

Complexity of Data Dependence problems for Program Schemas with Concurrency

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The problem of deciding whether one point in a program is data dependent upon another is fundamental to program analysis and has been widely studied. In this paper we consider this problem at the abstraction level of *program schemas* in which computations occur in the Herbrand domain of terms and predicate symbols, which represent arbitrary predicate functions, are allowed. Given a vertex l in the flowchart of a schema S having only equality (variable copying) assignments, and variables v, w , we show that it is PSPACE-hard to decide whether there exists an execution of a program defined by S in which v holds the initial value of w at at least one occurrence of l on the path of execution, with membership in PSPACE holding provided there is a constant upper bound on the arity of any predicate in S . We also consider the ‘dual’ problem in which v is required to hold the initial value of w at every occurrence of l , for which the analogous results hold. Additionally, the former problem for programs with non-deterministic branching (in effect, free schemas) in which assignments with functions are allowed is proved to be polynomial-time decidable provided a constant upper bound is placed upon the number of occurrences of the concurrency operator in the schemas being considered. This result is promising since many concurrent systems have a relatively small number of threads (concurrent processes), especially when compared with the number of statements they have.

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$$\begin{array}{l}
u := h(); \\
\text{if } p(u) \quad \text{then } v := f(u); \\
\quad \quad \quad \text{else } v := g();
\end{array}$$
Fig. 1. Schema S

1. INTRODUCTION

A schema represents the statement structure of a program by replacing real functions and predicates by symbols representing them. A schema, S , thus defines a whole class of programs which all have the same structure. Each program can be obtained from S via a domain D and an *interpretation* i which defines a function $f^i : D^n \rightarrow D$ for each function symbol f of arity n , and a predicate function $p^i : D^m \rightarrow \{\mathbf{T}, \mathbf{F}\}$ for each predicate symbol p of arity m . As an example, Figure 1 gives a schema S , and the program P of Figure 2 is defined from S by interpreting the function symbols f, g, h and the predicate symbol p as given by P , with D being the set of integers. The subject of schema theory is connected with that of program transformation and was originally motivated by the wish to compile programs effectively [Greibach 1975]. Many results on schema equivalence [Danicic et al. 2007; Laurence et al. 2004; 2003; Sabelfeld 1990; Luckham et al. 1970] and on applying schema formulation to program slicing [Laurence 2005; Danicic et al. 2005] have been published.

In this paper we are concerned with establishing complexity bounds for data dependence problems defined on schemas. We only consider schema interpretations over the Herbrand domain of terms in the variables and function symbols. We consider the problem of deciding the following two properties, defined using a schema S , a variable v , a variable or function symbol f and a vertex l in the flowchart of S .

- (Existential data dependence.) If there is an executable path through S that ends at l at which point the term defined by v contains the symbol f , then $\exists DD_S(f, v, l)$ is said to hold.
- (Universal data dependence.) If, for all executable paths through S , the term defined by v contains the symbol f whenever l is reached, then $\forall DD_S(f, v, l)$ is said to hold.

If S belongs to the class of schemas in which all assignments are equality assignments (that is, assignments of the form $v := w$; in which the value held by a variable w is copied to v), we prove the following.

- The problems defined by these properties are both PSPACE-hard, even when S is further required to belong to the class of schemas in which no concurrency

$$\begin{array}{l}
u := 1; \\
\text{if } u > 1 \quad \text{then } v := u + 1; \\
\quad \quad \quad \text{else } v := 2;
\end{array}$$
Fig. 2. Program P

$$\begin{array}{l}
x := f(); \\
\text{while } p(z) \text{ } x := g(); \\
l : \quad \quad \quad y := x;
\end{array}$$

Fig. 3. A schema demonstrating the greater precision obtainable by considering data dependence problems defined on program schemas compared with non-deterministic programs.

constructs are allowed and only two while loops are permitted in S , one of which lies in the body of the other, and no predicate symbol occurs more than once.

- If S is required to contain no loops or concurrency constructs, and each of its predicate symbols has zero arity, then $\exists DD_S(f, v, l)$ is NP-hard, and $\forall DD_S(f, v, l)$ is co-NP-hard.
- Both problems lie in PSPACE provided there is a constant upper bound on the arity of any predicate in S .

Additionally, we consider the existential data dependence problem in the case where assignments having function symbols are allowed, but where all schemas are free (that is, all paths are executable) and hence all branching is, in effect, non-deterministic. One possible application of data dependence on a function symbol f would be in the case where f corresponds to a call to a function or method that we are altering; we might then want to decide whether this change can propagate through to the value of a particular variable at a particular point. For the class of free schemas, we prove the following.

- Deciding existential data dependence is shown to be PSPACE-complete, owing to a reduction from the finite intersection problem for deterministic finite state automata.
- Under the further condition that a constant upper bound is placed upon the number of occurrences of the concurrency operator in the schemas being considered, existential data dependence then becomes decidable in polynomial time.

To the authors' knowledge, neither problem has been previously considered for arbitrary schemas. Both problems have been studied for programs of various types. In [Müller-Olm and Seidl 2001], it is proved that deciding existential data dependence (expressed in the paper as a slicing problem) is PSPACE-complete for programs having concurrency constructs, but only non-deterministic branching. Müller-Olm et al. have also considered a generalisation of our universal data dependence problem [2005a; 2005b], defined by testing for equality between two terms at particular program points, but their programs use term inequality guards on edges in flowcharts, and apart from this restriction, their programs are non-deterministic. In [Müller-Olm and Rütting 2001], an extensive classification of the complexity of deciding both our problems is given, but branching is non-deterministic and the domain is that of the integers in every case.

Schemas represent a significantly closer approximation to real-life programs than purely non-deterministic programs, even when these are very simple. To demonstrate this, consider the schema S in Figure 3, in which x, y and z are distinct variables.

Clearly $\exists DD_S(g, y, l)$ does not hold, since execution cannot enter the while loop in S and subsequently leave it, whereas if the while loop is replaced by the line

loop $x := g()$; to give a non-deterministic schema T , then $\exists DD_T(g, y, l)$ holds. This example motivates extending the study of data dependence problems to schemas, since the gain in precision may be considerable. Another justification for considering program schemas is given by the fact that they have precisely the same level of abstraction as is usually assumed in program slicing.

As an example of the use of establishing universal data dependence, consider a program which calculates the cost of a purchase - we would expect the overall price to depend always on the costs and amounts of the item(s) purchased. If this fails, then the program clearly contains a fault.

The complexity results for existential dependence are more promising than they might initially appear. This is because many concurrent systems have only relatively few threads even if they are quite large (in terms of lines of code). The results also suggest that it should be easier to ‘scale’ data dependence algorithms to large programs/schemas with only a few threads than to smaller programs/schemas with many threads. For schemas and programs that might not be free, data dependence calculated on the assumption that freeness holds provides a conservative abstraction of the actual data dependence. As a result, if existential data dependence does not hold under the freeness assumption then we know it does not hold even if the program or schema under consideration is not free. This is important in areas such as security where we wish to show that the value of one variable x , whose value is accessible, cannot depend on the value of another variable y whose value should be kept secret.

2. BASIC DEFINITIONS FOR SCHEMAS

Throughout this paper, \mathcal{F} , \mathcal{P} , \mathcal{V} and \mathcal{L} denote fixed infinite sets of *function symbols*, *predicate symbols*, *variables* and *labels* respectively. We assume a function

$$\text{arity} : \mathcal{F} \cup \mathcal{P} \rightarrow \mathbb{N}.$$

The arity of a symbol x is the number of arguments referenced by x . Note that in the case when the arity of a function symbol g is zero, g may be thought of as a constant.

