APPLICATIONS OF OPTIMIZATION TO SOVEREIGN DEBT ISSUANCE

A thesis submitted for the degree of Doctor of Philosophy

by

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April 2013

Abstract

This thesis investigates different issues related to the issuance of debt by sovereign bodies such as governments, under uncertainty about the future interest rates. Several dynamic models of interest rates are presented, along with extensive numerical experiments for calibration of models and comparison of performance on real financial market data. The main contribution of the thesis is the construction and demonstration of a stochastic optimisation model for debt issuance under interest rate uncertainty. When the uncertainty is modeled using a model from a certain class of single factor interest rate models, one can construct a scenario tree such that the number of scenarios grows linearly with time steps. An optimization model is constructed using such a one factor scenario tree. For a real government debt issuance remit, a multi-stage stochastic optimization is performed to choose the type and the amount of debt to be issued and the results are compared with the real issuance. The currently used simulation models by the government, which are in public domain, are also reviewed. Apparently, using an optimization model, such as the one proposed in this work, can lead to substantial savings in the servicing costs of the issued debt.

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Chapter 1

Introduction

Economies follow a complex business cycle, often moving from boom to depression. This leads to unpredictable variations in the amount of money collected by governments. The extra expenditure in the form of cash payments over receipts as well as the refinancing of maturing debt and its cash interest payments causes budget deficits for the governments. To make sure that the cash payment obligations are met, governments borrow funds in the form of public debt. The formulation of government debt strategies requires analyzing a complex dynamic inter-temporal problem. The future costs and risks depend on many factors including the size and structure of the existing debt and the evolution of the interest rates. When borrowing to finance the primary net funding requirement the government can choose from a number of different instruments. Examples include treasury bills, coupon bonds with fixed or index-linked (such as inflation, GDP and other similar indices) coupons and retail saving bonds in local or foreign currencies. The government wishes to select the composition and the maturity structure of its portfolio that minimize the cost of servicing the debt at a given level of risk. This involves designing the maturity structure of the sovereign portfolio in such a way that the governments financing costs are kept low and insulated from macroeconomic shocks. Most of the academic literature on optimal sovereign debt portfolio emphasizes on the role of the debt management in providing insurance against shocks as prescribed by the optimal taxation theory with the final goal of stabilizing the debt-to-GDP ratio. Debt managers should minimize the risk that tax rates will have to be changed in response to economic developments. While offering many insights, this approach has few empirical implications. In practice, the majority of government debt managers make no explicit reference to fiscal policy, focusing instead on the budget-smoothing objective.

The academic perspective on sovereign debt management problem has been tied to economies of the world since the late 19th century and onwards; see e.g. [13] and [37]. Some of the first academic papers published on public debt started to appear at the beginning of the twentieth century [44], discussing the Italian public debt problem. The first waves of academic papers on public debt problems came with the great depression era (from 1929 to the late 1930's) ([18],[72],[100]). The excessively large amount of public debt from the great depression has left economists to wonder what to do with all the debt [70]. Alternative solutions were thought with the use of monetary policies with debt management to spare the owners of the debt financial loss while sparing the population most of the pains due to high inflation; see [95], [105] and [97] for some early references on this strategies. The 1960's saw a sudden rise in the power of central banks in stabilizing inflation [106] and [85].

By the mid 1970's, the public debt problem changed to be seen as wealth and seen as a tool to promote budget smoothing. The first models were created then [8], [12], [9], [4] and [3]. In the 1990's, the concept of optimal debt management started appearing more frequently [68], [10] and [11]. As the public debts rose to new limits [79], new regulations made their way [78] , [107], [71] and [61]. With a sudden rise in computer processing power and memory available, along side increasingly more complex methods of mathematical programming, optimization with public debt management has become viable and appeared [104]. The South Korean debt management problem [52] and the Brazilian debt management problems [47] are both good examples the use of mathematical programming with public debt issuance. More complex constraints appeared to better model the Italian problems [1] and [22]. The Turkish debt problem was modeled in the form of a multi-objective problem [7]. As the thesis progresses, relevant papers will be cited to subsequent applications.

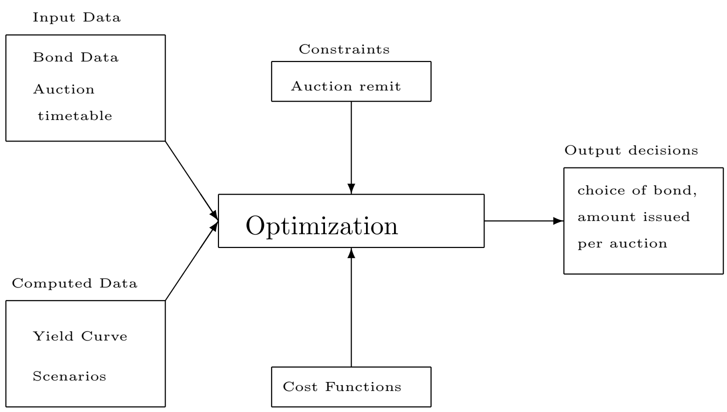

Figure 1.1: Optimization process used for the debt issuance problem

The objective of this thesis is to look at the applications of modelling and optimization paradigms from operations research for the issuance of sovereign debt. The general objective is to minimise the total cost of debt issuance under interest rate uncertainty. However this cannot be done without several measures of risk and restrictions on refinancing costs. Public debt management is a complex field and the issuance alone is very complicated task. As such, a number of models have been created and are still being created to better mimic the actions of debt management offices, with further focus on the underlying mathematics needed for the modeling of debt creation. We will focus on models of evolution of uncertain interest rates, their calibration and their use as an input to issuance optimization model. These models are based on secondary market data. We will also look at macroeconomic models based on primary economic data. The methodology followed is shown in figure 1.1.

The rest of this thesis is organized as follows (Figure 1.2 above outlines the relationships between various chapters):

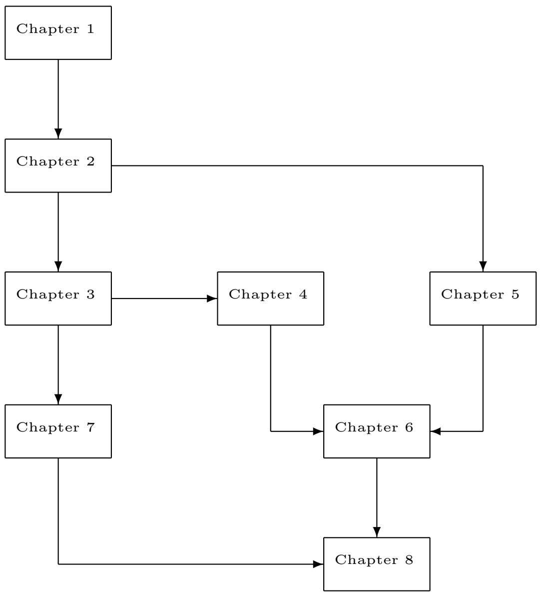

Figure 1.2: Organisation chart of the chapters

- The following chapter will cover some mathematical preliminaries regarding probability, stochastic processes and modeling of interest rates. While this chapter introduces static interest rate models, the next chapter 3 focuses on dynamic interest rate models which are used in the subsequent chapters.
- In chapter 4, we will use the interest rate models shown in the previous chapter to generate possible scenarios to forecast future values of bonds.

- Chapter 5 presents several methods to tackle the multi-stage issuance of public debt management.
- The mathematical programming using mixed-integer models are defined in chapter 6, using the scenarios created previously from the methods of chapter 4 and the method to re-evaluate and back-test the possible solutions from chapter 5.
- chapter 7 is dedicated to simulations for the sovereign debt problem.

Directions for further research as well as a list of my own contributions to the field are outlined in the concluding chapter.

Chapter 2

Mathematical preliminaries

In this chapter, we collect together several definitions and background material which is required for the subsequent chapters. The first section is mostly based on [88] and the second on [17]. Most of the material in this chapter can be found in many other standard graduate level textbooks.

2.1 Probability

We wish to define a random variable, for that we will need to define a sample space, a σ-field and a filtration. Let us begin by defining a sample space.

Definition 1. *A sample space is the set of all possible outcomes of an experiment. Well denote a sample space as* Ω.

Next, we define a class of subsets of Ω called a σ-field:

Definition 2. *Let* X *be some set, and* 2^X *symbolically represent its power set. Then a subset* $\Sigma \subset 2^X$ *is called a* σ*-field if it satisfies the following three properties:*

- Σ *is non-empty:* $\exists \mathcal{A} \subset X and \mathcal{A} \in \Sigma$.
- Σ *is closed under complementation: If* $\mathcal{A} \in \Sigma$*, then so is its complement,* $X \backslash \mathcal{A}$.
- Σ *is closed under countable unions: If* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, ... \in \Sigma$*, then so is* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \cdots$.

We should now define a probability measure next:

Definition 3. *A function* $\mathbb{P}$ *is a probability measure if it satisfies the following conditions:*

- *the function* $\mathbb{P}$ *returns values within the interval* $[0, 1]$,

- *the function* $\mathbb{P}$ *returns* 0 *for the empty set, and* 1 *for the entire space,*
- *for all countable collections* $\{A_i\}$ *of the space,* $\mathbb{P}(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mathbb{P}(A_i)$.

We can progress to define a probability space:

Definition 4. *Let* $\Omega \neq \emptyset$, $\mathcal{A} \subseteq 2^{\Omega}$ *a* σ*-field on* Ω *and* $\mathbb{P}$ *be a probability measure on* $\mathcal{A}$. *Then* $(\Omega, \mathcal{A}, \mathbb{P})$ *is called a probability space.*

Finally, we define a filtration:

Definition 5. *Let us assume a series of time steps* $t_1, t_2, \ldots$, *where we know more a later times. Therefore we obtain a successively larger* σ*-field at each time step:* $\sigma_1 < \sigma_2 < \sigma_3 < \ldots$. *The set of* σ*-fields is known as a filtration* $\mathcal{F}$.

Filtrations can exist in discrete time and continuous time. We are now in a position to define a random variables:

Definition 6. *If* $\Omega \neq \emptyset$, *a random variable* $X : \Omega \to \mathcal{E}$ *is a measurable function from a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *as long as the probability* $\mathbb{P}[X]$ *satisfies the following conditions:*

- $\{\omega \in \Omega : X(\omega) \in \mathcal{B}\} \in \mathcal{F}$ *holds when* $\mathcal{B} \in \varepsilon$,
- *the probability of the events* $X = +\infty$ *and* $X = -\infty$ *equals zero.*

$(\mathcal{E}, \varepsilon)$ is called a state space and $(\Omega, \mathcal{F})$ is the underlying space.

Definition 7. *A stochastic process is a family of random variables* $(X_t)_{t\in I}$ *from a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *into a state space* $(\mathcal{E}, \varepsilon)$. *The set* I *is the index set of discrete or continuous time in a discrete or continuous state space.*

Remark. A stochastic process (X_n) is said to be *adapted* to the filtration $\mathcal{F}_n$ if (X_n) is known at time t_n.

2.2 Brownian Motion

We will be using continuous time stochastic interest rate models in the subsequent work and some relevant definitions will be outlined here.

In continuous time, a Brownian motion (usually denoted by W_t) can be defined as such:

Definition 8. *A continuous stochastic process* $W = \{W_t : t \geq 0\}$ *is called a Brownian Motion with start in* $x \in \mathbb{R}$ *if the following statements hold:*

- $W_0 = x$,

- *the process has independent increments, i.e. for all times* $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ *the increments* $W_{t(n)} - W_{t(n-1)}, W_{t(n-1)} - W_{t(n-2)}, \cdots, W_{t(2)} - W_{t(1)}$ *are independent random variables.*

- $\forall t \geq 0$ *and* $h \geq 0$*, the increments* $W_{t+h} - W_t$ *are normally distributed with expectation zero and variance* h*.*

- *the function* $t \mapsto W_t$ *is almost surely continuous.*

The Brownian motion $\{W_t : t \geq 0\}$ *is said to be a standard Brownian motion if* $x = 0$*.*

Itô processes represent a more general class of class of stochastic processes than Brownian motion. They are usually represented by a stochastic differential equation:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \tag{2.1}$$

where dt and dW_t are, respectively, the infinitesimal time increment and the corresponding Wiener process increment. For a unique time continuous solution to exist for this stochastic differential equation, the coefficient functions $\mu(X_t, t)$ and $\sigma(X_t, t)$ must satisfy:

$$|\mu(\alpha, t)| + |\sigma(\alpha, t)| \leq C(1 + |\alpha|), \tag{2.2}$$

$$|\mu(\alpha, t) - \mu(\beta, t)| + |\sigma(\alpha, t) - \sigma(\beta, t)| \leq D|\alpha - \beta|, \tag{2.3}$$

for some constants C and D over any time t such that: $0 < t < T$ and X_0 a random variable.

