INTEGRAL ZEROES OF KRAWTCHOUK POLYNOMIALS

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Abstract

Krawtchouk polynomials appear in many various areas of mathematics starting from discrete mathematics (e.g., in coding theory), association schemes, and in the theory of graph representations. The existence/non-existence of integral zeroes of these polynomials is crucial for the existence/non-existence of combinatorial structures in the Hamming association schemes. The integer zeroes of Krawtchouk polynomials for $k = 4, 5, 6$ and 7 have been found using some very recent results on solvability of polynomial diophantine equations. Our aim is two-fold: Firstly, to verify these results using extensive computer calculations. This requires the solution of some of Pell's equations and the use of the symbolic mathematics software MATHEMATICA. Secondly, we numerically investigate a conjecture dealing with the integer zeroes of the Krawtchouk polynomials $P^{m^2}_{\binom{m}{2}}(x)$ and provide confirmation of the conjecture using a combination of approaches up to $m \leq 1000$, i.e., for the polynomials up to degree of about half a million.

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1. Introduction

1.1. **Historical Background.**

Upon checking the free encyclopedia Wikipedia one can find various transliterations of the name Krawtchouk, originating from the Ukrainian language Кравчук, also written as Kravchuk. The Krawtchouk polynomials are a special case of the Meixner polynomials of the first kind [1]. Meixner polynomials, or the *discrete Laguerre* polynomials, are a family of discrete orthogonal polynomials, which are given in terms of binomial coefficients and the rising symbol of Pochhammer, as

$$M_k(x,\beta,\gamma) = \sum_{j=0}^{k} \binom{x}{j}\binom{n}{j}(-1)^j j!\gamma^{-j}(x-\beta)_{(k-j)}. \tag{1}$$

In fact, Krawtchouk polynomials were named after Mikhail Krawtchouk who first designed them in his most famous work published in 1929, "Sur une generalisation des polynomes d'Hermite"[2]. In this, he introduced a system of discrete orthogonal polynomials associated with the binomial distribution. In the same year, whilst the first world financial crisis was in full swing, Krawtchouk was elected a member of the Ukrainian Academy of Sciences. He taught at the National Technical University of Ukraine, previously called the Kiev Polytechnic Institute, as chair of the mathematics department. Krawtchouk benefitted immensely from his exposure to famous mathematicians such as Courant, Hadamard, Hilbert and Tricomi. Some of the most elementary examples of Krawtchouk polynomials are:

(1) $P_0^n(x) = 1$;
(2) $P_1^n(x) = -2x + n$;
(3) $P_2^n(x) = 2x^2 - 2nx + \binom{n}{2}$;
(4) $P_3^n(x) = -\frac{4}{3}x^3 + 2nx^2 - (n^2 - n + \frac{2}{3})x + \binom{n}{3}$.

1.2. **Motivation for Studying Krawtchouk Polynomials.**

Krawtchouk polynomials are very important in combinatorial mathematics, and their properties are being continuously revealed in the literature [3]. Krawtchouk polynomials play a crucial role in various areas of mathematics, such as combinatorics, modular elliptic curves and coding theory [4, 5], and in the theory of graph representations [6, 7]. In particular, binary Krawtchouk polynomials serve a major role in developing Hamming codes and protocols. They are similarly important in graph theory and number theory [8].

The existence of integer zeroes is connected with the existence of combinatorial structures in the Hamming association schemes. In fact, integer zeroes of Krawtchouk polynomials have received a special attention due to their relation to several problems in combinatorics, e.g., the existing of perfect codes, the graph reconstruction problem, etc (see e.g. Ref. [9]).

1.3. **Definition and Properties of Krawtchouk Polynomials.**

Here we follow Ref. [3] to present some important/known properties of Krawtchouk polynomials. Unless stated otherwise, all the results mentioned in this paragraph can be found in Ref. [3].

General Krawtchouk polynomials are orthogonal with respect to the binomial probability measure supported on the set $\{0, 1, 2, ..., n\} = \mathbb{Z}_n$, defined for $0 \leq p \leq 1$, by,

$$\mu(x) = \binom{n}{x} p^x (1-p)^{n-x}. \tag{2}$$

For $p = \frac{1}{2}$ we find the binary Krawtchouk polynomials, $P_k^n(x)$, which can be obtained via the explicit formulas,

$$P_k^n(x) = \sum_{j=0}^{k} \binom{n-x}{k-j} \binom{x}{j} (-1)^j \tag{3}$$

$$P_k^n(x) = \sum_{j=0}^{k} \binom{n-j}{k-j} \binom{x}{j} (-2)^j, \tag{4}$$

and

$$P_k^n(x) = \sum_{j=0}^{k} \binom{n-k+j}{j} \binom{n-x}{k-j} (-1)^j 2^{k-j}. \tag{5}$$

The Krawtchouk binary polynomials can be also obtained through the generating function,

$$F_x^n(z) = (1-z)^x (1+z)^{(n-x)} = \sum_{k=0}^{\infty} P_k^n(x) z^k, \tag{6}$$

and the general Krawtchouk polynomials are also defined by

$$P_k^n(x,p) = \sum_{j=0}^{k} \binom{n-x}{k-j}\binom{x}{j}(-1)^{k-j}p^{k-j}(1-p)^j. \tag{7}$$

The following recurrent relations for Krawtchouk polynomials are known:

$$(k+1)P_{k+1}^n(x) = (n-2x)P_k^n(x) - (n-k+1)P_{k-1}^n(x), \tag{8}$$

$$(n-x)P_k^n(x+1) = (n-2k)P_k^n(x) - xP_k^n(x-1), \tag{9}$$

$$(n-k+1)P_k^{n+1}(x) = (3n-2k-2x+1)P_k^n(x) - 2(n-x)P_k^{n-1}(x), \tag{10}$$

$$P_k^n = P_k^n(x-1) - P_{k-1}^n(x) - P_{k-1}^n(x-1), \tag{11}$$

$$P_k^n = P_k^{n-2}(x-1) - P_{k-2}^{n-2}(x-1), \tag{12}$$

$$P_k^n = P_k^{n-1}(x) + P_{k-1}^{n-1}(x), \tag{13}$$

and

$$P_k^n = P_k^{n-1}(x-1) - P_{k-1}^{n-1}(x-1), \tag{14}$$

whilst the following are standard initial conditions:

$$P_0^n(x) = 1 \tag{15}$$

and

$$P_1^n(x) = n - 2x. \tag{16}$$

Krawtchouk polynomials of small degree are also known, taking the following forms:

$$P_2^n(x) = \frac{(n-2x)^2 - n}{2} \tag{17}$$

$$P_3^n(x) = \frac{(n-2x)((n-2x)^2 - 3n + 2)}{6} \tag{18}$$

Krawtchouk polynomials evaluated at 0 and 1 are also known:

$$P_k^n(0) = \binom{n}{k} \tag{19}$$

and

$$P_k^n(1) = \frac{n-2k}{n}\binom{n}{k}. \tag{20}$$

Very important symmetry properties of Krawtchouk polynomials are reflected in the following relations and formulae:

$$P_k^n(x)\binom{n}{x} = P_x^n(k)\binom{n}{k}, (for\ nonnegative\ integer\ x). \tag{21}$$

and

$$P_k^n(x) = P_k^n(n-x)(-1)^k, \tag{22}$$

That they are symmetric (or antisymmetric) with respect to $\frac{n}{2}$ leads to

$$P_k^n(x) = P_{n-k}^n(x)(-1)^x, (for\ integer\ x, 0 \le x \le n). \tag{23}$$

These symmetric properties allow us to only have to deal with integer zeroes for $k \le \frac{n}{2}$ (in fact, $k < \frac{n}{2}$).

1.4. **Coefficients of Krawtchouk Polynomials.**

Several coefficients of Krawtchouk polynomials can be calculated directly.

If $P_k^n(x) = a_k x^k + ... + a_0$ then,

$$a_k = \frac{-2^k}{k!}, \tag{24}$$

$$a_{k-1} = \frac{-2^{k-1}n}{(k-1)!}, \tag{25}$$

$$a_{k-2} = \frac{-2^{k-2}(3n^2 - 3n + 2k - 4)}{(k-2)!}, \tag{26}$$

and

$$a_0 = \binom{n}{k}. \tag{27}$$

1.5. **Integer Zeroes of Binary Krawtchouk Polynomials.**

From now on the term 'Krawtchouk polynomials' is restricted for describing for the binary Krawtchouk polynomials only.

Several infinite families of integral zeroes of Krawtchouk polynomials are known [9]. Evidently, for n even and k odd we always require integer multiples of $n/2$ to be zero, known as a trivial zero. The known values of k for which there exist non-trivial integer zeroes for infinitely many n are: $k = 2, 3, \frac{n-3}{2}, \frac{n-4}{2}, \frac{n-5}{2}, \frac{n-6}{2}, \frac{n-8}{2}$, provided k is an integer; For $k = 2, 3$ the results can be found from equations (16) and (17).

Lemma 1. *(see e.g Ref.* [9]*) Let* $y = n - 2k$*. Then*

(1) $P_k^n(2i) = 0 \Leftrightarrow \sum_{j=o}^{y/2} \binom{k}{i-j}\binom{y}{2i}(-1)^j = 0,$

(2) $P_k^n(2i+1) = 0 \Leftrightarrow \sum_{j=o}^{(y-1)/2} \binom{k}{i-j}\binom{y}{2i+1}(-1)^j = 0.$

Proof. Using equations (4) and (14), one can see that to find even and odd zeroes of $P_k^n(x)$ one should find zero coefficients with even and odd indices respectively of $(1-z)^k(1+z)^{n-k} = (1-z^2)^k(1+z)^y$. The result now follows from calculating the coefficient at z^{2i} and z^{2i+1} respectively.

The lemma tells us that for small y the even zeroes can be found from the following diophantine equation:(see Ref. [9])

(1) $y = 3 : 4x - n + 1 = 0;$
(2) $y = 4 : 8x^2 - 8nx + n^2 - 2n = 0;$
(3) $y = 5 : 16x^2 - 12nx + 4r + n^2 - 4n + 3 = 0;$
(4) $y = 6 : 16x^2 - 16nx + n^2 - 6n + 8 = 0,$

and for the non-trivial odd zeroes:

(1) $y = 3 : 4x - 3n - 1 = 0;$
(2) $y = 5 : 16x^2 - 20nx - 4r + 5n^2 + 3 = 0;$
(3) $y = 6 : 16x^2 - 16nx + 3n^2 - 2n + 8 = 0;$
(4) $y = 8 : 8x^2 - 8nx + n^2 - 2n + 16 = 0.$

The following two theorems were proven in Ref. [10] and [11], respectively.