Definition 2.1 schemas. We define the set of all *schemas* recursively as follows. $l : \text{skip}$ is a schema. An assignment $l : y := f(\mathbf{x})$; where $y \in \mathcal{V}$, $f \in \mathcal{F}$, $l \in \mathcal{L}$ and \mathbf{x} is a vector of $\text{arity}(f)$ variables, is a schema. Similarly an equality assignment $l : y := x$; for $y, x \in \mathcal{V}$ is a schema. From these all schemas may be ‘built up’ from the following constructs on schemas.

sequences. $S' = U_1 U_2 \dots U_r$ is a schema provided that each U_i for $i \in \{1, \dots, r\}$ is a schema.

if schemas. $S'' = l : \text{if } p(\mathbf{x}) \text{ then } T_1 \text{ else } T_2$ is a schema whenever $p \in \mathcal{P}$, $l \in \mathcal{L}$, \mathbf{x} is a vector of $\text{arity}(p)$ variables, and T_1, T_2 are schemas.

non-deterministic branches. $S'' = l : T_1 \sqcup T_2 \dots \sqcup T_m$ is a schema whenever $l \in \mathcal{L}$ and T_1, \dots, T_m are schemas.

while schemas. $S''' = l : \text{while } q(\mathbf{y}) T$ is a schema whenever $q \in \mathcal{P}$, $l \in \mathcal{L}$, \mathbf{y} is a vector of $\text{arity}(q)$ variables, and T is a schema.

non-deterministic loops. $S'''' = l : \text{loop } T$ is a schema if $l \in \mathcal{L}$ and T is a schema.

concurrent schemas. $S'''' = l : T_1 || T_2 \dots || T_m$ is a schema, where T_1, \dots, T_m are schemas.

We only consider schemas without repeated labels; for example, in the case of the ‘while’ schema $l : \text{while } q(\mathbf{y})T$, we assume that the label l does not occur in the recursive definition of T .

The semantics of schemas are defined by their flowcharts, which are finite directed graphs. A directed graph G is a pair (V, E) with $E \subseteq V \times V$. We define $V = \text{Vertices}(G)$, the set of *vertices of G*.

Definition 2.2. Given a schema S , we define a finite directed graph $\text{Flowchart}(S)$ with an edge labelling function edgeType_S that associates to each edge of $\text{Flowchart}(S)$ either ε , a triple (p, \mathbf{x}, X) for a predicate p , a vector \mathbf{x} of variables and $X \in \{\top, \text{F}\}$, or an assignment, as follows. Unless otherwise stated below, edgeType_S maps to ε .

- (1) If S is $l : \text{skip}$ or $l : y := f(\mathbf{x})$; or $l : y := x$; then $\text{Flowchart}(S)$ has vertex set $\{\mathbf{start}, l, \mathbf{end}\}$ and edges (\mathbf{start}, l) and (l, \mathbf{end}) . Here $\text{edgeType}_S(l, \mathbf{end}) = \varepsilon$, $y := f(\mathbf{x})$; or $y := x$;, respectively.
- (2) If $S = S_1 S_2$, then $\text{Flowchart}(S)$ has vertex set

$$\text{Vertices}(\text{Flowchart}(S_1)) \cup \text{Vertices}(\text{Flowchart}(S_2))$$

and contains every edge occurring in either S_1 or S_2 , with the function edgeType_S returning the same value as in S_1 or S_2 respectively, except that $\text{Flowchart}(S)$ does not have any edge (l, \mathbf{end}) for a vertex l in S_1 or (\mathbf{start}, l) for a vertex l in S_2 . Instead, it has an edge (l_1, l_2) for each pair of edges (l_1, \mathbf{end}) and (\mathbf{start}, l_2) in $\text{Flowchart}(S_1)$ and $\text{Flowchart}(S_2)$ respectively, with the function $\text{edgeType}_S(l_1, l_2) = \text{edgeType}_{S_1}(l_1, \mathbf{end})$.

- (3) If $S = l : S_1 \sqcup S_2 \dots \sqcup S_m$, then $\text{Flowchart}(S)$ has vertex set

$$\text{Vertices}(\text{Flowchart}(S_1)) \cup \dots \cup \text{Vertices}(\text{Flowchart}(S_m)) \cup \{l\}$$

and contains all edges (l', l'') lying in any $\text{Flowchart}(S_k)$ such that $l' \neq \mathbf{start}$, with the function edgeType_S returning the same value as edgeType_{S_k} in the appropriate $\text{Flowchart}(S_k)$, and also contains an edge (l, l'') for each edge (\mathbf{start}, l'') in any $\text{Flowchart}(S_k)$. Additionally, $\text{Flowchart}(S)$ contains the edge (\mathbf{start}, l) .

- (3') If $S = l : \text{if } p(\mathbf{x}) \text{ then } S_1 \text{ else } S_2$, then $\text{Flowchart}(S)$ is identical to $\text{Flowchart}(l : S_1 \sqcup S_2)$ except that the edges (l, l'') for each edge (\mathbf{start}, l'') in either $\text{Flowchart}(S_1)$ or $\text{Flowchart}(S_2)$ are mapped by edgeType_S to (p, \mathbf{x}, \top) or $(p, \mathbf{x}, \text{F})$ respectively.
- (4) If $S = l : \text{while } q(\mathbf{y})T$, then $\text{Flowchart}(S)$ has vertex set $\text{Vertices}(T) \cup \{l\}$ and contains all edges (l', l'') lying in $\text{Flowchart}(T)$ such that $l' \neq \mathbf{start}$ and $l'' \neq \mathbf{end}$, with the functions edgeType_S returning the same value as edgeType_T , and also contains an edge (l, l'') for each edge (\mathbf{start}, l'') in $\text{Flowchart}(T)$, with $\text{edgeType}_S(l, l'') = (q, \mathbf{y}, \top)$, and an edge (l'', l) for each edge (l'', \mathbf{end}) in $\text{Flowchart}(T)$, with $\text{edgeType}_S(l'', l) = \text{edgeType}_T(l'', \mathbf{end})$. Additionally, $\text{Flowchart}(S)$ contains the edges (\mathbf{start}, l) and (l, \mathbf{end}) , with $\text{edgeType}_S(l, \mathbf{end}) = (q, \mathbf{y}, \text{F})$.
- (4') If $S = l : \text{loop } T$, then $\text{Flowchart}(S)$ is identical to $\text{Flowchart}(\text{while } q(\mathbf{y})T)$, except that edges with l as initial vertex map to ε under edgeType_S .

(5) If $S = l : S_1 \| S_2 \| \dots \| S_m$, then $Flowchart(S)$ has vertex set

$$\times_{i=1}^m (Vertices(Flowchart(S_i)) \cup \{\mathbf{start}, l, \mathbf{end}\}),$$

and given any $r \leq m$ and vertices $l_i \in Vertices(Flowchart(S_i))$ for all $i \neq r$ and any edge (l', l'') in $Flowchart(S_r)$, the graph $Flowchart(S)$ has an edge

$$((l_1, \dots, l_{r-1}, l', l_{r+1}, \dots, l_m), (l_1, \dots, l_{r-1}, l'', l_{r+1}, \dots, l_m))$$

whose image under $edgeType_S$ is equal to $edgeType_{S_r}(l', l'')$. Additionally, $Flowchart(S)$ contains the edges (\mathbf{start}, l) , $(l, (\mathbf{start}, \dots, \mathbf{start}))$ and $((\mathbf{end}, \dots, \mathbf{end}), \mathbf{end})$.

2.1 Semantics of schemas

The symbols upon which schemas are built are given meaning by defining the notions of a state and of an interpretation. It will be assumed that variables take values in the set of terms built from the sets of variables and function symbols. This set, which we denote by $Term(\mathcal{F}, \mathcal{V})$, is usually called the Herbrand domain. It is formally defined as follows:

- each variable is a term,
- if $f \in \mathcal{F}$ is of arity n and t_1, \dots, t_n are terms then $f(t_1, \dots, t_n)$ is a term.

