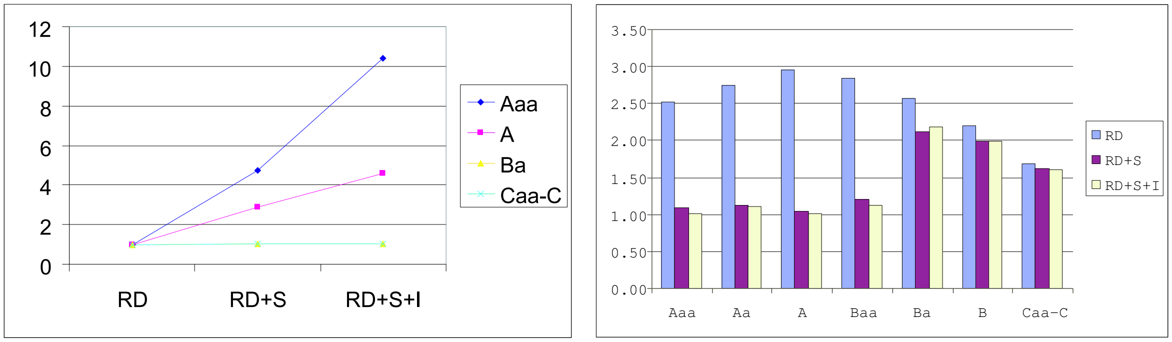


FIGURE 2. The structure of credit risk: spread and interest rate uncertainty and sensitivity to changes in asset correlation.

portfolio risk significantly. Conversely, not considering all risk factors underestimates the portfolio risk significantly.

The right panel of Figure 2 illustrates the impact of asset correlation between the underlying exposures on portfolio risk for different simulation models (RD, RD+S, RD+S+I) and different credit quality exposures.[2] As we can see, for high quality debt when considering all risk factors, increasing the asset correlation has no significant impact on the overall portfolio risk (factors close to 1). However when ignoring market and spread risk, asset correlation seems to be highly significant; hence ignoring all risk factors may lead to wrong conclusions about the structure of credit risk. On the other hand, for lower quality portfolios, an increase in asset correlation increases the overall portfolio risk across all simulation models. For a portfolio of B rated bonds, increasing asset correlations from ten to thirty percent approximately doubles the CVaR of the portfolio.

In summary, Figure 2 clearly indicated that modelling credit risk is highly complex and specific to the underlying class of credit risky securities. Monte-Carlo simulation approaches seem to be the only dependable approach to build many models that go even beyond the complexities of the approach presented in this paper. Such approaches (e.g. Schönbucher and Schubert (2001), Jarrow and Yu (2001) or Davis and Lo (2001)) allow a joint specification of default timing issues and dynamic aspects of default risk and spread changes. Default timing (and hence temporal) aspects may be extremely important for certain asset classes such as n^{th}-to-default baskets or CDOs. The *timing risk* leads therefore to a dynamic, multi-period risk assessment that captures not only the probability of defaults but also clustering effects of events within certain intervals. Also, one may think of extensions including regime-switching characteristics or multiple defaults and restructuring activities. Given that these temporal aspects are important for risk assessment, a quantitative risk management framework needs to capture such structures.

[2]These asset correlations denote the correlation between the latent random variables that drive joint default and migration events within a latent variable portfolio credit risk model. These correlations are therefore linked to the credit event correlation through a model structure that will be discussed in section 5.1.

3. An Optimum Decision Making Perspective

From earlier argument we can say consequently that

(i) a scenario based optimisation model is desired and also fits in our two-phased modelling framework,
(ii) alternative risk metrics should be potentially captured,
(iii) *timing risk* (and hence temporal aspects) may have to be incorporated.

Managing large portfolios using standard industry solutions (such as the model proposed by KMV-Corporation (1997), the CreditMetrics model proposed by the RiskMetrics group (1997), or the CreditRisk^{+} model proposed by Credit Suisse Financial Products (1997)) are based on Value-at-Risk calculations. However, as Frey and McNeil (2002) point out, the conceptual weaknesses of VaR (e.g. the lack of subadditivity in the framework of coherent risk measures) is exploited if one tries to maximise the expected return of a portfolio subject to some constraints on VaR. For an example of the inconsistency of VaR in credit portfolios and the dangers of mean-VaR portfolio optimisations, see Frey and McNeil (2002).

Hence, when developing optimisation approaches for credit risk management, we have to choose an adequate risk metric and carefully assess the problem and develop adequate simulation models that provide input to the optimisation models. We investigate these issues further in the following discussion (also see figures 3 and 4). The left panel of Figure 3 reports efficient frontiers from a MAD optimisation when applied to two different scenario sets; one set includes only interest rate and spread simulations (frontier obtained without simulated tails) whereas the other set explicitly simulates migrations and defaults as well (frontier obtained with simulated tails). The underlying portfolio was of mixed quality (for details, see Jobst and Zenios (2001a)) and the figure indicates that a misspecification of the underlying simulation model leads to a significant underestimation of the total portfolio risk. The importance of integrating alternative risk factors is further illustrated in Figure 4 where backtesting results with real world data are reported (see Jobst and Zenios (2001b)). The figure shows the performance of a portfolio optimisation tracking model (tracking the Merrill Lynch Eurodollar index) when the underlying scenario sets include defaults, migrations, spreads and interest rate simulations (left panel) and when we omit interest rate and spread scenarios (resulting in CreditMetrics type scenarios). The good tracking performance diminishes when specifying the scenario sets inadequately (the Eurodollar index is an investment grade index which requires the consideration of spreads and interest rate uncertainty as indicated in Figure 2). However as Jobst and Zenios point out, an adequate simulation model is not the only requirement to successfully develop quantitative credit risk management tools. In addition we need to choose an adequate risk metric as revealed in the right panel of Figure 3. The figure reports the mean-CVaR efficient frontier (solid line) for the same portfolio that is reported in the left panel. We also report the CVaR of the MAD-optimal portfolio (.99 CVaR(MAD*)). We can clearly see that the output of the MAD optimisation leads to highly inefficient portfolios in a CVaR perspective. Hence, given the importance of tails in the presence of credit risk, MAD optimisation models seem to be unsuitable.

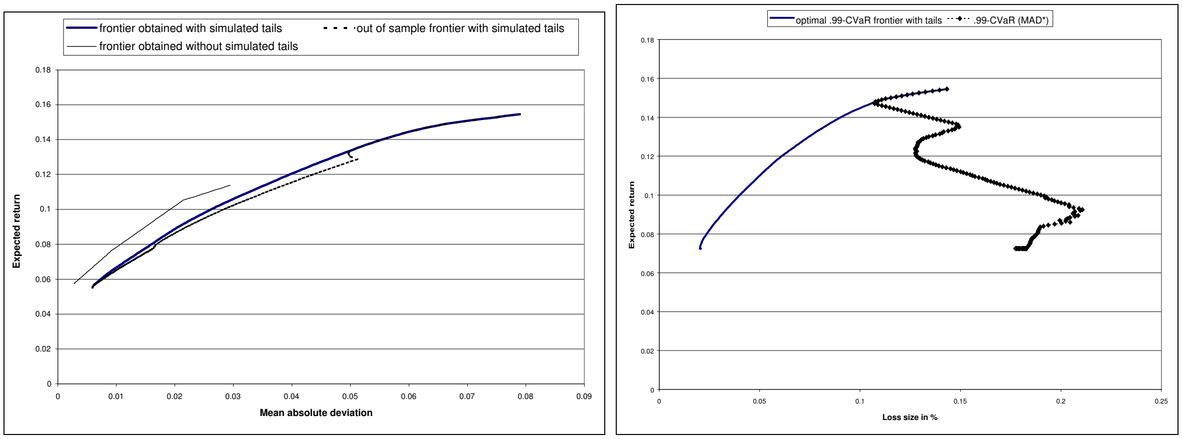


FIGURE 3. Performance of the tracking model vs the index and corresponding tracking errors when the scenario generation does not include uncertainty in interest rates and credit spreads.

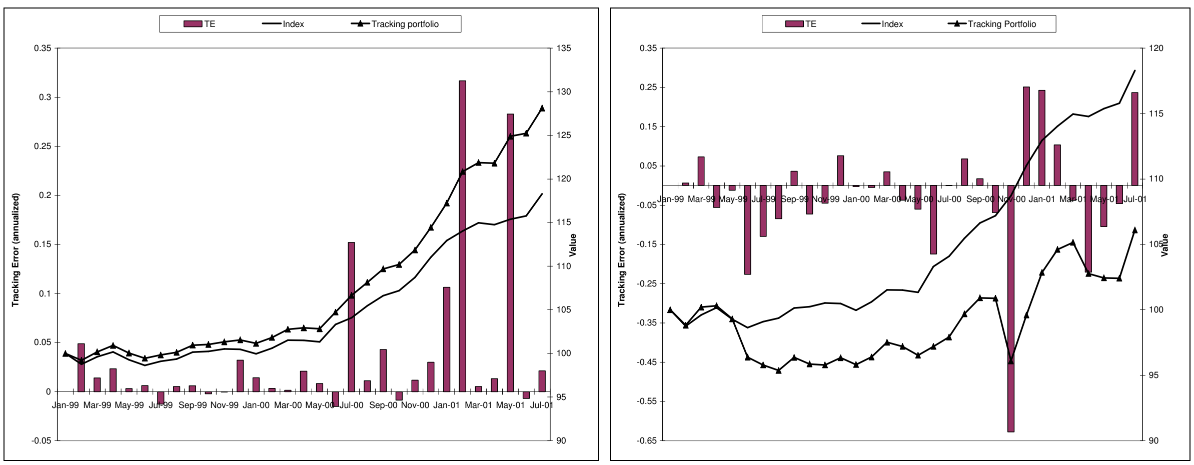


FIGURE 4. Performance of the tracking model vs the index and corresponding tracking errors when the scenario generation does not include uncertainty in interest rates and credit spreads.

So far, we have motivated the choice of simulation and optimisation models for quantitative credit risk management and pointed out the significance of adequate simulation and optimisation paradigms. Inline with most other approaches to credit risk optmisation[3] the models are single period in nature, hence failing to incorporate dynamic aspects in both, the simulation and decision (optimisation) process. In a credit risk context, dynamic aspects such as the default timing can be extremely important and should be adequately addressed.[4] For example, the optimisation models should be able to make corrective decisions such as liquidation of positions if defaults seem unavoidable under certain scenarios.

[3]An exception is Zagst et al. (2002) who consider a multi-timeperiod structure in the scenario representation. However the optimisation paradigm is still anticipative, hence, allowing only for one decision here and now.

[4]An example are CDO's where the timing of cashflows is mainly driven by the performance of the underlying collateral through time.

We develop such multistage stochastic programming models that incorporate recourse decisions (such as buying and selling at future periods) and hence provide a step forward in developing successful quantitative credit risk management (optimisation) tools.

4. Dynamic asset and liability management modelling under Credit Risk

For real world applications it is meaningful to study multi-time period models which capture the dynamical aspects of both pricing as well as the ALM investment decisions. Multistage stochastic programming in particular is well applied to process ALM models, for instance see Mulvey and Vladimirou (1992), Ziemba and Mulvey (1998), Consigli and Dempster (1998) and Kouwenberg and Zenios (2001).

We develop credit risk optimisation models within a stochastic programming framework and present a generic multistage stochastic ALM model that maximises the expected value of terminal wealth under limited CVaR risk constraints. These constraints are imposed on the portfolio value at future points in time, and in a liability matching context.

4.1. **Model structure and model development.** We show how the portfolio composition can be optimised by maximizing the expected final value or return of the portfolio under given constraints that ensure coverage of the liabilities of a company at a maximum tolerated risk. One set of restrictions is due to a minimum required cashflow per period to cover liabilities, the second set limits the risk of portfolio wealth. Both risks are measured by the conditional value at risk (CVaR) to account for the downside risk and extreme losses. These CVaR based reformulation of liability restrictions are essential in the presence of credit risk due to the default events which imply a stop in coupon income.

We consider a discrete time formulation with a planning horizon of $T \in [0, T^*]$. We divide the interval into m_T time periods, that is we define a set $\mathcal{T} = \{t_0, t_1, ..., t_{m_T}\}$, with $t_0 := 0$ and $t_{m_T} := T$, and denote the price of bond $i \in U$, $U = \{1, ..., n\}$ by $P_i(t) := P_i(t, T_i)$, where n denotes the number of bonds and T_i denotes the maturity of bond i. We consider a benchmark portfolio value or return and limit the risk of falling significantly below the benchmark at times $t \in \mathcal{T}_B = \{T_1^B, ..., T_{m_B}^B\} \subset \mathcal{T}$, where m_B denotes the number of timesteps at which we measure and restrict the risk. In addition, we consider liability payments at times $t \in \mathcal{T}_L = \{T_1^L, ..., T_{m_L}^L\} \subset \mathcal{T}$, which have to be covered by cashflows such as coupon payments. In the presence of default risk, these coupon payments may not be available, hence risk based liability constraints are introduced.

Anticipative versus multistage recourse models

Throughout this section we develop several multi-time period optimisation models; we distinguish particularly between simple anticipative and more sophisticated multi-stage recourse formulations.

Anticipative models take into consideration stochasticity in future prices and model parameters. Despite the multi-period stochastic (data) structure decisions are only possible initially, at time $t_0 = 0$, which is illustrated in the left panel of Figure 5. The dots rep-

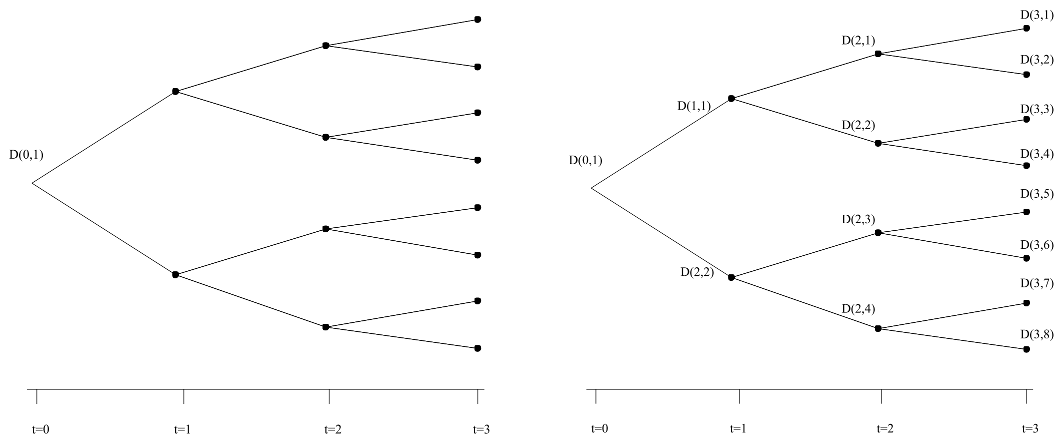


FIGURE 5. Anticipative and Multi-stage recourse decision tree structure.

resent the (scenario dependent) data (e.g. prices) and the $D(t, i)$ denotes the decision at time t in node i. Hence, the initial decision is not changed throughout the model horizon.