Theorem 1. *For each fixed $k \geq 4$, $P_k^n(x)$ can only have non-trivial integer zeroes for finitely many n.*

Theorem 2. *Let $y > 6$ be an odd number, a power of 2, or of the form $2pq$, where p is an odd prime, q is odd, and p does not divide q. Then for $k = \frac{n-y}{2}$, $P_k^n(x)$ can only possess non-trivial even zeroes for finitely many n.*

For the cases $P_4^n(x)$ and $P_5^n(x)$ (which will be discussed in detail in chapter 2) the equations with $x = (n-y)/2$ will have the form:

$$P_4^n(x) \Rightarrow y^4 - 6y^2n + 8y^2 + 3n^2 - 6n, \tag{28}$$

$$P_5^n(x) \Rightarrow y(y^4 - 10y^2n + 20y^2 + 15n^2 - 50n + 24). \tag{29}$$

and can be analysed using Pell's equation.

The formal solution of $P_4^n(x) = 0$ is for

$$x = \frac{1}{2}(n + \sqrt{-4 + 3n - \sqrt{2}\sqrt{8 - 9n + 3n^2}}, \tag{30}$$

to be an integer. The outer root

$$-4 + 3n - \sqrt{2}\sqrt{8 - 9n + 3n^2} \tag{31}$$

must thus be an integer, which means that the inner root

(32) $$8 - 9n + 3n^2$$

must be a perfect square. To satisfy the last condition we have to solve Pell's equations. Then we have to check that the outer root gives an integer only for a geometrical progression generated by Pell's equations.

In this way we can reduce the amount of numbers that need to be checked with our programme than simply going through the numbers in a row, $n = 1, 2, 3, ...$

For $k > 5$ we find equations of higher degree, e.g., for $k = 6$,

(33)
$$P_6^n(x) \Rightarrow y^6 - 15ny^4 + 40y^4 + 45n^2y^2 - 210ny^2 + 184y^2 - 15n^3 + 90n^2 - 120n.$$

1.6. **A Conjecture on Krawtchouk Polynomials.**

The coefficients of the following conjecture are precisely Krawtchouk polynomials in view of the generating function equation (6). That is here, and in general, the problem of finding integer zeroes is equivalent to the question of zero coefficients in equation (6).

In Ref. [9] the following conjecture is stated:

Let $n = m^2$ and $s = \binom{m}{2}$, $n - s = \binom{m+1}{2}$, *So*

$$\sum_{i=0}^{n} a_i z^i = (1-z)^{\binom{m}{2}}(1+z)^{\binom{m+1}{2}} \tag{34}$$

Then the only zero coefficients for equation (34) are: $a_2 = 0$, $a_{m^2-2} = 0$, *and* $a_{\frac{m^2}{2}} = 0,\ if\, m = 2\ mod\, 4.$

Let us now show that the coefficients a_2, a_{m^2-2} *and* $a_{\frac{m^2}{2}}$ if $m = 2$ mod 4 are indeed zero.

Proof:

Define

$$f(z) = (1-z)^{\binom{m}{2}}(1+z)^{\binom{m+1}{2}} = \sum_{n=0}^{m^2} a_n z^n. \tag{35}$$

Coefficient a_2**:**

Taking k derivatives and setting $z \to 0$ yields

$$f^{(k)}(0) = k! a_k \Rightarrow a_n = \frac{1}{n!} f^{(n)}(0). \tag{36}$$

The second derivative of $f(z)$ is

$$f''(z) = \frac{m\left(-1+m^2\right) z(-2+mz)}{\left(-1+z^2\right)^2} f(z). \tag{37}$$

Since $f(0) = 1$, we find

$$\frac{1}{2}f''(0) = 0 = a_2. \tag{38}$$

Expression for a_n:

We know from the binomial theorem, that

$$(1+z)^N = \sum_{k=0}^{N} \binom{N}{k} z^k, \tag{39}$$

so we can write

$$f(z) = \sum_{k=0}^{N} \sum_{j=0}^{M} \binom{N}{k} \binom{M}{j} (-1)^k z^{k+j}, \tag{40}$$

where we have denoted $N = \binom{m}{2}$ and $M = \binom{m+1}{2}$. Changing the order of summation, i.e. setting $k + j = n$, gives

$$f(z) = \sum_{n=0}^{N+M} z^n \sum_{k=n-m}^{n} \binom{N}{k} \binom{M}{n-k} (-1)^k \tag{41}$$

$$= (-1)^{n-M} \sum_{n=0}^{m^2} z^n \sum_{k=0}^{M} \binom{N}{n-M+k} \binom{M}{k} (-1)^k, \tag{42}$$

where we also changed the summation variable $k \to n - M + k$ and used the binomial symmetry $\binom{M}{n-k} = \binom{M}{k}$. Therefore

$$a_n = (-1)^{n-M} \sum_{k=0}^{M} \binom{N}{n-M+k} \binom{M}{k} (-1)^k. \tag{43}$$

Zero coefficients:

Consider now

$$(44)\quad a_{m^2-s} = (-1)^{m^2-s-M} \sum_{k=0}^{M} \binom{N}{m^2-s-M+k} \binom{M}{k} (-1)^k$$

$$(45)\quad = (-1)^{s-N} \sum_{k=0}^{M} \binom{N}{s-k} \binom{M}{k} (-1)^k,$$

where we have used the definitions for N, M and the binomial symmetry. By changing the summation to run from M to 0 by $k \to M-k$, we find

$$(46)\quad a_{m^2-s} = (-1)^{s-N} \sum_{k=0}^{M} \tbinom{N}{s-M+k} \tbinom{M}{k} (-1)^{k-M} = (-1)^N a_s.$$

Setting $s = 2$ yields $a_{m^2-2} = a_2 = 0$. Setting $s = m^2/2$ for even m^2 yields

$$(47)\quad a_{m^2/2} = (-1)^N a_{m^2/2}.$$

Then $(-1)^N = -1$ for $N = \frac{1}{2}m(m-1) = odd$ and $m = even$, i.e. for $m = 4s+2$ when s is a nonnegative integer, leading to $a_{m^2/2} = 0$.

1.7. **Numerical Investigation of the Conjecture.**

We are going to check now for as large an m as manageable that $(1-z)^{\binom{m}{2}}(1+z)^{\binom{m+1}{2}}$ has only 3 *zero coefficients* for $m \geq 3$. We shall use the MATHEMATICA software package. We begin by describing an algorithm to verifying the above conjecture for small values of m. We start with a straightforward approach which will modified later to a more efficient method.

There is a problem with verifying the above conjecture for large m because the degree of those polynomials grows very fast, as $\frac{m^2}{2}$. An example is given in Appendix A.

In fact, we do not need to calculate all of the coefficients to check the validity of the conjecture. Our programme can be modified to account for the fact that $a_2 = 0$, i.e., we can restrict our calculation to $a_i \neq 0$ for $3 \leq i \leq \frac{m^2-1}{2}$. An example is given in Appendix B.

With the above in mind, the values of the smallest coefficients for the conjecture of Krawtchouk polynomials are shown in the following table. We present the smallest coefficients between a_3 and $a_{\frac{m^2}{2}-1}$, for $3 \leq m \leq 100$.

Table 1. Smallest Coefficients for Krawtchouk conjecture

m	Smallest Coefficient	m	Smallest Coefficient	m	Smallest Coefficient
3	6	31	9920	59	68440
4	20	32	10912	60	71980
5	40	33	11968	61	75640
6	70	34	13090	62	79422
7	112	35	14280	63	83328
8	168	36	15540	64	87360
9	240	37	16872	65	91520
10	330	38	18278	66	95810
11	440	39	19760	67	100232
12	572	40	21320	68	104788
13	728	41	22960	69	109480
14	910	42	24682	70	114310
15	1120	43	26488	71	119280
16	1360	44	28380	72	124392
17	1632	45	30360	73	129648
18	1938	46	32430	74	135050
19	2280	47	34592	75	140600
20	2660	48	36848	76	146300
21	3080	49	39200	77	152152
22	3542	50	41650	78	158158
23	4048	51	44200	79	164320
24	4600	52	46852	80	170640
25	5200	53	49608	81	177120
26	5850	54	52470	82	183762
27	6552	55	55440	83	190568
28	7308	56	58520	84	197540
29	8120	57	61712	85	204680
30	8990	58	65018	86	211990
87	219472	92	259532	97	304192
88	227128	93	268088	98	313698
89	234960	94	276830	99	323400
90	242970	95	285760	100	333300
91	251160	96	294880		

1.8. **Using Modular Arithmetics.**

In this section we show how one can extend the preceding calculations to greater values of m by using modular arithmetics.

No zero coefficients have been found in the range $m = 3$ to 239, so the conjecture of Krawtchouk polynomials holds here.

Thus, using MATHEMATICA we have checked all $m \leq 239$ and the conjecture is true. (there is insufficient memory for the present code to check beyond $m > 239$)

The idea is as follows: Let $s = \Pi_{i=1}^{k} p_i^{\alpha_i}$ be the prime factorization of s (in my programme $s = h[n]$). We have

$$s = \Pi_{i=1}^{k} p_i^{\alpha_i} \geq \Pi_{i=1}^{k} p_i \geq 2^k, \tag{48}$$

hence $k \leq \log_2 s$, i.e., the number of prime factors $k = k(n)$ which n can have does not exceed $\log_2 s$. The coefficients of $(1-x)^{\binom{n}{2}}(1+x)^{\binom{n+1}{2}}$ do not exceed the corresponding coefficients of $(1+x)^{\binom{n}{2}}(1+x)^{\binom{n+1}{2}} = (1+x)^{n^2}$, i.e., they are definitely less than 2^{n^2}.

Thus, $h[n]$ has at most $\log_2 2^{n^2} = n^2$ prime factors. This means that if we take any n^2 primes $p_1, p_2, ..., p_{n^2}$, then $h[n] = 0$ iff $h[n] = 0$ mod p_i, $i = 1, 2, ..., n^2$.

It turns out that most values of m can be excluded just by one large prime $p \simeq 10^6$. Exceptional cases were excluded by using one extra prime modulo.