The function symbols represent the ‘natural’ functions with respect to the set of terms; that is, each function symbol f defines the function $(t_1, \dots, t_n) \mapsto f(t_1, \dots, t_n)$ for all n -tuples of terms (t_1, \dots, t_n) . A *state* is a function from \mathcal{V} into the set of terms. An interpretation i defines, for each predicate symbol $p \in \mathcal{P}$ of arity m , a function $p^i : Term(\mathcal{F}, \mathcal{V})^m \rightarrow \{\mathbf{T}, \mathbf{F}\}$. We define the *natural state* $e : \mathcal{V} \rightarrow Term(\mathcal{F}, \mathcal{V})$ by $e(v) = v$ for all $v \in \mathcal{V}$.

Definition 2.3 state associated with a path through Flowchart(S) for schema S. Given a state d , a schema S and a path ν through $Flowchart(S)$ whose first element is \mathbf{start} , we define the state $\mathcal{M}[\nu]_d$ recursively as follows.

- $\mathcal{M}[\mathbf{start}]_d(v) = d(v)$ for all variables v .
- If $\nu = \mu l l'$ for vertices l, l' in $Flowchart(S)$ and $edgeType(l, l')$ is not an assignment, then $\mathcal{M}[\nu]_d = \mathcal{M}[\mu]_d$.
- If $\nu = \mu l l'$ for $l, l' \in Labels(S)$ and S and $edgeType(l, l') = y := f(x_1, \dots, x_n)$;; then $\mathcal{M}[\nu]_d(z) = \mathcal{M}[\mu]_d(z)$ for all variables $z \neq y$, and

$$\mathcal{M}[\nu]_d(y) = f(\mathcal{M}[\mu]_d(x_1), \dots, \mathcal{M}[\mu]_d(x_n)),$$

and the case of equality assignments is treated analogously.

Definition 2.4 executable paths and free schemas. Given a schema S and an interpretation i and a path ν through $Flowchart(S)$ whose first element is \mathbf{start} , we say that ν is compatible with i if given any prefix $\mu l l'$ of ν such that $edgeType_S(l, l') = (p, x_1, \dots, x_n, X)$, $p^i(\mathcal{M}[\mu]_e(x_1), \dots, \mathcal{M}[\mu]_d(x_n)) = X$ holds. A path whose first element is \mathbf{start} is said to be executable if there exists an interpretation with which it is compatible. A schema is said to be free if every path whose first element is \mathbf{start} is executable.

Since a schema S may contain the non-deterministic **loop**, \sqcup and \parallel constructions, an initial state d and an interpretation i need not define a unique executable path in $Flowchart(S)$ from **start** to **end**. In the event that only one executable path exists, we denote it by $\pi_S(i, d)$, and write $\mathcal{M}\llbracket S \rrbracket_d^i$ to denote the state $\mathcal{M}\llbracket \pi_S(i, d) \rrbracket_d^i$. If S is merely a sequence of assignments, so that the interpretation i is irrelevant, then we simply write $\mathcal{M}\llbracket S \rrbracket_d$.

2.2 The data dependence problems

We now formalise the two data dependence conditions with which we are concerned in this paper.

Definition 2.5. Let S be a schema and let $v \in \mathcal{V}$, let $l \in Vertices(Flowchart(S))$ and let $f \in \mathcal{F} \cup \mathcal{V}$. The predicate $\exists DD_S(f, v, l)$ is defined to hold if there is an executable path μ through $Flowchart(S)$ which starts at **start** and ends at l such that the term $\mathcal{M}\llbracket \mu \rrbracket_e(v)$ contains f ; and the predicate $\forall DD_S(f, v, l)$ is defined to hold if for every executable path μ through $Flowchart(S)$ that starts at **start** and ends at l , the term $\mathcal{M}\llbracket \mu \rrbracket_e(v)$ contains f .

3. COMPLEXITY RESULTS FOR SCHEMAS HAVING ONLY EQUALITY ASSIGNMENTS

In this section, we prove that even if we restrict ourselves to the class of schemas without concurrency constructs and having only equality assignments, both the existential and universal data dependence problems are PSPACE-hard, and become NP-hard and co-NP-hard respectively if schemas are also required to be loop-free. We also show that if we keep the restriction to equality assignments but allow concurrency constructs, and add the further assumption of a constant bound on the arity of any predicate symbol, both problems lie in PSPACE.

3.1 Notational conventions

- In the proof of Theorems 3.1 and 3.5, we will define schemas without indicating labels, and indicate paths simply by using sequences of predicates and **end**. These schemas do not have the concurrency \parallel symbol and hence all vertices in the appropriate graph $Flowchart(S)$ lie in $Labels(S) \cup \{\mathbf{start}, \mathbf{end}\}$. In the cases where this convention is used, paths in the sense of Definition 2.2 are defined unambiguously.
- We will need to refer to finite sets of non-negative integers ‘without gaps’. Thus we define the set

$$[m, n] = \{m, m + 1, \dots, n\}$$

for any $m \leq n$.

- In order to save space, we will sometimes abbreviate schemas consisting of sequences of equality assignments by using the quantifier \forall . For example, in Fig. 5, the line $\forall k \in [0, m_j] t_k := s_{j,k}$ is intended as a shorthand for the sequence

$$t_0 := s_{j,0}; t_1 := s_{j,1}; \dots; t_{m_j} := s_{j,m_j};$$

The lines $\forall j \in [1, m] \forall k \in [0, m_j] s_{j,k} := u_{bad}$; and $\forall s \in \bigcup_{j \in [1, m]} F_j s := u_{good}$; in Fig. 6 have analogous meanings. We only use this notation in cases where the

$$D_l \equiv C_{l,y_{l,1}} C_{l,y_{l,2}} C_{l,y_{l,3}}$$

$$\text{where } C_{l,y} \equiv \begin{cases} \text{if } q_j() \text{ then } u_l := u_{l-1}; & \text{if } y = x_j \\ \text{if } q_j() \text{ then skip else } u_l := u_{l-1}; & \text{if } y = \neg x_j \end{cases}$$

Fig. 4. The definition of the schema D_l used in the proof of Theorem 3.1.

order of the assignments is immaterial, since no variable occurs on both the left side of one assignment in the sequence and the right side of another, and so the assignments commute.

- In Lemma 3.4 and Theorem 3.5, we will define finite state automata for which the word ‘state’ has its usual meaning; however we will also define schemas having variables which are the states of the automata, and thus the word state has the distinct meaning of a function from variables (automata-theoretic states) to elements of the domain (variables, in the case of schemas having only equality assignments). This should not cause confusion.

3.2 NP-hardness of data dependence problems for loop-free schemas without concurrency constructions

Our main NP-hardness result follows.

THEOREM 3.1. *For a schema S , $v \in \mathcal{V}$ and $f \in \mathcal{V}$, the problem of deciding $\exists DD_S(f, v, \mathbf{end})$ is NP-hard and that of deciding $\forall DD_S(f, v, \mathbf{end})$ is co-NP-hard, even when (in the case of both problems) S is restricted to membership of the class of schemas satisfying the following conditions.*

- S has no concurrency or non-deterministic branching constructions and has only equality assignments.
- S contains no loops.
- Each predicate in S has zero arity.

PROOF. We consider $\exists DD_S$ first, and then indicate the proof for $\forall DD_S$. To show NP-hardness of deciding whether $\exists DD_S(f, v, l)$ holds, we use a polynomial-time reduction from 3SAT, which is known to be an NP-hard problem [Cook 1971]. An instance of 3SAT comprises a set $X = \{x_1, \dots, x_n\}$ and a propositional formula $\rho = \bigwedge_{k=1}^m y_{k,1} \vee y_{k,2} \vee y_{k,3}$, where each $y_{i,j}$ is either x_k or $\neg x_k$ for some $k \leq n$. The problem is satisfied if there exists a valuation $\delta : X \cup \neg X \rightarrow \{\mathbf{T}, \mathbf{F}\}$ such that for each $x \in X$, $\{\delta(x), \delta(\neg x)\} = \{\mathbf{T}, \mathbf{F}\}$, under which ρ evaluates to \mathbf{T} . Given this instance of 3SAT we will construct a schema S that satisfies the conditions given in the statement of the Theorem and contains variables u_{bad}, u_0, \dots, u_n such that $\exists DD_S(u_0, u_n, \mathbf{end})$ holds if and only if ρ is satisfiable. The schema S is

$$\forall j \in [1, n] u_j := u_{bad}; D_1 \dots D_m,$$

where D_l is as defined in Figure 4. Clearly S can be constructed in polynomial time from the given instance of 3SAT, as required.