A generalization of this model is the multi-stage recourse formulation, where additional recourse decisions are taken into consideration at the preceeding time periods. In particular, a set of scenarios that share the same history up to a certain point in time, share also the same decision. This is known as non-anticipativity and is visualized in the right panel of Figure 5.

A special case of this multistage recourse formulation is a two stage model, with the corresponding decision tree represented in the left panel of Figure 6. This two-stage tree

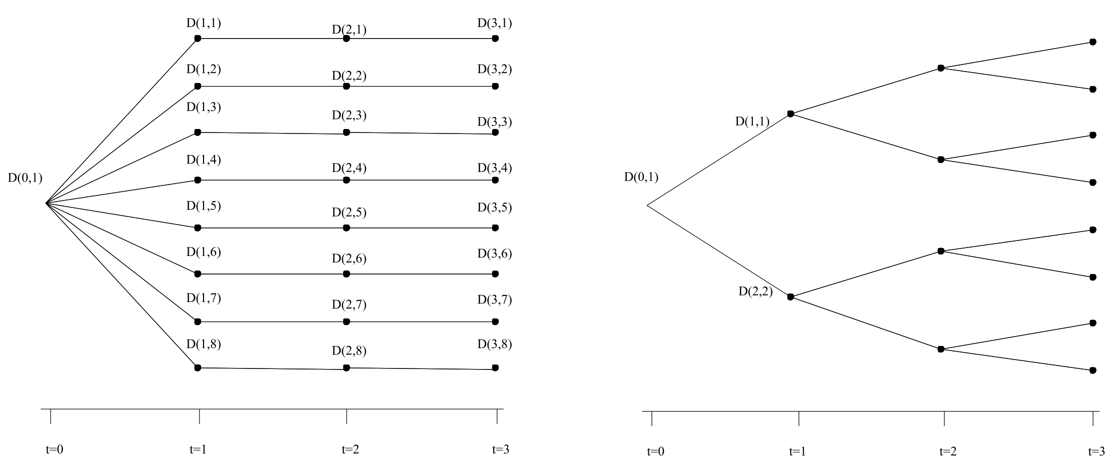


FIGURE 6. Two-stage recourse decision trees: A natural two stage model and an approximation of multi-stage structure.

presents an approximation of the multi-stage recourse model of Figure 5 (right panel) in such a way that non-anticipativity is relaxed for latter periods. Hence, the model is simpler in its decision structure, however, more efficient algorithms for solving two-stage models exist that make two-stage models more tractable from a computational viewpoint. This description of the two-stage formulation can be seen as a two-stage approximation of the (true) multi-stage model, where the difference is only in the decision structure in later periods.

Of course, an alternative two-stage formulation can be constructed by allowing for only one set of recourse decisions at $t > 0$. An example is presented in the right panel of Figure 6 where we allow one set of recourse decisions at $t = 1$ and assume that the first-stage decision and recourse variables are fixed from that time-period onwards. This model is of course more flexible than the anticipative model and less flexible the multi-stage recourse model (or its two-stage equivalent), due to the number or possible recourse actions.

Cashflows and recovery payments
Coupon payments $F_i(t)$ between liability dates and time periods are put into a cash account, and reinvested at a continuous interest rate r^{ca}, that is

$$\hat{F}_i(t, t_k) := F_i(t) \cdot e^{r^{ca}(t,t_k)(t_k - t)}, \quad t \in (t_{k-1}, t_k].^5 \tag{1}$$

In the scenario generation, we handle recovery payments $\Upsilon_i(\tau)$ in a similar way. Given default at time $\tau \in (t_{k-1}, t_k]$, we assume to receive a cashflow

$$\hat{\Upsilon}_i(\tau, t_k) := \Upsilon_i(\tau) \cdot e^{r^{ca}(\tau,t_k)(t_k - \tau)}, \tag{2}$$

where $\Upsilon_i(\tau)$ either specified as a constant amount, a random variable or a fraction of the pre-default value of the bond. Hence, in the scenario generation we use indicator functions to denote the cashflow at time $t \in (t_{k-1}, t_k]$, that is

$$F_i^d(t, t_k) := \mathbf{1}_{\{\tau > t\}} \hat{F}_i(t, t_k) + \mathbf{1}_{\{\tau \le t_k\}} \hat{\Upsilon}_i(t, t_k). \tag{3}$$

Multiple cashflows of bond i in the interval $(t_{k-1}, t_k]$ are treated correspondingly, that is

$$\bar{F}_i(t_k) := \left(\sum_{j=1 \ldots n_i, t_{ij} \in (t_{k-1}, t_k]} \mathbf{1}_{\{\tau > t_{ij}\}} \hat{F}_i(t_{ij}, t_k) \right) + \mathbf{1}_{\{\tau \le t_k\}} \hat{\Upsilon}_i(t, t_k), \tag{4}$$

where t_{ij} denotes the time of the j's coupon payment of bond i, and n_i denotes the number of scheduled coupon payments (see figure 7).

Given default at time $\tau \in (t_{k-1}, t_k]$ and the corresponding cash recovery payment, we assume that no further cashflows $F_i(t)$, for $t > \tau$ will be received. We assume that the recovery payment is put into the cash account, and furthermore $P_i(t, T_i) := 0, \quad t \ge \tau$. Hence, the value of the cash account given portfolio cash inflows between t_{k-1} and t_k is given by

$$v(t_k) = v(t_{k-1}) e^{r^{ca}(t_{k-1},t_k)(t_k - t_{k-1})} + \sum_{i=1}^{n} x_i \bar{F}_i(t_k) - Liab(t_k), \tag{5}$$

where $Liab(t_k)$ denotes the desired liability payments and x_i denote the portfolio holding in asset i, which is assumed to be constant throughout this planning horizon in the anticipative model, i.e. $x_i := x_i(t_0)$.

[5] In the numerical implementation, we assume that that r^{ca} is given by the short interest rate r.

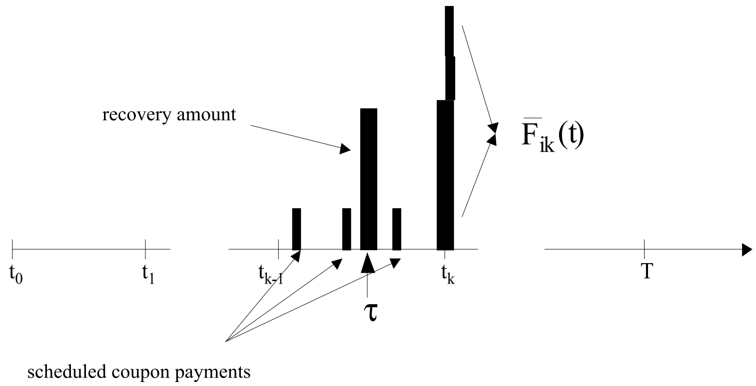


FIGURE 7. Cashflow handling in discrete time-period setup.

In the multistage recourse models, we also need to include the corresponding rebalancing decisions, which amounts to recourse actions, that is, (5) has to be replaced by

$$\begin{aligned} v(t_k) &= v(t_{k-1})e^{r^{ca}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i(t_{k-1})\bar{F}_i(t_k) - Liab(t_k) \\ &+ \sum_{i=1}^{n} x_i^s(t_k)P_i(t_k,T_i)(1-tc_i^s) - \sum_{i=1}^{n} x_i^b(t_k)P_i(t_k,T_i)(1+tc_i^b), \end{aligned} \tag{6}$$

where x_i^s denotes the units of bond i sold and x_i^b denote the units of bond i bought, and where we assume constant transaction costs tc_i^s and tc_i^b as a fraction of the bonds value.[6]

Scenario index

In the anticipative as well as multistage recourse model we use ω to denote scenario index which is identified as one of many enumerations of the stochastic data paths which are presented as a set Ω such that $\omega \in \Omega$ and $|\Omega|$ the total number of datapaths in this collection (set).

In section 5 we explain in detail our scenario generation procedure that incorporates (i) economic (interest rate and credit spread) as well as (ii) credit (defaults, rating migrations and recovery rates) scenarios. The anticipative model includes an initial decision $x_i(t_0)$ which is fixed from time 0 to the end of the planning horizon T.

Portfolio wealth

Then, the future portfolio wealth, given price scenarios $P_i^\omega(t,T_i)$, $t \in (t_{k-1}, t_k]$, is given in the anticipative model by

$$W_{t_k}^\omega := \sum_{i \in U} x_i(t_0)P_i^\omega(t_k,T_i) + v_{t_k}^\omega, \tag{7}$$

[6]Alternative transaction cost assumptions could be easily introduced. If, for example, an fixed amount should be charged as soon as a transaction takes place, we can employ binary variables following the modelling principles presented in Jobst et al. (2001).

where $v^\omega_{t_k}$ is given according to equation (5) as

$$v^\omega_{t_k} := v^\omega_{t_{k-1}} e^{r^{ca,\omega}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i(t_0)\bar{F}^\omega_i(t_k) - Liab^\omega_{t_k}.$$

In the corresponding multistage recourse model, we assume that the portfolio wealth is given as

$$W^\omega_{t_k} := \sum_{i\in U} x^\omega_i(t_{k-1})P^\omega_i(t_k,T_i) + v^\omega_{t_{k-1}} e^{r^{ca,\omega}_{t_{k-1}}(t_k-t_{k-1})} + \sum_{i\in U} x^\omega_i(t_{k-1})\bar{F}^\omega_i(t_k), \tag{8}$$

which corresponds to the value just before portfolio rebalancing takes place at time t_k. After rebalancing, the portfolio value is given by

$$\widetilde{W}^\omega_{t_k} := \sum_{i\in U} x^\omega_i(t_k)P^\omega_i(t_k,T_i) + v^\omega_{t_k}. \tag{9}$$

It is easily seen that (8) and (9) are equivalent, if no transaction costs are considered.

Risk constraints

In both, the anticipative and the recourse models downside risk constraints are introduced. To restrict the downside risk of the future portfolio value at times $t \in \mathcal{T}_B$, we consider CVaR constraints with respect to a specific benchmark $B(t) \in \mathbb{R}$.[7] Hence, the portfolio loss under a given scenario is given by

$$Loss^\omega_B(t_k) := B(t_k) - W^\omega(t_k), \tag{10}$$

or in relative terms

$$RLoss^\omega_B(t_k) := \frac{B(t_k) - W^\omega(t_k)}{B(t_k)}, \tag{11}$$

where $W^\omega(t_k)$ is defined by (7) in the anticipative model and by (8) in the multistage recourse model. We introduce a set of auxiliary variables, y^ω_B ($y^\omega_B \geq 0$), as

$$y^\omega_B(t) \geq Loss^\omega_B(t) - \zeta^B_t,$$

and define the set of CVaR constraints as

$$\zeta^B_t + \frac{\sum_{\omega\in\Omega} \pi^\omega y^\omega_B(t)}{1-\alpha_B} \leq B^{CVaR}(t),$$

where $B^{CVaR}(t)$ denotes the maximum level of risk (CVaR) tolerated, and α_B is the percentile. For further details, see Rockafellar and Uryasev (2000).

To restrict the risk in the liability stream we introduce $Loss^\omega_L(t)$ and variables $y^\omega_L(t)$, $t \in \mathcal{T}_L$. The loss at time $t \in \mathcal{T}_L$ is defined by

$$Loss^\omega_L(t) := 0 - v^\omega(t) = -v^\omega(t). \tag{12}$$

---

[7]If we are given a benchmark return $r^b(t)$ for the period $(0,t]$, the corresponding benchmark value is given by $B(t) = W_0(1 + r^b(t))$.

The following constraints are added

$$y_L^{\omega}(t) \geq Loss_L^{\omega}(t) - \zeta_t^{\omega} \tag{13}$$

$$y_L^{\omega}(t) \geq 0, \tag{14}$$

$$\zeta_t^L + \frac{\sum_{\omega \in \Omega} \pi^{\omega} y_L^{\omega}(t)}{1 - \alpha_L} \leq L^{CVaR}(t), \tag{15}$$

where $L^{CVaR}(t)$ denotes the maximum allowed tolerance risk of not matching the liabilities. These restrictions are not necessary when only market risk is considered, in particular when treasury coupon bonds are used to finance the liabilities. In that case, the problem relaxes to exact cashflow matching. However, give the risk of default the restrictions ensure that liabilities will be covered up to some probabilistic tolerance (confidence level).

Non-anticipativity constraints
In the multistage recourse model, we need also a set of constraints that ensure that scenarios sharing a common history up to any point in time must also share common decisions up to that point in time. This requirement is known as non-anticipativity, and greatly complicates the processing of SP models as it ties together optimisation problems pertaining to separate scenarios. In order to ensure consistency of the solution, if two scenarios ω_a and ω_b, $a \neq b$, are indistinguishable up to a given time period t, then, the related decisions up to that period must be the same. In order to define these constraints we need the information embodied in the scenario structure. Let S_o be the bundle of scenarios passing through node o and let $\mathcal{N}_t$ define the set of all notes at time t, $t = t_1, ..., T$. Then, non-anticipativity constraints for a decision vector x_t, $t \in \mathcal{T}$, can be formalised as

$$x_{t,\omega_a} = x_{t,\omega_b}, a \neq b, \forall \omega_a, \omega_b \in S_o, \tag{16}$$

with $o \in \mathcal{N}_t$, $t = 1, ..., T - 1$.

4.2. **The optimisation models.** We can now state the anticipative as well as the recourse model after introducing the relevant sets and indices.