An example of the MATHEMATICA programme is shown in Appendix C for the results up to $m \leq 1000$, i.e., for the polynomials up to degree of approximately half a million.

1.9. **Krawtchouk Polynomials Proximity to Zero.**

In this section (using MATHEMATICA) we attempt to estimate how close $P_k^n(x)$ can approach zero for $k = m(m-1)/2$ and $n = m^2$.

Since Krawtchouk polynomials are conjectured to be nonzero at integer points, we wrote a programme (shown in Appendix D) which gives these values for normalised Krawtchouk polynomials.

Normalisation: It was conjectured (I. Krasikov, *private communication*) that,

$$F_m(x) = \frac{\sqrt{\frac{\pi}{2}\binom{n}{x}}\sqrt[4]{x(n-x)} \cdot P^n_{\binom{m}{2}}(x)}{\sqrt{\binom{n}{k} \cdot 2^n}} \tag{49}$$

behaves as a sinus of some function, i.e., $|F_m(x)| \preceq 1$ and is almost an equioscillatory function.

Below are the plots of $F_m(x)$ of degree $\binom{6}{2}$ and $\binom{7}{2}$ corresponding to $m = 6, 7$.

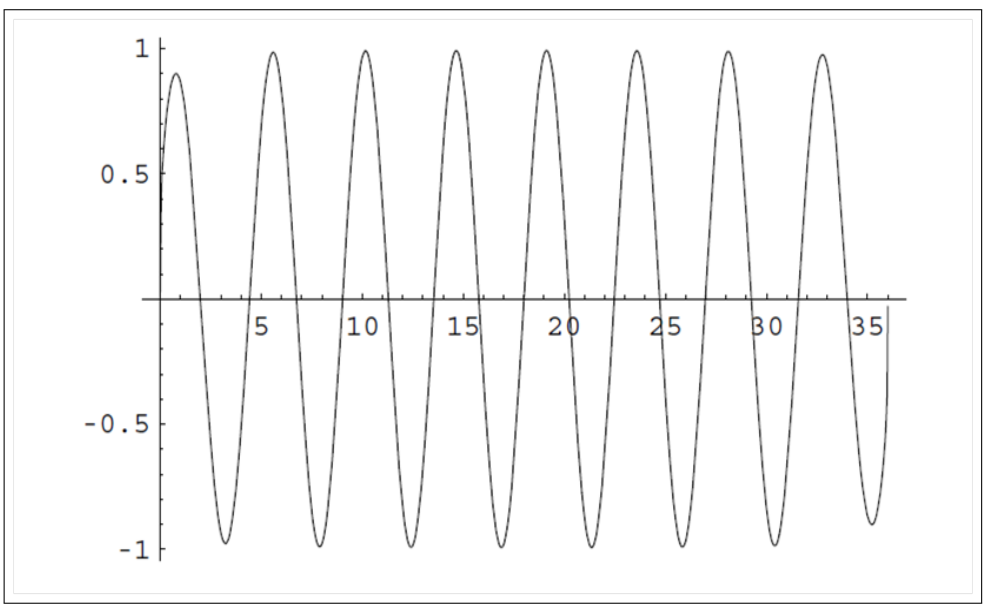

FIGURE 1. Plot of $F_6(x)$ of degree $\binom{6}{2}$.

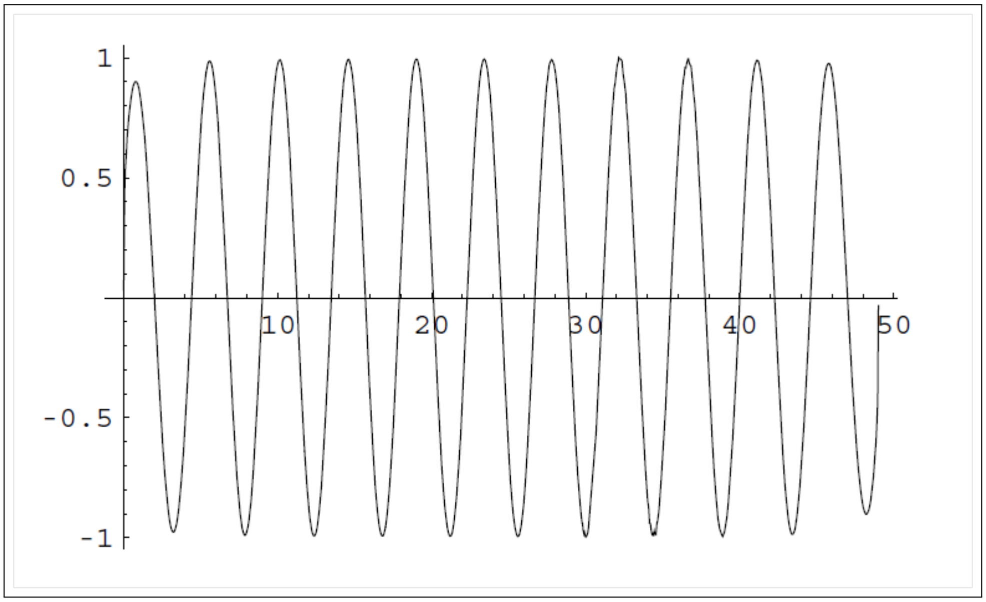

FIGURE 2. Plot of $F_7(x)$ of degree $\binom{7}{2}$.

We calculated the value of $F_m(x)$ at integer points $0, 1, 2, ..., m^2$ and found the minima for $m = 3$ to 225, as given in the following table.

TABLE 2. The Decreasing Subsequence of $m^2 \cdot F$ and the value of $m^3 \cdot F$

m	$m^2 \cdot F$	$m^3 \cdot F$
5	2.49	12.45
6	1.46	8.76
7	0.41	2.87
9	0.376	3.384
56	0.321	17.976
69	0.129	8.901
77	0.0468	3.603
137	0.0228	3.123

We were unable to calculate the data in this table to larger values of m, so the results are inconclusive. However, it seems plausible that $F_m \cdot m^3 > const$. An example is given in Appendix D.

2. Integer Zeroes of Krawtchouk Polynomials of Degree 4 and 5

2.1. On Integral Zeroes of Krawtchouk Polynomials of Small Degree.

In the next sections we study the integral zeroes of Krawtchouk polynomials of small degree and try to derive them through new methodology.

It will be convenient to define here non-trivial integral zeroes of $P_k^n(x)$ to be an ordered triple of positive integers (k, x, n), with $4 \leq k \leq n/2, x \leq n/2$ and $P_k^n(x) = 0$.

Chihara and Stanton show in Ref. [12] that the integral zeroes for degree 1,2, and 3 are:

(1) $(1, k, 2k), k \geq 1$,
(2) $(2, k(k-1)/2, k^2), k \geq 3$, and
(3) $(3, k(3k \pm 1)/2, 3k^2 + 3k + 3/2 \pm (k + 1/2)), k \geq 2$.

The following theorem describes the non-trivial integral zeroes of the binary Krawtchouk polynomials (see equation (3)) using t and r as parameters.

Theorem 3. [13] *The Krawtchouk polynomials $P_k^n(x)$ have inequivalent non-trivial integer zeroes, (k, x, n), for the following values:*

(1) for $t \geq 2$ *and* $r = 3 + 2\sqrt{2}$,

$$\begin{aligned} k &= (r^t + r^{-t} - 20) \\ x &= n/2 - (r^t - r^{-t})/2\sqrt{2} \\ n &= 2k + 4 \Rightarrow k = \tfrac{n}{2} - 2. \end{aligned}$$

(2) for $t \geq 2$ *and* $r = 9 + 4\sqrt{5}$,

$$\begin{aligned} k &= ((\sqrt{5} \pm 1)r^t + (\sqrt{5} \mp r^{-t}))/2\sqrt{5} - 3 \\ x &= (3k + 7)/4 \mp ((\sqrt{5} \pm 1)r^t - (\sqrt{5} \mp 1)r^{-t})/8 \\ n &= 2k + 5 \Rightarrow k = \tfrac{n}{2} - \tfrac{5}{2}. \end{aligned}$$

(3) for $t \geq 2$ *and* $r = 9 + 4\sqrt{5}$,

$$\begin{aligned} k &= ((2\sqrt{5} \pm 4)r^t - (2\sqrt{5} \mp 4)r^{-t})/2\sqrt{5} - 3 \\ x &= (3k+7)/4 \mp ((2\sqrt{5} \pm 4)r^t + (2\sqrt{5} \mp 4)r^{-t})/8 \\ n &= 2k + 5 \Rightarrow k = \tfrac{n}{2} - \tfrac{5}{2}. \end{aligned}$$

(4) for $t \geq 2$ *odd and* $r = 2 + \sqrt{3}$,

$$\begin{aligned} k &= ((2\sqrt{3} \pm 1)r^t + (2\sqrt{3} \mp 1)r^{-t})/4\sqrt{3} - 7/2 \\ x &= k + 3 - (2\sqrt{3} \pm 1)r^t + (2\sqrt{3} \mp 1)r^{-t})/8 \\ n &= 2k + 6 \Rightarrow k = \tfrac{n}{2} - 3. \end{aligned}$$

(5) for $t \geq 2$ *and* $r = 3 + 2\sqrt{2}$,

$$\begin{aligned} k &= ((5 \pm 2\sqrt{2})r^t + (5 \mp 2\sqrt{2})r^{-t})/4 - 9/2 \\ x &= k + 4 - ((5 \pm 2\sqrt{2})r^t + (5 \mp 2\sqrt{2})r^{-t})/4\sqrt{2} \\ n &= 2k + 8 \Rightarrow k = \tfrac{n}{2} - 4. \end{aligned}$$

Here is a list of the some Krawtchouk polynomial inequivalent non-trivial integer zeroes, (k, x, n), at the first values of t.

k	178	1134	6760	39182
x	19	159	975	5731
n	360	2272	13416	73868

k	230	4178	75022	1346266
x	44	798	14328	257114
n	465	8361	150049	2692537

k	607	10943	196415	3524575
x	116	2090	37512	673134
n	1219	21891	392835	7049155

k	30	463
x	4	62
n	66	932

k	103	622	3647	21278
x	31	138	1069	6233
n	214	1252	7302	42564

2.2. **Pell's Equation.**

Here we describe some classical results of Pell's equation.