Assume first that there exists a valuation $\delta : X \rightarrow \{\mathbf{T}, \mathbf{F}\}$ under which ρ evaluates to \mathbf{T} . Define the interpretation i to map $q_j()$ to $\delta(x_j)$ for each q_j . Then the path

$\pi_S(i, e)$ clearly passes through at least one assignment $u_l := u_{l-1}$; within each D_l in S , proving $\exists DD_S(u_0, u_n, \mathbf{end})$ holds. Conversely, if $\exists DD_S(u_0, u_n, \mathbf{end})$ holds, then there is an interpretation i such that the path $\pi_S(i, e)$ passes through the sequence of assignments $u_1 := u_0; \dots, u_n := u_{n-1}$; in turn, and hence passes through $u_l := u_{l-1}$; at least once within each D_l . Define the valuation δ as follows; $\delta(x_j) = \top$ if and only if i maps $q_j()$ to \top . Clearly ρ evaluates to \top . Thus we have proved the Theorem for $\exists DD_S$.

To prove co-NP-hardness of deciding the $\forall DD_S$ relation under the restricted conditions given, observe that the final value of the variable u_n always lies in $\{u_0, u_{bad}\}$ and so $\exists DD_S(u_0, u_n, \mathbf{end}) \iff \neg \forall DD_S(u_{bad}, u_n, \mathbf{end})$ holds. Thus deciding $\forall DD_S(f, v, \mathbf{end})$ is co-NP-hard. \square

3.3 PSPACE-hardness result for data dependence problems for schemas without concurrency constructions

The main theorem of this subsection, Theorem 3.5, uses a polynomial-time reduction from the following automata-theoretic problem.

Definition 3.2. Consider a set of deterministic finite state automata A_1, \dots, A_m for some $m \geq 0$, all using an alphabet Σ . The finite state automata intersection problem is that of deciding whether there exists a word in Σ^* that is accepted by every automaton A_j .

THEOREM 3.3 [KOZEN 1977]. *The finite state automata intersection problem is PSPACE-complete.*

Given a deterministic finite state automaton A and a member σ of its alphabet, we wish to construct a schema consisting only of a sequence of assignments whose variables are the states of A and such that for any transition $s \xrightarrow{\sigma} s'$ in A , $\forall DD_S(s', s, \mathbf{end})$ holds. The schema

$$\begin{aligned} \forall k \in [0, a] \ t_k &:= s_k; \\ \forall k \in [0, a] \ s_k &:= t_{\chi(k)}; \end{aligned}$$

satisfies this requirement if A has state set $\{s_0, \dots, s_a\}$ and its σ -transitions all have the form $s_k \xrightarrow{\sigma} s_{\chi(k)}$ for a function $\chi : [0, a] \rightarrow [0, a]$, with new variables t_k disjoint from the variables s_l . It may be worth mentioning that the simpler schema

$$\begin{aligned} s_0 &:= s_{\chi(0)}; \\ &\vdots \\ s_a &:= s_{\chi(a)}; \end{aligned}$$

does *not* satisfy the required data dependence condition because the assignments may ‘interfere’ with one another; for example, if A has only two states s_0, s_1 and has transitions $s_1 \xrightarrow{\sigma} s_0$ and $s_0 \xrightarrow{\sigma} s_1$, then if S is the schema

$$\begin{aligned} s_0 &:= s_1; \\ s_1 &:= s_0; \end{aligned}$$

then $\forall DD_S(s_1, s_1, \mathbf{end})$ rather than the required $\forall DD_S(s_0, s_1, \mathbf{end})$ holds. Thus it is necessary to introduce the ‘copying’ variables t_k .

$$\begin{aligned} \forall k \in [0, m_j] \ t_k &:= s_{j,k}; \\ \forall k \in [0, m_j] \ s_{j,k} &:= t_{\chi_j(l,k)}; \end{aligned}$$

Fig. 5. The schema $U_{j,l}$ of Lemma 3.4. Here the t_k are new variables used solely for copying and the function χ_j is defined by the state transition function η_j of the automaton A_j as follows; for any letter α_l and state $s_{j,k}$, $\eta_j(\alpha_l, s_{j,k}) = s_{j,\chi_j(l,k)}$. Observe that the value defined by a variable $s_{j,k}$ after execution of $U_{j,l}$ is the same as that defined by the variable $\eta_j(\alpha_l, s_{j,k})$ before execution, since $\forall DD_{U_{j,l}}(\eta_j(\alpha_l, s_{j,k}), s_{j,k}, \mathbf{end})$ holds.

The motivation for constructing a schema in this way from a given finite state automaton is shown by Lemma 3.4.

LEMMA 3.4. *Consider a set of m deterministic finite state automata A_1, \dots, A_m for some $m \geq 0$, all using an alphabet $\Sigma = \{\alpha_1, \dots, \alpha_n\}$, with each automaton A_j having state set $S_j = \{s_{j,0}, \dots, s_{j,m_j}\}$ and total transition function $\eta_j : \Sigma \times S_j \rightarrow S_j$. For each automaton A_j and each letter $\alpha_l \in \Sigma$, let $U_{j,l}$ be the predicate-free schema in Fig. 5 and define $V_l = U_{1,l} \dots U_{m,l}$. Let $l_1, l_2, \dots, l_r \in [1, n]$ and define $\gamma = \alpha_{l_r} \alpha_{l_{r-1}} \dots \alpha_{l_1} \in \Sigma^*$.*

- (1) *For every $j \in [1, m]$ and any $s \in S_j$, $\forall DD_{V_{l_1} \dots V_{l_r}}(\eta_j(\gamma, s), s, \mathbf{end})$ holds.*
- (2) *Assume each automaton A_j has initial state $s_{j,0}$ and final state set $F_j \subseteq S_j$. Let e_{final} be the state (in the program sense)*

$$\begin{cases} s_{j,k} \mapsto u_{bad} & s_{j,k} \in S_j - F_j \\ s_{j,k} \mapsto u_{good} & s_{j,k} \in F_j \end{cases}$$

for new variables u_{bad}, u_{good} . Then

$$\mathcal{M}[[V_{l_1} \dots V_{l_r}]_{e_{final}}(s_{j,0}) = u_{good}$$

for all j if and only if the word γ is accepted by every automaton A_j .

PROOF. (1) can be straightforwardly proved by induction on r . (2) follows immediately from (1) using the fact that for any j , A_j accepts γ if and only if $\eta_j(\gamma, s_{j,0}) \in F_j$ holds. \square

We now give the main PSPACE-hardness theorem of the paper, Theorem 3.5. The proof of this Theorem will construct a schema in which solving an existential data dependence problem corresponds to solving a given instance of the finite state automata intersection problem. Parts of the schema constructed will ‘simulate’ state transitions of the automata.

THEOREM 3.5. *For a schema S , $v \in \mathcal{V}$ and $f \in \mathcal{V}$, the problems of deciding whether $\exists DD_S(f, v, \mathbf{end})$ and $\forall DD_S(f, v, \mathbf{end})$ hold are both PSPACE-hard, even when S is restricted to membership of the class of schemas satisfying the following conditions.*

- *S has no concurrency or non-deterministic branching constructions and has only equality assignments,*
- No predicate occurs more than once in S .*

— S contains two while predicates, one of which lies in the body of the other.

PROOF. We consider $\exists DD_S$ first, and then indicate the proof for $\forall DD_S$. We prove the Theorem using a reduction from the intersection problem for finite state automata, given in Definition 3.2, which is PSPACE-complete by Theorem 3.3. Thus we assume an instance of this problem comprising a set of m deterministic finite state automata A_1, \dots, A_m for some $m \geq 0$, all using an alphabet $\Sigma = \{\alpha_1, \dots, \alpha_n\}$, with each A_j having state set $S_j = \{s_{j,0}, \dots, s_{j,m_j}\}$, total transition function $\eta_j : \Sigma \times S_j \rightarrow S_j$, initial state $s_{j,0}$ and final state set $F_j \subseteq S_j$, as in the statement of Lemma 3.4. The problem is satisfied if there is a word in Σ^* which is accepted by every automaton A_j .