Remark. μ and σ may or may not be independent of X_t.

The solution of the stochastic differential equation is:

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{2.4}$$

where the second integral is a stochastic integral with respect to Wiener process. More information of the construction of stochastic integrals with respect to Brownian motion can be found in [58]. In particular, if σ is independent of X_t and is a deterministic function of time, $\int_0^t \sigma(s)dW_s$ is itself a Gaussian random variable for a fixed t, with mean zero and variance given by $\int_0^t \sigma^2(s)ds$.

2.2.1 Arbitrage and market assumptions

In mathematical finance, one of the basic assumptions is the perfect market assumption, as described in [31]:

Definition 9. *A market is called perfect if:*

- *there are enough financial assets to be traded.*

- *financial contracts can be enforced.*

- *the market allows competitive trading.*

- *the market is free to access.*

- *there are no financial constraints.*

It is rather obvious that there are no markets that are really perfect in practice. However for the purpose of modelling financial assets, it is useful to give assumptions on the markets. There are several fundamental assumptions of mathematical finance (FAMF), that have been created on perfect markets and complement each other to model asset prices. Three FAMF are needed in this work, the first one is the arbitrage-free market or no-arbitrage assumption:

Definition 10. *An arbitrage is an opportunity to obtain an instantaneous risk-free profit by exploiting price discrepancies.*

It is assumed that there are **no-arbitrage** in the secondary markets, as First FAMF. If arbitrage opportunities arise, they are quickly exploited and cleared out by arbitrageur. There are many models that allow for arbitrage in the market and some that are arbitrage-free, as explained in chapter 1. The no-arbitrage assumption implies that two different assets with identical payoffs and risks must have the same price. The price of an asset is the preoccupation for an investor seeking a return on investment, which leads us the next FAMF.

Definition 11. *Second FAMF: Every investor attempts to maximize his return on investment while reducing his risk.*

Optimization concerns a particular asset and should not be confused with preference which is logical choice of an asset out of several. Optimization implies some form of active asset management and reinvesting to obtain a better return. Optimization may lead to a portfolio of assets that reduce the overall risk while maintaining a good return on investment.

Definition 12. *Third FAMF: Market equilibrium dictates a fair price or equilibrium price for any future cash flow depending on supply and demand as well as an investor's preference for that asset.*

This is a principle that implies that the market will revert to an equilibrium as time goes by. It also enables us to use mean reverting models for risk-free rate estimation.

2.2.2 Time value of money

Another basic concept in finance relevant in this thesis is the idea of time value of money. Assume that there exist a risk-free interest rate r per year, and the interest is paid, for a unit of money, n times along the year. Then the value of that unit of money in t years will be:

$$x = \left(1 + \frac{r}{n}\right)^{nt}, \tag{2.5}$$

if the interest is paid continuously along the t years then:

$$x = e^{rt}, \tag{2.6}$$

where x is the value of that unit of money in t years. The value of a unit of money today, payable t years from now, will be:

$$d = \left(1 + \frac{r}{n}\right)^{-nt}, \tag{2.7}$$

where d is the current value of that unit of money, if interest is compounded n times a year over t years. If it is continuously compounded, then the discount factor is given by the limit of the expression in equation 2.7:

$$d = \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{-nt} = e^{-rt}. \tag{2.8}$$

2.2.3 Bond pricing and interest rate dynamics

Bonds are the most commonly traded assets in the world and as of 2010, the bond market is larger then the equity market representing $95 trillion, of which 43% are government bonds [76]. Bonds have an average traded volume of $822 billion per trading day in the US alone [93].

Definition 13. *A bond is a debt security issued by governments, corporations or other entities, where the issuer pays an interest or coupon at regular intervals in time and returns the principal when the bond is about to expire or maturity.*

A bond that doesn't pay coupons is called zero-coupon bond. Coupons may have a fixed or floating value, floating coupons are linked to an index such as an inflation rate or GDP growth rate. They are usually used to finance important projects or activities.

There is a need for issuing public debt regardless of any deficit, in order to support the need for low risk assets from the financial sector. Bonds are nowadays standard, highly liquid and tradable securities. Governments bonds can be issued in different currencies as well:

Definition 14. *Sovereign debt is debt issued by a national government. It is often considered a risk-free, as governments have an array of tools to guarantee repayment such as raising taxes or printing money.*

Remark. Debt issued by a non-national body, such as a municipal, regional, state debt is called sub-sovereign debt. Debt created by supranational institutions such as the World Bank, Kreditanstalt für Wiederaufbau, Asian Development Bank, European Investment Bank or the European Financial Stability Facility are also considered sub-sovereign or quasi-sovereign as the sovereign guarantees remain absent.

Definition 15. *A government that issues bonds in the country's domestic currency is called a government bond, otherwise it is called a sovereign bond.*

There are several ways of measuring the financial return earned by holding a bond:

- a *coupon yield* is the annual return of owning a bond over a year. Fixed income bonds have a fixed coupon yield at issuance.
- a *current yield* is the return the owner of a bond will receive as a percentage of the current price of the bond.
- a *yield to maturity* is an estimate of what an investor will receive if the bond is held to its maturity date relative to its current price.

Let us use the following notations for this section:

- $P(t, \tau_k)$ is the price of a bond maturing at τ_k at time t.
- $R(t, \tau_k)$ is the annualised interest rate of a unit of currency due at τ_k at time t.
- r_t is the short rate, a continuously compounded, annualized interest rate at which an entity can borrow money for an infinitesimally short period of time from time t,
- $f(t, \tau_j, \tau_k)$ is the forward rate of a unit of currency lent at τ_j and due at τ_k at time t.
- L is the face value of the bond at issuance.

In an arbitrage-free market the following property holds:

$$(1 + R(t, \tau_k))^{\tau_k} = (1 + R(t, \tau_j))^{\tau_j} \times (1 + f(t, \tau_j, \tau_k))^{\tau_k - \tau_j} \tag{2.9}$$

A zero-coupon bond price, compounded $n-$times a year, is defined by:

$$P_n(t, \tau_k) = L \times (1 + \frac{r}{n})^{-n(\tau_k - t)} \tag{2.10}$$

as described in section 2.2.2. By taking the limit on the frequency as in the previous section 2.8, the price of a zero-coupon bond becomes:

$$P(t, \tau_k) = \lim_{n \to \infty} L \times (1 + \frac{r}{n})^{-n(\tau_k - t)} = L \times e^{-r(\tau_k - t)} \tag{2.11}$$

Assuming that r_t is a stochastic process, so is the following integral given by $e^{-\int_t^T r_s ds}$. The price of a zero coupon bond is then given by:

$$P(t, \tau_k) = L \times \mathbb{E}(e^{-\int_t^T r_s ds} | \mathcal{F}_t) \tag{2.12}$$

where the expected value is computed under appropriate risk neutral measure. This can also be written as:

$$P(t, \tau_k) = L \times e^{-R(t, \tau_k)\tau_k} \tag{2.13}$$

which gives us a formula for the *yield* in terms of bond price:

$$R(t, \tau_k) = -\frac{log(P(t, \tau_k))}{\tau_k}. \tag{2.14}$$

It is easy to see that a coupon bond is just a succession of zero-coupon bonds that discount the coupon at every payment and the face value L of the bond at maturity. For a class of models called *exponential affine models*, $R(t, \tau_k)$ happens to be an affine function of the instantaneous interest rate r_t (also called the short rate) in equation 2.12. We will look in chapter 3 at exponential affine models in more details.

Definition 16. *A yield curve is the relation between the annualized interest rates and the corresponding time to maturity. In other words, each point on the yield curve gives an annualized, continuously compounded rate of interest which an investor can expect for a given time to maturity.*

Remark. Yield curves exhibit different shapes in practice:

- Normal yield curve is a monotonous ascending yield curve in time to maturity. A normal yield curve reflects the market's expectation to have a greater yield in return of investing for a longer term and is a sign of a growing economy and rising inflation. Investors will ask for a higher rate of return on securities with longer maturity dates, expecting higher interest rates in the near future.

- Flat yield curve is a yield curve where all the maturities have very similar returns. A flat yield curve reflects uncertainty in the near and long term interest rates. The market investors are willing to get out of their position in longer investments at the value of the shorter yield.

- Humped yield curve is a yield curve where the shorter and longer maturities are similar but the yields in between vary. Another sign of uncertainty in the markets. Similarly to the flat yield curve, it is usually a yield curve between a normal and inverted yield curve.

- Inverted yield curve when the short term yields are higher then the long term yields. They occur when interest rates are high and expected to fall or when the long term investment is seen as a lower risk compared to the shorter ones. They reflect an expectation for the short rate and inflation to drop. It's usually a negative sign for investors, with certain economies as exceptions.

- Steep yield curve is a normal yield curve where there is higher rise in returns than found normally. It reflects a high growth and high inflation economy or a recently recovering economy. Investors are generally wary of such yield curves as it is unsustainable and seen as a sign of high risk investments.

Flat and humped yield curves have become rare since the later 1990s as central banks adopted a policy of pre-announcing interest rate moves several weeks before.

Suppose that a vector of bond prices with maturities $\tau_1, \tau_2, \cdots, \tau_N$ are available in the market at each time t_i, $i = 1, 2, \cdots, M$. Now, one can construct a model for future bond price dynamics simply by looking at the prices at a fixed time instant. Alternatively, one can look at a time series of a single bond price and construct a model for interest rates. In markets where prices are consistent with each other and there is no arbitrage, the two models differ via a process called the price of risk explained in the next subsection. We will try to model how the yield curves evolve through time in the next chapter.

2.2.4 Price of risk

The price of an asset can be seen as a function of a deterministic part and a stochastic part. The deterministic part will be called the drift. The drift is the underlying trend of the asset, usually redirected by some kind of random walk or Brownian motion. The stochastic part of the underlying asset will be represented by the volatility of the price of the asset. The volatility is the annual standard deviation of continuously compounded returns of a financial asset, it can be seen as the intensity of the Brownian motion in the model used to price.

Definition 17. *The price of risk, or risk premium, is the excess return given to the investor for bearing the risk involved by owning the underlying asset, in excess of the risk-free rate.*

Let us denote the drift of the underlying asset price at time t as $\mu(t, \tau_k)$, the price of risk as $\lambda(t, \tau_k)$ and the volatility as σ_t. for a specific market instrument. In a risk-free

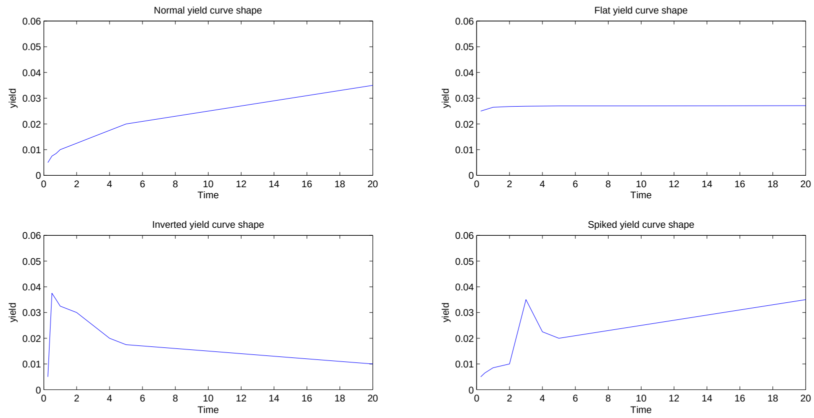

Figure 2.1: Plot of four types of yield curves

investment, the price of risk is nil, so the drift of an asset maturing at τ_k at time t is:

$$\mu(t, \tau_k) = r_t, \tag{2.15}$$

if r_t is the risk-free rate. If the investment carries some risk then:

$$\mu(t, \tau_k) = r_t + \lambda(t, \tau_k) \times \sigma_t. \tag{2.16}$$

where σ_t is the volatility of the market instrument at time t. In practice, this results in using $\mu(t, \tau_k)$ as drift in objective measure (or while looking at data for a fixed τ_k along a time series) but using $\mu(t, \tau_k) + \lambda\sigma$ as drift in risk neutral measure (i.e., while looking at prices of different τ_k, at the same t). The price of risk will be used in the next section 2.3 and in chapter 3 to calibrate several different dynamic models.

There are models of static yield curves, which are simply parametrized functional relationships between yields and corresponding times to maturity. One of the very popular ones is the Nelson Siegel model. We will concentrate our attention on dynamic interest rate models in this thesis, i.e. models which describe the evolution of a yield curve through time. However, we first look at one of the most popular static models, viz. the Nelson-Siegel model, and its dynamic generalizations.