Sets and Indices

i	index defining each bond $(i = 1, .., n)$
ω	scenario index $\omega \in \Omega$
$\mathcal{T}$	discrete time steps $\mathcal{T} = \{t_0, t_1, ..., t_{m_T}\}$, with $t_0 := 0$ and $t_{m_T} := T$
U	universe of n bonds: $U = \{1, ..., n\}$
$\mathcal{T}_B$	set of timesteps with benchmark restrictions$\mathcal{T}_B = \{T_1^B, ..., T_{m_B}^B\} \subset \mathcal{T}$
$\mathcal{T}_L$	set of timesteps with liability restrictions $\mathcal{T}_L = \{T_1^L, ..., T_{m_L}^L\} \subset \mathcal{T}$
$\mathcal{N}_t$	set of nodes at time t in scenario tree
D_o	bundle of scenarios passing through node $o \in \mathcal{N}_t$, $t = t_1, ..., T - 1$

Data Parameters

π^ω	probability of scenario ω
r_i^ω	holding period return of bond i under scenario ω
β_i, (β_i^c)	weight of security i in the index (in class c)
I^ω, (I_c^ω)	index (asset class index) return under scenario ω
b_i	initial face value holding of bond i
P_{i0}	initial price of bond i
P_{it}^ω	price of bond i at time t under scenario ω, $P_{it}^\omega = P_i^\omega(t, T_i)$
r_0^ω	riskless rate of return under scenario ω
tc_i^s, tc_i^b	transaction costs for selling, buying of security i
W_0	initial portfolio wealth
W^ω (W_t^ω)	portfolio wealth under scenario ω (at time t)
$\mathcal{T}$	discrete time steps $\mathcal{T} = \{t_0, t_1, ..., t_{m_T}\}$, with $t_0 := 0$ and $t_{m_T} := T$
T_i	the maturity of bond i
r^{ca}	continuous cash rate
$\bar{F}_i(t, t_k)$	default adjusted cashflow (multiple coupons / recovery) at time t_k, $t \in (t_{k-1}, t_k]$
$Liab^\omega(t)$	liability at time t under scenario l
$B(t)$	value of benchmark at time t with benchmark return $r^B(t)$
$B^{CVaR}(t)$	tolerated benchmark CVaR (right hand side) at time t
$L^{CVaR}(t)$	tolerated liability CVaR (right hand side) at time t

Decision Variables

c_0	initial cash holding
x_i	weight of (or face value holding in) security i in the portfolio
v (v_t^ω)	amount invested in cash (at time t under scenario ω)
x_i^b	face value purchased of security i
x_i^s	face value sold of security i
ζ	Value-at-Risk
$\zeta_t^B(\zeta_t^L)$	Value at Risk at time t for benchmark (liability) risk
$y_B^\omega(t)$ $(y_L^\omega(t))$	auxiliary variable in CVaR calculation for benchmark (liability) risk
α_B (α_L)	confidence level in benchmark (liability) risk restrictions

Anticipative (ALM) model

$$\text{Max} \quad \sum_{\omega\in\Omega} \pi^\omega W_T^\omega \tag{17}$$

s.t.

$$x_i = b_i + x_i^b + x_i^s, \quad i \in U \tag{18}$$

$$v_{t_0} = c_{t_0} + \sum_{i\in U} \left(x_i^s P_{it_0}(1 - tc_i^s) - x_i^b P_{it_0}(1 + tc_i^b) \right) \tag{19}$$

$$v_{t_k}^\omega = v_{t_{k-1}}^\omega e^{r_{t_{k-1}}^\omega (t_k - t_{k-1})} + \sum_{i\in U} x_i \bar{F}_{it_k}^\omega - Liab_{t_k}, \quad \omega \in \Omega, k = 1, ..., m_T \tag{20}$$

$$W_{t_k}^\omega = \sum_{i\in U} x_i P_{it_k}^\omega + v_{t_k}^\omega, \quad \omega \in \Omega, k = 1, ..., m_T \tag{21}$$

$$y_{Lt_L}^\omega \geq -v_{t_L}^\omega - \zeta_{t_L}^L, \quad \omega \in \Omega, t_L \in \mathcal{T}_L \tag{22}$$

$$L_{t_L}^{CVaR} \geq \zeta_{t_L}^L + \frac{\sum_{\omega\in\Omega} \pi^\omega y_{Lt_L}^\omega}{1 - \alpha_L}, \quad t_L \in \mathcal{T}_L \tag{23}$$

$$y_{Bt_B}^\omega \geq \frac{B(t_B) - W_{t_B}^\omega}{B(t_B)} - \zeta_{t_B}^B, \quad \omega \in \Omega, t_B \in \mathcal{T}_B \tag{24}$$

$$B_{t_B}^{CVaR} \geq \zeta_{t_B}^B + \frac{\sum_{\omega\in\Omega} \pi^\omega y_{Bt_B}^\omega}{1 - \alpha_B}, \quad t_B \in \mathcal{T}_B \tag{25}$$

$$y_{Lt_L}^\omega \geq 0, \quad \omega \in \Omega, t_L \in \mathcal{T}_L \tag{26}$$

$$y_{Bt_B}^\omega \geq 0, \quad \omega \in \Omega, t_L \in \mathcal{T}_L \tag{27}$$

$$v_0^\omega = v_{t_0}, \quad \omega \in \Omega. \tag{28}$$

Multistage recourse (ALM) model

$$
\begin{aligned}
(29)\quad & \text{Max} && \sum_{\omega\in\Omega} \pi^{\omega} W_T^{\omega} \\
& \text{s.t.} && \\
(30)\quad & x_{it_0} &=& \; b_i + x_{it_0}^{b} + x_{it_0}^{s}, \quad i \in U \\
(31)\quad & v_{t_0} &=& \; c_{t_0} + \sum_{i\in U}\left(x_{it_0}^{s} P_{it_0}(1-tc_i^{s}) - x_{it_0}^{b} P_{it_0}(1+tc_i^{b})\right) \\
(32)\quad & x_{it_k}^{\omega} &=& \; x_{it_{k-1}}^{\omega} + x_{it_k}^{b\omega} + x_{it_k}^{s\omega}, \quad \omega\in\Omega, i\in U, k=1,...,m_T \\
& v_{t_k}^{\omega} &=& \; v_{t_{k-1}}^{\omega} e^{r_{t_{k-1}}^{\omega}(t_k-t_{k-1})} + \sum_{i\in U} x_{it_{k-1}}^{\omega} \bar{F}_{it_k}^{\omega} - Liab_{t_k} \\
(33)\quad & && + \sum_{i\in U}\left(x_{it_k}^{s\omega} P_{it_k}^{\omega}(1-tc_i^{s}) - x_{it_k}^{b\omega} P_{it_k}^{\omega}(1+tc_i^{b})\right) \quad \omega\in\Omega, k=1,...,m_T \\
(34)\quad & W_{t_k}^{\omega} &=& \; \sum_{i\in U} x_{it_{k-1}}^{\omega}(P_{it_k}^{\omega} + \bar{F}_{it_k}^{\omega}) + v_{t_{k-1}}^{\omega} e^{r_{t_{k-1}}^{\omega}(t_k-t_{k-1})}, \quad \omega\in\Omega, k=1,...,m_T \\
(35)\quad & y_{Lt_L}^{\omega} &\geq& \; -v_{t_L}^{\omega} - \zeta_{t_L}^{L}, \quad \omega\in\Omega, t_L\in\mathcal{T}_L \\
(36)\quad & L_{t_L}^{CVaR} &\geq& \; \zeta_{t_L}^{L} + \frac{\sum_{\omega\in\Omega}\pi^{\omega} y_{Lt_L}^{\omega}}{1-\alpha_L}, \quad t_L\in\mathcal{T}_L \\
(37)\quad & y_{Bt_B}^{\omega} &\geq& \; \frac{B(t_B) - W_{t_B}^{\omega}}{B(t_B)} - \zeta_{t_B}^{B}, \quad \omega\in\Omega, t_B\in\mathcal{T}_B \\
(38)\quad & B_{t_B}^{CVaR} &\geq& \; \zeta_{t_B}^{B} + \frac{\sum_{\omega\in\Omega}\pi^{\omega} y_{Bt_B}^{\omega}}{1-\alpha_B}, \quad t_B\in\mathcal{T}_B \\
(39)\quad & y_{Lt_L}^{\omega} &\geq& \; 0, \quad \omega\in\Omega, t_L\in\mathcal{T}_L \\
(40)\quad & y_{Bt_B}^{\omega} &\geq& \; 0, \quad \omega\in\Omega, t_L\in\mathcal{T}_L \\
(41)\quad & v_0^{\omega} &=& \; v_{t_0}, \quad \omega\in\Omega \\
(42)\quad & x_{i0}^{\omega} &=& \; x_{it_0}, \quad \omega\in\Omega \\
(43)\quad & x_{it_k}^{b\omega_a} &=& \; x_{it_k}^{b\omega_b}, \quad a\neq b, \forall\omega_a,\omega_b\in\mathcal{S}_o, o\in\mathcal{N}_{t_k}, k=1,...,m_T-1 \\
(44)\quad & x_{it_k}^{s\omega_a} &=& \; x_{it_k}^{s\omega_b}, \quad a\neq b, \forall\omega_a,\omega_b\in\mathcal{S}_o, o\in\mathcal{N}_{t_k}, k=1,...,m_T-1
\end{aligned}
$$

Equations (43) and (44) define the non-anticipativity constraints for the buying and selling variables, which implies also non-anticipativity to the portfolio holding variables.

Remarks:

(i) In the following case studies, we have implemented a small modification of this model. We assume that (5) and (6) are given by

$$v(t_k) = v(t_{k-1})e^{r^{ca}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i \bar{F}_i(t_k)$$

and

$$\begin{aligned} v(t_k) &= v(t_{k-1})e^{r^{ca}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i(t_{k-1})\bar{F}_i(t_k) \\ &+ \sum_{i=1}^{n} x_i^s(t_k)P_i(t_k,T_i)(1-tc_i^s) - \sum_{i=1}^{n} x_i^b(t_k)P_i(t_k,T_i)(1+tc_i^b), \end{aligned}$$

respectively, i.e. we do not assume that the liabilities are paid out directly. We define the loss function (12) correspondingly as

$$Loss_L^{\omega}(t) := Liab_L^{\omega}(t) - v^{\omega}(t), \quad t \in \mathcal{T}_L.$$

The two formulations are (almost) identical , when $Liab_L^{\omega}(t)$ in the latter formulation incorporates the actual liability in period t plus reinvestment of cash which match the liabilities from pervious periods.

(ii) In the multistage recourse context, the model and CVaR constraints can be seen as a path-independent (regulator's) view of risk, that is, risk is defined from the initial state over multiple time periods, which is illustrated in the right panel of Figure 8 Alternatively, we could have a short term view (traders view) and employ risk restrictions depending on the state of the world at future time periods (path-dependent), which is shown in the left panel of Figure 8.

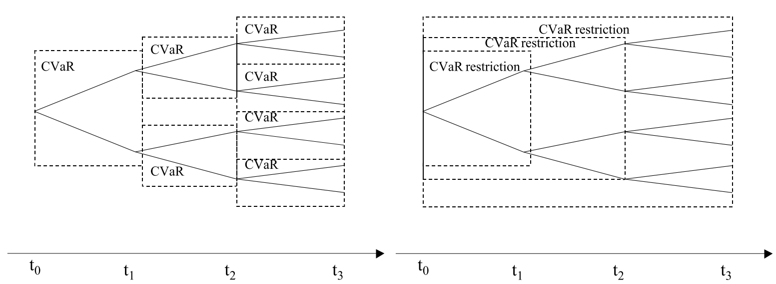


FIGURE 8. Left Panel: Path-dependent (traders) view on risk constraints (short term state dependent) - Right panel: path-independent (regulators) view on risk constraints(long term view).

5. CASE STUDY

5.1. **Multi-stage scenario generation: Integrating market and credit risk.** We employ the simulation model of Jobst and Zenios (**?**) and develop multi-stage event tree extensions next. We consider a filtered probability space (or stochastic basis) $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$

where $\mathcal{P}$ denotes the real probability measure and the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T^*}$ describes the information flow, and $\mathcal{F}_t$ represents the information available at time t. We consider n firms simultaneously. The migration process κ^j describes the evolution of the credit rating of firm j, where $\kappa^j \in S$ given the state space $S = \{1, 2, \ldots, K\}$, with K denoting the default state. Given this setup, we develop a simulation model where the risk of the future value of a credit risk sensitive instrument can be decomposed into

(i) the risk that a firm's rating changes (including the risk of default)
(ii) the correlation between credit events
(iii) the risk that changes occur in the average spread of exposures with the same final rating as the firm[8]
(iv) the effect of interest rate uncertainty.

5.1.1. *Credit events: Rating migrations and defaults.* The future credit rating $\kappa_{\bar{T}} \in S$ of each bond is simulated according to the actual migration process under $\mathcal{P}$. We assume that the probability of changes from rating l to rating m over one time period is a constant π_{lm} and let the migration matrix be denoted by $Q := [\pi_{lm}]_{l,m\in S}$. These matrices are published on a regular basis by rating agencies, such as Standard and Poor's or Moody's. Defaults are modelled when the process hits the absorbing state K, hence, $\pi_{Nl} = 0$, $l = 1, \ldots, N-1$, and $\pi_{NN} = 1$. Figure 9 illustrates the branching scheme between the rating classes. Since our primary concern is in the simulation of a portfolio of credit exposures

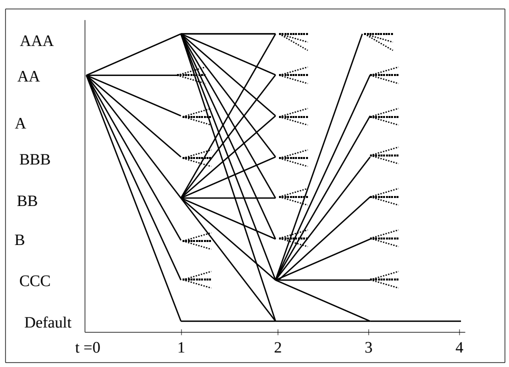


FIGURE 9. Transitions between different rating classes.