Given these automata, consider the schema S given in Fig. 6. Clearly S satisfies the conditions listed in the statement of the Theorem and S can be constructed in polynomial time from the set of automata A_j as input. We now show that $\exists DD_S(u_{good}, a_m, \mathbf{end})$ holds if and only if the intersection of the acceptance sets of all the automata A_j is non-empty, thus proving the Theorem.

```

 $\forall j \in [1, m] a_j := u_{bad};$ 
 $\forall j \in [1, m] b_j := u_{bad};$ 
while  $Q_1(a_m)$ 
{
   $\forall j \in [1, m] a_j := u_{bad};$ 
   $\forall j \in [1, m-1] b_j := u_{bad};$ 
   $c := u_{bad};$ 
   $\forall j \in [1, m] \forall k \in [0, m_j] s_{j,k} := u_{bad};$ 
  if  $Q_2(b_m)$ 
    then  $c := u_{good};$ 
    else {
       $\forall s \in \bigcup_{j \in [1, m]} F_j s := u_{good};$ 
      while  $Q_3(s_{1,0}, \dots, s_{m,0}) T_n$ 
      }
  if  $p_1(s_{1,0})$ 
    then  $a_1 := b_m;$ 
    else  $b_1 := c;$ 
  if  $p_2(s_{2,0})$ 
    then  $a_2 := a_1;$ 
    else  $b_2 := b_1;$ 
  :
  :
  if  $p_m(s_{m,0})$ 
    then  $a_m := a_{m-1};$ 
    else  $b_m := b_{m-1};$ 
}

```

Fig. 6. The schema S used in the proof of Theorem 3.5. The schema T_n is defined in Fig. 7.

—(\Leftarrow). Assume first that there is a word $\gamma = \alpha_{d_z} \alpha_{d_{z-1}} \dots \alpha_{d_1}$ that is accepted by every automaton A_j , for minimal z . We will prove that $\exists DD_S(u_{good}, a_m, \mathbf{end})$ holds. Define the interpretation i on the predicates Q_1, Q_2, Q_3 and each p_j as

$$\begin{aligned}
T_i &\equiv \quad \text{if } q_i(\mathbf{s}) && \text{then } V_i \\
&&& \text{else } T_{i-1} \\
T_1 &\equiv V_1 \\
\text{where } & V_l \equiv U_{1,l} \dots U_{m,l}
\end{aligned}$$

Fig. 7. The recursive definition of the schema T_l . Here \mathbf{s} is a vector whose entries are all the variables $s_{j,k}$, in any fixed order, and $U_{j,l}$ is the schema in Fig. 5. Observe that an execution of T_n entails an execution of one schema V_l , for some $l \in [1, n]$.

follows.

$$\begin{cases}
Q_1(u_{bad}) \mapsto \top, & Q_1(u_{good}) \mapsto \text{F} \\
Q_2(u_{bad}) \mapsto \top, & Q_2(u_{good}) \mapsto \text{F} \\
Q_3(v_1, \dots, v_m) \mapsto \text{F} \text{ iff every } v_j = u_{good} \\
p_j(u_{bad}) \mapsto \text{F}, & p_j(u_{good}) \mapsto \top
\end{cases}$$

We now indicate how i is defined on the predicates q_l . Define the path

$$\begin{aligned}
\mu &= Q_3 q_n q_{n-1} \dots q_{d_1} Q_3 q_n q_{n-1} \dots q_{d_2} Q_3 \dots Q_3 q_n q_{n-1} \dots q_{d_z} Q_3 \\
&\in \Pi(\text{while}(Q_3(s_{1,0}, \dots, s_{m,0})) T_n).
\end{aligned}$$

We wish $\pi_S(i, e)$ to follow the path μp_1 whenever it encounters $\text{while}(Q_3(s_{1,0}, \dots, s_{m,0})) T_n$, in effect executing the schema $V_{d_1} \dots V_{d_z}$. We now show that this is possible. First observe that by Part (2) of Lemma 3.4 applied to the suffices of γ , every variable $s_{j,0}$ defines the value u_{good} at the last occurrence of Q_3 along μ , but this does not hold at any earlier occurrence of Q_3 , since this would imply that a strict suffix of γ was accepted by every automaton A_j , contradicting the minimality of z . Thus the definition of i on Q_3 given above ensures that $\pi_S(i, e)$ follows the path μp_1 where required, provided that i can be defined appropriately on each predicate q_l .

Suppose that this is impossible; that is, that there is a repeated q_l -predicate term along μ for some q_l , which i would have to map to both \top and F . Thus we can write $\mu = \mu' q_l \mu'' q_l \mu'''$ such that every variable $s_{j,k}$ defines the same value at the two occurrences of q_l . Assume that Q_3 occurs z' times in μ' and z'' times in μ'' ; clearly $z'' \geq 1$. Since no variable apart from the variables $s_{j,k}$ occurs in the while schema guarded by Q_3 , every variable $s_{j,0}$ defines the same value after the path $\mu' q_l \mu''$ as after μ , namely u_{good} . Thus by Part (2) of Lemma 3.4, the word $\alpha_{d_z} \alpha_{d_{z-1}} \dots \alpha_{d_{z'+z''}} \dots \alpha_{d_{z'-1}} \dots \alpha_{d_1}$ is accepted by every automaton A_j , contradicting the minimality of z .

Thus we have shown that the interpretation i can be defined so that $\pi_S(i, e)$ always follows the path μ whenever $\text{while}(Q_3(s_{1,0}, \dots, s_{m,0})) T_n$ is reached, and furthermore, every variable $s_{j,0}$ defines the value u_{good} at the end of μ , and so p_1 is the next symbol through which $\pi_S(i, e)$ passes.

We now prove that $\mathcal{M}[S]_e^i(a_m) = u_{good}$ holds. The definition of i on Q_1 ensures that $\pi_S(i, e)$ passes at least once through the body of Q_1 , and since i maps $Q_2(b_m)$ to \top and each $p_j(u_{bad})$ to F , on the first passing of $\pi_S(i, e)$ through the body of Q_1 , the assignment $c := u_{good}$; and all assignments to every b_j occur, and hence

b_m defines the value u_{good} when Q_1 is reached for the second time along $\pi_S(i, e)$. Since i maps $Q_1(u_{bad})$ to \top , the path $\pi_S(i, e)$ then enters the body of Q_1 a second time, and since i maps $Q_2(u_{good})$ to \top , this time $\pi_S(i, e)$ passes through Q_3 . As proved above, $\pi_S(i, e)$ terminates within $while(Q_3(s_{1,0}, \dots, s_{m,0}))T_n$ and every $s_{j,0}$ defines u_{good} when $\pi_S(i, e)$ then reaches p_1 , and so $\pi_S(i, e)$ then passes through all the assignments $a_1 := b_m$; and $a_j := a_{j-1}$; after which a_m defines the value u_{good} . Since i maps $Q_2(a_m)$ to \top , $\exists DD_S(u_{good}, a_m, \mathbf{end})$ holds, as required.