Definition 1. [14] *Pell's equation is a diophantine equation of the form* $x^2 - dy^2 = 1$*,* $x, y \in \mathbb{Z}$*, where* d *is a given natural number which is not a square. An equation of the form* $x^2 - dy^2 = a$ *for an integer* a *is usually referred to as a Pell-type equation.*

For $d = c^2, c \in \mathbb{Z}$, the equation $x^2 - dy^2 = a$ can be factored as $(x - cy)(x + cy) = a$ and therefore solved without using any further theory. So, unless stated otherwise, d will always be assumed to not be a square.

The equation $x^2 - dy^2 = a$ can still be factored as

$$(x + y\sqrt{d})(x - y\sqrt{d}) = a.$$

In order to be able to make use of this factorisation, we must deal with numbers of the form $x + y\sqrt{d}$, where x, y are integers.

Definition 2. [14] *The conjugate of the number* $z = x+y\sqrt{d}$ *is defined as* $\bar{z} = x - y\sqrt{d}$*, and its norm as* $N(n) = z\bar{z} = x^2 - dy^2 \in \mathbb{Z}$*.*

Theorem 4. [14] *The norm and the conjugate are multiplicative in* $z : N(z_1z_2) = N(z_1)N(z_2)$ *and* $\overline{z_1z_2} = \bar{z_1} \cdot \bar{z_2}$*.*

Theorem 5. [14] *If* z_0 *is the minimal element of* $\mathbb{Z}[\sqrt{d}]$ *with* $z_0 > 1$ *and* $N(z_0) = 1$*, then all the elements* $z \in \mathbb{Z}[\sqrt{d}]$ *with* $Nz = 1$ *are given by* $z = \pm z_0^n, n \in \mathbb{Z}$

Corollary 1. *If* (x_0, y_0) *is the smallest solution of Pell's equation with* d *given, then all natural solutions* (x, y) *of the equation are given by* $x + y\sqrt{d} = \pm(x_0 + y_0\sqrt{d})^n, n \in \mathbb{N}$*.*

Note that $z = x+y\sqrt{d}$ *determines* x *and* y *by the formulae* $x = \frac{z+\bar{z}}{2}$ *and* $y = \frac{z-\bar{z}}{2\sqrt{d}}$*. Thus, all the solutions of the Pell's equation are given by the formulae*

$$x = \frac{z_0^n + \bar{z_0^n}}{2} \ \ and \ \ y = \frac{z_0^n - \bar{z_0^n}}{2\sqrt{d}}.$$

2.3. **Pell-type Equation.**

A Pell-type equation (i.e., an equation of the form $x^2 - dy^2 = -1$) may, in general, not have integer solutions. When it does, it is possible to describe the general solution.

Theorem 6. [14] *Equation* $x^2 - dy^2 = -1$ *has an integral solution if and only if there exists* $z_1 \in \mathbb{Z}[\sqrt{d}]$ *with* $z_1^2 = z_0$.

Theorem 7. [14] *If* a *is an integer such that the equation* $N(z) = x^2 - dy^2 = a$ *has an integer solution, then there is a solution with* $|x| \leq \frac{z_0+1}{2\sqrt{z_0}}\sqrt{|a|}$ *and the corresponding upper bound for* $y = \sqrt{\frac{x^2-a}{d}}$.

2.4. **Applying Pell's Equation to Our Two Cases.**

Here we will demonstrate how to use Pell's equation to solve equations (80) and (82), which are needed in the subsequent sections.

(1) **General remarks**

Let us consider a Diophantine equation of the form

$$Ax^2 - By^2 = C \tag{50}$$

with integer A, B, and C. By multiplying both sides with A one can rewrite it as

$$A^2x^2 - ABy^2 = AC, \quad t^2 - ABy^2 = AC, \quad t = Ax. \tag{51}$$

The solutions of equation (51) are connected with the solutions of the corresponding Pell's equation,

$$t^2 - ABy^2 = 1 \tag{52}$$

in the following way. Let us assume one knows one solution t_0, y_0 of equation (52) and T_0, Y_0 of equation (51). Then, multiplying equations (51) and (52) one finds:

$$AC = (T_0^2 - ABY_0^2) \cdot (t_0^2 - ABy_0^2) =$$
$$= (T_0 + \sqrt{AB}Y_0)(T_0 - \sqrt{AB}Y_0) \cdot (t_0 + \sqrt{AB}y_0)(t_0 - \sqrt{AB}y_0) =$$
$$= [(T_0 + \sqrt{AB}Y_0)(t_0 + \sqrt{AB}y_0)] \cdot [(T_0 - \sqrt{AB}Y_0)(t_0 - \sqrt{AB}y_0)] =$$
$$= [T_0t_0 + ABY_0y_0 + \sqrt{AB}(t_0Y_0 + T_0y_0)] \cdot [T_0t_0 +$$
$$ABY_0y_0 - \sqrt{AB}(t_0Y_0 + T_0y_0)] =$$
$$= (T_0t_0 + ABY_0y_0)^2 - AB(t_0Y_0 + y_0T_0).$$

This is equivalent to (remember that $T_0 = AX_0$)

$$A^2(X_0t_0 + By_0Y_0)^2 - AB(t_0Y_0 + AX_0y_0)^2 = AC \tag{53}$$

Comparing this equation with equation (51) one can see that if X_0,Y_0 is a solution of equation (50) and t_0,y_0 is a solution of Pell's equation (51) then

$$X_1 = X_0 t_0 + B y_0 Y_0, \quad Y_1 = t_0 Y_0 + A X_0 y_0 \tag{54}$$

is also solution of equation (50). Equation (54) can be used recursively.

(2) **Solution of Pell's equation**

Pell's equation

$$t^2 - \gamma y^2 = 1, \quad \gamma = AC \tag{55}$$

has one trivial solution $t = 1$, $y = 0$ (not important) and an infinite number of non-trivial solutions. It also has a property that if t_i, y_i and t_j, y_j are solutions of equation (55), then

$$t_k = t_i t_j + \gamma y_i y_j, \quad y_k = t_i y_j + t_j y_i \tag{56}$$

is also solution of equation (55), whilst

$$(t_i t_j + \gamma y_i y_j)^2 - \gamma(t_i y_j + t_j y_i)^2 = (t_i t_j)^2 + \cancel{2\gamma(t_i t_j)(y_i y_j)} + \gamma^2 (y_i y_j) -$$
$$-\gamma[(t_i y_j)^2 + \cancel{(t_i t_j)(y_i y_j)} + (y_i t_j)^2] = (t_i^2 - \gamma y_i^2) \cdot (t_j^2 - \gamma y_j^2) = 1$$

In general, all solutions of equation (55) can be obtained with successive "multiplication" (56) of the first non-trivial solution by itself, i.e.,

$$\begin{aligned} t_1 &= t_0^2 + \gamma y_0^2, \quad &, y_1 = 2 t_0 y_0 \\ t_2 &= t_1 t_0 + \gamma y_1 y_0, \quad &, y_2 = t_0 y_1 + t_1 y_0 \\ t_n &= t_{n-1} t_0 + \gamma y_{n-1} y_0, \quad &, y_n = t_0 y_{n-1} + t_{n-1} y_0 \end{aligned} \tag{57}$$

(58)

It is convenient to write the solution of equation (55) in the other form. From equation (57) one can easily see that

$$(t_0 + y_0\sqrt{\gamma})^2 = [t_0^2 + \gamma y_0^2] + 2t_0 y_0\sqrt{\gamma} = t_1 + y_1\sqrt{\gamma}$$
$$(t_0 - y_0\sqrt{\gamma})^2 = t_1 - y_1\sqrt{\gamma}$$

and this chain can be continued

$$(t_0 + y_0\sqrt{\gamma})^3 = (t_0 - y_0\sqrt{\gamma})^2 \cdot (t_0 - y_0\sqrt{\gamma}) =$$
$$= (t_1 + y_1\sqrt{\gamma}) \cdot (t_0 - y_0\sqrt{\gamma}) =$$
$$= (t_1 t_0 + \gamma y_1 y_0) + (t_0 y_1 + t_1 y_0)\sqrt{\gamma} = t_2 + y_2\sqrt{\gamma}$$
$$(t_0 - y_0\sqrt{\gamma})^3 = t_2 - y_2\sqrt{\gamma}$$
$$...etc$$
$$(t_0 + y_0\sqrt{\gamma})^n = (t_0 + y_0\sqrt{\gamma})^{n-1} \cdot (t_0 + y_0\sqrt{\gamma}) =$$
$$= (t_{n-2} + y_{n-2}\sqrt{\gamma}) \cdot (t_0 + y_0\sqrt{\gamma}) = t_{n-1} + y_{n-1}\sqrt{\gamma}$$
$$(t_0 - y_0\sqrt{\gamma})^n = t_{n-1} - y_{n-1}\sqrt{\gamma},$$

Ultimately, one can explicitly express the $n-$th solution over the first non-trivial solution as

$$t_{n-1} = \frac{1}{2}[(t_0 + y_0\sqrt{\gamma})^n + (t_0 - y_0\sqrt{\gamma})^n] \tag{59}$$

$$y_{n-1} = \frac{1}{2\sqrt{\gamma}}[(t_0 + y_0\sqrt{\gamma})^n - (t_0 - y_0\sqrt{\gamma})^n] \tag{60}$$

(3) $A = 2$, $B = 3$, $C = 5$

Firstly, substitute $A = 3$, $B = 2$ into Pell's equation (55) (remember $t = Ax$),

$$t^2 - 6y^2 = 1 \tag{61}$$

The first solution is clearly

(62) $$t_0 = \pm 5, \quad y_0 = \pm 2$$

($\pm$ because of the symmetry $t \leftrightarrow -t, y \leftrightarrow -y$). Then

(63) $$t_{n-1} = \pm \frac{1}{2}[(5+2\sqrt{6})^n + (5-2\sqrt{6})^n]$$

(64) $$y_{n-1} = \pm \frac{1}{2\sqrt{6}}[(5+2\sqrt{6})^n - (5-2\sqrt{6})^n].$$

After substitution of A, B, and C into equation (50), we have to solve

(65) $$2x^2 - 3y^2 = 5.$$

The first solution is clearly

(66) $$X_0 = \pm 2, \quad Y_0 = \pm 1$$

and, after substituting equations (63) and (66) into (54), the general solution is