(corporate bonds), we are also interested in the joint distribution of multiple exposures. We incorporate correlation between migrations and defaults following the latent variable approach employed in the CreditMetrics methodology of JP Morgan (1997). This method was also applied in empirical studies, such as Nickel et al. (2000) and Kiesel et al. (2001). The approach is based on latent factors z^j for firm j, which are assumed to be standard normally distributed, driving the transitions of a single exposure. To capture dependence

[8]We don't model the risk that the gap between the idiosyncratic spread and the average spread changes. A similar assumption is taken in Kiesel et al. (2001). Given a certain exposure in rating l, we apply the OAS methodology to match market prices and assume the same OAS forward in time if the bond stays in the same rating class. Given a rating change, we assume the bond is priced at the average spread of the new rating class (fair market value).

between the transitions of different exposures, we introduce correlation between the latent normal random variables. Conditional on the initial rating $l \in S$, we let Z_{lm}, $m = 1, \dots, N-1$ represent cut-off points (or barriers) such that

$$\begin{aligned}
\pi_{lK} &= \mathbf{P}\left[\, z^j \leq Z_{lK} \,\right] = \Phi(Z_{lK}), \\
\pi_{lK-1} &= \mathbf{P}\left[\, z^j \leq Z_{lK-1} \cap z^j > Z_{lK} \,\right] = \Phi(Z_{lK-1}) - \Phi(Z_{lK}) \\
&\vdots \\
\pi_{l1} &= \mathbf{P}\left[\, z^j \leq Z_{l1} \cap z^j > Z_{l2} \,\right] = 1 - \Phi(Z_{l2}),
\end{aligned}$$

where Φ denotes the standard normal distribution function. Given the probabilities π_{lm}, then Z_{lm} can be easily derived by solving the above equations recursively. Hence, rating transitions and default events can be simulated by sampling the multivariate latent random factors z^j and determining whether or not the process crosses one of the calibrated barriers. We denote simulated scenario sets as *credit* or *rating scenarios*. CreditMetrics suggests using the correlation coefficients of the firms equity or asset returns.[9] Given the distribution of ratings at the risk horizon $\bar{T}$, the value of a given bond is derived by adding the distribution of interest rates and credit spreads for the new rating class. JP Morgan suggests using currently observable forward interest rates and credit spreads, and hence, assumes a deterministic interest rate and spread environment. In our approach, we employ and simulate a dynamic term structure model of credit spreads (and interest rates), as developed in the reduced form modelling literature for pricing credit risk sensitive securities.

5.1.2. *The term structure of interest rates and credit spreads.* Interest rates and credit spread processes are modelled via correlated stochastic processes of the form

$$dx(t) = \hat{\mu}_x(x,t)dt + \sigma_x(x,t)d\hat{W}_x(t) \tag{45}$$

where $d\hat{W}_x(t)$ denotes the standard Brownian motion under the real measure $\mathcal{P}$. The risk neutral dynamics (under the measure $\mathcal{Q}$) can be obtained by introducing a market price of risk $\gamma_x(t)$ with $\hat{\mu}_x(x,t) = \mu_x(x,t) + \gamma_x(t)\sigma_x(x,t)$. The risk-neutral process is then given as

$$dx(t) = \mu_x(x,t)dt + \sigma_x(x,t)dW_x(t), \tag{46}$$

with $dW_x(t) = d\hat{W}_x(t) + \gamma_x(t)dt$.

Such a process is considered for the short interest rate, i.e., $x = r$, and for the credit spreads, i.e., $x = h_k$ of each rating class $k = 1, ..., K-1$, where 1 denotes the highest (Aaa) and $K-1$ denotes the lowest pre-default rating (Caa-C). Correlation between the processes can be captured by correlating the Wiener terms, i.e.

$$dW_i(t)dW_j(t) = \rho_{ij}dt \tag{47}$$

where $i, j \in \{r, h_1, ..., h_{K-1}\}$.

[9]CreditMetrics does not employ the actual equity correlation, instead it constructs pseudo equity returns and correlation coefficients based on weighted averages of national and sector equity indices. The weights have to be attached on a judgemental basis.

Given these specifications for the short rate and the credit spread process with a given short spread $h(t) := h_k(t)$, we follow Duffie and Singleton (1999) and Schönbucher (2000) to derive the value of credit risky bonds, in particular, zero-coupon bonds. The price $\bar{p}(t,T)$ of a defaultable zero-coupon bond n at time t in rating $\kappa_t^n = k \in \{1,\dots,K-1\}$ with maturity T is given as

$$\bar{p}^{\kappa_t}(t,T) = \mathbf{E}^{\mathcal{Q}}\left[e^{-\int_t^T (r(s)+h_k(s))ds}\right], \tag{48}$$

where the bond face value of 1 is discounted with a risk adjusted short rate $r_t + h_{kt}$.[10] The corresponding price of a risky coupon bond is given as

$$P_n^{\kappa_0}(0,T_n) = \sum_{\tau=0}^{T_n} F_n(\tau)\bar{p}^{\kappa_0}(0,\tau) \tag{49}$$

where $F_n(t)$ denotes the coupon payments (plus principal at maturity $t = T_n$). If the model price is not equal to the observed bond price, as the bond trades above or below the average spread of the rating class, we can employ the option adjusted spread (OAS) methodology to correct for risk factors that were not explicitly taken into consideration. Hence, we introduce a firm or bond specific factor that gives insight into the relative value of a specific bond with respect to a (hypothetical) bond that is priced at the average spread of the corresponding rating class.

5.1.3. *Current and future portfolio valuation.* Given the dynamics of the risk free rates and spreads above we define the present value of the portfolio as

$$W_0 = \sum_{n=1}^{N} w_n\ P_n^{\kappa_0^n}(0,T_n), \tag{50}$$

where w_n denote the holdings of bond n in the portfolio. $P_n^{\kappa_0^n}(0,T_n)$ denotes the current price of bond n

Similarily, the value of the portfolio at a risk horizon $\bar{T}$ is given as

$$W_{\bar{T}} = \sum_{n=1}^{N} w_n\ P_n^{\kappa_{\bar{T}}}(\bar{T},T_n) \tag{51}$$

where $\bar{T} \le \max_n T_n$. $P_n^{\kappa_{\bar{T}}}(\bar{T},T_n)$ now denotes the price of coupon bond n at time $\bar{T}$ and is given as

$$P_n^{\kappa_{\bar{T}}}(\bar{T},T_n) = \sum_{t=\bar{T}}^{T_n} F_n(t)\bar{p}_n^{\kappa_{\bar{T}}}(\bar{T},t). \tag{52}$$

The corresponding zero coupon bond price $\bar{p}_n^{\kappa_{\bar{T}}}(\bar{T},T_n)$ depends on the level of interest rates $r(\bar{T})$ and credit spreads $h_{\kappa_{\bar{T}}}(\bar{T})$ at the risk horizon $\bar{T}$, the further evolution of $r(t)$ and $h_{\kappa_{\bar{T}}}(t)$ under $\mathcal{Q}$ after $\bar{T}$ and until maturity, i.e. $\bar{T} \le t \le T_n$, and the credit rating $\kappa_{\bar{T}}$

[10] The bond index superscript n for κ_t^n is dropped when there is no ambiguity.

of bond n at the risk horizon $\bar{T}$. This price is given for $h(t) := h_{\kappa_{\bar{T}}}(t)$ as in equation (48) as

$$\bar{p}_n^{\kappa_{\bar{T}}}(\bar{T}, T_n) = \mathbf{E}_{\bar{\mathbf{T}}}^{\mathcal{Q}} \left[\, e^{-\int_{\bar{T}}^{T_n} r(s)+h(s)ds} \, \right] \tag{53}$$

If a bond matures before the risk horizon $\bar{T}$ (i.e. $T_n \leq \bar{T}$), we need to impose an assumption on reinvesting the received cashflows, such as reinvesting in a default free money market account.

These steps are summarized as follows: Given input data on the individual securities, term structures of default free and defaultable bonds of different ratings and the rating transition probabilities, we simulate a set of economic scenarios for interest rates and credit spreads. In addition we simulate a set of rating and default scenarios reflecting migrations. Given that bond n at time $\bar{T}$ is in rating $\kappa_{\bar{T}}$ we need to obtain the price of this bond conditional on the state of the economic scenarios at $\bar{T}$ and according to the evolution of the state variables under the risk neutral measure $\mathcal{Q}$ from time $\bar{T}$ onwards.

5.1.4. *Multi-period extensions and Multi-stage event trees.* In order to apply the model in a multi-stage stochastic programming context, we need to generate a corresponding (credit) event tree. In order to generate a multi-period event tree, we need to sample many subtrees, conditional on the state of the risk factors in each root node.

Generating the interest and spread scenarios according to a multi-stage event tree is straightforward, when we sample from the underlying diffusion processes. We start in the root note and sample the first stage scenarios. We then move on to the next time period and conditionally on the state of the world in a give node, we sample the next subtree. We repeat this setup, until the overall event tree is generated.

Generating an event tree from the CreditMetrics migration and default model is slightly more complex. Firstly, we have to calculate time-dependent migration matrices from the historical (usually one year) transition matrix. In a multi-period setup, considering $\bar{K}$ periods, we consider the time horizon $[t_0, t_1, ..., t_{\bar{K}}]$. Hence, at time t_k, we need to calculate the corresponding transition matrix over the next $t_{k+1} - t_k$ years. Secondly, we need to extend the calibration of the migration barriers into a multi-period setting. We have developed two different approaches, a *discrete extension* and a *diffusion driven extension*, which are based on the default model extensions presented in Finger (2000).

Time dependent transition matrices

Given the one year constant historical transition matrix, we generate time dependent matrices in the following way (assuming that the historical matrix Q can be diagonalised). We assume that a nonsingular transformation matrix M and a diagonal matrix $D_Q = diag\{d_1, ..., d_K\}$ exist, such that

$$Q = MD_QM^{-1}.$$

This diagonalisation is possible if Q has a complete set of distinct eigenvalues. Assuming that Q can be diagonalised, then a time dependent transition matrix $Q_k := Q(t_{k-1}, t_k)$ can be stated as follows,

$$Q(t_{k-1}, t_k) := MD(t_{k-1}, t_k)M^{-1},$$

where $D(t_{k-1},t_k) = diag\{d_1 \cdot (t_k - t_{k-1}), ..., d_K \cdot (t_k - t_{k-1})\}$. Note that the diagonalisation also allows us to derive the generator matrix Λ as

$$\Lambda = M^{-1} D_\Lambda M,$$

where $D_\Lambda = diag\{\ln d_1, ..., \ln d_K\}$ in a computationally cheap way.

Discrete CreditMetrics extension

In the discrete extension, we assume that migrations in consecutive periods are driven by independent random variables $z^j_{t_k}$, $k = 1, ..., \bar{K}$ for company j (independent through time). This corresponds to the simplest possible extension, where the one period model is simply repeated. For the first time period, we assign standard normal random variables z^j_1 to each exposure j, with a correlation between two distinct exposures z^i_1 and z^j_1 given by ρ^{CM}_{ij}.

For the assets that survive the first period, we assign a second set of standard normal variables z^j_2, such that the correlation between z^i_2 and z^j_2 is again ρ^{CM}_{ij}. The variables are independent from one period to the next.

Given $Q_1 := Q(t_0,t_1)$, the migration and default thresholds for the first period can be obtained for each initial rating $l \in S$ by iteratively solving the following system of equations:

$$\begin{aligned}
\pi^{01}_{lK} &= \mathbf{P}\left[\, z^j_1 \leq Z^1_{lK} \,\right] = \Phi(Z^1_{lK}),\\
\pi^{01}_{lK-1} &= \mathbf{P}\left[\, z^j_1 \leq Z^1_{lK-1} \cap z^j_1 > Z^1_{lK} \,\right] = \Phi(Z^1_{lK-1}) - \Phi(Z^1_{lK})\\
&\vdots\\
\pi^{01}_{l1} &= \mathbf{P}\left[\, z^j_1 \leq Z^1_{l1} \cap z^j_1 > Z^1_{l2} \,\right] = 1 - \Phi(Z^1_{l2}).
\end{aligned}$$

For example, default happens for company j initially rated l when $z^j_1 \leq Z^j_{lK}$. For the next period, the Markov property implies that given the time t_1 rating of asset j, the probability of the next migration depends only on the current state and is independent of the history. Hence, firm j, initially rated l that survives the first period in the same rating will migrate to a rating $\bar{l}$ over the second period with probability $\pi^{12}_{l\bar{l}}$, $\bar{l} \in S$, where π^{12}_{lm} denotes the elements of $Q^2 := Q(t_1,t_2)$. Hence, given the simulated standard normal random factors z^j_2 and the current rating l, the migrations after the thresholds are derived from the following set of equations:

$$\begin{aligned}
\pi^{12}_{lK} &= \mathbf{P}\left[\, z^j_2 \leq Z^2_{lK} \,\right] = \Phi(Z^2_{lK}),\\
\pi^{12}_{lK-1} &= \mathbf{P}\left[\, z^j_2 \leq Z^2_{lK-1} \cap z^j_2 > Z^2_{lK} \,\right] = \Phi(Z^2_{lK-1}) - \Phi(Z^2_{lK})\\
&\vdots\\
\pi^{12}_{l1} &= \mathbf{P}\left[\, z^j_2 \leq Z^2_{l1} \cap z^j_2 > Z^2_{l2} \,\right] = 1 - \Phi(Z^2_{l2}).
\end{aligned}$$

An extension to further periods is straightforward, given the assumed Markov chain dynamics.

Diffusion driven CreditMetrics extension

The discrete CreditMetrics extension does not allow for any correlation of migration and default rates through time. For example, if a high default rate is realized in the first period, this has no bearing on the default rate in the second period since the drivers z_2^j are independent of the first period. Intuitively this behavior is not expected in the market. A high default rate in one period is likely to cause at least a decrease in credit quality to those obligers that did not default. The implication would then be that the default rates for the second period would have high tendency, too. In order to capture this behavior, we introduce a CreditMetrics extension where defaults in consecutive periods are not driven by independent random variables, but rather by a single diffusion process. Our diffusion driven extension is described by

(i) default migration threshold $Z_{lm}^1, ..., Z_{lm}^{\bar{K}}$, $l, m \in S$,
(ii) to each obligor j, assign a Wiener process W^j, $W_0^j = 0$, where the instantaneous correlation between distinct W^i, W^j is ρ_{ij}^{CM},
(iii) obligor j migrates in the first year according to W_1^j and the barrier Z_{lm}^1, given the initial rating $l \in S$. For example, default happens if $W_1^j \leq Z_{lK}^1$,
(iv) for time t_k, obligor i migrates in the period $[t_{k-1}, t_k]$ to rating $\bar{l}$, if it survives the previous t_{k-1} periods, i.e. $W_1^j > Z_{lK}^1, ..., W_{t_{k-1}}^j > Z_{lK}^{t_{k-1}}$, and if $W_{t_l}^j \leq Z_{l\bar{l}}^{t_k}$ and $W_{t_l}^j > Z_{l\bar{l}+1}^{t_k}$, where $Z_{lm}^{t_k}$ is the migration threshold as seen from today (time t_0) for a bond initially rated $l \in S$.