2.3 The Nelson Siegel class of interest rate models

The Nelson-Siegel model is a type of yield curve model and was introduced in 1987 [81], in the following form:

$$f(\tau) = L + Se^{-\frac{\lambda}{\tau}} + C\frac{\lambda}{\tau}e^{-\frac{\lambda}{\tau}}, \tag{2.17}$$

where $f(\tau)$ is the instantaneous forward rate, maturing in τ. The constants L, S, C can be seen as the level L, slope S and curvature C and λ a constant price of risk. Integrating the instantaneous forward rate will give us the yield curve:

$$y(\tau, T) = \frac{1}{\tau}\int_0^\tau f(u, T)du, \tag{2.18}$$

The corresponding yield of a zero coupon bond at t maturing at T is:

$$y(t, T) = L_t + S_t\left(\frac{1 - e^{-\lambda_t(T-t)}}{\lambda_t(T-t)}\right) + C_t\left(\frac{1 - e^{-\lambda_t(T-t)}}{\lambda_t(T-t)} - e^{-\lambda_t(T-t)}\right), \tag{2.19}$$

The price of a unit zero-coupon bond with maturity T at time t would simply be:

$$P(t, T) = e^{-y(t,T)(T-t)}, \tag{2.20}$$

which is identical to equation 2.13. The Dynamic Nelson-Siegel model (DNS) is characterised by four time-dependent parameters L_t, S_t, C_t and λ_t. The price is simply a discounted continuously compounded asset, as described in equation 2.8 and 2.13. At any given time t, it is possible to find the values of these parameters to fit a wide variety of possible shapes of yield curve from a given set of yields or bond prices. This has made this four parameter model (where the parameters are independent of time) very popular in practice. Rather than assuming constant parameters and then re-calibrating the model at each time t (e.g. daily basis), the dynamic Nelson-Siegel model [24] provides an extension of the original three factor Nelson Siegel model [80]. This model is then further extended in [36] so that the transition of the level, slope and curvature are autoregressive and becomes:

$$\begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix} \tag{2.21}$$

where γ_{ij} is the real, i^{th} and j^{th} column of the transition matrix Γ, η_t is the white noise of the transitions. The individual yields can be computed as:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_t(\tau_1-t)}}{\lambda_t(\tau_1-t)} & \left(\frac{1-e^{-\lambda_t(\tau_1-t)}}{\lambda_t(\tau_1-t)} - e^{-\lambda(\tau_1-t)}\right) \\ 1 & \frac{1-e^{-\lambda_t(\tau_2-t)}}{\lambda_t(\tau_2-t)} & \left(\frac{1-e^{-\lambda_t(\tau_2-t)}}{\lambda_t(\tau_2-t)} - e^{-\lambda(\tau_2-t)}\right) \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda_t(\tau_N-t)}}{\lambda_t(\tau_N-t)} & \left(\frac{1-e^{-\lambda_t(\tau_N-t)}}{\lambda_t(\tau_N-t)} - e^{-\lambda(\tau_N-t)}\right) \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{t,\tau_1} \\ \varepsilon_{t,\tau_2} \\ \vdots \\ \varepsilon_{t,\tau_N} \end{pmatrix} \tag{2.22}$$

where $\varepsilon_{t,\tau}$ is the independent and identically distributed measurement noise for a security maturing at τ at time t.

A simplified version was later created to estimate multi-country yield curve dynamics in [34]. The simplified model will be called from here on out, basic dynamic Nelson-Siegel (basic DNS). It is a two factor version of the standard model described in equations (2.21)-(2.22) where the transition evolves as:

$$\begin{pmatrix} L_t \\ S_t \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} L_{t-1} \\ S_{t-1} \end{pmatrix} + \begin{pmatrix} U_t(L) \\ U_t(S) \end{pmatrix} \tag{2.23}$$

where ϕ_{ij} is a real number and $U_t(X)$ are the disturbances such that:

$$\mathbb{E}[U_g(i)U_h(j)] = \begin{cases} (\sigma_i)^2, & \text{if } g = h \text{ and } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.24}$$

σ_i corresponds to the standard deviation attributed to the i^{th} factor. The yields are obtained from the following equation:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_t(\tau_1-t)}}{\lambda_t(\tau_1-t)} \\ 1 & \frac{1-e^{-\lambda_t(\tau_2-t)}}{\lambda_t(\tau_2-t)} \\ \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda_t(\tau_N-t)}}{\lambda_t(\tau_N-t)} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{t,\tau_1} \\ \varepsilon_{t,\tau_2} \\ \vdots \\ \varepsilon_{t,\tau_N} \end{pmatrix} \tag{2.25}$$

where $\varepsilon_{t,\tau}$ is the independent and identically distributed measurement noise for a security maturing at τ at time t.

One of the issues of the Nelson-Siegel model is that the yield obtained is not arbitrage free [25]. Some results have been obtained from the original model regarding U.S. treasury yield curves [82] and more extensive results using the dynamic Nelson-Siegel model ([33], [89]). A more detailed description of advantages and problems of the Nelson-Siegel class of models can be found in [45].

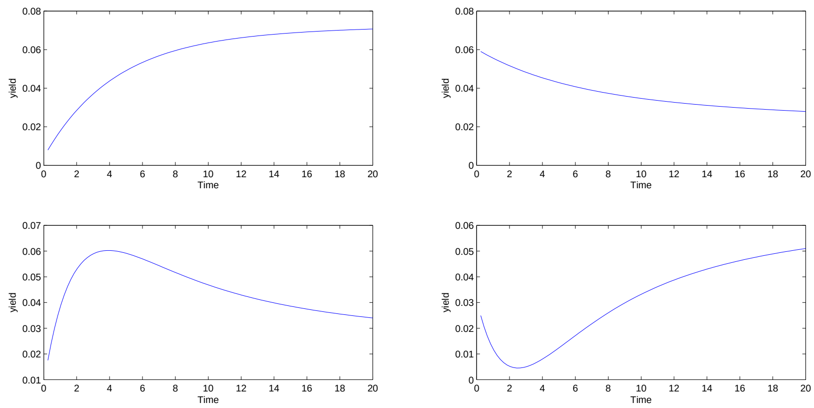

Figure 2.2: Examples of Nelson-Siegel yield curves

2.3.1 Arbitrage-free Nelson-Siegel model

Dynamic Nelson Siegel model described above allows for arbitrage. An arbitrage-free version of the Nelson-Siegel model is given in [20]. Assuming the state variable L_t, S_t and C_t are Markov processes defined on a set of $\mathbb{R}$ that satisfies the stochastic differential equation:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} K^{Q_1(t)} \\ K^{Q_2(t)} \\ K^{Q_3(t)} \end{pmatrix} \left[\begin{pmatrix} \theta^{Q_1(t)} \\ \theta^{Q_1(t)} \\ \theta^{Q_1(t)} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} \right] dt + \Sigma_t dW_t^Q, \tag{2.26}$$

where W^Q is a standard Brownian motion in $\mathbb{R}^n$ on the filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$, θ^Q is the drift term and K^Q are bounded, continuous functions on $\mathbb{R}^{n\times n}$. Σ_t is the bounded and continuous volatility matrix at time t. More than three factors are rarely needed to explain the movements in interest rates, see e.g. [30]. We now prove that risk-free rate is an affine function:

$$r_t = L_t + S_t, \tag{2.27}$$

Proof. The short risk-free rate r_t is the equivalent to the instantaneous yield $y(t,t)$. Let's consider the Taylor expansion:

$$\begin{aligned} e^{-\lambda_t(T-t)} &= \sum_{i=0}^{\infty} \frac{(-\lambda_t(T-t))^i}{i!}, \\ &= 1 - \lambda_t(T-t) + \frac{(\lambda_t(T-t))^2}{2} + O(\lambda_t^3(T-t)^3), \end{aligned} \tag{2.28}$$

The Taylor expansion of the Nelson-Siegel yield $y(t,T)$ from 2.19 is:

$$y(t,T) = L_t + S_t\left(1 - \frac{\lambda_t(T-t)}{2}\right) + C_t\left(\frac{\lambda_t(T-t) - \lambda_t^2(T-t)^2}{2}\right) + O(\lambda_t^3(T-t)^3), \quad (2.29)$$

So the limit of the yield $y(t,T)$ as T tends to t is:

$$\lim_{T\to t} y(t,T) = L_t + S_t. \quad (2.30)$$

□

The authors of [20] also propose two arbitrage free models. The first one is the Independent factor Arbitrate Free Nelson-Siegel (AFDNSi) model where the factors evolve as:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 \\ 0 & \kappa_{22}^P & 0 \\ 0 & 0 & \kappa_{33}^P \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix}\right] dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix} \quad (2.31)$$

and the yields are:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2}\right) \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \left(\frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N}\right) \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} - \begin{pmatrix} \frac{\varkappa_i(\tau_1)}{\tau_1} \\ \frac{\varkappa_i(\tau_2)}{\tau_2} \\ \vdots \\ \frac{\varkappa_i(\tau_N)}{\tau_N} \end{pmatrix} + \begin{pmatrix} \varepsilon_{t,\tau_1} \\ \varepsilon_{t,\tau_2} \\ \vdots \\ \varepsilon_{t,\tau_N} \end{pmatrix} \quad (2.32)$$

where $\varepsilon_{t,\tau}$ is the independent and identically distributed measurement noise for a security maturing at τ at time t and $\varkappa_i(\tau)$ is defined by:

$$\begin{aligned} -\frac{\varkappa_i(\tau)}{\tau} =& \frac{\sigma_{11}^2\tau^2}{6} - \sigma_{22}^2\left[\frac{1}{2\lambda^2} - \frac{1-e^{-\lambda\tau}}{\lambda^3\tau} + \frac{1-e^{-2\lambda\tau}}{4\lambda^3\tau}\right] \\ & - \sigma_{33}^2\left[\frac{1}{2\lambda^2} + \frac{e^{-\lambda\tau}}{\lambda^2} - \tau\frac{e^{-2\lambda\tau}}{4\lambda} - 3\frac{e^{-2\lambda\tau}}{4\lambda^2}\right. \\ & \left. - 2\frac{1-e^{-\lambda\tau}}{\lambda^3\tau} + \frac{5}{8}\frac{1-e^{-2\lambda\tau}}{\lambda^3\tau}\right] \end{aligned} \quad (2.33)$$

The second model, the Correlated factor Arbitrage Free Nelson-Siegel (AFDNSc) model, evolves as:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & \kappa_{12}^P & \kappa_{13}^P \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ \sigma_{21} & \sigma_2 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix} \tag{2.34}$$

and the yields are:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2}\right) \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \left(\frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N}\right) \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} - \begin{pmatrix} \frac{\varkappa_c(\tau_1)}{\tau_1} \\ \frac{\varkappa_c(\tau_2)}{\tau_2} \\ \vdots \\ \frac{\varkappa_c(\tau_N)}{\tau_N} \end{pmatrix} + \begin{pmatrix} \varepsilon_{t,\tau_1} \\ \varepsilon_{t,\tau_2} \\ \vdots \\ \varepsilon_{t,\tau_N} \end{pmatrix} \tag{2.35}$$

where $\varepsilon_{t,\tau}$ is the independent and identically distributed measurement noise for a security maturing at τ at time t.
$\varkappa_c(\tau)$ is defined as follows:

$$\begin{aligned} -\frac{\varkappa_c(\tau)}{\tau} = & -\frac{\sigma_1^2\tau^2}{6} - (\sigma_{21}^2 + \sigma_2^2)\left[\frac{1}{2\lambda^2} - \frac{1-e^{-\lambda\tau}}{\lambda^3\tau} + \frac{1-e^{-2\lambda\tau}}{4\lambda^3\tau}\right] \\ & - (\sigma_{31}^2 + \sigma_{32}^2 + \sigma_3^2)\left[\frac{1}{2\lambda^2} + \frac{e^{-\lambda\tau}}{\lambda^2} - \tau\frac{e^{-2\lambda\tau}}{4\lambda} - 3\frac{e^{-2\lambda\tau}}{4\lambda^2}\right. \\ & \left. - 2\frac{1-e^{-\lambda\tau}}{\lambda^3\tau} + 5\frac{1-e^{-2\lambda\tau}}{8\lambda^3\tau}\right] - \sigma_1\sigma_{21}\left[\frac{\tau}{2\lambda} + \frac{e^{-\lambda\tau}}{\lambda^2} - \frac{1-e^{-\lambda\tau}}{\lambda^3\tau}\right] \\ & - \sigma_1\sigma_{31}\left[\frac{3e^{-\lambda\tau}}{\lambda^2} + \frac{\tau}{2\lambda} + \frac{\tau e^{-\lambda\tau}}{\lambda} - 3\frac{1-e^{-\lambda\tau}}{\lambda^3\tau}\right] \\ & - (\sigma_{21}\sigma_{31} + \sigma_2\sigma_{32})\left[\frac{1}{\lambda^2} + \frac{e^{-\lambda\tau}}{\lambda^2} - \frac{e^{-2\lambda\tau}}{2\lambda^2}\right. \\ & \left. - 3\frac{1-e^{-\lambda\tau}}{\lambda^3\tau} + 3\frac{1-e^{-2\lambda\tau}}{4\lambda^3\tau}\right] \end{aligned} \tag{2.36}$$

An empirical study of implementation of this model is reported in [36], where the authors calibrate the model to U.S. Treasury yields with maturity of 3, 6, 9, 12, 18, 24, 36, 48, 60, 84, 96, 108, 120, 180, 240 and 360 months. The yields are taken from the end of month bid/ask average price quotes from January 1987 to December 2002. The arbitrage free models fit and forecast as well as the regular dynamic Nelson-Siegel model and appears to perform better for the medium to long maturities; further, they offer the rigor of the arbitrage free assumption. The main disadvantage of independent factor and correlated factor arbitrage-free Nelson-Siegel model is that it increases the number of parameters by 4 and 13 respectively. This makes it difficult to calibrate and use for forecasting or optimization.