(67)
$$X_{n-1} = \pm[(5+2\sqrt{6})^n + (5-2\sqrt{6})^n] \pm \frac{3}{2\sqrt{6}}[(5+2\sqrt{6})^n - (5-2\sqrt{6})^n],$$

and

(68)
$$Y_{n-1} = \pm \frac{1}{2}[(5+2\sqrt{6})^n + (5-2\sqrt{6})^n] \pm \frac{1}{2\sqrt{6}}[(5+2\sqrt{6})^n - (5-2\sqrt{6})^n].$$

(4) $A = 2$, $B = 5$, $C = 27$

Following the above route, Pell's equation is

(69) $$t^2 - 10y^2 = 1.$$

The initial solution is clearly

(70) $$t_0 = \pm 19, \quad y_0 = \pm 6,$$

with the general solution of Pell's equation being

$$t_{n-1} = \pm\frac{1}{2}[(19+6\sqrt{10})^n + (19-6\sqrt{10})^n] \tag{71}$$

$$y_{n-1} = \pm\frac{1}{2\sqrt{10}}[(19+6\sqrt{10})^n - (19-6\sqrt{10})^n]. \tag{72}$$

The equation to solve is then

$$2x^2 - 5y^2 = 27, \tag{73}$$

with the initial solution

$$X_0 = \pm 4, \quad Y_0 = \pm 1. \tag{74}$$

Finally, substituting equations (63) and (66) into (54) gives the overall solution

(75)

$$X_{n-1} = \pm 2[(19+6\sqrt{10})^n+(19-6\sqrt{10})^n] \pm \frac{5}{2\sqrt{10}}[(19+6\sqrt{10})^n-(19-6\sqrt{10})^n],$$

and

(76)

$$Y_{n-1} = \pm\frac{1}{2}[(19+6\sqrt{10})^n+(19-6\sqrt{10})^n] \pm \frac{4}{\sqrt{10}}[(19+6\sqrt{10})^n-(19-6\sqrt{10})^n].$$

2.5. **Krawtchouk Polynomials of Degree 4.**

The quartic equation $P_4^n(x) = 0$ has a finite number of non-trivial solutions. This follows from a well-known theorem on hyperelliptic equations [15]. We can write

$$\sum_{k=0}^{\infty} P_k^n(x) z^k = (1 - z^2)^4 (1 + z)^n \tag{77}$$

and

$$\sum_{k=0}^{\infty} P_k^n(x) z^k = (1 - 4z^2 + 6z^4 - 4z^6 + z^8)(1 + z)^n. \tag{78}$$

Using MATHEMATICA, the values of solutions that help us to find the zeroes of Krawtchouk polynomials of degree 4 are shown in Figure 3. From this, we can say that for degree 4 the equation must be as follows:

$$2(y^2 - 3n + 4)^2 - 3(2n - 3)^2 = 5 \tag{79}$$

where $y = n - 2x$, i.e., $y = n \bmod 2$.

Also, we can see that by reducing equation (79) to the integer zeroes, we get the following solutions:

For, $n = 0$, $y = 0$, and for, $n = 1$, $y = \pm 1$.

For, $n \geq 2$, $y = \pm\sqrt{-4 + 3n \pm \sqrt{16 - 18n + 6n^2}}$.

To solve these equations we will use the technique to solve Pell's equation given in Ref. [16].

Krawtchouk polynomials, k=4.nb 1

```
In[1]:= p[-1] = 0

Out[1]= 0

In[2]:= p[0] = 1

Out[2]= 1

In[3]:= p[k_Integer] := p[k] = Expand[((n - 2 x) p[k - 1] - (n - k + 2) p[k - 2]) / k]

In[4]:= p[4]
```

Out[4]= $-\frac{n}{4}+\frac{11\,n^2}{24}-\frac{n^3}{4}+\frac{n^4}{24}-\frac{4\,n\,x}{3}+n^2\,x-\frac{n^3\,x}{3}+\frac{4\,x^2}{3}-n\,x^2+n^2\,x^2-\frac{4\,n\,x^3}{3}+\frac{2\,x^4}{3}$

```
In[5]:= % /. x → (n - y) / 2
```

Out[5]= $-\frac{n}{4}+\frac{11\,n^2}{24}-\frac{n^3}{4}+\frac{n^4}{24}-\frac{2}{3}\,n\,(n-y)+\frac{1}{2}\,n^2\,(n-y)-\frac{1}{6}\,n^3\,(n-y)+$
$\frac{1}{3}\,(n-y)^2-\frac{1}{4}\,n\,(n-y)^2+\frac{1}{4}\,n^2\,(n-y)^2-\frac{1}{6}\,n\,(n-y)^3+\frac{1}{24}\,(n-y)^4$

```
In[6]:= Factor[%]
```

Out[6]= $\frac{1}{24}\,(-6\,n+3\,n^2+8\,y^2-6\,n\,y^2+y^4)$

```
In[7]:= Solve[% == 0, y]
```

Out[7]= $\left\{\left\{y\to-\sqrt{-4+3\,n-\sqrt{2}\,\sqrt{8-9\,n+3\,n^2}}\right\},\left\{y\to\sqrt{-4+3\,n-\sqrt{2}\,\sqrt{8-9\,n+3\,n^2}}\right\},\right.$
$\left.\left\{y\to-\sqrt{-4+3\,n+\sqrt{2}\,\sqrt{8-9\,n+3\,n^2}}\right\},\left\{y\to\sqrt{-4+3\,n+\sqrt{2}\,\sqrt{8-9\,n+3\,n^2}}\right\}\right\}$

FIGURE 3. Krawtchouk polynomials of degree 4.

2.6. **Application of Pell's Equation to Krawtchouk Polynomials of Degree 4.**

Firstly, let $x_i = y^2 - 3n + 4$ and $y_i = 2n - 3$ so that equation (43) becomes:

$$2x_i^2 - 3y_i^2 = 5 \tag{80}$$

Using the solutions of Pell's equation (67) and (68), all the integer solutions for equation (80) should follow the following solutions:

let $s \in \mathbb{Z}$, such that $s \geq 0$

$$x_1 = \frac{1}{4}(-4(5-2\sqrt{6})^s + \sqrt{6}(5-2\sqrt{6})^s - 4(5+2\sqrt{6})^s - \sqrt{6}(5+2\sqrt{6})^s),$$
$$x_2 = \frac{1}{4}(4(5-2\sqrt{6})^s - \sqrt{6}(5-2\sqrt{6})^s + 4(5+2\sqrt{6})^s + \sqrt{6}(5+2\sqrt{6})^s),$$
$$x_3 = \frac{1}{4}(4(5-2\sqrt{6})^s + \sqrt{6}(5-2\sqrt{6})^s + 4(5+2\sqrt{6})^s - \sqrt{6}(5+2\sqrt{6})^s),$$
$$x_4 = \frac{1}{4}(-4(5-2\sqrt{6})^s - \sqrt{6}(5-2\sqrt{6})^s - 4(5+2\sqrt{6})^s + \sqrt{6}(5+2\sqrt{6})^s).$$

and solution for y_i are:

$$y_1 = \frac{1}{6}(-3(5-2\sqrt{6})^s + 2\sqrt{6}(5-2\sqrt{6})^s - 3(5+2\sqrt{6})^s - 2\sqrt{6}(5+2\sqrt{6})^s),$$
$$y_2 = -\frac{1}{6}(-3(5-2\sqrt{6})^s + 2\sqrt{6}(5-2\sqrt{6})^s - 3(5+2\sqrt{6})^s - 2\sqrt{6}(5+2\sqrt{6})^s),$$
$$y_3 = \frac{1}{6}(3(5-2\sqrt{6})^s + 2\sqrt{6}(5-2\sqrt{6})^s + 3(5+2\sqrt{6})^s - 2\sqrt{6}(5+2\sqrt{6})^s),$$
$$y_4 = -\frac{1}{6}(3(5-2\sqrt{6})^s + 2\sqrt{6}(5-2\sqrt{6})^s + 3(5+2\sqrt{6})^s - 2\sqrt{6}(5+2\sqrt{6})^s),$$

Example 1. *Let $s = 0$, in the solutions x_1 and y_1, then,*

$$x_1 = \frac{1}{4}(-4 + \sqrt{6} - 4 - \sqrt{6}) = -2$$

and,

$$y_1 = \frac{1}{6}(-3 + 2\sqrt{6} - 3 - 2\sqrt{6}) = -1$$

so, from equation (80) *we get,*

$$2(-2)^2 - 3(-1)^2 = 5$$

Other examples of integer solutions for equation (80):

x_2	y_2	x_2	y_2
2	1	1517078	1238689
16	13	15017524	12261757
158	129	148658162	121378881
1564	1277	1471564096	1201527053
15482	12641	14566982798	11893891649
153256	125133	144198263884	117737389437

It may become clearer if we use an alternate form of equation (79), such as

$$y^4 - 6y^2n + 8y^2 + 3n^2 - 6n = 0$$

To find the integer solutions we can follow the proceeding route:

Possible intermediate steps:

$$3n^2 - 6ny^2 - 6n + y^4 + 8y^2 = 0$$

Expanding terms on the left hand side:

$$3n^2 + n(-6y^2 - 6) + y^4 + 8y^2 = 0$$

Solving the quadratic equation by computing the square, then dividing both sides by 3:

$$n^2 + \frac{1}{3}n(-6y^2 - 6) + \frac{1}{3}(y^4 + 8y^2) = 0$$

Subtracting $\frac{1}{3}(y^4 + 8y^2)$ from both sides:

$$n^2 + \frac{1}{3}n(-6y^2 - 6) = \frac{1}{3}(-y^4 - 8y^2)$$

Adding $\frac{1}{36}(-6y^2-6)^2$ to both sides:

$$n^2+\frac{1}{3}n(-6y^2-6)+\frac{1}{36}(-6y^2-6)^2=\frac{1}{36}(-6y^2-6)^2+\frac{1}{3}(-y^4-8y^2)$$

Factoring the left hand side:

$$(n+\frac{1}{6}(-6y^2-6))^2=\frac{1}{3}(2y^4-2y^2+3)$$

Taking the square root of both sides:

$$|n+\frac{1}{6}(-6y^2-6)|=\frac{\sqrt{2y^4-2y^2+3}}{3}$$

Eliminating the absolute value:

$$n+\frac{1}{6}(-6y^2-6)=\pm\frac{\sqrt{2y^4-2y^2+3}}{3}$$

Adding $\frac{1}{6}(6y^2+6)$ to both side:

$$n=\frac{1}{3}(3+3y^2\pm\sqrt{3}\sqrt{3-2Y62+2y^4})$$

and all the integer solutions n will be found from the following:

$$n=y^2\pm\frac{\sqrt{2y^4-2y^2+3}}{\sqrt{3}}+1\ and\ y,n\in\mathbb{Z}$$

Then we check if x_i and y_i give an integer solution for equation (79). In Ref. [17] it was conjectured that the only non-trivial integral zeroes are (17,7), (66,30), (1521,715), (15043,7476). It was proven in Ref. [9] that the list is complete.