At time t_0, given an exposure j with initial rating $l \in S$, the migration probabilities over the next interval are given directly from the migration matrix $Q_{01} := Q(t_0, t_1)$, i.e.

$$\begin{aligned}
\pi_{lK}^{01} &= \mathbf{P}\left[\, W_1^j \leq Z_{lK}^1 \,\right] \\
\pi_{lK-1} &= \mathbf{P}\left[\, W_1^j \leq Z_{lK-1}^1 \cap W_1^j > Z_{lK}^1 \,\right] \\
&\vdots \\
\pi_{l1} &= \mathbf{P}\left[\, W_1^j \leq Z_{l1}^1 \cap W_1^j > Z_{l2}^1 \,\right] = 1 - \mathbf{P}\left[\, W_1^j > Z_{l2}^1 \,\right].
\end{aligned}$$

For subsequent periods, we can derive similarly the probability for a bond j initially rated l to migrate to rating $\bar{l}$ at time t_k, given no default before t_k. For time period two, we have

$$\begin{aligned}
\mathbf{P}\left[\, W_2^j \leq Z_{l\bar{l}}^2 \cap W_2^j > Z_{l\bar{l}+1}^2 \cap W_1^j > Z_{lK}^1 \,\right] &= \sum_{m=1}^{K-1} \pi_{lm}^{01}\pi_{m\bar{l}}^{12} \\
&= \sum_{m=1}^{K} \pi_{lm}^{01}\pi_{m\bar{l}}^{12} - \pi_{lK}^{01}\pi_{K\bar{l}}^{12}. \qquad (54)
\end{aligned}$$

Given the absorbing default state in our Markov Chain representation, we obtain the following description for the probability of a migration to rating $\bar{l}$ in time period two given survival in the first period, i.e.

$$\mathbf{P}\left[\, W_2^j \leq Z_{l\bar{l}}^2 \cap W_2^j > Z_{l\bar{l}+1}^2 \cap W_1^j > Z_{lK}^1 \,\right] = \begin{cases} \pi_{l\bar{l}}^{02} & \bar{l} \neq K \\ \pi_{l\bar{l}}^{02} - \pi_{l\bar{l}}^{01} & \bar{l} = K \end{cases} \qquad (55)$$

Similarly, we can show that for subsequent periods ($k > 2$),

$$\mathbf{P}\left[\ W_k^j \le Z_{l\bar{l}}^k \cap W_k^j > Z_{l\bar{l}+1}^k \cap W_{k-1}^j > Z_{lK}^{k-1} \cap ... \cap W_1^j > Z_{lK}^1\ \right] = \begin{cases} \pi_{l\bar{l}}^{0k} & \bar{l} \neq K \\ \pi_{l\bar{l}}^{0k} - \pi_{l\bar{l}}^{0k-1} & \bar{l} = K \end{cases} \tag{56}$$

Given these probabilities, we can derive the migration boundaries of all future periods iteratively. The boundaries for time period one, given a bond initially rated l can be derived given Q_{01} as in the original single period model, i.e.

$$\begin{aligned} \pi_{lK}^{01} &= \mathbf{P}\left[\ W_1^j \le Z_{lK}^1\ \right] = \Phi(Z_{lK}^1/\sqrt{t_1}), \\ \pi_{lK-1}^{01} &= \mathbf{P}\left[\ W_1^j \le Z_{lK-1}^1 \cap W_1^j > Z_{lK}^1\ \right] = \Phi(Z_{lK-1}^1/\sqrt{t_1}) - \Phi(Z_{lK}^1/\sqrt{t_1}) \\ &\vdots \\ \pi_{l1}^{01} &= \mathbf{P}\left[\ W_1^j \le Z_{l1}^1 \cap W_1^j > Z_{l2}^1\ \right] = 1 - \Phi(Z_{l2}^1/\sqrt{t_1}). \end{aligned}$$

To calibrate the model to the second threshold at time t_2, we start once more with the default boundary, that is

$$\begin{aligned} &\mathbf{P}\left[\ W_2^j \le Z_{lK}^2 \cap W_1^j > Z_{lK}^1\ \right] \\ &= \mathbf{P}\left[\ W_2^j \le Z_{lK}^2\ \right] - \mathbf{P}\left[\ W_2^j \le Z_{lK}^2 \cap W_1^j \le Z_{lK}^1\ \right] \\ &= \pi_{lK}^{02} - \pi_{lK}^{01}. \end{aligned} \tag{57}$$

Since W^j is a Wiener process, we know the standard deviation of W_t^j is $\sqrt{t}$, and that for $s < t$, the correlation between W_s^j and W_t^j is $\sqrt{\frac{s}{t}}$. Thus, given Z_{lK}^1, we can find Z_{lK}^2 such that

$$\Phi\left(\frac{Z_{lK}^2}{\sqrt{t_2}}\right) - \Phi_2\left(\frac{Z_{lK}^1}{\sqrt{t_1}}, \frac{Z_{lK}^2}{\sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}}\right) = \pi_{lK}^{02} - \pi_{lK}^{01},$$

where $\Phi_2(\cdot)$ denotes the distribution function of a bivariate standard normal variable. Z_{lK}^2 can be found numerically, e.g. by applying the Newton-Raphson method. Given Z_{lm}^1, $l, m \in S$, and Z_{lK}^2, we can solve for Z_{lK-1}^2. Generally, for $\bar{l} \neq K$, we get

$$\begin{aligned} \pi_{l\bar{l}}^{02} &= \mathbf{P}\left[\ W_2^j \le Z_{l\bar{l}}^2 \cap W_2^j > Z_{l\bar{l}+1}^2 \cap W_1^j > Z_{lK}^1\ \right] \\ &= \mathbf{P}\left[\ W_2^j \le Z_{l\bar{l}}^2\ \right] - \mathbf{P}\left[\ W_2^j \le Z_{l\bar{l}+1}^2\ \right] \\ &- \mathbf{P}\left[\ W_2^j \le Z_{l\bar{l}}^2 \cap W_1^j \le Z_K^1\ \right] + \mathbf{P}\left[\ W_2^j \le Z_{l\bar{l}+1}^2 \cap W_1^j \le Z_K^1\ \right]. \end{aligned} \tag{58}$$

Therefore, given Z_{lm}^1, $m \in S$, and $Z_{lK}^2, ..., Z_{l\bar{l}+1}^2$, we can solve

$$\begin{aligned} \pi_{l\bar{l}}^{02} &= \Phi\left(\frac{Z_{l\bar{l}}^2}{\sqrt{t_2}}\right) - \Phi\left(\frac{Z_{l\bar{l}+1}^2}{\sqrt{t_2}}\right) \\ &- \Phi_2\left(\frac{Z_{lK}^1}{\sqrt{t_1}}, \frac{Z_{l\bar{l}}^2}{\sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}}\right) + \Phi_2\left(\frac{Z_{lK}^1}{\sqrt{t_1}}, \frac{Z_{l\bar{l}+1}^2}{\sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}}\right), \end{aligned} \tag{59}$$

for $Z_{l\bar{l}}^2$ numerically.

For subsequent periods $t_k > t_2$, we can calibrate the model by solving

$$\mathbf{P}\left[\ W_k^j \le Z_{lK}^k \cap W_{k-1}^j > Z_{lK}^{k-1} \cap ... \cap W_1^j > Z_{lK}^1\ \right] = \pi_{lK}^{0k} - \pi_{lK}^{0k-1}$$

for Z^k_{lK}, given $Z^{k-1}_{lm}, ..., Z^1_{lm}$, $m \in S$, where we utilize the properties of the Wiener processes W^j to compute the probability on the left hand side. Similarly, for $\bar{l} \neq K$, given in addition the thresholds $Z^k_{l\bar{l}+1}, ..., Z^k_{lK}$, we can calibrate $Z^k_{l\bar{l}}$ by solving

$$\mathbf{P}\left[\, W^j_k \leq Z^k_{l\bar{l}} \cap W^j_k > Z^k_{l\bar{l}+1} \cap W^j_{k-1} > Z^{k-1}_{lK} \cap ... \cap W^j_1 > Z^1_{lK} \,\right] = \pi^{0k}_{l\bar{l}}.$$

5.1.5. *Implementation details.* Details on the actual data, the stochastic processes and estimation can be found in Jobst and Zenios (**?**). A typical transition matrix regularly published by rating agencies is employed. The extended Vasicek model of Hull and White (1990) is estimated for the short term interest rate and credit spread processes (see also Kijima and Muromachi (2000)). Given this dynamics, closed form solutions for simple security prices can be obtained at the nodes in the tree. The simple discrete CreditMetrics extension is employed throughout this case study.

5.2. **Ex ante decision making: An illustrative ALM model.** In this section we implement the generic model for asset and liability management and investigate the behaviour of the model. We have chosen a planning horizon of $T = 18$ month starting on the June 30, 2001. We also consider 16 (hypothetical) coupon bonds with annual coupon payments and a notional of 100. We have considered 4 investment grade ratings and 4 bonds in each of these rating classes leading to altogether 16 bonds. Table 1 contains summary details of these bonds. With respect to the time horizon, we

Rating	Maturity	Coupon	Rating	Maturity	Coupon
Aaa	30 Jun 2002	5.0%	*A*	30 Jun 2002	7.0%
	31 Dec 2002	5.0%		31 Dec 2002	7.0%
	30 Jun 2004	5.0%		30 Jun 2004	7.0%
	31 Dec 2006	5.0%		31 Dec 2006	7.0%
Aa	30 Jun 2002	6.0%	*Baa*	30 Jun 2002	8.0%
	31 Dec 2002	6.0%		31 Dec 2002	8.0%
	30 Jun 2004	6.0%		30 Jun 2004	8.0%
	31 Dec 2006	6.0%		31 Dec 2006	8.0%

TABLE 1. Bond details used throughout the asset and liability case studies.

set $\mathcal{T}_B = \mathcal{T}_L = \{6m, 9m, 12m, 18m\}$ and choose timesteps of 3 months, hence $\mathcal{T} = \{0, 3m, 6m, 9m, 12m, 15m, 18m\}$. In the liability stream we set minimum limits to the cash account at times $t \in \mathcal{T}_L$, in particular we require to hold $(5, 10, 20, 40)$ at times $t \in \mathcal{T}_L$, given an initial budget of $c_0 = 100$. We also assume a short cash rate $r^{ca}(t,T)$ to be given by the short interest rate, i.e. $r^{ca}(t,T) = r \cdot (T-t)$, and we don't consider transaction costs, i.e. $tc^b_i = tc^s_i = 0$, for all $i \in U$. We have chosen a total of 1000 scenarios, with the following number of nodes leaving a parent node at each time step $t \in \mathcal{T}$. At time $t_0 = 0$ we have 10 branches, at t_1 we have 1 branch, at t_2 we have 10 branches, at t_3 we have 1 branch, at t_4 we have 10 and finally at t_5 we have 1 branch leaving each parent node at the previous time step.[11] In the scenario generation, we assume a correlation of 20% between the latent random variables. Furthermore, we choose

[11]When 10 branches are considered, we sample 2 economic and 5 credit scenarios.

a benchmark return $r^b = 4\%$, $\alpha_L = \alpha_B = 99\%$, $L_{t_L}^{CVaR} = 0$, $t_L \in \mathcal{T}_L$, and $B_{t_B}^{CVaR} = 0.5\%$, $t_B \in \mathcal{T}_B$. This choice of scenarios and parameters is for illustrative purposes, aiming to show the potential (effectiveness) of the developed ALM model, and giving insights into multi-stage versus anticipative models. However, in order to employ the model in a real world setting, a much larger number of scenarios is required to obtain stable results, leading to very large models which are challenging from a computational point of view (curse of dimensionality).

5.2.1. *Anticipative models.* The following four examples correspond to the anticipative (or no-recourse) model introduced earlier.

Example 1: Anticipative, unconstrained

In the first example, we do not consider any risk constraints and simply maximize the expected value of terminal wealth at the planning horizon, that is the dirty price of the portfolio plus the value of the cash account at $T = 18m$. The model invests entirely in the Baa rated, longest maturity (December 31, 2006) bond, leading to an expected value of 108.69. The simulated value of the portfolio and the value of the cash account compared to the desired cash amounts (liability stream) are shown in Figure 10. We observe that

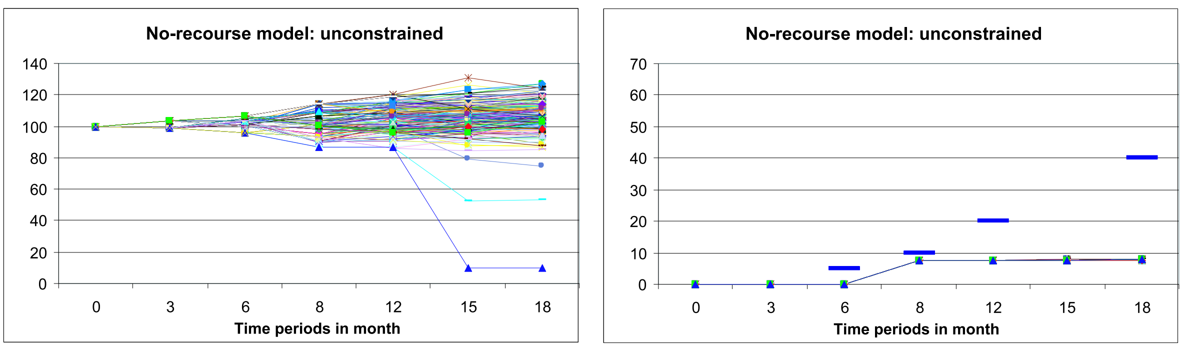


FIGURE 10. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

the optimal portfolio is hit hard under some scenarios after 12 month, leading to severe losses in the portfolio. We also observe that the liability stream (amount of cash required in the cash account) is not covered. In the following, we gradually add constraints in order to meet the targets.