—(\Rightarrow). Conversely, suppose that $\exists DD_S(u_{good}, a_m, \mathbf{end})$ holds. Thus $\mathcal{M}[\llbracket S \rrbracket_e^i(a_m)] = u_{good}$ holds for some interpretation i . The only sequence of assignments which could copy u_{good} at the start of S to a_m at the end consists, in order, of the assignment $c := u_{good}$; and those referencing every b_j for $j < m$ followed by those referencing b_m and every a_j for $j < m$, and so $\pi_S(i, e)$ must pass through all of these in turn. Furthermore, owing to the assignments setting c and b_1, \dots, b_{m-1} to u_{bad} , the assignments referencing c and every b_j for $j < m$ must occur in a single passing through the body of Q_1 , during which every $s_{j,0}$ defines u_{bad} when p_j is reached. Thus i must map every $p_j(u_{bad})$ to \top . Similarly, owing to the assignments $a_j := u_{bad}$; the assignments referencing every a_j for $j < m$ must also occur in a single passing through the body of Q_1 , and so the predicate term defined by each $p_j(s_{j,0})$ must map to \top , and so every $s_{j,0}$ must define a value distinct from u_{bad} simultaneously. The only possibility is u_{good} , and so at some point the path $\pi_S(i, e)$, must reach p_1 with each $s_{j,0}$ defining u_{good} , and thus must have passed through Q_3 since the last occurrence of Q_2 . Let $V_{d_1} \dots V_{d_z}$ be the sequence of schemas V_k occurring on $\pi_S(i, e)$ since this occurrence; then by Part (2) of Lemma 3.4, the word $\alpha_{d_z} \alpha_{d_z-1} \dots \alpha_{d_1}$ is accepted by every automaton A_j , as required.

To prove PSPACE-hardness of deciding the $\forall DD_S$ relation, observe that the final value of the variable a_m always lies in $\{u_{good}, u_{bad}\}$ and so $\exists DD_S(u_{good}, a_m, \mathbf{end}) \iff \neg \forall DD_S(u_{bad}, a_m, \mathbf{end})$ holds. Thus deciding $\forall DD_S(f, v, \mathbf{end})$ is co-PSPACE-hard and hence PSPACE-hard. \square

3.4 Membership in PSPACE of data dependence problems for the class of schemas having a bound on the arity of all predicates and having only equality assignments, but without restrictions on concurrency constructs

In order to prove that our problems lie in PSPACE, we need to show that the successors of a vertex in $Flowchart(S)$ can be enumerated in polynomial time. This motivates Theorem 3.6.

THEOREM 3.6. *Let S be a schema.*

- (1) *The vertices of $Flowchart(S)$ can be encoded as words in the alphabet $Labels(S) \cup \{\mathbf{start}, \mathbf{end}\}$ in which no element of $Labels(S)$ occurs more than once and \mathbf{start} and \mathbf{end} each occur not more than $|Labels(S)|$ times.*
- (2) *Given any $l' \in Vertices(Flowchart(S))$, the set of all $l'' \in Vertices(Flowchart(S))$ for which (l', l'') is an edge in $Flowchart(S)$, and the corresponding values of $edgeType(l', l'')$, can be computed in polynomial time.*

PROOF. (1) We indicate the encoding by assuming that S has the form $S = l : S_1 \parallel S_2 \parallel \dots \parallel S_m$; the encoding in the case of the other constructions given in

Definition 2.2 is straightforward to infer. In the concurrent case, $Flowchart(S)$ has vertex set $\times_{i=1}^m (Vertices(Flowchart(S_i)) \cup \{\mathbf{start}, l, \mathbf{end}\})$ and a vertex of $Flowchart(S)$ can be encoded either by an element of $\{\mathbf{start}, l, \mathbf{end}\}$ (representing themselves) or by a word $w = w_1 \dots w_m$, where each w_i represents an element

$l_i \in Vertices(Flowchart(S_i))$ and w represents (l_1, \dots, l_m) . The conditions given on the frequency of letters in w follow easily from those for each w_i and the fact we assume that no label occurs more than once in S .

- (2) This follows easily by induction on the structure of S , using the encoding given in Part (1) of this Theorem. \square

Our other main theorem of this Section follows.

THEOREM 3.7. *Let S be a schema and let $v \in \mathcal{V}$, let l be a vertex of $Flowchart(S)$ and let $f \in \mathcal{V}$. Assume that all assignments in S are equality assignments. Assume that there is a constant upper bound on the arity of any predicate symbol occurring in S . Then the problems of deciding whether $\exists DD_S(f, v, l)$ or $\forall DD_S(f, v, l)$ hold both lie in PSPACE.*

PROOF. We first prove decidability of $\exists DD_S(f, v, l)$ in PSPACE. We do this by constructing the following algorithm, which lies in NPSPACE. We non-deterministically guess a path beginning at \mathbf{start} through the schema S that realises the copying of the initial value of the variable f onto v at the vertex l . At each point in the algorithm we store not just the vertex and the state (with the domain restricted to the set of variables referenced in S) reached, but also a finite, initially empty set of equations of the form $p(\mathbf{y}) = X$ for predicate p occurring in S , variable vector \mathbf{y} whose components are referenced in S and $X \in \{\mathsf{T}, \mathsf{F}\}$. If n is an upper bound on the total number of predicates and variables occurring in S and b is the assumed constant upper bound on the arity of any predicate in the class of schemas under consideration, then the number of equations of this form is bounded by $2n^{b+1}$ and thus the data stored at any point in the execution of the algorithm is polynomially bounded.

Whenever the algorithm crosses an edge (l', l'') in $Flowchart(S)$ satisfying $edgeType_S(l', l'') = (q, \mathbf{x}, X)$, the equation $q(\mathbf{y}) = X$ is added to the set, where the vector $\mathbf{y} = \mathcal{M}[\mu]_e \mathbf{x}$, with μ being the path traced by the algorithm up to the vertex l' . No equation is added to the set when an edge for which $edgeType_S$ returns ϵ or an assignment is crossed. Thus this equation set encodes the set of interpretations which are compatible with the path followed, in the sense that an interpretation i is compatible with this path if and only if $p(\mathbf{y}) = X$ is a consequence of i for all equations $p(\mathbf{y}) = X$ in the set.

The algorithm terminates and returns *false* if the equation set acquires a pair of contradictory equations (that is, a pair $p(\mathbf{w}) = \mathsf{T}$, $p(\mathbf{w}) = \mathsf{F}$) at any point. It terminates and returns *true* if l is reached with the state mapping v to f without two contradictory equations having occurred in the set. By Theorem 3.6, this algorithm lies in NPSPACE. Since PSPACE = NPSPACE holds, the problem of deciding $\exists DD_S(f, v, l)$ is thus in PSPACE.

To prove decidability of $\forall DD_S(f, v, l)$ in co-NPSPACE = PSPACE instead, we modify the algorithm as follows; termination with output *true* occurs if l is reached with the state *not* mapping v to f . \square

4. COMPLEXITY RESULTS FOR FREE SCHEMAS

If we allow assignments with function symbols, and not just equalities, to occur in schemas, then deciding data dependence becomes harder, and the proof of membership in PSPACE for both problems in Theorem 3.7 does not appear to generalise. However, under restriction to the class of free schemas, we prove in Theorem 4.5 that deciding existential data dependence is PSPACE-complete, using Müller-Olm’s result [Müller-Olm and Seidl 2001] for non-deterministic programs. Additionally, we prove in Theorem 4.11 that under the further condition that a constant bound is placed on the number of subschemas occurring in parallel, this problem becomes polynomial-time decidable.

Recall that a schema is *free* if every path through its flowchart is executable. As an example, the schema

$$\textit{while } q(z) \textit{ do } z := h(z);$$

(we have omitted labels from its definition) is free, whereas $\textit{while } q(z) \textit{ do } z := g();$ is not free, since there is no interpretation and initial state such that the path so defined enters the body of q exactly once.

4.1 PSPACE-completeness of the existential data dependence problem for free schemas

Theorem 4.5 is the main result of this subsection.