Using MATHEMATICA we have checked up to the $20,000,000,000$ integer solution for equation (80). The only integral zeroes for equation (79) are (17,7), (66,30), (1521,715), (15043,7476) (see Appendix E).

2.7. **Krawtchouk Polynomials of Degree 5.**

The same system that applied to Krawtchouk polynomials of degree 4 can also be applied for degree 5.

Krawtchouk polynomials, k=5.nb

```
In[8]:= p[-1] = 0

Out[8]= 0

In[9]:= p[0] = 1

Out[9]= 1

In[10]:= p[k_Integer] := p[k] = Expand[((n - 2 x) p[k - 1] - (n - k + 2) p[k - 2]) / k]

In[11]:= p[5]
```

Out[11]= $\frac{n}{5}-\frac{5n^2}{12}+\frac{7n^3}{24}-\frac{n^4}{12}+\frac{n^5}{120}-\frac{2x}{5}+\frac{5nx}{6}-\frac{5n^2x}{4}+\frac{n^3x}{2}-\frac{n^4x}{12}+2nx^2-n^2x^2+\frac{n^3x^2}{3}-\frac{4x^3}{3}+\frac{2nx^3}{3}-\frac{2n^2x^3}{3}+\frac{2nx^4}{3}-\frac{4x^5}{15}$

```
In[12]:= % /. x → (n - y) / 2
```

Out[12]= $\frac{n}{5}-\frac{5n^2}{12}+\frac{7n^3}{24}-\frac{n^4}{12}+\frac{n^5}{120}+\frac{5}{12}n(n-y)-\frac{5}{8}n^2(n-y)+\frac{1}{4}n^3(n-y)-\frac{1}{24}n^4(n-y)+\frac{1}{2}n(n-y)^2-\frac{1}{4}n^2(n-y)^2+\frac{1}{12}n^3(n-y)^2-\frac{1}{6}(n-y)^3+\frac{1}{12}n(n-y)^3-\frac{1}{12}n^2(n-y)^3+\frac{1}{24}n(n-y)^4-\frac{1}{120}(n-y)^5+\frac{1}{5}(-n+y)$

```
In[13]:= Factor[%]
```

Out[13]= $\frac{1}{120}y\,(24-50n+15n^2+20y^2-10ny^2+y^4)$

```
In[14]:= Solve[% == 0, y]
```

Out[14]= $\{\{y\to 0\},\{y\to -\sqrt{-10+5n-\sqrt{2}\sqrt{38-25n+5n^2}}\},\{y\to \sqrt{-10+5n-\sqrt{2}\sqrt{38-25n+5n^2}}\},\{y\to -\sqrt{-10+5n+\sqrt{2}\sqrt{38-25n+5n^2}}\},\{y\to \sqrt{-10+5n+\sqrt{2}\sqrt{38-25n+5n^2}}\}\}$

FIGURE 4. Krawtchouk polynomials of degree 5.

It is clear from figure 4 that the equation to be solved will take the form:

$$2(y^2-5n+10)^2-5(2n-5)^2=27. \tag{81}$$

By reducing equation (81) to integer zeroes we find the following solutions:

For, $n = 1$, $y = \pm 1$, and for, $n = 2$, $y = \pm 2$,

For, $n \geq 3$, $y = \pm\sqrt{-10 + 5n \pm \sqrt{76 - 50n + 10n^2}}$.

Applying Pell's equation, using the substitituion $x_i = y^2 - 5n + 10$ and $y_i = 2n - 5$ in equation (81), we find:

$$2x_i^2 - 5y_i^2 = 27. \tag{82}$$

Using the solutions of Pell's equations given by equations (75) and (76), the integer solutions for equation (82) are:

Let $s \in \mathbb{Z}$ such that $s \geq 0$, then

$$x_1 = \frac{1}{4}(-8(19-6\sqrt{10})^s+\sqrt{10}(19-6\sqrt{10})^s-8(19+6\sqrt{10})^s-\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_2 = \frac{1}{4}(8(19-6\sqrt{10})^s-\sqrt{10}(19-6\sqrt{10})^s+8(19+6\sqrt{10})^s+\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_3 = \frac{1}{4}(8(19-6\sqrt{10})^s+\sqrt{10}(19-6\sqrt{10})^s+8(19+6\sqrt{10})^s-\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_4 = \frac{1}{4}(-8(19-6\sqrt{10})^s-\sqrt{10}(19-6\sqrt{10})^s-8(19+6\sqrt{10})^s+\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_5 = \frac{3}{4}(-4(19-6\sqrt{10})^s+\sqrt{10}(19-6\sqrt{10})^s-4(19+6\sqrt{10})^s-\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_6 = -\frac{3}{4}(-4(19-6\sqrt{10})^s+\sqrt{10}(19-6\sqrt{10})^s-4(19+6\sqrt{10})^s-\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_7 = \frac{3}{4}(4(19-6\sqrt{10})^s+\sqrt{10}(19-6\sqrt{10})^s+4(19+6\sqrt{10})^s-\sqrt{10}(19+6\sqrt{10})^s),$$
$$x_8 = -\frac{3}{4}(4(19-6\sqrt{10})^s+\sqrt{10}(19-6\sqrt{10})^s+4(19+6\sqrt{10})^s-\sqrt{10}(19+6\sqrt{10})^s).$$

and solutions for y_i are:

$$y_1 = \frac{1}{10}(-5(19-6\sqrt{10})^s+4\sqrt{10}(19-6\sqrt{10})^s-5(19+6\sqrt{10})^s-4\sqrt{10}(19+6\sqrt{10})^s),$$

$$y_2 = -\frac{1}{10}(-5(19-6\sqrt{10})^s+4\sqrt{10}(19-6\sqrt{10})^s-5(19+6\sqrt{10})^s-4\sqrt{10}(19+6\sqrt{10})^s),$$
$$y_3 = \frac{1}{10}(5(19-6\sqrt{10})^s+4\sqrt{10}(19-6\sqrt{10})^s+5(19+6\sqrt{10})^s-4\sqrt{10}(19+6\sqrt{10})^s),$$
$$y_4 = -\frac{1}{10}(5(19-6\sqrt{10})^s+4\sqrt{10}(19-6\sqrt{10})^s+5(19+6\sqrt{10})^s-4\sqrt{10}(19+6\sqrt{10})^s),$$
$$y_5 = \frac{3}{10}(-5(19-6\sqrt{10})^s+2\sqrt{10}(19-6\sqrt{10})^s-5(19+6\sqrt{10})^s-2\sqrt{10}(19+6\sqrt{10})^s),$$
$$y_6 = -\frac{3}{10}(-5(19-6\sqrt{10})^s+2\sqrt{10}(19-6\sqrt{10})^s-5(19+6\sqrt{10})^s-2\sqrt{10}(19+6\sqrt{10})^s),$$
$$y_7 = \frac{3}{10}(5(19-6\sqrt{10})^s+2\sqrt{10}(19-6\sqrt{10})^s+5(19+6\sqrt{10})^s-2\sqrt{10}(19+6\sqrt{10})^s),$$
$$y_8 = -\frac{3}{10}(5(19-6\sqrt{10})^s+2\sqrt{10}(19-6\sqrt{10})^s+5(19+6\sqrt{10})^s-2\sqrt{10}(19+6\sqrt{10})^s).$$

Example 2. *Let $s = 0$, in the solutions x_1 and y_1, then,*

$$x_1 = \frac{1}{4}(-8+\sqrt{10}-8-\sqrt{10}) = -4,$$

and,

$$y_1 = \frac{1}{10}(-5+4\sqrt{10}-5-4\sqrt{10}) = -1,$$

so, from equation (82) *we find,*

$$2(-4)^2 - 5(-1)^2 = 27$$

Other examples of integer solutions for equation (82) are given in the following table.

x_2	y_2	x_2	y_2
4	1	8367350944	5291977393
106	67	317738989726	200955781795
4024	2545	12065714258644	7631027730817
152680	96643	458179402838746	289778097989251
5802604	3669889	17398751593613704	11003936695860721
220346146	139359139	660694381154482006	417859816344718147

Using MATHEMATICA we have checked up to the $20,000,000,000$ integer solution for equation (82). The only integral zeroes for equation (81) are (17,3), (36,14), (67,28), (289,133), (10882,5292), (48324,24013) (see Appendix F).

3. Integer zeroes of Krawtchouk polynomials of degree 6 and 7

Roelof J. Stroker has provided a complete set of integral zeroes of the binary Krawtchouk polynomials of degree 6 and 7 [18]. The zeroes of these polynomials correspond to points on certain rational elliptic curves. The results are obtained by applying estimates of associated linear forms of elliptic logarithms.