Example 2: Anticipative, liability constrained

In this example, we add CVaR constraints on covering the liabilities as times $t \in \mathcal{T}_L$. To hold the liability constraints the optimal portfolio shifts some of the long maturity Baa bonds to shorter maturity Baa bonds and a small initial cash holding. The asset and maturity allocation is displayed in Figure 11. This portfolio has an expected value of 108.25. The corresponding portfolio value and cash account evolution are displayed in Figure 12. We observe that all liability (cash account) requirements are met. The first requirement of 5 at time $t_2 = 6m$ is mainly covered by the initial cash holding, the t_3

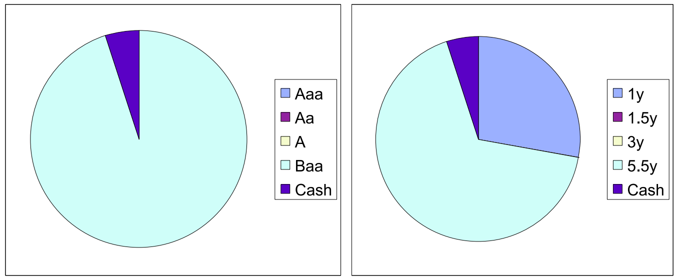


FIGURE 11. Asset class and maturity allocation in the no-recourse (anticipative) model with liability constraints.

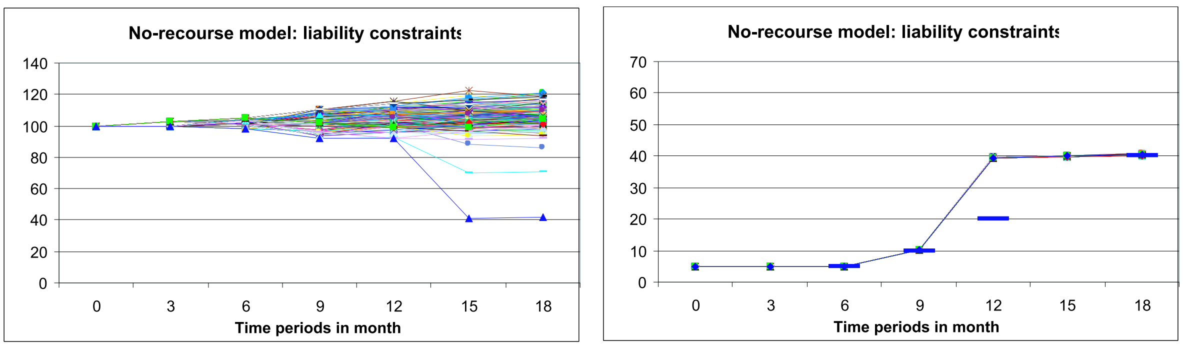


FIGURE 12. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

requirement of 10 is covered by coupon payments and the remaining two cash requirements at months 12 and 18 are covered by the cashflow received at maturity (and subsequent re-investment in cash) of the 1 year *Baa* bond. It is somewhat surprising that the model invests in the short *Baa* rated bond, leading to a large cash holding after one year. We also see that the portfolio is still hit quite severely under some scenarios, with the value deteriorating to just above 40. The given portfolio, however, satisfied all constraints.

Example 3: Anticipative, wealth constrained
For the third example, we add CVaR constraints to the portfolio value at times $t \in \mathcal{T}_B$ to the model of Example 1. Hence, we control the wealth growth of the portfolio without restrictions on liabilities. The expected value of the optimal portfolio is 107.68, and in order to satisfy the CVaR constraints, parts of the investment in *Baa* rated bonds is shifted to slightly less risky *A* rated bonds. We can also observe that none of the initial capital is invested in cash (see Figure 13). If we take a closer look at the portfolio evolution (Figure 14), we observe that the portfolio wealth shows less volatility. In particular extreme loss scenarios are eliminated as a result of the CVaR restrictions. On the other hand, the cash account does not capture the desired level for the liability payments at times 6m and 9m, as shown in the right panel. This is a result of not considering any constraints on liabilities. The coverage of the 12 and 18m cash levels are a result of the high expected value of the 1 year bond, leading to principal repayment at maturity $t_4 = 12m$.

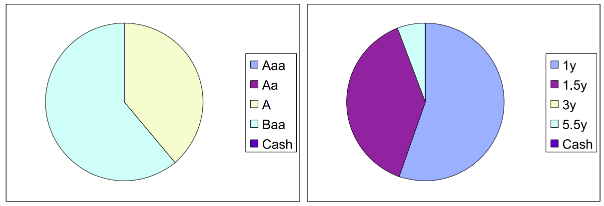


FIGURE 13. Asset class and maturity allocation.

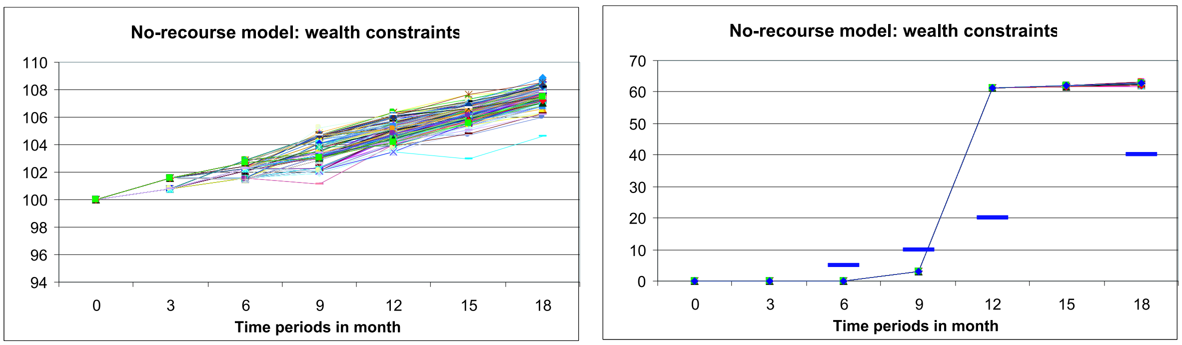


FIGURE 14. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

Example 4: Anticipative, liability and wealth constrained
In this example, we add CVaR constraints on portfolio wealth and liability targets to the model of example 1. The optimal portfolio invests 37.72% in A rated 1.5 year maturity bonds, 49.77% in Baa rated 1 year bonds and 6.89% in cash. Figure 15 shows the portfolio value and cash account evolution. This results in a expected portfolio value of 105.58 at

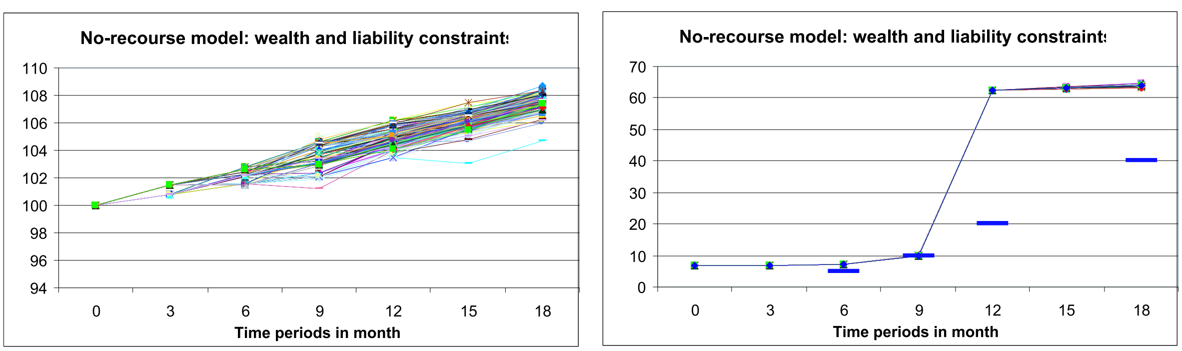


FIGURE 15. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

the end of the planning horizon. We also observe that extreme loss situations are omitted and that all liability requirements (desired cash account levels) are covered in all periods. This is due to a significant initial cash investment, coupon payments and bond maturity.

Overall, these four examples highlight the behaviour of the model and its capability of controlling portfolio value as well as liability risk. In particular in the presence of credit risk, these CVaR constraints on liabilities are important and essential for risk control.

5.2.2. *Multistage recourse models.* The multistage recourse model is more sophisticated than the simple anticipative model. In our investigation we repeat the above analysis in a more general six stage implementation of the multistage stochastic recourse model. Hence, these models allow for corrective decisions (recourse decisions). The aim is to highlight the application of CVaR constraints in a multistage framework and to gain some insight into multistage stochastic programs. In this six stage model, we allow for rebalancing decisions at all times $t \in \mathcal{T}$.

Example 5: 6-stage model, unconstrained
Example 5 assumes like Example 1 no risk constraints on either liabilities or portfolio wealth. The portfolio wealth is 118.72 and considerably higher than in the previous example. The model invests all the capital in the *Baa* rated 1 year bond, with scenario dependent recourse decisions thereafter. Hence, given the current scenario set, the 1 year *Baa* bond has the highest expected value over the first three month period. Figure 16 (left panel) shows that the portfolio value almost never drops below 95, however in some scenarios, the portfolio is hit and the value drops to just above 85. The cash account

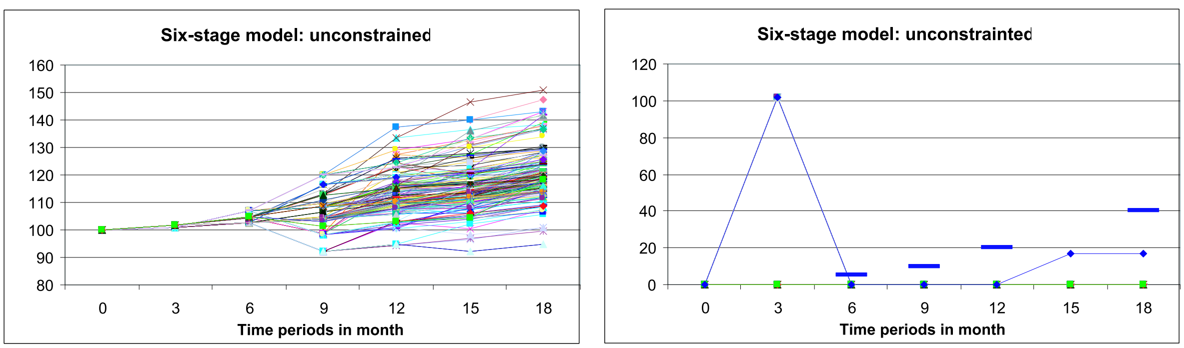


FIGURE 16. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

statistic is shown in the right panel of Figure 16. The desired cashflow stream is not covered, and under some scenarios we observe that the model chooses to sell all bonds (at month 3) and invest it in cash, before re-investing it in bonds thereafter.

Example 6: 6-stage model, liability constrained
Example 6 generates the results for the six-stage liability constrained model. The capital is initially invested entirely in the *Baa*, 1 year bond, leading to an expected portfolio wealth of 117.19. We observe in Figure 17 that the liability stream is covered under all scenarios.

Example 7: 6-stage model, wealth constrained
In this example we consider wealth constraints, only. Initially, the optimal solution is

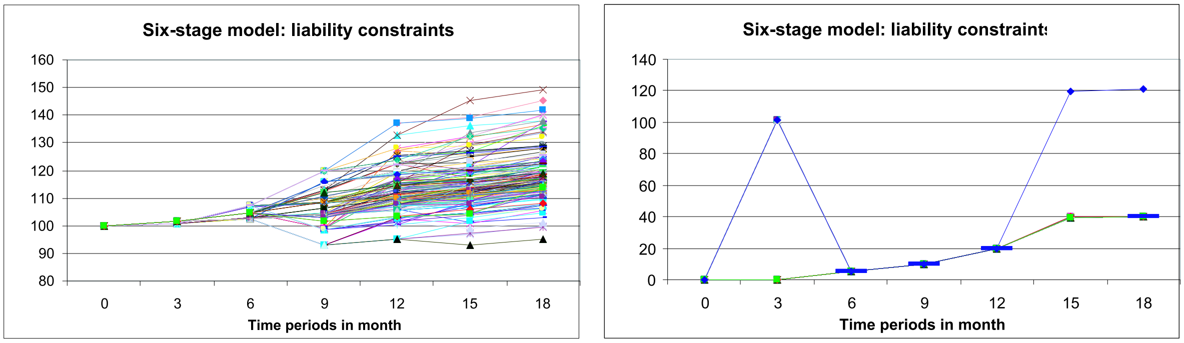


FIGURE 17. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

to invest the entire capital in the risky *Baa*, 1 year bond, leading to an expected value of 118.70 at the end of the planning horizon. Figure 18 shows the portfolio and cash account value under the scenarios, prevailing a portfolio that is limited in the downside risk, however without covering the desired cash account stream.

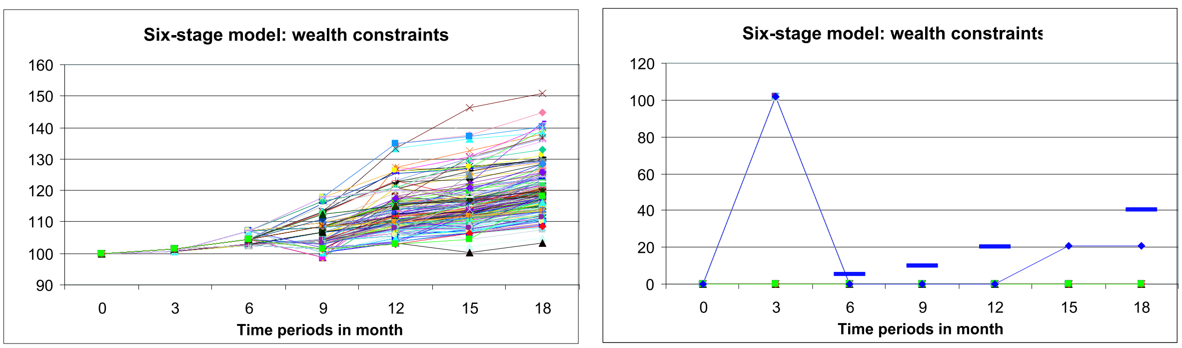
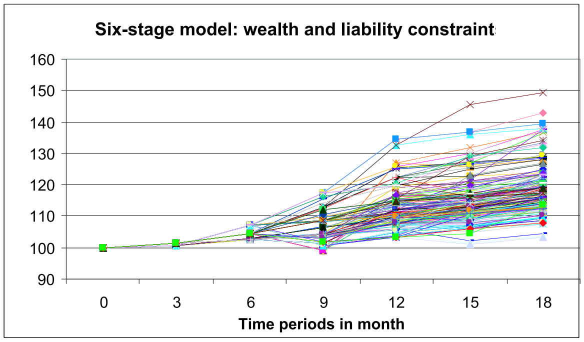


FIGURE 18. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

Example 8: 6-stage model, liability and wealth constrained
This final example shows the results for the six-stage model when both, portfolio value and liability constraints are added. The initial investment does not change to the previous models, and the expected portfolio wealth is 117.17. Figure 19 shows that all cashflows are covered and the portfolio value prevails limited downside risk.