LEMMA 4.1. *Given any schema S without predicates, a variable v and $f \in \mathcal{V} \cup \mathcal{F}$, the problem of deciding whether $\exists DD_S(f, v, \mathbf{end})$ holds is PSPACE-hard.*

PROOF. This is [Müller-Olm and Seidl 2001, Theorem 2]. \square

LEMMA 4.2. *Given any free schema S , a vertex l in $\textit{Flowchart}(S)$, a variable v and $f \in \mathcal{V} \cup \mathcal{F}$, with l, v and f all occurring in S , there exists a free schema S' which does not contain any **loop** or \sqcup constructions, and such that $\exists DD_S(f, v, l)$ holds if and only if $\exists DD_{S'}(f, v, l)$ does. Furthermore, S' can be constructed in polynomial time from S .*

PROOF. Given S , we replace **loop** or \sqcup constructions with while and nested if statements respectively, in the following way. Let z be a variable not occurring in S and not equal to v or f , let h be any function symbol and let q be any predicate symbol. Suppose that $m : \mathbf{loop } T$ occurs in S ; then we replace it by $m' : z := h(z); m : \textit{while } q(z) \textit{ do } \{m'' : z := h(z); T\}$, for new labels m', m'' . Similarly, an occurrence of $m : T_n \sqcup \dots \sqcup T_1$ in S can be replaced by the schema $m : P_n$, where we recursively define $P_1 \equiv z := h(z); T_1$ and $P_r \equiv \textit{if } q(z) \textit{ then } z := h(z); T_r \textit{ else } z := h(z); P_{r-1}$ for $r > 1$, where we have omitted labels in the definitions of each P_r . Let S' be the schema obtained from S after all the **loop** or \sqcup constructions have been replaced. Since z is never referenced in the original schema S , the new assignments to z cannot interfere with the existing data dependence relations in S , and the length of any term defined by z along a path through S' must successively increase at each assignment to z , hence the introduction of the new while and if statements cannot cause repeated predicate terms to occur. Thus S' is free if S is. There is a natural correspondence between paths in S and in S' , and thus $\exists DD_S(f, v, l)$ holds if and only if $\exists DD_{S'}(f, v, l)$ follows. Also, S' can be constructed in polynomial time from S , proving the Lemma. \square

Definition 4.3. Given a schema S , $l, l' \in \text{Vertices}(\text{Flowchart}(S))$ and variables v, v' , we define the relation $(l, v) \rightsquigarrow_S^*(l', v')$ to hold if either $\text{edgeType}(l, l')$ is an assignment to v' that references v , or $v = v'$ and $\text{edgeType}(l, l')$ is not an assignment to v' .

LEMMA 4.4. *For any free schema S , a vertex l in $\text{Flowchart}(S)$, a variable v and $f \in \mathcal{F}$, $\exists DD_S(f, v, l)$ holds if and only if there exist $m, n \in \text{Vertices}(\text{Flowchart}(S))$ and a variable w such that $\text{edgeType}(m, n)$ is an assignment to w with function symbol f and $(n, w) \rightsquigarrow_S^*(l, v)$ holds.*

PROOF. This follows immediately from the definition of $\exists DD_S(f, v, l)$. \square

THEOREM 4.5. *Given any free schema S , a vertex l in $\text{Flowchart}(S)$, a variable v and $f \in \mathcal{V} \cup \mathcal{F}$, the problem of deciding whether $\exists DD_S(f, v, l)$ holds is PSPACE-complete, and is PSPACE-hard even if $l = \mathbf{end}$ and S does not contain any **loop** or \sqcup symbols.*

PROOF. The PSPACE-hardness result follows immediately from Lemmas 4.1 and 4.2.

To show membership in PSPACE, we first assume that $f \in \mathcal{F}$, since if $f \in \mathcal{V}$ then we can replace S by the schema $S' \equiv f := g(); S$ for a function symbol g not occurring in S , for then $\exists DD_S(f, v, l) \iff \exists DD_{S'}(g, v, l)$ holds, and S' can be constructed in polynomial time from the input. The result then follows from Lemma 4.4 as follows. We non-deterministically guess an edge (m, n) in $\text{Flowchart}(S)$ and a variable w such that $\text{edgeType}(m, n)$ is an assignment to w with function symbol f and then decide whether $(n, w) \rightsquigarrow_S^*(l, v)$ holds. This can be done by guessing a path from (n, w) to (l, v) in the digraph whose vertices are pairs (l', v') for $l' \in \text{Vertices}(\text{Flowchart}(S))$ and variables v' occurring in S and whose edges are given by the \rightsquigarrow_S relation. At any point in the algorithm, only the current pair (l', v') is stored, rather than the entire graph. By Theorem 3.6, only polynomial space in the input is required for this, thus proving that the problem lies in NPSpace = PSPACE. \square

4.2 Polynomial-time complexity of the existential data dependence problem for the class of free schemas with a bound on the number of concurrency constructs

We now consider the existential data dependence problem in which a constant upper bound is placed on the number of occurrences of \parallel in the schemas. Owing to the freeness assumption on the class of schemas under consideration, $\exists DD_S$ can be defined by an iterative data flow analysis. Lemma 4.7 provides the crucial result in showing that in this case, the problem is polynomial-time bounded. This result relies on Lemma 4.6, which follows from the inductive definition of a schema flowchart in Definition 2.2.

LEMMA 4.6. *Let B be a non-negative integer and suppose that there are non-decreasing functions $P_B : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following conditions.*

- (0). $P_{B+1}(n) \geq P_B(n)$ if $n \geq 1$.
- (1). $P_B(n) \geq 3$ if $n \geq 1$.

(2, 3, 3'). $P_B(n_1 + \dots + n_m) \geq P_{C_1}(n_1) + \dots + P_{C_m}(n_m) + 1$ if $m \geq 2$, $C_1 + \dots + C_m = B$ and $n_i \geq 1 \forall i$.

(4, 4'). $P_B(n + 1) \geq P_B(n) + 1$ if $n \geq 1$.

(5). $P_B(n_1 + \dots + n_m) \geq P_{C_1}(n_1) \dots P_{C_m}(n_m) + 3$ if $C_1 + \dots + C_m = B - m + 1$ and $B \geq m - 1 \geq 1$ and $n_i \geq 1 \forall i$.

Then for every schema S encoded by a word of length n , in which \parallel occurs not more than B times, $Flowchart(S)$ has not more than $P_B(n)$ vertices.

PROOF. This follows by induction on the structure of S . Each Condition in the statement of the Lemma apart from (0) is labelled with the number of the case in Definition 2.2 that requires it. As an example, consider Condition (5). Assume that $S = l : S_1 \parallel S_2 \parallel \dots \parallel S_m$; then $Flowchart(S)$ has vertex set $\times_{i=1}^m (Vertices(Flowchart(S_i)) \cup \{\mathbf{start}, l, \mathbf{end}\})$. Assume that \parallel occurs not more than B' times in S and exactly C_i times in each S_i . Define $B = C_1 + \dots + C_m + m - 1$. Suppose each schema S_i is encoded by a word of length n_i and S is encoded by a word of length n , then

$$n \geq n_1 + \dots + n_m$$

holds. By the inductive hypothesis, each $Flowchart(S_i)$ has not more than $P_{C_i}(n_i)$ vertices. Hence $Flowchart(S)$ has not more than $P_{C_1}(n_1) \dots P_{C_m}(n_m) + 3$ vertices, and hence by (5) and the monotonicity Condition (4), not more than $P_B(n)$ vertices. Thus since clearly $B' \geq B$ holds, it follows from (0) that $Flowchart(S)$ has not more than $P_{B'}(n)$ vertices, proving the Lemma in this case. Other cases are treated analogously. \square

LEMMA 4.7. *Given any integer $B \geq 0$, let χ_B be the set of all schemas in which \parallel occurs not more than B times. Then there exists an algorithm that when given a schema S in χ_B as input, constructs the graph $Flowchart(S)$ and is polynomial-time bounded.*

PROOF. For each $B \geq 0$, it suffices to prove that the set containing $|Vertices(Flowchart(S))|$ for every schema S in χ_B is polynomially bounded in terms of the number of letters needed to encode S . The conclusion of the Lemma then follows from Part (2) of Lemma 3.6. Consider the functions $P_B : n \mapsto \max(3, n^{6(B+1)})$. We will show that they satisfy Conditions (0–5) of Lemma 4.6, and hence that $P_B(n)$ is an upper bound for the number of vertices in $Flowchart(S)$ for any schema in χ_B encoded by a word of length n . The existence of the polynomial bound required will follow immediately.