Remark. For simplicity, we will assume that $\sigma_1, \sigma_2, \sigma_3$ for the independent factor arbitrage free dynamic Nelson Siegel model to be equal to a single constant value σ. Similarly, $\sigma_1, \sigma_2, \sigma_3$ for the correlated-factor arbitrage-free dynamic Nelson Siegel model are also assumed to be equal to a single constant σ.

The numerical calibrations of the arbitrage-free models are included in chapter 3.

2.4 Macroeconomic models

In more recent bond pricing models, several attempts have been made to add macroeconomic variables to better model the economic outlook of markets as a whole with the asset prices. This is a better methodology to model when the economy is expected to change within the time frame of the life of an underlying asset. We will look at the macroeconomic Nelson-Siegel model in detail as an example.

Authors of [35] and [36] add to the dynamic Nelson-Siegel model explained previously three more macroeconomic variables: the manufacturing capacity utilization (CU_t), the federal funds rate (FFR_t) and the annual price inflation (normally the consumer price index [74], denoted here as $INFL_t$) and test with 1-month yield, 12-month yield and 60-month yield in the US. Let f_t be the vector of factors L_t,S_t,C_t,CU_t,FFR and $INFL_t$. Let Γ be the transition matrix:

$$\Gamma = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n,1} & \gamma_{n,2} & \cdots & \gamma_{n,n} \end{pmatrix} \tag{2.37}$$

and let μ be the vector of mean state of each factor:

$$\mu = \begin{pmatrix} \mu_L \\ \mu_S \\ \mu_C \\ \mu_{CU} \\ \mu_{FFR} \\ \mu_{INFL} \end{pmatrix} \tag{2.38}$$

To obtain the yields of n bonds for at each time steps:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \Lambda f_t + \varepsilon_t. \tag{2.39}$$

where Λ is the matrix:

$$\Lambda = \begin{pmatrix} 1 & \frac{1-e^{-\tau_1\lambda}}{\tau_1\lambda} & \frac{1-e^{-\tau_1\lambda}}{\tau_1\lambda} - e^{-\tau_1\lambda} & 0 & 0 & 0 \\ 1 & \frac{1-e^{-\tau_2\lambda}}{\tau_2\lambda} & \frac{1-e^{-\tau_2\lambda}}{\tau_2\lambda} - e^{-\tau_2\lambda} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\tau_N\lambda}}{\tau_N\lambda} & \frac{1-e^{-\tau_N\lambda}}{\tau_N\lambda} - e^{-\tau_N\lambda} & 0 & 0 & 0 \end{pmatrix} \tag{2.40}$$

The 3 right most columns are filled with 0's to remain consistent with the view that only 3 factors are needed to distill the information in the yield curve [36]. This means that although the macroeconomic variables are not used directly to model yields, they affect the transitions of the variables that do. η_t and ε_t are white noise found in the data:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} = WN \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right]. \tag{2.41}$$

This can now be simplified to:

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t \ , \tag{2.42}$$

$$y_t = \Lambda f_t + \varepsilon_t \ , \tag{2.43}$$

Numerical results of the models can be found in [36]. This model is not arbitrage-free either and has 6 factors, making the model incredibly difficult to calibrate, with at least 42 non zero variables that need calibration. This model is even harder to use for forecasting purposes. We will return to the subject of simulation models in chapter 7.

2.5 Summary

This concludes the mathematical preliminaries chapter. Here, we have covered:

- basic probability theory and Brownian motion,
- basic mathematical financial assumptions such as: arbitrage, perfect markets, time value of money, bond pricing, the price of risk and interest rate dynamics,

- The Nelson-Siegel class of yield curve models,
- The macroeconomic extension of the dynamic Nelson-Siegel model.

The preliminaries will be of use in the next chapter 3, and in chapter 5.

Chapter 3

Dynamic interest rate models

In this chapter, we will describe several models used for modelling the changes in interest rate and methods of their calibration. Some of these models are relevant in the next chapter to generate possible scenarios for future interest rates.

3.1 Models of short rate

As discussed earlier in chapter 1, public debt is created out of bonds and bond price movement can be explained by short rate models. The short rate, denoted as r_t in sections 2.2.1-2.3, is continuously compounded, annualized interest rate to borrow money for an infinitesimally short period of time from time t. Let us first look at a simple single factor model, i.e. models in which there is a single source of uncertainty driving all the interest rates and then move on to more modern multifactor models. Those models will be calibrated to actual bond data to compare with each other, in terms of likelihood attained and forecasting performance. The first single factor model examined is an Ornstein-Uhlenbeck process. Ornstein-Uhlenbeck process is a stationary, Gaussian and Markovian mean reverting stochastic process [108], in the form:

$$dx_t = \alpha(\beta - x_t)dt + \sigma dW_t \tag{3.1}$$

where W_t is the standard Brownian motion defined in section 2.2. α, β are constants, σ is a positive constant and $t > 0$. x_t has the following mean and variance, if x_0 is a constant:

$$\mathbb{E}[x_t|\mathcal{F}_0] = \beta + (x_0 - \beta)e^{-\alpha t}. \tag{3.2}$$

$$Var[x_t|\mathcal{F}_0] = \frac{\sigma^2}{2\alpha}. \tag{3.3}$$

The expected value expression explains the *mean reverting* terminology; starting at any x_0, x_t tends to revert to its long run mean β.

3.1.1 Single factor models

We will consider first the Vasicek Model introduced in [109] and then the Cox, Ingersoll and Ross (CIR) Model first described in [26].

Vasicek Model

Assuming that the instantaneous spot rate behaves as an Ornstein-Uhlenbeck process with constant coefficients we can write:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \tag{3.4}$$

W_t is the standard Brownian motion, κ the rate of reversion, θ mean of the drift, σ the volatility of the short rate r_t and r_0 is a constant. In the new notation and conditioning on time s (instead of $t = 0$), the conditional expected value and the conditional variance are given by:

$$\mathbb{E}[r_t|\mathcal{F}_s] = r_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}). \tag{3.5}$$

$$Var[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2\kappa}[1 - e^{-2\kappa(t-s)}]. \tag{3.6}$$

A zero-coupon bond that will mature at T will be worth, according to the Vasicek model, at time t:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{3.7}$$

where

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4\kappa}B(t,T)^2\right\}, \tag{3.8}$$

$$B(t,T) = \frac{1}{\kappa}[1 - e^{-\kappa(T-t)}]. \tag{3.9}$$

This model can be slightly modified to include the market price of risk λ_t as λr_t (as mentioned in section 2.2.4), where λ is a constant and r_t the instantaneous spot rate as before [19].

$$dr_t = \kappa(\theta - \frac{\lambda\sigma}{\kappa} - r_t]dt + \sigma dW_t \tag{3.10}$$

with the same conditions as with equation (3.4).

Cox Ingersoll Ross (CIR) Model

The CIR model is similar to the Vasicek model, but with an added square root term to the Brownian motion. Unlike the Vasicek model, the instantaneous short rate r_t remains always positive. The model is given by the following equations:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{3.11}$$

$\kappa, \theta, \sigma, x_0$ are defined as previously in section 3.1.1. There is also an extra condition:

$$2\kappa\theta > \sigma^2 \tag{3.12}$$

to keep the short rate positive. The mean and variance conditional on filtration $\mathcal{F}_s$ will be:

$$\mathbb{E}[r_t|\mathcal{F}_s] = r_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}), \tag{3.13}$$

$$Var[r_t|\mathcal{F}_s] = r_s\frac{\sigma^2}{\kappa}\left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) + \theta\frac{\sigma^2}{2\kappa}\left(1 - e^{-\kappa(t-s)}\right)^2. \tag{3.14}$$

The price of a zero-coupon bond maturing at time T will be at time t with the CIR model as described in [19]:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{3.15}$$

where

$$A(t,T) = \left[\frac{2h\ \exp\{(\kappa+h)(T-t)/2\}}{2h + (\kappa+h)(\exp^{(T-t)h} - 1)}\right]^{2\kappa\theta/\sigma^2}, \tag{3.16}$$

$$B(t,T) = \frac{2(\exp\{(T-t)h\} - 1)}{2h + (\kappa+h)(\exp\{(T-t)h\} - 1)}, \tag{3.17}$$

$$h = \sqrt{\kappa^2 + 2\sigma^2}. \tag{3.18}$$

with the addition of the market price of risk λ as defined in section 2.2.4, the CIR model becomes:

$$dr_t = [\kappa\theta - (\kappa + \lambda\sigma)r_t]dt + \sigma\sqrt{r_t}dW_t, \tag{3.19}$$

from equation (3.11) and is still required to satisfy the inequality (3.12). The pricing formula changes to:

$$\tilde{A}(t,T) = \left[\frac{2h\ \exp\{(\kappa+\lambda+h)(T-t)/2\}}{2h + (\kappa+\lambda+h)(\exp\{(T-t)h\} - 1)}\right]^{2\kappa\theta/\sigma^2}, \tag{3.20}$$

$$\tilde{B}(t,T) = \frac{2(\exp\{(T-t)h\} - 1)}{2h + (\kappa+\lambda+h)(\exp\{(T-t)h\} - 1)}, \tag{3.21}$$

$$\tilde{h} = \sqrt{\kappa^2 + \lambda^2 + 2\sigma^2}, \tag{3.22}$$

and

$$P(t,T) = \tilde{A}(t,T)e^{-\tilde{B}(t,T)r_t}. \tag{3.23}$$

3.1.2 Multifactor models

The most important limitation of single factor models lie in the fact that for multiple bonds maturing at different times with the same model will be perfectly correlated. So someone could simply hedge a bond with another bond, as all the bond yields will move in parallel. Another important limitation is the yield curve which lacks slopes and has a constant yield as the maturity tends to infinity. Having a multifactor model means that the bond yields are no longer perfectly correlated and their yields will be able to match more interesting and realistic term structures. Multifactor models offer "humped" shaped yield curves, and are smooth curves over long horizons. Normally, interest rate dynamics is adequately described by two factors; see, e.g.[30] for empirical evidence. We describe two common two factor models next.

The two factor Vasicek model

Similarly to the single factor Vasicek model (represented henceforth as Vas1), the two factor Vasicek model (represented henceforth as Vas2) is:

$$r_t = x_t + y_t, \tag{3.24}$$

$$dx_t = \kappa_x(\theta_x - x_t)dt + \sigma_x dW_t^1, \tag{3.25}$$

$$dy_t = \kappa_y(\theta_y - y_t)dt + \sigma_y dW_t^2. \tag{3.26}$$

where $x(0) = x_0$ and $y(0) = y_0$, $\kappa_z, \theta_z, \sigma_z$ are constants for a factor $z \in \{x, y\}$, x_0 and y_0 are constants and W_t^1, W_t^2 are independent Brownian motions as defined in section 2.2.

The price of a zero coupon bond will be:

$$P(t,T) = A_x(t,T)A_y(t,T)e^{(-B_x(t,T)x_t - B_y(t,T)y_t)}, \tag{3.27}$$

where

$$A_z(t,T) = \exp\left\{\left(\theta_z - \frac{\sigma_z^2}{2\kappa_z^2}\right)[B_z(t,T) - T + t] - \frac{\sigma_z^2}{4\kappa_z}B_z(t,T)^2\right\},\ \forall z \in (x,y), \tag{3.28}$$

$$B_z(t,T) = \frac{1}{\kappa_z}[1 - e^{-\kappa_z(T-t)}],\ \forall z \in (x,y). \tag{3.29}$$

More details on the two factor model, including a derivation for the formulae above can be found in [19].