5.2.3. *Remarks and Summary.*

(i) We have studied a flexible dynamic multistage stochastic recourse model in which anticipativity and risk constraints are progressively introduced. We have shown the ability of these multistage models to constrain the two types of risk using a small set of (in-sample) scenarios. A distinguishing feature of the model is the use



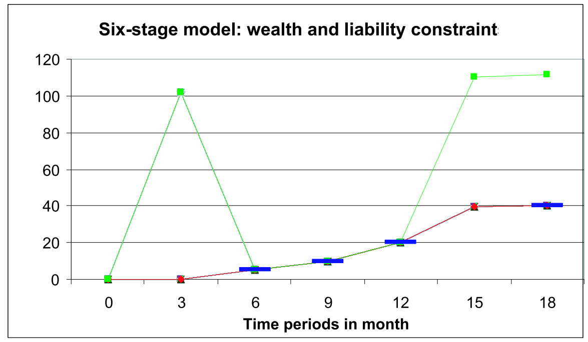


FIGURE 19. Left Panel: Portfolio value under different scenarios. Right Panel: Cash account under different scenarios (marked lines) compared to the desired cash amounts (horizontal bars) at times $\mathcal{T}_L$.

of CVaR (risk constraints) on liabilities in addition to the constraints on portfolio wealth. These are required in the presence of credit risk as the investor is faced with the chance of default, and the corresponding losses in coupon payments (and notional).

(ii) In order to highlight the effect of recourse actions, we summarize in Table 2 the observed results. We also add results for a 3-stage implementation, where recourse decisions are possible at times t_2 and t_4. Overall we observe an increasing objective

Example	Constraints	$\mathbf{E}[\ W_T\ ]$	Init Port.	Comment
1	no	108.69	Baa	severe loss scenarios cashflows (CFs) not covered
2	L	108.25	cash + Baa	significant losses still present, CFs covered
3	W	107.68	A + Baa	significant loss scenarios eliminated, CFs not covered
4	W+L	105.58	A + Baa + cash	significant loss scenarios eliminated, CFs covered
5	no	118.72	Baa	no severe loss scenarios, CFs not covered
6	L	117.19	Baa	no significant losses, CFs covered
7	W	118.70	Baa	no significant loss scenarios , CFs not covered
8	W+L	117.70	Baa	no significant loss scenarios, CFs covered
3-stage	no	112.10	Baa	no severe loss scenarios, CFs not covered
3-stage	L	111.10	Baa	no significant losses, CFs covered
3-stage	W	111.14	A + Baa	no significant loss scenarios , CFs not covered
3-stage	W+L	110.25	A + Baa	no significant loss scenarios, CFs covered

TABLE 2. Summary results of no-recourse and recourse models.

function values with increasing number of stages. This indicates that flexibility pays, and stems from higher yielding investments in subperiods where the initial portfolio choice performs poorly. We also see that the rebalancing decisions eliminate portfolios with initial holdings in cash to cover early liabilities. These results, however, were obtained without considering transaction costs and for a very small illustrative scenario set.

(iii) These simple examples highlight also the importance of scenario generation. In the multistage context, the model goes for an investment strategy that heavily depends on the small scenario set under consideration. These high return, low risk strategies over different subperiods may not exist under larger sample sets. Such experiments need to be conducted and require significantly higher computational costs.

5.3. **Ex post decision analysis: A bond index tracking experiment.** Whereas the last section investigated the behaviour of the multistage asset and liability model conceptually, we present a case study based on real market data in this section. We consider the problem of index tracking at an asset allocation level and compare a no-recourse formulation with a two-stage implementation of the model.

We develop a model to determine the allocation in certain asset classes in order to track the Merrill Lynch Eurodollar index. We represent these asset classes by synthetic bonds, approximated from the index data. In particular, we consider the four investment grade ratings and four different maturity buckets; thus for each rating, we aggregate bonds in four groups with 1–3 years, 3–5 years, 5–7 years and 7+ years of maturity. For example, on January 31, 1999 we obtain the following asset class details which correspond to the market weighted average of all bonds belonging to each asset class (Table 3). Instead of

Rat.	Mat.	Mat.Bucket	CPN	Dur.	Yield	OAS	Hold.	Price	HPR
Aaa	12/4/00	1	6.24	1.69	5.29	58.44	11.85	102.62	0.00
	11/16/02	2	6.05	3.34	5.21	52.45	19.16	104.08	-0.01
	2/24/05	3	6.56	4.91	5.30	53.04	6.13	109.28	-0.02
	2/7/08	4	5.99	6.90	5.36	40.38	6.48	107.34	-0.03
Aa	12/14/00	1	6.47	1.71	5.45	74.17	13.01	102.61	0.00
	9/12/02	2	6.48	3.14	5.45	73.21	19.50	105.83	-0.01
	12/23/04	3	6.59	4.80	5.51	73.60	5.18	106.04	-0.02
	11/27/07	4	6.48	6.62	5.67	71.76	6.72	106.73	-0.03
A	11/22/00	1	6.77	1.65	5.66	94.13	3.11	103.18	0.00
	10/30/02	2	6.50	3.25	5.70	97.95	5.39	104.26	-0.01
	9/12/05	3	6.84	5.24	6.16	136.88	0.96	106.26	-0.02
	9/19/07	4	6.91	6.31	7.86	298.18	0.37	96.67	0.01
Baa	3/3/01	1	7.68	1.94	7.89	322.80	0.34	102.75	0.01
	1/11/03	2	6.82	3.36	7.45	275.97	1.24	98.25	-0.01
	3/1/06	3	8.44	5.09	10.16	533.19	0.15	94.96	0.01
	10/4/08	4	7.48	6.68	7.52	255.31	0.42	102.14	-0.02

TABLE 3. Asset class details (synthetic bonds) on January 31, 1999. The holding period return (HPR) corresponds to the return over the next one month.

using the quoted prices we use the exact prices implied from the current term structures and assume that these asset classes or synthetic bonds are priced in the market at the fair value. In the backtesting experiment we will however use the real, observed holding period return for each of these synthetic bonds to assess the model performance. Similarly, we can obtain such statistics for the index one month later, on February 28, 1999. Of course the details may differ significantly from the previous month, which also implies different prices. Hence at the end of the month the price for an asset class differs from the price at the beginning of the next month. We adjust the next month holding after pricing the synthetic bonds such that the total market value in each asset class at the beginning of the new month is equal to the end of the month market value from the previous month.

Figure 20 plots the quoted option adjusted spreads (OAS) for the shortest and longest maturity buckets in every rating class over a period of 2.5 years. We can observe a

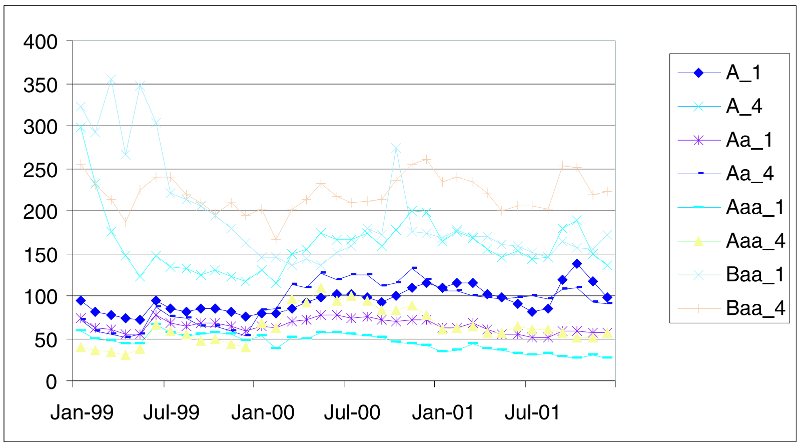


FIGURE 20. OAS of eight asset classes from January 31, 1999 to December 31, 2001.

period of spread widening (especially for all long maturity bonds) between December 1999 and November/December 2000, and a long period of spread tightening afterwards until July/August 2001.

5.3.1. *The CVaR-Index Tracking Model.* Portfolio value or loss distributions of a given portfolio in the presence of credit risk exhibit fat tails and non-normality. When we are tracking a bond index, we are less interested in the absolute risk, but more concerned about the relative risk, that is, losses with respect to the performance of a bond index, which is a random variable itself. The following Figure 21 plots the loss distribution (at a risk horizon of one year) for an investment grade portfolio when we overweight *Baa* rated bonds. We observe that the downside risk is significantly higher, compared to the upside potential (indicating a maximum loss of 10%). We therefore need to develop optimisation models that account for the tail events in the tracking context. We have chosen once more the CVaR methodology. Instead of defining the losses with respect to the initial portfolio value or expected value or with respect to a pre-specified (deterministic) benchmark (as in the previous section) we define the losses in this section with respect to the random index, that is

$$ILoss_t^{\omega} := I_t^{\omega} - W_t^{\omega}, \quad t \in \mathcal{T}, \omega \in \Omega, \tag{60}$$

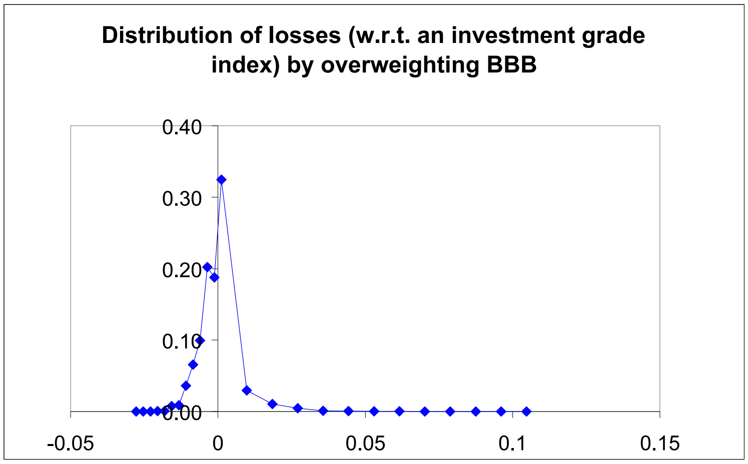


FIGURE 21. Distribution of losses with respect to the index.

or in relative terms as

$$RILoss_t^\omega := \frac{I_t^\omega - W_t^\omega}{I_t^\omega}, \quad t \in \mathcal{T}, \omega \in \Omega, \tag{61}$$

where I_t^ω denotes the index value and W_t^ω denotes the value of the tracking portfolio (including reinvested cash payments) at time t under scenario ω. In the following case study, we only consider one set of constraints at the end of the horzion $T = 6m$. Given this loss definition, and the definitions of section 5.2, we derive the tracking models from the ALM models introduced in the previous section.

We can derive the anticipative CVaR tracking model from the ALM model of the previous section by setting $Liab_{t_k}$ equal to zero and deleting the corresponding liability risk constraints (22, 23 and 26) in the Anticipative ALM model. We also have to adjust constraints (24, 25 and 27) and replace $y_{L_{t_L}}$ by y_{IT} where

$$y_{IT} \geq \frac{I_T^\omega - W_T^\omega}{I_T^\omega} - \zeta_T^I.$$

These constraints model the CVaR of the portfolio with respect to the index at the end of the horizon T.

Similar adjustment can be made to the MS-recourse model, equations (29) to (44), resulting in the multi-stage recourse index tracking (asset allocation) model.

5.3.2. *Empirical investigation.* In the following we investigate the tracking performance of the Merrill Lynch Eurodollar index when we implement the suggested optimal portfolio suggested by the anticipative and the two-stage model ($\mathcal{T} = \{0, 3m, 6m\}$). Of course, in the two-stage model, we only implement the first stage decision. We also investigate the tracking of a government bond index with US Treasury securities, only. Finally, we extend the government bond investment universe by corporate bonds and conduct the experiments once again.

Firstly, we present backtesting results and highlight some interesting results. In the following we consider transaction costs which are rating class dependent, that is 5 bp for Aaa, 10 bp for Aa, 20 bp for A and 40 bp for Baa.[12] We implement the CVaR constraint at a 95% level and allow for 100bp CVaR ($I_T^{CVaR} = 1\%$), that is, the expected losses with respect to the index below the 95%−VaR have to be less than one percent. The anticipative model allows for only one decision at time $t_0 = 0$, whereas the two-stage model also allows for portfolio rebalancing at $t_1 = 3m$. The backtesting is conducted by implementing the suggested portfolio and re-running the simulation/optimisation models after one month.