3.1. **Krawtchouk Polynomials of Degree 6.**

The Krawtchouk polynomial for degree 6 is given by

$$\begin{aligned} y^6 - 15y^4n + 40y^4 + 45y^2n^2 - 210y^2n \\ -15n^3 + 184y^2 + 90n^2 - 120n = 0. \end{aligned} \tag{83}$$

This can be dealt with using Diophantine Equations. In this case, let $U = n$ and $V = y^2$ in equation (83), so that the following binary diophantine equation emerges:

$$\begin{aligned} -15U^3 + 45U^2V - 15UV^2 + V^3 + 90U^2 - \\ 210UV + 40V^2 - 120U + 184V = 0. \end{aligned} \tag{84}$$

Solutions for U and V in equation (84) are given by (with $M = -2V^6 + 30V^5 - 351V^4 + 620V^3 - 897V^2 + 600V - 400$):

$U_1 = \frac{1}{3^{2/3}}\sqrt[3]{\frac{2}{5}}(12V^3 - 15V^2 + \sqrt{3}\sqrt{M} + 3V)^{1/3}$
$-(-1350V^2 + 1350V - 2700)/(135 \times 5^{2/3}\sqrt[3]{6}(12V^3 - 15V^2 + \sqrt{3}\sqrt{M} + 3V)^{1/3} + V + 2,$

$U_2 = -\frac{1}{\sqrt[3]{5}6^{2/3}}(1 - i\sqrt{3}(12V^3 - 15V^2 + \sqrt{3}\sqrt{M} + 3V)^(1/3) + ((1 + i\sqrt{3})(-1350V^2 + 1350V^2 + 1350V - 2700))/(270 \times 5^{2/3}\sqrt[3]{6}(12V^3 - 15V^2 + \sqrt{3}\sqrt{M} + 3V)^{1/3}) + V + 2,$

and

$U_3 = -\frac{1}{\sqrt[3]{5}6^{2/3}}(1 + i\sqrt{3}(12V^3 - 15V^2 + \sqrt{3}\sqrt{M} + 3V)^(1/3) + ((1 - i\sqrt{3})(-1350V^2 + 1350V^2 + 1350V - 2700))/(270 \times 5^{2/3}\sqrt[3]{6}(12V^3 - 15V^2 + \sqrt{3}\sqrt{M} + 3V)^{1/3}) + V + 2.$

The first few integer solutions for equation (83) are then:
$n = 1, \ \ y = 1$
$n = 2, \ \ y = 0, 2$
$n = 3, \ \ y = 1, 3$

The zeroes of the Krawtchouk polynomials of degree 6 are then found from the following route:

Theorem 8. [18] *The diophantine equation (84) has integral solutions (U, V) as given in Table 3, below. In addition to the solutions, corresponding values of x,n,y are also given in the Table. Symmetry about x=n/2 permits us to reduce the required solutions to $x \leq n/2$.*

The complete set of integer zeroes for equation (83) was given in Ref. [18].

TABLE 3. Solutions of equation (83).

Solutions (U, V) of (47), $U = n$, $V = y^2$, $x \leq n/2$											
(U, V)	x	n	y	(U, V)	x	n	y	(U, V)	x	n	y
(-14,-56)				(3,1)	1	3	1	(9,25)	2	9	5
(-4,-20)				(3,9)	0	3	3	(12,4)	5	12	2
(-1,-9)				(4,0)	2	4	0	(12,36)	3	12	6
(0,0)	0	0	0	(4,4)	1	4	2	(12,100)	1	12	10
(1,1)	0	1	1	(4,16)	0	4	4	(16,144)	2	16	12
(2,-14)				(5,1)	2	5	1	(25,9)	11	25	3
(2,0)	1	2	1	(5,9)	1	5	3	(67,25)	31	67	5
(2,4)	0	2	2	(5,25)	0	5	5	(345,1225)	155	345	35
(3,-5)											

3.2. **Krawtchouk Polynomials of Degree 7.**

Modified Krawtchouk polynomial for degree 7 is:

$$y(y^6 - 21y^4n + 70y^4 + 105y^2n^2 - 630y^2n - 105n^3 \\ +784y^2 + 840n^2 - 1764n + 720) = 0. \tag{85}$$

To deal with this using Diophantine Equations we let $U = n - 1$ and $V = y^2 - 1$ in equation (85) to find

$$-105U^3 + 105U^2V - 21UV^2 + V^3 + 630U^2 \\ -462UV + 52V^2 - 840U + 360V = 0. \tag{86}$$

Solutions for U and V in equation (86) are then found through the following route:

Let $H = -2V^6 + 58V^5 - 1269V^4 + 6264V^3 - 38175V^2 + 88200V - 294000$

$U_1 = \frac{1}{3\times 5^{2/3}}\sqrt[3]{\frac{2}{7}}(20V^3 - 45V^2 + 3\sqrt{5}\sqrt{H} - 45V)^{1/3} - (-4410V^2 + 13230V - 132300)/(945\times 7^{2/3}\sqrt[3]{10}(20V^3 - 45V^2 + 3\sqrt{5}\sqrt{H} - 45V)^{1/3}) + \frac{V+6}{3}$,

$U_2 = -\frac{1}{3\sqrt[3]{7}10^{2/3}}(1 - i\sqrt{3})(20V^3 - 45V^2 + 3\sqrt{5}\sqrt{H} - 45V)^{1/3} + ((1 + i\sqrt{3})(-4410V^2 + 13230V - 132300))/(1890\times 7^{2/3}\sqrt[3]{10}(20V^3 - 45V^2 + 3\sqrt{5}\sqrt{H} - 45V)^{1/3}) + \frac{V+6}{3}$,

and

$U_3 = -\frac{1}{3\sqrt[3]{7}10^{2/3}}(1 + i\sqrt{3})(20V^3 - 45V^2 + 3\sqrt{5}\sqrt{H} - 45V)^{1/3} + ((1 - i\sqrt{3})(-4410V^2 + 13230V - 132300))/(1890\times 7^{2/3}\sqrt[3]{10}(20V^3 - 45V^2 + 3\sqrt{5}\sqrt{H} - 45V)^{1/3}) + \frac{V+6}{3}$.

The first few integer solutions for equation (85) are then:

$n = 1,\ \ y = 0, 1$

$n = 2,\ \ y = 0, 2$

$n = 3,\ \ y = 0, 1, 3$

$n = 4,\ \ y = 0, 2, 4$

The zeroes of Krawtchouk polynomials of degree 7 are found as follows:

Theorem 9. [18] *The diophantine equation (86) has integral solutions (U,V) as given in Table 4 below. In addition to the solutions, the table also gives the corresponding values of x,n,y. Similar to the Krawtchouk polynomials of degree 6, symmetry about x=n/2 permits the restriction to* $x \leq n/2$.

The complete set of integer zeroes for equation (85) was given in Ref. [18].

TABLE 4. Solutions of equation (85)

Solutions (U,V) of (49), $U=n$, $V=y^2$, $x \leq n/2$											
(U,V)	x	n	y	(U,V)	x	n	y	(U,V)	x	n	y
(-22,-132)				(3,-7)				(5,35)	0	6	6
(-6,-42)				(3,3)	1	4	2	(8,8)	3	9	3
(-3,-25)				(3,15)	0	4	4	(13,15)	5	14	4
(0,0)	0	1	1	(4,0)	2	5	1	(13,63)	3	14	8
(1,3)	0	2	2	(4,8)	1	5	3	(13,143)	1	14	12
(2,-18)				(4,24)	0	5	5	(16,80)	4	17	9
(2,0)	1	3	1	(5,3)	2	6	2	(21,255)	4	22	16
(2,8)	0	3	3	(5,15)	1	6	4	(1028,1368)	469	1029	37

The solution process again employs recent developments in the estimation of linear forms in elliptic logarithms. Extensive coverage of this method is given in Ref. [19], [20], & [21]. A detailed proof of Theorem 8 is found in Ref. [18], whilst the proof of Theorem 9 has an entirely similar structure.

Appendices

A. Mathematica programme

Firstly:

We should use the Expand[*expr*] to expands out products and positive integer powers in *expr*.

Secondly:

The Binomial [n,m] gives the binomial coefficient $\binom{n}{m}$.

Example:

When m=10 the result will be as follows:

```
Expand[(1-z)^Binomial[10,2](1+z)^Binomial[11,2]]
```

```
(1 + 10z - 330z^3 - 825z^4 + 4752z^5 + 21120z^6 -
    34320z^7 - 291060z^8 + 31240z^9 + 2708992z^10 +
    2204280z^11 - 18480540z^12 - 29306640z^13 + 95230080z^14 +
    233465232z^15 - 365945910z^16 - 1382588460z^17 +
    939642880z^18 + 6534277420z^19 - 585397098z^20 -
    25482402000z^21 - 9454193280z^22 + 83415992400z^23 +
    65482791660z^24 - 230728139928z^25 - 280152829440z^26 +
    537151192600z^27 + 932243618020z^28 - 1030675892400z^29 -
    2580943314048z^30 + 1528017910320z^31 + 6123319096455z^32 -
    1339395298410z^33 - 12640577986560z^34 - 1047608424918z^35 +
    22883390635105z^36 + 8025093350560z^37 - 36428580714240z^38 -
    22304274057120z^39 + 50888231592792z^40 + 45596171546640z^41 -
    61841242383360z^42 - 76943873141520z^43 + 64030757427720z^44 +
    111691296518752z^45 - 53669770668800z^46 - 142121681174880z^47 +
    30769808424300z^48 + 160003003806360z^49 - 160003003806360z^51 -
    30769808424300z^52 + 142121681174880z^53 + 53669770668800z^54 -
    111691296518752z^55 - 64030757427720z^56 + 76943873141520z^57 +
    61841242383360z^58 - 45596171546640z^59 - 50888231592792z^60 +
```

22304274057120z^61 + 36428580714240z^62 - 8025093350560z^63 -
22883390635105z^64 + 1047608424918z^65 + 12640577986560z^66 +
1339395298410z^67 - 6123319096455z^68 - 1528017910320z^69 +
2580943314048z^70 + 1030675892400z^71 - 932243618020z^72 -
537151192600z^73 + 280152829440z^74 + 230728139928z^75 -
65482791660z^76 - 83415992400z^77 + 9454193280z^78 +
25482402000z^79 + 585397098z^80 - 6534277420z^81 -
939642880z^82 + 1382588460z^83 + 365945910z^84 -
233465232z^85 - 95230080z^86 + 29306640z^87 + 18480540z^88 -
2204280z^89 - 2708992z^90 - 31240z^91 + 291060z^92 +
34320z^93 - 21120z^94 - 4752z^95 + 825z^96 + 330z^97 -
10z^99 - z^100)

B. Modification to the Mathematica programme

We will use the following commands in Mathematica:

(1) For[*start, test, incr, body*] – to make the programme run from 3 to 239.
(2) Print[*expr1, expr2,..*] – to print the *expr1*, followed by a new line.
(3) Min[$x_1, x_2, ...$] – yields the numerically smallest value for x_i (our case is 0).
(4) Table[*expr, imax*] – to tabulate the results.
(5) Abs[z] – to find the absolute value of the real or complex number z.
(6) Coefficient[*expr, form*] – gives the coefficient of *form* in the polynomial *expr*.
(7) Expand[].
(8) Binomial[].
(9) Floor[x] gives the greatest integer less than or equal to x.