The two factor CIR model

The two factor Cox Ingersoll Ross (represented henceforth as CIR2) model is a simple extension to the previous CIR model:

$$r_t = x_t + y_t, \tag{3.30}$$
$$dx_t = \kappa_x(\theta_x - x_t)dt + \sigma_x\sqrt{x_t}dW_t^1, \tag{3.31}$$
$$dy_t = \kappa_y(\theta_y - y_t)dt + \sigma_y\sqrt{y_t}dW_t^2. \tag{3.32}$$

With $x(0), y(0), \kappa_i, \theta_i, \sigma_i$ are similar to the previous definition in section 3.1.2 and again W_t^1 and W_t^2 are independent Brownian motions as defined in section 2.2. The price of a zero coupon bond will be:

$$P(t,T) = A_x(t,T)A_y(t,T)e^{-B_x(t,T)x_t - B_y(t,T)y_t}, \tag{3.33}$$

where

$$A_z(t,T) = \left[\frac{2h_z\ \exp\{(\kappa_z + \lambda_z + h_z)(T-t)/2\}}{2h_z + (\kappa_z + \lambda_z + h_z)(\exp\{(T-t)h_z\} - 1)}\right]^{2\kappa_z\theta_z/\sigma_z^2}, \tag{3.34}$$
$$B_z(t,T) = \frac{2(\exp\{(T-t)h_z\} - 1)}{2h_z + (\kappa_z + \lambda_z + h_z)(\exp\{(T-t)h_z\} - 1)}, \tag{3.35}$$
$$h_z = \sqrt{\kappa_z^2 + \lambda_z^2 + 2\sigma_z^2}, \tag{3.36}$$

where $z \in x, y$ and all other variables are defined as previously in section 3.1.2. Both models can be extended to a n factor model in a similar fashion [19].

Remark. 2-factor models seem to provide a good compromise in terms of flexibility of fitting a variety of shapes of term structures [84]. It also has the added advantage of having a manageable number of parameters for calibration [5] and [30]. A single factor CIR model has 5 parameters to calibrate and a factor to evaluate at each time step to obtain the short rate. The 2-factor CIR model has 11 parameters to calibrate and 2 factors to evaluate at each time step to get the short rate.

3.2 Kalman filter to calibrate models from bond price data

The calibration of models is essential to forecasting the value of bonds in future scenarios but also to get the parameter values to use the models with. In practice, we don't have access to short rate r_t or its components x_t, z_t. We can only measure bond prices $P(t_i, T_j)$ at discrete times t_i, $i = 1, 2, \ldots, N$ and for discrete maturities $j = 1, 2, \ldots, J$. To calibrate the interest rate model as well as to forecast bond prices, we need to be able to infer the values of x_t, z_t from the measured bond prices, in a computationally tractable manner. We can use the fact that the system is entirely affine in x_t, z_t since the yields are affine in x_t, z_t. For affine state space systems, the most common way of inferring the values of latent or hidden variables from measured variables is using a recursive moment estimator called the Kalman filter.

Let us denote, x_n here is a generic latent state variable and its relationship with factors x_t, y_t mentioned earlier can be clarified as follows. x_n is the short rate r_n at time t_n for a single factor model and x_n is a vector of the two latent states at time t_n for a two factor model. z_n is the yield vector at time t_n. Note that yields are affine in the short rate for both CIR and Vasicek models. Kalman filtration [65], is a recursive filter used to approximate the latent or unobserved states of linear dynamic systems from noisy measurements. To describe linear filtering in a general set up, let us consider in a discrete time the following linear dynamic state space system:

$$x_{n+1} = \Psi x_n + \Pi + \varepsilon_{n+1} \qquad \text{Evolution Equation} \tag{3.37}$$

$$z_n = \Upsilon x_n + \Xi + \eta_n \qquad \text{Measurement Equation} \tag{3.38}$$

where ε and η are Gaussian, uncorrelated, white noise from the evolution and measurements with mean zero. Finding x_n at time t_n may be of interest to predict the yield curve at time t_{n+1} or to find a discounted factor at a time to maturity where there might not be an observable measurement. The matrices $\Psi, \Pi, \Upsilon, \Xi, \mathbb{E}(\varepsilon_n \varepsilon_n^\top) = Q > 0, \mathbb{E}(\eta_n \eta_n^\top) = R \geq 0$ are all constants with respect to every iterations. For calibration and forecasting in interest rate models, the matrices Ψ, Π, Υ, Ξ are expressed in terms of the model parameters, as will later be explained in section 3.2.1. The following set of recursive equations is normally referred to as the Kalman filter:

$$\text{innovations } v_n = z_n - (\Upsilon \widehat{x}_{n|n-1} + \Xi), \tag{3.39}$$

$$\text{variance of innovations } \Sigma_n = \Upsilon P_{n|n-1} \Upsilon^\top + R, \tag{3.40}$$

$$\text{Kalman gain } K_n = \Psi P_{n|n-1} \Upsilon^\top \Sigma_n^{-1}, \tag{3.41}$$

$$\text{conditional mean } \widehat{x}_{n+1|n} = \Psi x_{n|n-1} + \Pi + K_n v_n, \tag{3.42}$$

$$\text{conditional variance } P_{n+1|n} = \Psi P_{n|n-1} \Psi^\top + Q - \Psi P_{n|n-1} \Upsilon^\top \Sigma_n^{-1} \Upsilon P_{n|n-1} \Psi^\top. \tag{3.43}$$

Our objective here will be to predict future values of our short rate to evaluate the corresponding yields. Once x_{n+1} has a predicted value then a value for z_{n+1} can be obtained from equation (3.38). Vice versa, from the observation z_n, a value can be found for the unobservable x_n. For the purpose of bond issuance, x_n will correspond to the instantaneous interest rate (or short rate) at time t_n and z_n will be the bond yield at time t_n. The short rate vector x_n is obtained from the vector of discounted factor $e^{-R(t,\tau_n)\tau_n}$ for any $\tau_n > t$ given $z_1, z_2, \cdots, z_M$. Since the innovations are jointly Gaussian with variance Σ_n, the prediction error decomposition of the logarithm of likelihood function will be:

$$L(z_n, \Theta) = \sum_{n=1}^{M} L(z_n|z_{n-1}, \Theta), \tag{3.44}$$

$$= -\sum_{n=1}^{M} \left[\frac{dim(z_n)}{2} log(2\pi) + \frac{1}{2} log(|\Sigma_{n|n-1}|) + \frac{1}{2} v_n^\top \Sigma_n^{-1} v_n \right], \tag{3.45}$$

Afterwards a sequence of innovations v_n can be constructed and maximise the likelihood of observations over the set of parameters. Maximising the likelihood is equivalent however to maximising its logarithm which, in turn, is equivalent to minimizing the following function:

$$L(z_n, \Theta) = \sum_{n=1}^{M} (\log \det(\Sigma_n) + v_n^\top \Sigma_n^{-1} v_n). \tag{3.46}$$

Given a sequence of observations $z_1, \ldots, z_M$ and a parameter vector Θ which characterizes the system matrices, Kalman filter equations (3.39)-(3.43) generate a sequence of innovations from which $L(z_n, \Theta)$ can be computed. In our implementation, this function of Θ is then minimized over Θ by using Nelder-Mead method, via the fminsearch routine in MATLAB. Note that the set of equations (3.39)-(3.43) are run over the entire data set for each new value of parameter vector Θ to calculate the innovations and hence the cost function $L(z_n, \Theta)$. Maximum likelihood is a generic technique and can be used provided the expressions for the conditional mean $\mathbb{E}(x_{n|n-1})$ and the covariance of $x_{n|n-1}$ can be found. This may be found using Kalman filters as described above or it may be found more easily if x_n is measurable. We will use the Kalman filter to find the best initial values for the parameters and use the Kalman filter again to estimate the best values of the time changing variables. A review of the use of Kalman filtration to financial mathematics can be found in [29] which also discusses some of the generalizations of the above framework for nonlinear and non-Gaussian systems. More detailed exposition on the use of filtering in time series models is given in [40].

3.2.1 Numerical Results on calibration on interest rate models

To set up a calibration problem formally in discrete time, we need to discretise the pricing models considered. To begin, we will use a natural discretisation of the Vasicek model (see [58]) which preserves the conditional mean and variance of r_{n+1} at time $t_{n+1} = t_n + \Delta t$ and is given by:

$$r_{t+\Delta t} = \mathbb{E}(r_{t+\Delta t}|r_t) + \sqrt{Var(r_{t+\Delta t}|r_t)}\sqrt{\Delta}\epsilon_{t+\Delta t}, \tag{3.47}$$

where the mean and variance are given in equations (3.2)-(3.3) and more specifically for the Vasicek Model, given in equations (3.5)-(3.6). The Euler discretisation (as shown in [48]) will be:

$$r_{t+\Delta t} - r_t = \kappa(\theta - r_t)\Delta t + \sigma\varepsilon_{t+\Delta t}\sqrt{\Delta t}, \tag{3.48}$$

where $\{\varepsilon_t\}$ is a sequence of scalar i.i.d. Gaussian random variables with zero mean and unit variance and $\Delta t = t_{n+\Delta t} - t_n$ is assumed to be a constant for all n. r_n is an unobserved variable as there is no observable security which pays return instantaneously. When using the conditional mean and variance equations for the CIR model (3.13) - (3.14), the Euler discretisation is in the form:

$$r_{t+\Delta t} - r_t = \kappa(\theta - r_t)\Delta t + \sigma\sqrt{r_t}\varepsilon_{t+\Delta t}\sqrt{\Delta t}, \tag{3.49}$$

where $\varepsilon_{t+\Delta t}$ is defined as before. We use a similar discretisation scheme for a two factor Vasicek model:

$$x_{t+\Delta t} - x_t = \kappa_1(\theta_1 - x_t)\Delta t + \sigma_1\varepsilon^x_{t+\Delta t}\sqrt{\Delta t}, \tag{3.50}$$

$$z_{t+\Delta t} - z_t = \kappa_2(\theta_2 - z_t)\Delta t + \sigma_2\varepsilon^z_{t+\Delta t}\sqrt{\Delta t}, \tag{3.51}$$

and for a two factor CIR model:

$$x_{t+\Delta t} - x_t = \kappa_1(\theta_1 - x_t)\Delta t + \sigma_1\sqrt{x_t}\varepsilon^x_{t+\Delta t}\sqrt{\Delta t}, \tag{3.52}$$

$$z_{t+\Delta t} - z_t = \kappa_2(\theta_2 - z_t)\Delta t + \sigma_2\sqrt{z_t}\varepsilon^z_{t+\Delta t}\sqrt{\Delta t}. \tag{3.53}$$

while x_t and y_t are not observable, one may observe yields from zero coupon bonds at each time t_n which we will be denoted by $z(t_n, T_i)$, for different maturities $T_1, T_2, \ldots T_N$, $T_i > t_n$. To describe the observation equation, denote by $\mathbf{z}_n$ a vector in $\mathbb{R}^N$ whose i^{th} element is $z(t_n, T_i)$. Further, one may assume that our model of the short rate is imperfect and the vector of observed yields at time t_n is given by:

$$\mathbf{R}_n = \mathbf{z}_n + \sigma_z\mathbf{e}_n, \tag{3.54}$$

where $\{\mathbf{e}_n\}$ is a vector valued, i.i.d. Gaussian sequence with zero mean and identity matrix as covariance and $\sigma_z > 0$ is a constant indicating the dispersion of the observed yields from their value given by the model. Our time discretisation form a linear state space system. At this point, we should remind that every unit zero coupon bond price obtained from dynamic interest rate model was declared in the form:

$$P(t,T) = A_t e^{-B_t r_t}, \tag{3.55}$$

and from 2.20,the yield of a unit zero coupon bond price is:

$$P(t,T) = e^{-yield(t,T)(T-t)}. \tag{3.56}$$

This means that we can describe the yield as:

$$yield(t,T) = \frac{B_t r_t - log(A_t)}{T-t}. \tag{3.57}$$

Let us denote: $\Xi_t(\tau)$, the vector of individual $log(A)_t/\tau$ and $\Upsilon_t(\tau)$, the vector of individual Υ_t/τ. For the Nelson-Siegel class of yield curves, the yield curves are already given by the model. We can use Kalman filter outlined in equations (3.39)-(3.43) to calibrate the model from observed time series $\mathbf{R}_n$, $n = 1, 2, \ldots, M$. The recursive equations for Kalman filter are repeated below for easy reference. The optimal estimate of r_{n+1} based on measurement $\mathbf{R}_n$ (respectively, based on $\mathbf{R}_{n+1}$) is denoted as $\hat{r}_{n+1|n}$ (respectively, $\hat{r}_{n+1|n+1}$). $\mathbf{v}_n$ denotes the innovations vector at time t_n while Σ_n denotes the covariance matrix of innovations at time t_n. The set of equations given below outline the recursive propagation of estimates from $\hat{r}_{n|n-1}, P_{n|n-1}$ to $\hat{r}_{n+1|n}, P_{n+1|n}$ after measuring $\mathbf{R}_n$.