Clearly the functions P_B satisfy Conditions (0, 1, 4, 4'). We now prove that they satisfy Condition (2, 3, 3') under the stated assumptions. Observe that

$$\begin{aligned} P_B(n_1 + \dots + n_m) &= (n_1 + \dots + n_m)^{6(B+1)} = \left(\left(\sum_{i \leq m} n_i \right)^2 \right)^{3(B+1)} \\ &\geq \left(\sum_{i \leq m} n_i^2 + 2n_1 n_2 \right)^{3(B+1)} \geq \left(\sum_{i \leq m} n_i^2 + 1 \right)^{3(B+1)} + 1 \geq \sum_{i \leq m} n_i^{6(B+1)} + 3m + 1 \end{aligned}$$

(since each $n_i \geq 1$ and $m \geq 2$)

$$\geq \sum_{i \leq m} P_{C_i}(n_i) + 1$$

since each $C_i \leq B$. It now remains to prove (5). We have

$$P_B(n_1 + \dots + n_m) = (n_1 + \dots + n_m)^{6(B+1)} = \prod_{j \leq m} (\sum_{i \leq m} n_i)^{6(C_j+1)}$$

(since $\sum_{j \leq m} (C_j + 1) = B + 1$)

$$\geq \prod_{j \leq m} ((\max_{i \leq m} n_i + 1)^2)^{3(C_j+1)} \geq \prod_{j \leq m} (\max_{i \leq m} n_i^2 + 2 + 1)^{3(C_j+1)}$$

(since each $n_i \geq 1$ and $m \geq 2$)

$$\geq \prod_{j \leq m} (\max_{i \leq m} n_i^2 + 2)^{3(C_j+1)} + 3 \geq \prod_{j \leq m} (n_j^{6(C_j+1)} + 2^3) \geq \prod_{j \leq m} P_B(n_j) + 3,$$

thus proving the Lemma. \square

Definition 4.8. Let S be a schema. We define the set W_S to be the subset of $(\mathcal{V} \cup \mathcal{F}) \times \mathcal{V}$ for which both components occur in S .

Definition 4.9 recursive definition of $\exists \text{DatDep}_S$ for a schema S . Let S be a schema. Then $\exists \text{DatDep}_S$ is the function H from $W_S \times \text{Vertices}(\text{Flowchart}(S))$ to $\{\mathsf{T}, \mathsf{F}\}$ satisfying the following

- (1) $H(v, v, \mathbf{start}) = \mathsf{T}$ for all $(v, v) \in W_S$.
- (2) If w is a variable, (l, l') is an edge in $\text{Flowchart}(S)$ and $\text{edgeType}(l, l')$ is not an assignment to the variable w , then $H(f, w, l) = \mathsf{T} \Rightarrow H(f, w, l') = \mathsf{T}$ holds.
- (3) If $x, y \in \mathcal{V}$ and (l, l') is an edge in $\text{Flowchart}(S)$ and $\text{edgeType}(l, l')$ is an assignment to the variable y that references x , then $H(f, x, l) = \mathsf{T} \Rightarrow H(f, y, l) = \mathsf{T}$ holds. If in addition, the assignment $\text{assign}_S(l, l')$ has function symbol h , then $H(h, y, l) = \mathsf{T}$ holds,

for which the set $H^{-1}(\mathsf{T})$ is minimal.

THEOREM 4.10. *Let S be a free schema and let $(f, v) \in W_S$. Let $l \in \text{Vertices}(\text{Flowchart}(S))$. Then $\exists \text{DD}_S(f, v, l) \iff \exists \text{DatDep}_S(f, v, l)$ holds.*

PROOF. Define the function $K : W_S \times \text{Vertices}(\text{Flowchart}(S)) \rightarrow \{\mathsf{T}, \mathsf{F}\}$ as follows; $K(f, v, l) = \mathsf{T}$ if and only if there is a path μ through S from \mathbf{start} to l such that the term $\mathcal{M}[\mu]_e(v)$ contains f . Since S is free, $\exists \text{DD}_S = K$ holds. Thus it suffices to show that $K = \exists \text{DatDep}_S$ holds, and this follows from the fact that Definition 4.9, with K in place of $\exists \text{DatDep}_S$, gives an equivalent definition of K . \square

The main Theorem of this subsection follows.

THEOREM 4.11. *Let $B \geq 0$ and let S be a free schema in which every \parallel construction occurs not more than B times, and let $f \in \mathcal{V} \cup \mathcal{F}$ and $v \in \mathcal{V}$. Let $l \in \text{Vertices}(\text{Flowchart}(S))$. Then it can be decided in polynomial time whether $\exists \text{DD}_S(f, v, l)$ holds.*

PROOF. From Theorem 4.10 it suffices to prove that it can be decided in polynomial time whether $\exists \text{DatDep}_S(f, v, l)$ holds, under the restriction given on \parallel constructions. We compute $\exists \text{DatDep}_S(f, v, l)$ as follows, using the graph $\text{Flowchart}(S)$. We may assume that $(f, v) \in W_S$, since otherwise $\exists \text{DD}_S(f, v, l)$ can clearly be decided in polynomial time.

We approximate $\exists \text{DatDep}_S$ on the domain $W_S \times \text{Vertices}(\text{Flowchart}(S))$ by a sequence of functions $H_1, H_2, \dots : W \rightarrow \{\text{T}, \text{F}\}$. Firstly, let H_1 satisfy Condition (1) of Definition 4.9 for every $(v, v) \in W_S$ and let $H_1(f, v, l) = \text{F}$ whenever $(f, v, l) \neq (v, v, \text{start})$. Given a function H_i that does not satisfy every instance of Condition (2) or (3) of Definition 4.9, obtain the function H_{i+1} by altering H_i on one such instance, so that $H_{i+1}^{-1}(\text{T})$ contains every element of $H_i^{-1}(\text{T})$, plus an additional one. Therefore a maximal function H_n is eventually reached with $n \leq W_S \times \text{Vertices}(\text{Flowchart}(S))$, which is polynomially bounded in terms of S , by Lemma 4.7. In addition, each function H_i can be encoded by listing the elements of $H_i^{-1}(\text{T})$, thus H_n is computable in polynomial time. By induction on i , every set $H_i^{-1}(\text{T}) \subseteq \exists \text{DatDep}_S^{-1}(\text{T})$, and H_n satisfies all three conditions in Definition 4.9, hence the minimality condition in the definition of $\exists \text{DatDep}_S$ implies $H_n = \exists \text{DatDep}_S$, thus proving the Theorem. \square

5. CONCLUSIONS

We have extended conventional data dependency problems to arbitrary schemas and have shown that both the existential and universal data dependence problems lie in PSPACE for schemas without concurrency constructs and having only equality assignments, provided that there is a constant upper bound on the arity of any predicate symbol occurring in the schemas. We have also shown that without this upper bound, both problems are PSPACE-hard. This PSPACE-hardness result, Theorem 3.5, entails constructing a schema without this arity restriction; see the predicates Q_3 and q_l in Figs. 6 and 7. This suggests that assuming this restriction may result in a lower complexity bound than PSPACE. Since schemas with predicates approximate the behaviour of real programs much more accurately than wholly non-deterministic programs which are normally used in program analysis, a reasonable class of schemas for which our two problems could be decided tractably would be of considerable interest.

In addition, we have proved that for free schemas, existential data dependence is decidable in polynomial time provided that a constant upper bound is placed on the number of occurrences of \parallel in the schemas being considered. We have not attempted to prove an analogous result for the universal data dependence relation. This would be an interesting subject for future investigation.

As mentioned in the Introduction, many concurrent systems have only relatively few threads even if they have many lines of code, and therefore the bound on the number of occurrences of \parallel is not particularly restrictive. The freeness hypothesis (equivalent to assuming non-deterministic branching) is common in program analysis, and its use ensures that no false positives for data dependence are computed. This is important in areas such as security where we wish to show that the value of one variable x , whose value is accessible, cannot depend on the value of another variable y whose value should be kept secret.

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