The programme is as follows:

```
For[m = 3, m <= 239, m++,
  Print[
    Min[
      Table[
        Abs[
          Coefficient[
            Expand[
              (1 - z)^Binomial[m, 2]
                (1 + z)^Binomial[m + 1, 2]]
            , z^i]], {i, 3,
          Floor[(m^2 - 1)/2]}]]]]]
```

C. Mathematica for Modular Arithmetics

The following MATHEMATICA programme shows the results up to $m \leq 1000$. Even if we find $h[n] = 0$ for some values of n, we have also checked the results separately using a different prime number.

```
(* u[n] is a vector of the coefficients of
(1-x^2)^Binomial[n,2];
w[n] is a vector of the coefficients of
f[n]=(1-x^2)^Binomial[n,2]
(1+x)^n=(1-x)^Binomial[n,2] (1+x)^Binomial[n+1,2];
w1[n] is a vector of the coefficients of
w[n] from x^3 to the middle,
namely those we have to check that
they are not zeroes. All the coefficients at each
step of the calculations are reduced mod (pr).
The number h[n] is the minimal coefficient of
w1[n]; m[n] and gr[n] are calculated in advance
to avoid repeated computation of them at each step.*)

Clear[u, w, w1, h, pr, m, gr]
m[n_Integer] := m[n] = Binomial[n, 2];
gr[n_Integer] := gr[n] = (n^2 + Mod[n, 2])/2;
pr = Prime[100000]
1299709
u[n_Integer] := (g = {1};
  Do[g = Mod[Join[g, {0, 0}] - Join[{0, 0}, g], pr],
 {i, 1, m[n]}]; g)

w[n_Integer] := (g = u[n];
```

```
  Do[g = Mod[Join[g, {0}] + Join[{0}, g], pr],
{i, 1, n}]; g)
w1[n_Integer] := Take[w[n], {4, gr[n]}]
h[n_Integer] := Min[w1[n]]
A = 240; B = 1000;
Do[Print[{n, h[n]}], {n, A, B}]
```

D. Proximity to Zero Using Mathematica

```
m2[m_Integer] := m2[m] = m^2
m21[m_Integer] := m21[m] = (m2[m] - m)/2;
m22[m_Integer] := m22[m] = m21[m] + m
q[m_Integer] := q[m] = Expand[(1 - x^2)^m21[m] (1 + x)^m]
b[m_Integer, 0] = 1;
b[m_Integer, k_Integer] := b[m, k] = b[m, k - 1]
(m2[m] - k + 1)/k
u[m_Integer, i_Integer] :=
u[m, i] = Sqrt[b[m, i]]/(i (m2[m] - i))^(1/4)
rn[m_Integer] :=
Sqrt[Pi/2] Min[
Table[N[Abs[Coefficient[q[m], x^i]]/u[m, i], 40], {i, 3,
m21[m]}]]/2^(m2[m]/2) Sqrt[Binomial[m2[m], m21[m]]]
Do[Print[{m, Timing[rn[m] m^2]}], {m, 5, 50}]
```

E. Integer zeroes for Krawtchouk Polynomials of degree 4

```
Clear[y1, n1, y2, n2, yy1, yy2, n3, y3, yy3]
For [s = 1, s <= 20000000000, s++;

 y1 = Simplify[
   1/6 (3 (5 - 2 Sqrt[6])^s - 2 Sqrt[6] (5 - 2 Sqrt[6])^s +
      3 (5 + 2 Sqrt[6])^s + 2 Sqrt[6] (5 + 2 Sqrt[6])^s)];
 n1 = (y1 + 3)/2;
 yy1 = Sqrt[-4 + 3 n1 - Sqrt[16 - 18 n1 + 6 n1^2]];
 If[IntegerQ[yy1],
  Print["INTEGER ZERO (N,Y1) = ", {n1, (n1 - yy1)/2}]];

 y2 = Simplify[
   1/6 (-3 (5 - 2 Sqrt[6])^s - 2 Sqrt[6] (5 - 2 Sqrt[6])^s -
      3 (5 + 2 Sqrt[6])^s + 2 Sqrt[6] (5 + 2 Sqrt[6])^s)];
 n2 = (y2 + 3)/2;
 yy2 = Sqrt[-4 + 3 n2 + Sqrt[16 - 18 n2 + 6 n2^2]];
 If[IntegerQ[yy2],
  Print["INTEGER ZERO (N,Y2) = ", {n2, (n2 - yy2)/2}]];

 y3 = Simplify[
   1/6 (-3 (5 - 2 Sqrt[6])^s - 2 Sqrt[6] (5 - 2 Sqrt[6])^s -
      3 (5 + 2 Sqrt[6])^s + 2 Sqrt[6] (5 + 2 Sqrt[6])^s)];
 n3 = (y3 + 3)/2;
 yy3 = Sqrt[-4 + 3 n2 - Sqrt[16 - 18 n2 + 6 n2^2]];
 If[IntegerQ[yy3],
  Print["INTEGER ZERO (N,Y3) = ", {n3, (n3 - yy3)/2}]];]
```

```
INTEGER ZERO (N,Y1) = {66,30}

INTEGER ZERO (N,Y3) = {17,7}

INTEGER ZERO (N,Y2) = {1521,715}

INTEGER ZERO (N,Y3) = {15043,7476}
```

F. Integer zeroes for Krawtchouk Polynomials of degree 5

```
Clear[y1, n1, y2, n2, yy1, yy2, n3, y3, yy3, n4, y4, yy4, n5, y5,
yy5, n6, y6, yy6, n7, y7, yy7, n8, y8, yy8]

For [s = 0, s <= 20000000000, s++;

 y1 = Simplify[
   1/10 (5 (19 - 6 Sqrt[10])^s - 4 Sqrt[10] (19 - 6 Sqrt[10])^s +
      5 (19 + 6 Sqrt[10])^s + 4 Sqrt[10] (19 + 6 Sqrt[10])^s)];
 n1 = (y1 + 5)/2;
 yy1 = Sqrt[-10 + 5 n1 - Sqrt[76 - 50 n1 + 10 n1^2]];
 If[IntegerQ[yy1],
  Print["INTEGER ZERO (N,Y1) = ", {n1, (n1 - yy1)/2}]];

 y2 = Simplify[-3/
     10 (-5 (19 - 6 Sqrt[10])^s + 2 Sqrt[10] (19 - 6 Sqrt[10])^s -
      5 (19 + 6 Sqrt[10])^s - 2 Sqrt[10] (19 + 6 Sqrt[10])^s)];
 n2 = (y2 + 5)/2;
 yy2 = Sqrt[-10 + 5 n2 - Sqrt[76 - 50 n2 + 10 n2^2]];
 If[IntegerQ[yy2],
  Print["INTEGER ZERO (N,Y2) = ", {n2, (n2 - yy2)/2}]];

 y3 = Simplify[
   1/10 (5 (19 - 6 Sqrt[10])^s + 4 Sqrt[10] (19 - 6 Sqrt[10])^s +
      5 (19 + 6 Sqrt[10])^s - 4 Sqrt[10] (19 + 6 Sqrt[10])^s)];
 n3 = (y3 + 5)/2;
 yy3 = Sqrt[-10 + 5 n3 + Sqrt[76 - 50 n3 + 10 n3^2]];
 If[IntegerQ[yy3],
```

```
 Print["INTEGER ZERO (N,Y3) = ", {n3, (n3 - yy3)/2}]];

y4 = Simplify[
  1/10 (-5 (19 - 6 Sqrt[10])^s - 4 Sqrt[10] (19 - 6 Sqrt[10])^s -
     5 (19 + 6 Sqrt[10])^s + 4 Sqrt[10] (19 + 6 Sqrt[10])^s)];
n4 = (y4 + 5)/2;
yy4 = Sqrt[-10 + 5 n4 + Sqrt[76 - 50 n4 + 10 n4^2]];
If[IntegerQ[yy4],
 Print["INTEGER ZERO (N,Y4) = ", {n4, (n4 - yy4)/2}]];

y5 = Simplify[
  3/10 (5 (19 - 6 Sqrt[10])^s + 2 Sqrt[10] (19 - 6 Sqrt[10])^s +
     5 (19 + 6 Sqrt[10])^s - 2 Sqrt[10] (19 + 6 Sqrt[10])^s)];
n5 = (y5 + 5)/2;
yy5 = Sqrt[-10 + 5 n5 - Sqrt[76 - 50 n5 + 10 n5^2]];
If[IntegerQ[yy5],
 Print["INTEGER ZERO (N,Y5) = ", {n5, (n5 - yy5)/2}]];

y6 = Simplify[-3/
    10 (5 (19 - 6 Sqrt[10])^s + 2 Sqrt[10] (19 - 6 Sqrt[10])^s +
     5 (19 + 6 Sqrt[10])^s - 2 Sqrt[10] (19 + 6 Sqrt[10])^s)];
n6 = (y6 + 5)/2;
yy6 = Sqrt[-10 + 5 n6 - Sqrt[76 - 50 n6 + 10 n6^2]];
If[IntegerQ[yy6],
 Print["INTEGER ZERO (N,Y6) = ", {n6, (n6 - yy6)/2}]];

y7 = Simplify[
  1/10 (-5 (19 - 6 Sqrt[10])^s + 4 Sqrt[10] (19 - 6 Sqrt[10])^s -
     5 (19 + 6 Sqrt[10])^s - 4 Sqrt[10] (19 + 6 Sqrt[10])^s)];
n7 = (y7 + 5)/2;
yy7 = Sqrt[-10 + 5 n7 + Sqrt[76 - 50 n7 + 10 n7^2]];
```

```
If[IntegerQ[yy7],
 Print["INTEGER ZERO (N,Y7) = ", {n7, (n7 - yy7)/2}]];

y8 = Simplify[
  3/10 (-5 (19 - 6 Sqrt[10])^s + 2 Sqrt[10] (19 - 6 Sqrt[10])^s -
     5 (19 + 6 Sqrt[10])^s - 2 Sqrt[10] (19 + 6 Sqrt[10])^s)];
n8 = (y8 + 5)/2;
yy8 = Sqrt[-10 + 5 n8 - Sqrt[76 - 50 n8 + 10 n8^2]];
If[IntegerQ[yy8],
 Print["INTEGER ZERO (N,Y8) = ", {n8, (n8 - yy8)/2}]];]
```

```
INTEGER ZERO (N,Y1) = {36,14}

INTEGER ZERO (N,Y2) = {67,28}

INTEGER ZERO (N,Y4) = {17,3}

INTEGER  ZERO  (N,Y3) = {10882,5292}

INTEGER ZERO (N,Y6) = {289,133}

INTEGER ZERO (N,Y1) = {48324,24013}
```

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