$$\mathbf{v}_n = \mathbf{R}_n - \Xi_n(\tau) + \Upsilon_n(\tau)\hat{r}_{n|k-1}, \tag{3.58}$$

$$\Sigma_n = \Upsilon_n(\tau) P_{n|n-1} \Upsilon_n(\tau)^\top + R, \tag{3.59}$$

$$K_n = \Psi_n P_{n|n-1} \Upsilon_n(\tau)^\top \Sigma_n^{-1}, \tag{3.60}$$

$$\hat{r}_{n+1|n} = \Psi_n \hat{r}_{n|n} + \Pi_n + K_n v_n, \tag{3.61}$$

$$P_{n+1|n} = \Psi_n P_{n|n-1} \Psi_n^\top + Q_n - \Psi_n P_{n|n-1} \Upsilon_n(\tau)^\top \Sigma_n^{-1} \Upsilon_n(\tau) P_{n|n-1} \Psi_n^\top, \tag{3.62}$$

(3.61) is the normal state space evolution adjusted with the Kalman gain and innovation. Ψ_n and Π_n are derived from the moment matching/ mean-variance preserving discretisations 3.47, [19]. They are for the Vasicek and CIR single factor models:

$$\Psi_n = e^{-\kappa \Delta t}, \tag{3.63}$$

$$\Pi_n = (\theta - \frac{\lambda\sigma}{\kappa})(1 - e^{-\kappa\Delta t}) \tag{3.64}$$

as described in equations (3.5)-(3.6) and (3.13)-(3.14). For the multi-factor models:

$$\Psi_n = \begin{pmatrix} e^{-\kappa_1\Delta t} & 0 & \cdots & 0 \\ 0 & e^{-\kappa_2\Delta t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{-\kappa_n\Delta t} \end{pmatrix}, \tag{3.65}$$

$$\Pi_n = \begin{pmatrix} (\theta_1 - \frac{\lambda_1\sigma_1}{\kappa_1})(1-e^{-\kappa_1\Delta t}) \\ (\theta_2 - \frac{\lambda_2\sigma_2}{\kappa_2})(1-e^{-\kappa_2\Delta t}) \\ \vdots \\ (\theta_n - \frac{\lambda_n\sigma_n}{\kappa_n})(1-e^{-\kappa_n\Delta t}) \end{pmatrix}, \tag{3.66}$$

and where $\Upsilon_n(\tau)$ and $\Xi_n(\tau)$ are for a k-factor Vasicek model for m bonds considered:

$$\Upsilon_n(\tau) = \begin{pmatrix} \frac{(1-e^{-\kappa_1\tau_1})}{\kappa_1} & \cdots & \frac{(1-e^{-\kappa_k\tau_1})}{\kappa_k} \\ \frac{(1-e^{-\kappa_1\tau_2})}{\kappa_1} & \cdots & \frac{(1-e^{-\kappa_k\tau_2})}{\kappa_k} \\ \vdots & \vdots & \vdots \\ \frac{(1-e^{-\kappa_1\tau_m})}{\kappa_1} & \cdots & \frac{(1-e^{-\kappa_k\tau_m})}{\kappa_k} \end{pmatrix}, \tag{3.67}$$

$$\Xi_n(\tau) = \begin{pmatrix} (\vartheta_1 - \frac{\sigma_1^2}{2\kappa_1^2})[C_n(\tau_1)-\tau_1] - \frac{\sigma_1^2}{4\kappa_1}(C_n(\tau_1))^2 + \cdots + (\vartheta_k - \frac{\sigma_k^2}{2\kappa_k^2})[C_n(\tau_1)-\tau_1] - \frac{\sigma_k^2}{4\kappa_k}(C_n(\tau_1))^2 \\ (\vartheta_1 - \frac{\sigma_1^2}{2\kappa_1^2})[C_n(\tau_2)-\tau_2] - \frac{\sigma_1^2}{4\kappa_1}(C_n(\tau_2))^2 + \cdots + (\vartheta_k - \frac{\sigma_k^2}{2\kappa_k^2})[C_n(\tau_2)-\tau_2] - \frac{\sigma_k^2}{4\kappa_k}(C_n(\tau_2))^2 \\ \vdots \\ (\vartheta_1 - \frac{\sigma_1^2}{2\kappa_1^2})[C_n(\tau_m)-\tau_m] - \frac{\sigma_1^2}{4\kappa_1}(C_n(\tau_m))^2 + \cdots + (\vartheta_k - \frac{\sigma_k^2}{2\kappa_k^2})[C_n(\tau_m)-\tau_m] - \frac{\sigma_k^2}{4\kappa_k}(C_n(\tau_m))^2 \end{pmatrix} \tag{3.68}$$

where $\vartheta_i = \theta_i - \lambda_i\sigma_i/\kappa_i$ and τ is the difference between the maturity of the bond and the current time t_n. For the k-factor CIR model:

$$\Upsilon_n(\tau) = \begin{pmatrix} \frac{2(e^{h_1\tau_1}-1)}{2h_1+(\kappa_1+\lambda_1+h_1)(e^{h_1\tau_1}-1)} & \cdots & \frac{2(e^{h_k\tau_1}-1)}{2h_k+(\kappa_k+\lambda_k+h_k)(e^{h_k\tau_1}-1)} \\ \vdots & \vdots & \vdots \\ \frac{2(e^{h_1\tau_m}-1)}{2h_1+(\kappa_1+\lambda_1+h_1)(e^{h_1\tau_m}-1)} & \cdots & \frac{2(e^{h_k\tau_m}-1)}{2h_k+(\kappa_k+\lambda_k+h_k)(e^{h_k\tau_m}-1)} \end{pmatrix}, \tag{3.69}$$

$$\Xi_n(\tau) = \begin{pmatrix} log(\left[\frac{2h_1\, e^{(\kappa_1+\lambda_1+h_1)\tau_1/2}}{2h_1+(\kappa_1+\lambda_1+h_1)(e^{h_1\tau_1}-1)}\right]^{\frac{2\kappa_1\theta_1}{\sigma_1^2}} + \cdots + \left[\frac{2h_k\, e^{(\kappa_k+\lambda_k+h_k)\tau_1/2}}{2h_k+(\kappa_k+\lambda_k+h_k)(e^{h_k\tau_1}-1)}\right]^{\frac{2\kappa_k\theta_k}{\sigma_k^2}}) \\ \vdots \\ log(\left[\frac{2h_1\, e^{(\kappa_1+\lambda_1+h_1)\tau_m/2}}{2h_1+(\kappa_1+\lambda_1+h_1)(e^{h_1\tau_m}-1)}\right]^{\frac{2\kappa_1\theta_1}{\sigma_1^2}} + \cdots + \left[\frac{2h_k\, e^{(\kappa_k+\lambda_k+h_k)\tau_m/2}}{2h_k+(\kappa_k+\lambda_k+h_k)(e^{h_k\tau_m}-1)}\right]^{\frac{2\kappa_k\theta_k}{\sigma_k^2}}) \end{pmatrix} \tag{3.70}$$

with $h = \sqrt{\kappa^2 + \lambda^2 + 2\sigma^2}$. The covariance matrix Q_n is defined for the two factor Vasicek model as:

$$Q_n = \sigma_v^2 I, \tag{3.71}$$

and for the CIR models as:

$$Q_n = \sigma_v^2 \phi_n, \tag{3.72}$$

where σ_v is the standard deviation of measurement equation noise and ϕ_n is r_n for the single factor CIR model and:

$$\phi_n = \begin{pmatrix} |x_n| & 0 \\ 0 & |y_n| \end{pmatrix} \tag{3.73}$$

for the two factor CIR model. To find estimates of parameters, the joint probability density function (also called the likelihood function) of observations is maximized over the parameter vector, which in the single factor case is:

$$\Theta = \begin{bmatrix} \kappa & \lambda & \theta & \sigma & r_0 & \sigma_z \end{bmatrix}^\top \tag{3.74}$$

Here λ is the price of risk which is assumed to be constant through time. Since the forecast errors are i.i.d. and Gaussian as shown in [6], the log likelihood function is expressed by:

$$L(\Theta) = \sum_{n=1}^{M} \log p(\mathbf{R}_n | \mathcal{F}_{n-1}, \Theta). \tag{3.75}$$

where $p(\mathbf{R}_n | \mathcal{F}_{n-1}, \Theta)$ is multivariate Gaussian with mean 0 and variance Σ_n, and maximising a concave function is the same as maximising its logarithm. Hence maximising $L(\Theta)$ is the same as minimizing $-L(\Theta)$.

$$-L(\mathbf{R}_n, \Theta) = \frac{1}{2} \sum_{n=1}^{M} \left(\log \det(\Sigma_n) + \mathbf{v}_n^T \Sigma_n^{-1} \mathbf{v}_n \right), \tag{3.76}$$

where the constant terms are ignored. This smooth nonlinear cost function can be minimized over the set of parameters using any standard nonlinear solver. We use MATLAB's " off-the-shelf " optimizer *fminsearch* which uses the Nelder-Mead method and seemed to perform satisfactorily. It should be noted that as the bond prices are real, i.e. they are not zero coupon bonds, the process of calibration is done by stripping coupons. We use the above procedure to calibrate various term structure models to UK government bond data. In particular, we will calibrate and compare eight different models: one factor Vasicek (Vas1), two factor Vasicek (Vas2), one factor CIR (CIR1), two factor CIR (CIR2) and certain models belonging to the class of dynamic Nelson-Siegel models. The class of dynamic Nelson-Siegel models used here have been described in greater detail earlier in section 2.3 and there will be four different models from this class:

- a basic dynamic Nelson-Siegel model (Basic DNS), a two factor version of the standard three factor model found in equations (2.23)-(2.25),
- the standard three factor dynamic Nelson-Siegel model (DNS) defined in equation (2.21)-(2.22),
- the arbitrage free dynamic Nelson-Siegel with independent factors (AFDNSi) defined in equation (2.31),
- the arbitrage free dynamic Nelson-Siegel with correlated factors (AFDNSc) defined in equation (2.34),

The basic DNS and DNS models are described in more detail in [36] and the arbitrage free DNS models in [20].Gilt yields from 2006 to 2008, obtained from the UK debt management office, were used for the numerical experiments. We use daily data from April 2006 to March 2008. The model is calibrated every quarter from March 2007 to March 2008, based on the past data stretching back one year. In other words, we move the one year calibration window forward through time as the year unfolds into the next financial year. The re-calibration takes into account the fact that the interest rate model parameters may not be constant and drift through time, *e.g.* due to the impact of the earlier issuance and due to the changes in the market sentiment. The choice of re-calibration every quarter corresponds to quarterly review. This gives us five calibrations. The initial guess for the single factor models are in table (3.1) and the two factor models in table (3.2). The results of the 1^{st} iteration are then used to guess the initialization parameters for the next iterations. The parameter values obtained through calibration for various models are reported in tables (3.3)-(3.9).

time	Vas1	CIR1
κ	0.10	0.045
λ	0.01	0.015
θ	0.05	0.04
σ	0.04	0.05
r_0	0.05	0.06
σ_v	0.01	0.01
P_0	100	100
Q_0	0.1	0.5

Table 3.1: Initial guess for the single factor models (UK 2006-2008 data).

time	Vas2	CIR2
κ_1	0.1	0.08
λ_1	0.07	0.005
θ_1	0.047	0.05
σ_1	0.045	0.045
x_0	0.05	0.05
σ_v	0.02	0.02
κ_2	0.055	0.013
λ_2	0.03	0.005
θ_2	0.047	0.05
σ_2	0.045	0.045
σ_w	0.02	0.02
z_0	0.03	0.03
P_0	100	100

Table 3.2: Initial guess for the two factor models (UK 2006-2008 data).

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
κ	0.118437	0.107130	0.103352	0.106248	0.109917
λ	0.010318	0.010415	0.010613	0.010378	0.010245
θ	0.049894	0.051202	0.051649	0.050243	0.050013
σ	0.044564	0.045321	0.045792	0.045540	0.045085
r_0	0.065100	0.066458	0.068616	0.068484	0.066599
σ_v	0.002555	0.002659	0.002599	0.002652	0.002744
P_0	100.597148	104.309313	104.080436	107.308930	109.870553
Q_0	0.103255	0.102218	0.103585	0.103874	0.105143

Table 3.3: Parameters of Vas1 model (UK 2006-2008 data).

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
κ	0.061925	0.051571	0.062338	0.064651	0.064603
λ	0.005209	0.010409	0.004168	0.004082	0.005628
θ	0.003070	0.008765	0.011169	0.011535	0.009663
σ	1.912993e-8	3.083262e-8	2.459200e-8	2.545363e-8	3.230975e-8
r_0	0.000592	0.001660	0.002860	0.002867	0.000769
σ_v	0.019869	0.020032	0.020101	0.019815	0.019726
P_0	15.373186	25.575654	22.567362	22.333447	26.955768
Q_0	0.687923	0.757067	0.499557	0.491960	0.536613

Table 3.4: Parameters of CIR1 model (UK 2006-2008 data).