Scenario Sets We generate all scenarios with a 20% correlation between the latent random variables triggering default. We simulate 12000 scenarios over a 6 month period. The event tree has 200 branches in the root node (10 economic and 20 credit scenarios), and 60 branches (6 economic and 10 credit scenarios) at time step 3m at each parent node.[13]

Case study 1: Corporate bond index tracking

Example 1: Anticipative AA model
In this example, we study the tracking of the Eurodollar index by investing in the asset classes according to the optimal solution of the anticipative model. Figure 22 plots the portfolio value versus the index value over the backtesting period of January 1999 to December 2001. We also plot the implemented or optimal asset allocation by rating class,

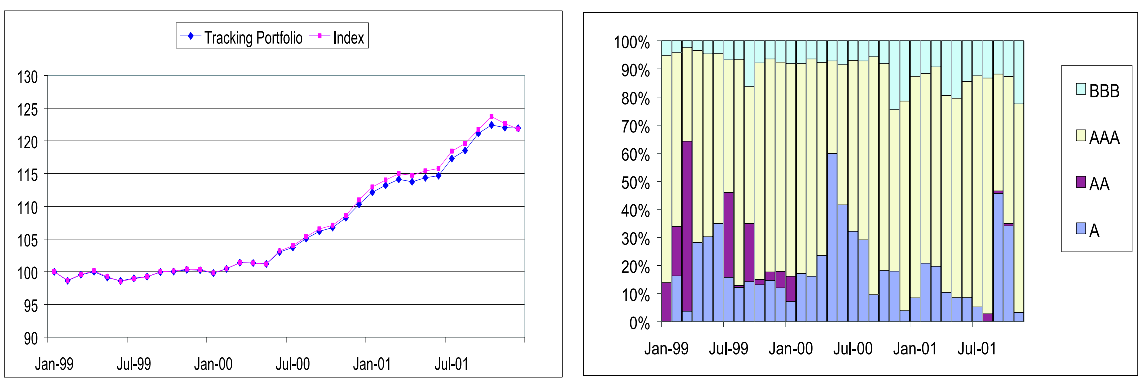


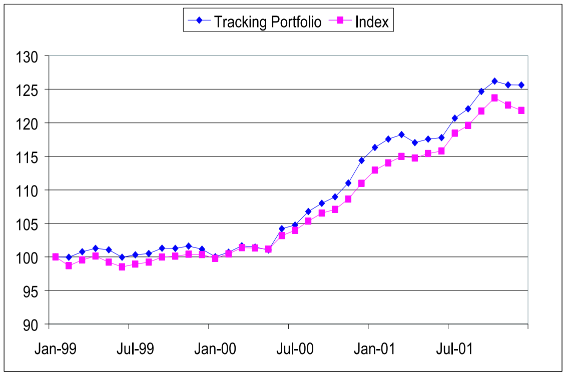
FIGURE 22. Backtesting results anticipative model: Portfolio value versus the index value (Eurodollar). Investment in corporate bonds is possible, only.

as suggested by the model (right panel).

Overall we observe a good tracking performance with a large holding in high quality, Aaa rated bonds. Surprisingly, apart from the first few months, hardly any capital is invested in Aa rated bonds.

[12]This transaction costs assumption stems from discussions with market participants.
[13]Solving the two-stage model with 12000 scenarios on a Pentium 4 2GHz processor, 512 MB RAM, using FortMP (2002), takes approximately 20 min to 1 hour.

Example 2: Two-stage recourse AA model
We now repeat the previous example, however we allow for recourse decisions at month 3 in the model. We plot the portfolio value evolution compared to the index, and the first stage investment decision in Figure 23.



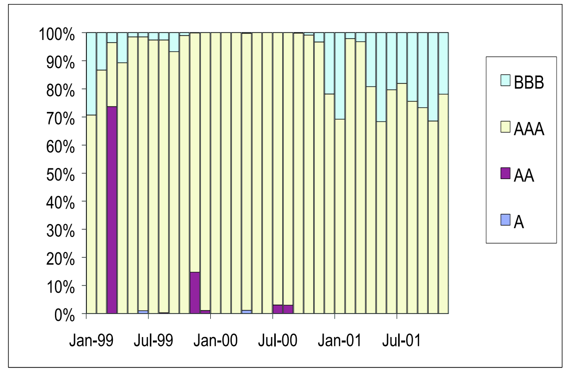


FIGURE 23. Backtesting results two-stage model: Portfolio value versus the index value (Eurodollar). Investment in corporate bonds is possible, only. Backtesting results: Asset allocation according to the optimal first stage solution of the two-stage stochastic program.

We observe a significantly larger final portfolio value compared to the one stage model. Especially interesting is to take a closer look at the implemented optimal portfolio. We observe overall a significant holding in *Aaa* rated bonds. In particular, during the period of spread widening (December 1999 to November 2000), the model shifts the investment out of *Baa* rated bonds to *Aaa* assets, which was the right decision as *Baa* spreads widened more compared to *Aaa* spreads. After that period, we can observe a shift of a significant part of the portfolio value to *Baa* rated securities (going long credit), during the period of spread tightening. Overall, the model does what a portfolio manager should have done. Knowing that the model can correct initial decisions in subsequent periods, leads to portfolios that differ more significantly from the index structure, however as explained above, this does not always imply more risky portfolios. In both examples, we observe as good tracking performance and especially in the two-stage implementation we observe a good reaction to market developments. Overall this supports the choice of our simulation and optimisation paradigm.

Case study 2: Tracking a government bond index

Example 3: Anticipative AA model
In this example, we aggregate the Merrill Lynch Government Bond index into six different maturity buckets, with 1–3, 3–5, 5–7, 7–9, 9–21, and 21–30 years of maturity. We implement the CVaR tracking model with no recourse decisions first. Figure 24 plots the portfolio value and asset allocation through time.

Example 4: Two-stage recourse AA model
We conduct the same experiment as in example 3 by applying the two-stage model. Figure 25 plots the portfolio value and first stage asset allocation decisions through time.

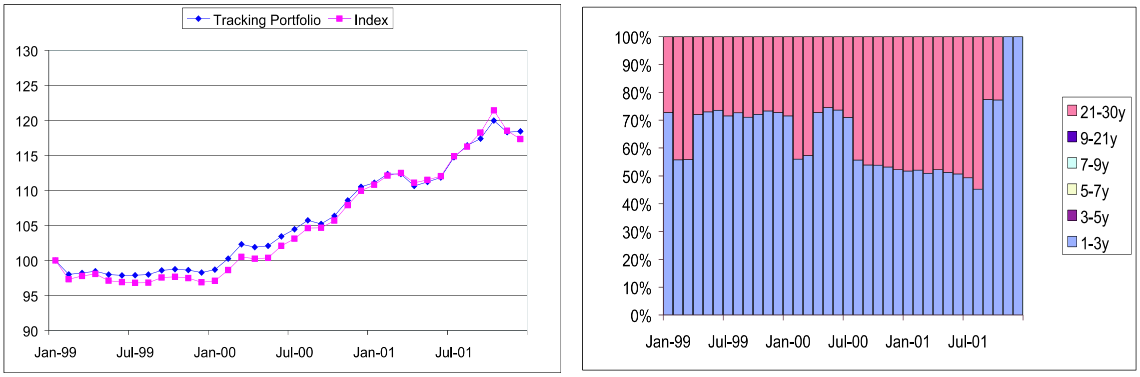


FIGURE 24. Backtesting results anticipative model: Portfolio value versus the index value (Government Index). Investment in treasury bonds is possible, only. Backtesting results: Asset allocation according to the optimal solution of the anticipative model.

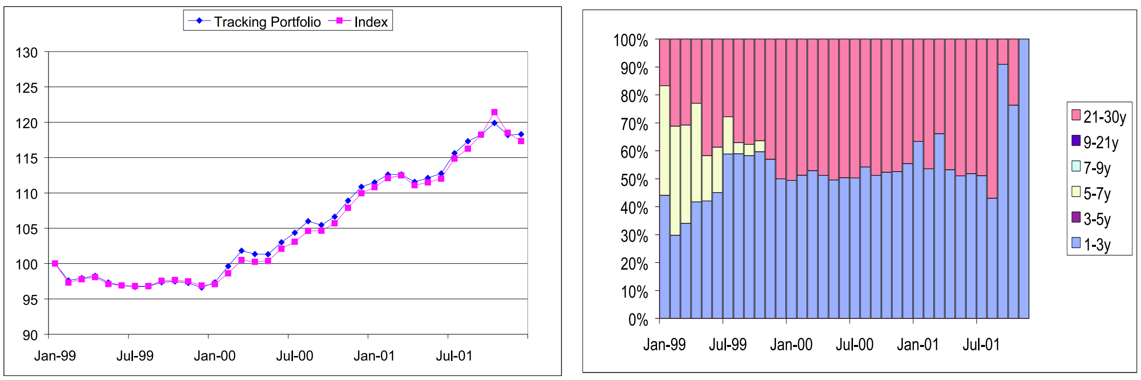


FIGURE 25. Backtesting results two-stage model: Portfolio value versus the index value (Government Index). Investment in treasury bonds is possible, only. Backtesting results: Asset allocation according to the optimal first stage solution of the two-stage stochastic program.

The performance of the model is very similar to the anticipative model, which is a result of similar asset allocation decisions. Only during the first few months, the decisions are slightly different, and indeed, the anticipative model performs better ex-post.

Overall we observe that the models invest mainly in long and short bonds throughout the period, leading to a good tracking performance, however without generating significant extra value.

Example 5: Anticipative AA model
In this example, we re-run the anticipative model of example 3 with an enlarged portfolio universe. We add the 16 synthetic corporate bonds representing investment grade assets. Figures 26 plots the portfolio performance and asset allocation throughout the backtesting period.

Overall, we observe once more a good tracking performance, with a very significant holding of 70%-80% in high quality (mainly *Aaa*) corporate bonds. This is somewhat surprising,

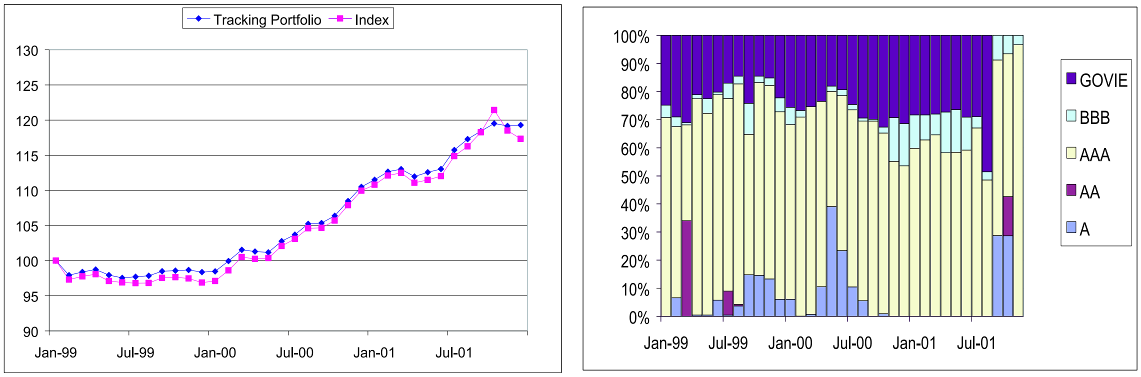


FIGURE 26. Backtesting results anticipative model: Portfolio value versus the index value (Government Index). Investment in treasury and corporate bonds is possible. Backtesting results: Asset allocation according to the optimal solution of the anticipative model.

however consistent with market practice. Also, over the short risk horizon and given the nature of the transition matrix employed, default events are extremely rare for the high rating class exposures. Therefore, all constraints can still be satisified. We can also observe a complete shift to corporate products at the end of the backtesting period, which makes sense due to the extreme tightening in corporate spreads.

Example 6: Two-stage recourse AA model
In this example, we re-run the previous experiment by applying the two-stage model, that is, allowing for a corrective recourse decision after 3 months. Figures 27 reports the portfolio performance and asset allocation throughout the backtesting period, which shows a significantly improved performance with respect to final portfolio wealth.

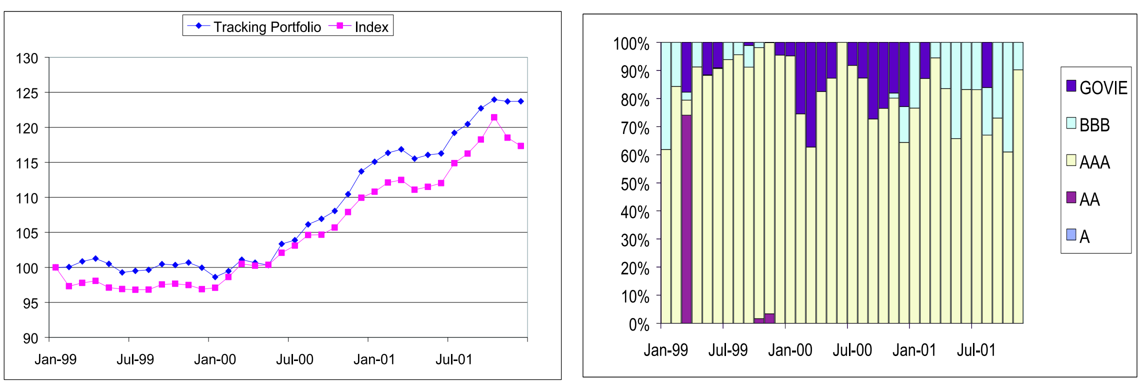


FIGURE 27. Backtesting results two-stage model: Portfolio value versus the index value (Government Index). Investment in treasury and corporate bonds is possible. Backtesting results: Asset allocation according to the optimal first-stage solution of the two-stage model.

Particularly interesting is once more that the flexibility of recourse decisions leads to a portfolio that differs more significantly from the index structure, while satisfying all risk constraints. This flexibility leads to decisions that respond to market developments by rebalancing (or recourse) actions. We observe that during the spread widening period (beginning of 1999 to the end of 2000), the model suggests significant investment in government bonds, whereas in the spread tightening periods (before and after the widening), the model invests almost entirely in corporate products. Once more, the model reacts to the market developments in an intuitive way.

5.3.3. *Stability of results.* In order to get some insight into the stability of these results, we repeated ten times Experiment 1 with alternative scenario sets. These scenario sets differ only in the initial random seeds, hence keeping the input data and branching structure unchanged. We observe an average monthly return of 61bp with an average standard deviation of 85bp. This compares favourably to the corresponding index statistic of 57bp and 90bp. A closer examination of the individual runs reveal that in eight out of ten runs the performance is extremely close. The underperformance under the remaining two scenario sets highlights and reveals the presence of some sampling error that needs to be considered and investigated further. Preliminary results along these lines are given in Jobst (2002) and go beyond the scope of this paper.

6. Conclusion and further Research

We have considered an important problem of financial planning involving fixed income assets which are subject to credit risk. In this paper we have introduced a number of innovations which we list below:

(i) We have integrated models of credit risk and market risk which makes our generation of time varying prices of the *fixed income* assets relatively more accurate compared to other approaches which do not jointly take into account these multiple sources of risk.
(ii) Within the decision making perspective we extend our earlier work and those of other researchers based on a simulation only paradigm of prices. We use a multistage decision making framework which fully takes into consideration (a) dynamical behaviour of the assets and (b) recourse decisions which respond appropriately to *the timing of default*.
(iii) Finally, our framework reveals one of the most powerful aspects of stochastic optimisation, that is, bringing together the optimum decision models with the descriptive approach of simulation models. This topic is discussed in section 1.4 and their application in the case studies is described in section 5 underline the value of this approach.
(iv) Future research needs to address the interaction and incorporation of modern default risk models with the optimisation paradigm as well as further investigation of the stability of the optimisation/simulation results; hence, methods of limiting the inherent model risk need further development.

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