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
κ_1	0.097654	0.096200	0.096038	0.095993	0.095504
λ_1	0.070736	0.074596	0.074988	0.075305	0.075412
θ_1	0.045791	0.045140	0.045067	0.045249	0.045243
σ_1	0.046015	0.047224	0.047209	0.047261	0.047326
x_0	0.050230	0.032652	0.032956	0.033191	0.033696
σ_v	0.020197	0.029862	0.030192	0.030272	0.030162
κ_2	0.054964	0.054218	0.054158	0.054218	0.054492
λ_2	0.030225	0.033708	0.033905	0.034027	0.034333
θ_2	0.047065	0.044503	0.044370	0.044163	0.043956
σ_2	0.046091	0.048904	0.049052	0.049191	0.049454
σ_w	0.020056	0.016743	0.016888	0.017008	0.017184
z_0	0.030200	0.025263	0.025463	0.025633	0.025933
P_0	100.339884	99.526279	100.288024	101.060924	100.794342

Table 3.5: Parameters of Vas2 model (UK 2006-2008 data).

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
κ_1	0.052408	0.052661	0.052800	0.052800	0.052951
λ_1	0.004951	0.004973	0.004987	0.004987	0.005002
θ_1	0.045660	0.045829	0.045939	0.045939	0.046070
σ_1	0.045427	0.045575	0.045686	0.047970	0.048107
x_0	0.052993	0.053217	0.053378	0.053378	0.053530
σ_v	0.023259	0.023362	0.024019	0.024019	0.024087
κ_2	0.010644	0.010636	0.010649	0.010649	0.010620
λ_2	0.006387	0.006427	0.006434	0.006434	0.006436
θ_2	0.033062	0.033069	0.033106	0.033106	0.033050
σ_2	0.054799	0.054876	0.054557	0.054557	0.054586
σ_w	0.021544	0.021635	0.021690	0.021690	0.021752
z_0	0.032677	0.032827	0.032921	0.032921	0.033015
P_0	105.977270	106.482148	106.796219	106.796221	109.771486

Table 3.6: Parameters of CIR2 model (UK 2006-2008 data).

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
λ_0	0.138662	0.138662	0.138662	0.138232	0.138232
l_0	0.147847	0.148217	0.148687	0.146613	0.147071
s_0	0.037379	0.037479	0.037595	0.037164	0.037281
c_0	0.014667	0.014674	0.014719	0.014713	0.014759
σ	0.000592	0.000592	0.000592	0.000597	0.000597
σ_v	0.035564	0.035564	0.035564	0.035492	0.035492
$P0$	-0.337769	-0.337769	-0.337769	-0.337040	-0.337040
$Q0$	-0.007052	-0.007088	-0.007111	-0.007120	-0.007142
$meanl$	-0.011207	-0.011255	-0.011290	-0.011181	-0.011216
$means$	0.013115	0.013488	0.013868	0.013817	0.014206
$meanc$	0.004050	0.004061	0.004074	0.004158	0.025933
a_{11}	0.042646	0.042646	0.042646	0.042772	0.004171
a_{12}	0.000853	0.000853	0.000853	0.000860	0.000860
a_{13}	0.010656	0.010656	0.010656	0.010766	0.010766
a_{21}	0.008927	0.008927	0.008927	0.009003	0.009003
a_{22}	0.036299	0.036299	0.036299	0.036263	0.036263
a_{23}	0.003066	0.003066	0.003066	0.003116	0.003116
a_{31}	-0.011675	-0.011675	-0.011675	-0.011684	-0.011684
a_{32}	-0.005959	-0.005959	-0.005959	-0.006002	-0.006002
a_{33}	0.012517	0.012517	0.012517	0.012638	0.012638

Table 3.7: Parameters of the DNSl model (UK 2006-2008 data)

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
λ_0	0.033079	0.033079	0.033207	0.033207	0.033207
l_0	0.209335	0.209335	0.212645	0.212644	0.212646
s_0	0.019013	0.019013	0.018933	0.018968	0.019034
c_0	0.022531	0.022531	0.022382	0.022422	0.022498
σ	-0.000128	-0.000128	-0.000128	-0.000128	-0.000128
σ_v	0.034758	0.034758	0.034865	0.034865	0.034865
$P0$	-0.513027	-0.513027	-0.513887	-0.513887	-0.513887
$Q0$	0.333635	0.333635	0.334279	0.334886	0.336060
$meanl$	-0.002335	-0.002335	-0.002331	-0.002335	-0.002343
$means$	0.002601	0.002731	0.002719	0.002826	0.002906
$meanc$	0.009749	0.009749	0.009667	0.009685	0.009719
a_{11}	0.012938	0.012938	0.012976	0.012976	0.012976
a_{12}	0.026422	0.026422	0.026508	0.026508	0.026508
a_{13}	-0.014772	-0.014772	-0.01495	-0.014952	-0.014952
a_{21}	0.013967	0.013967	0.014017	0.014017	0.014017
a_{22}	0.000355	0.000355	0.000356	0.000356	0.000356
a_{23}	0.032897	0.032897	0.033324	0.033324	0.033324
a_{31}	0.012712	0.012712	0.012716	0.012716	0.012716
a_{32}	0.004821	0.004821	0.004854	0.004854	0.004854
a_{33}	0.041492	0.041492	0.041902	0.041902	0.041902

Table 3.8: Parameters for AFDNSi model (UK 2006-2008 data)

time	$t = 03/2007$	$t = 06/2007$	$t = 09/2007$	$t = 12/2007$	$t = 03/2008$
λ_0	0.138549	0.138549	0.138549	0.140731	0.140731
l_0	0.089577	0.089577	0.089577	0.089470	0.089471
s_0	0.102094	0.102094	0.102094	0.101318	0.101318
c_0	0.029198	0.029198	0.029198	0.029290	0.029290
σ	0.011546	0.011546	0.011546	0.011431	0.011431
σ_{21}	0.001176	0.001176	0.001176	0.001188	0.001188
σ_{31}	0.001016	0.001016	0.001016	0.001018	0.001018
σ_{32}	0.001092	0.001092	0.001092	0.001105	0.001105
σ_v	0.037580	0.037580	0.037580	0.037743	0.037743
$P0$	96.283627	96.283627	96.283627	97.616635	97.616635
$Q0$	-0.000094	-0.000094	-0.000094	-0.000095	-0.000096
$meanl$	0.010909	0.010909	0.011454	0.011288	0.011853
$means$	0.011427	0.011427	0.011427	0.011448	0.011448
$meanc$	0.008496	0.008496	0.008496	0.008392	0.008392
a_{11}	0.009852	0.009852	0.009852	0.010007	0.010007
a_{12}	0.009829	0.009829	0.009829	0.009984	0.009984
a_{13}	0.009428	0.009428	0.009428	0.009064	0.009064
a_{21}	0.011303	0.011303	0.011303	0.011233	0.011233
a_{22}	0.011150	0.011150	0.011150	0.011183	0.011183
a_{23}	0.007872	0.007872	0.007872	0.007856	0.007856
a_{31}	0.011634	0.011634	0.011634	0.011817	0.011817
a_{32}	0.014319	0.014319	0.014319	0.014400	0.014400
a_{33}	0.009800	0.009800	0.009800	0.009955	0.009955

Table 3.9: Parameters for AFDNSc model (UK 2006-2008 data)

time	Vas1	CIR1	Vas2	CIR2
$t = 03/2007$	-10931.822	-6924.240	-292197.983	-7160.666
$t = 06/2007$	-10527.814	-6899.337	-13232.745	-7160.795
$t = 09/2007$	-10908.696	-6910.331	-13474.658	-7189.122
$t = 12/2007$	-10547.430	-6931.456	-503400.291	-8670.995
$t = 03/2008$	-9614.797	-6925.629	-579432.451	-8261.713

Table 3.10: Achieved likelihood values for the models after maximization (UK 2006-2008 data).

time	Basic DNS	DNS	AFDNSi	AFDNSc
$t = 03/2007$	-25165.672	-1.014447e+015	-5.845408e+014	-1.753180e+16
$t = 06/2007$	-25180.714	-8.806014e+015	-2.943862e+014	-9.275356e+15
$t = 09/2007$	-25222.765	-1.635689e+014	-2.130715e+014	-1.068158e+15
$t = 12/2007$	-25250.622	-1.711841e+014	-4.868823e+014	-3.273659e+15
$t = 03/2008$	-25274.040	-2.585992e+013	-1.044413e+014	-1.224788e+15

Table 3.11: Achieved likelihood values for the models after maximization (UK 2006-2008 data)

To compare the quality of yield curve matching, we measure the out-of-sample 2-norm errors achieved by each calibrated model. These errors are computed by:

$$\sum_{n=1}^{N} \sqrt{\sum_{t=0}^{T} (y_t^{real}(\tau_n) - y_t^{model}(\tau_n))^2}, \tag{3.77}$$

where n corresponds to the n^{th} bond modeled out of the N considered and τ_n is the time to maturity T. y_{real} corresponds to the actual UK yield and y_{model} corresponds to the model generated yield. The results obtained are found in tables (3.12)-(3.13).

A Jarque-Bera test is a goodness of fit statistical test for a normally distributed sample. It is performed on the short rates obtained from the calibrations to check whether the skewness and kurtosis match a normal distribution. It corresponds to:

$$T = \frac{n}{6}\left(S^2 + \frac{1}{4}(K-3)^2\right), \tag{3.78}$$

where the skewness S is defined as:

$$S = \frac{\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^2\right)^{3/2}}, \tag{3.79}$$

and the kurtosis K is defined as:

$$K = \frac{\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^2\right)^{2}}, \tag{3.80}$$

with r_i the short rate at time i and $\bar{r}$ is the short rate sample mean. Table (3.14) shows the Jarque-Bera test of the short rate distribution obtained from the calibration. The value 1 corresponds to a non-normally distributed short rate and 0 corresponds to a normally distributed short rate for each of the 5 calibrations performed per model. The table indicates that the normal distribution assumption for the short rate is inaccurate for Vasicek type models, even though the out of sample performance of Vas2 is better than most other models.

For example, the first row $(1,1,1,1,1)$ indicates that all the five re-calibrations indicate that the short rate is not normally distributed. Note that Vasicek model is a linear Gaussian process and inherently assumes Gaussian distribution for the short rate. The short rate in CIR model is known to be non-central chi-squared distributed, and the results of Jarque-Bera test in case of CIR1, CIR2 are consistent with the expectation that the rate is not normally distributed.

Figures (3.1)-(3.4) demonstrate the evolution of actual yields to maturity, after each calibration experiment for the different models from this chapter and figures (3.5)-(3.7) the models from the previous chapter. The figures in Appendix B correspond to the calibrations for the 4 next horizons. The models are more accurate in the earlier quarters with absolute errors below 0.3% when the interest rates were more stable.

Remark. The errors shown on the figures (3.1)-(3.4) are absolute and not percentage errors.

time	Vas1	CIR1	Vas2	CIR2
$t = 03/2007$	33.320036	34.475752	14.877411	22.446065
$t = 06/2007$	35.371241	30.595162	15.127590	22.551451
$t = 09/2007$	44.337066	39.098780	21.405030	27.009136
$t = 12/2007$	68.676484	64.209287	30.131410	35.667145
$t = 03/2008$	115.623473	111.613957	44.931169	52.011107

Table 3.12: Values of the out-of-sample 2-norm errors.

time	Basic DNS	DNS	AFDNSi	AFDNSc
$t = 03/2007$	23.023559	25.540732	24.569861	21.329200
$t = 06/2007$	23.173706	26.822766	24.379365	22.693189
$t = 09/2007$	27.492665	32.537907	29.941303	31.833671
$t = 12/2007$	36.488645	41.494472	39.246977	34.778105
$t = 03/2008$	51.526804	56.107529	57.407175	55.616764

Table 3.13: Values of the out-of-sample 2-norm errors.

Model	Jarque-Bera test
Vas1	(1,1,1,1,1)
CIR1	(1,1,1,1,1)
Vas2	(1,1,1,1,1)
CIR2	(1,1,1,1,1)
DNS	(0,0,1,1,1)
AFDNSi	(1,1,1,1,1)
AFDNSc	(1,1,1,1,1)

Table 3.14: Jarque-Bera test on all models

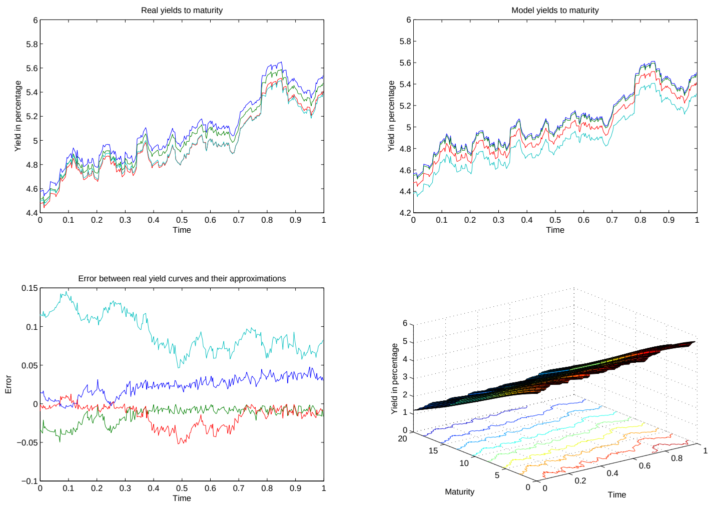

Figure 3.1: Vasicek Model calibrated with Kalman filtration at $t = 03/2007$

Remarks. The values of parameters seem fairly stable for the Vasicek model over time. These values also seem to make sense with a short rate around 5% and a low market volatility. The price of risk is nearly 1% along the year. The CIR model renders a downward yield curve which is closer to the real yield curve at the time despite having a very small mean rate. The short rate is also around 5%. Vas2 or CIR2 factor and the basic version of the dynamic Nelson-Siegel models (without curvature) approximate much better the real yield curves compared to the one factor models in terms of out-of-sample 2-norm errors and the maximum likelihood estimators converge to the same maximum of a variety of starting points along the year. It should be noticed that the 3-DNS (and other three factor models) have the best maximum likelihood estimators and have better yield curve approximations, in terms of 2-norm errors as reported in tables (3.12)-(3.13). In 4 out of 5 cases, the 3 factor DNS model with correlated factors outperforms the other three factor versions out-of-sample. Interestingly, basic DNS model performs better out-of-sample than the arbitrage-free model with independent factors or the three factor DNS model. The 3 factor models are also computationally quite expensive, and take significantly longer to calibrate, e.g. a three factor DNS model takes approximately 4600 seconds to calibrate in Matlab R2011b while a 2 factor model takes approximately 700 seconds per calibration, under similar software and hardware settings.

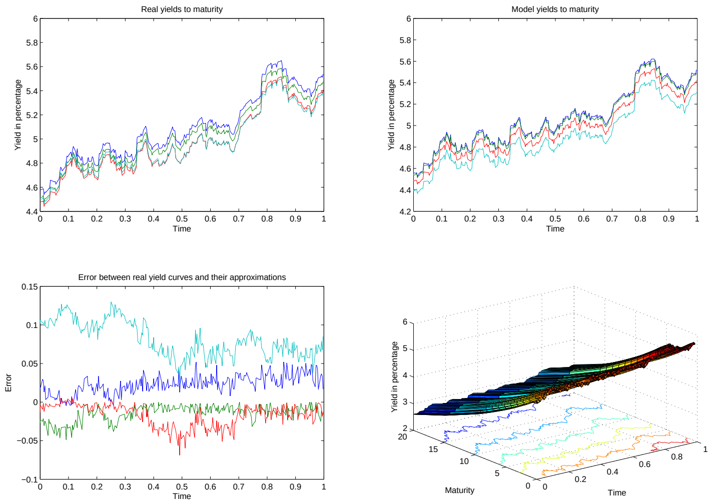

Figure 3.2: CIR Model calibrated with Kalman filtration at $t = 03/2007$

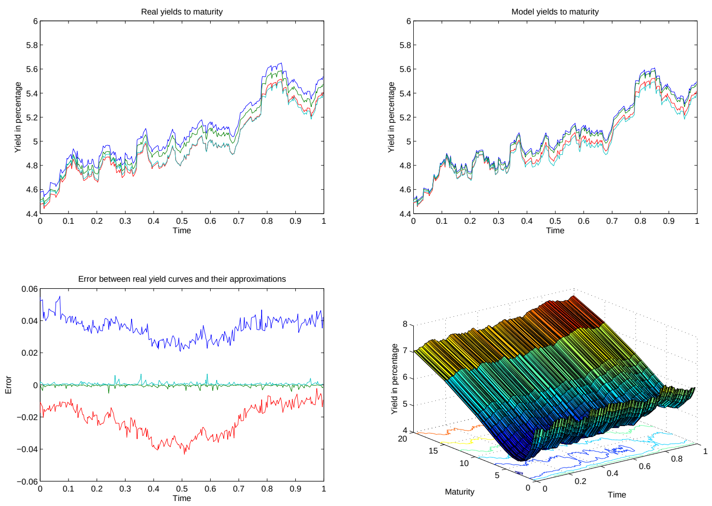

Figure 3.3: 2 factor Vasicek Model calibrated with Kalman filtration at $t = 03/2007$

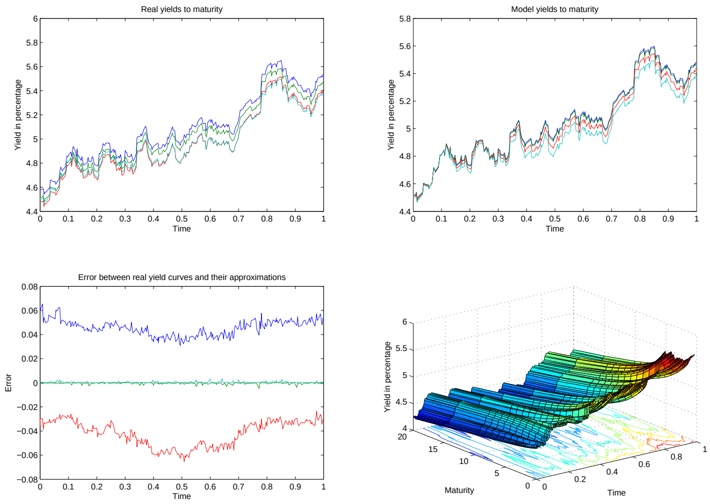

Figure 3.4: 2 factor CIR Model calibrated with Kalman filtration at $t = 03/2007$

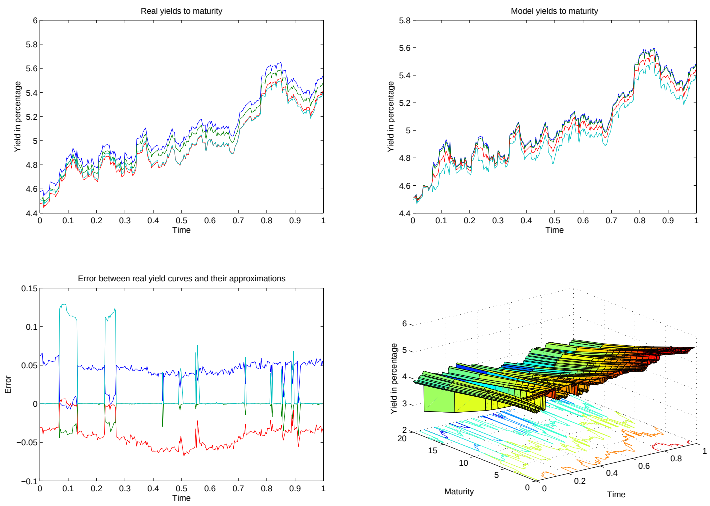

Figure 3.5: DNS model calibrated with Kalman filtration at $t = 03/2007$

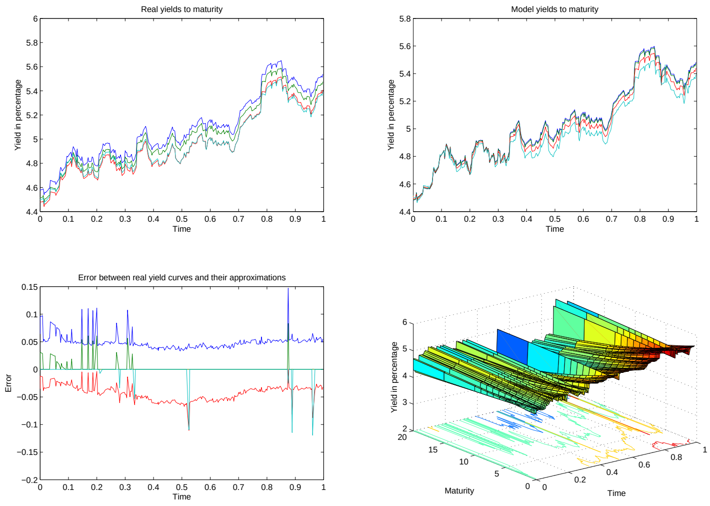

Figure 3.6: AFDNSi model calibrated with Kalman filtration at $t = 03/2007$

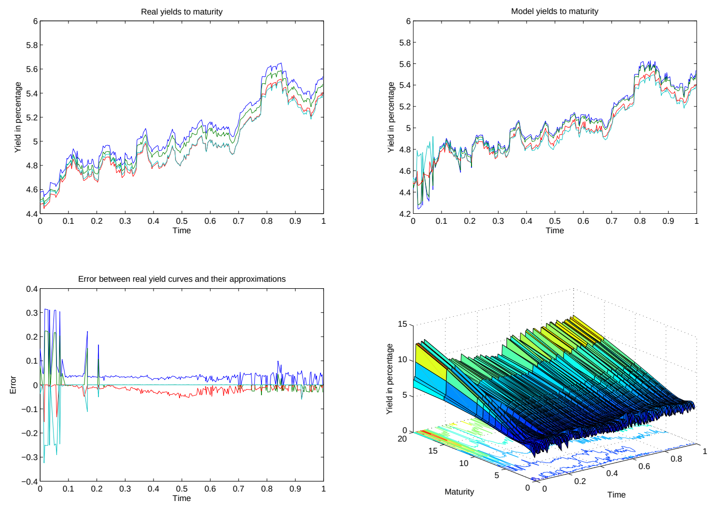

Figure 3.7: AFDNSc model calibrated with Kalman filtration at $t = 03/2007$

3.2.2 Interest rate risk measure

Simply taking the 1^{st} derivative of the price with respect to the short rate r_t will give us an indication of the movement bond prices corresponding to a small change in the short rate. In the case of the single factor models :

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{3.81}$$

where A, B are functions of time to maturity $T-t$. They are used in the Vasicek single factor model, described in section 3.1.1 and defined as:

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right], \tag{3.82}$$

$$A(t,T) = exp\left\{\left(\theta - \frac{\sigma^2}{2a^2}\right)[B(t,T) - \Delta] - \frac{\sigma^2}{4a}B(t,T)^2\right\}. \tag{3.83}$$

and in the case of the Cox Ingersoll Ross single factor model, described in section 3.1.1, are defined as:

$$A(t,T) = \left[\frac{2h\ \exp\{(\kappa+h)(T-t)/2\}}{2h + (\kappa+h)(\exp^{(T-t)h} - 1)}\right]^{2\kappa\theta/\sigma^2}, \tag{3.84}$$

$$B(t,T) = \frac{2(\exp\{(T-t)h\} - 1)}{2h + (\kappa+h)(\exp\{(T-t)h\} - 1)}, \tag{3.85}$$

$$h = \sqrt{\kappa^2 + 2\sigma^2}. \tag{3.86}$$

Taking the 1^{st} derivative with respect to the short rate r_t:

$$\frac{\partial P(t,T)}{\partial r_t} = -A(t,T)B(t,T)e^{(-B(t,T)r_t)}, \tag{3.87}$$

This expression will be used in chapter 6 for the debt issuance optimization problem under interest rate risk constraint.

Similarly the same be done for the two factor models considered in section 3.1.2 and section 3.1.2:

$$P(t,T) = A_x(t,T)A_y(t,T)e^{-B_x(t,T)x_t - B_y(t,T)y_t}, \tag{3.88}$$

The derivative with respect to the short rate $r_t = x_t + y_t$ would become:

$$\frac{\partial P(t,T)}{\partial r_t} = A_x(t,T)A_y(t,T)(-B_x(t,T) - B_y(t,T))e^{-B_x(t,T)x_t - B_y(t,T)y_t}, \tag{3.89}$$

For the Dynamic Nelson Siegel models described in section 2.3, we assumed in equation

(2.27) that the short rate $r_t = L_t + S_t$ and the price of a zero-coupon bond is:

$$P(t,T) = e^{-y(t,T)(T-t)}, \tag{3.90}$$

with $y(t,T)$ defined as:

$$y(t,T) = L_t + S_t \left(\frac{1 - e^{-\lambda(t)T}}{\lambda(t)T} \right) + C_t \left(\frac{1 - e{-}\lambda(t)T}{\lambda(t)T} - e^{-\lambda(t)T} \right), \tag{3.91}$$

so the derivative with respect to the short rate r_t is:

$$\frac{\partial P(t,T)}{\partial r_t} = -(1 + \frac{1 - e^{-\lambda(t)T}}{\lambda(t)T})(T-t)e^{-y(t,T)(T-t)}, \tag{3.92}$$

Those derivatives with respect to the short rate will be used as an interest rate risk measure, as they correspond to the potential gain or loss of the bond price with the gain or loss of one percent in the short rate.

3.3 Summary

This chapter has shown how interest rate models can be used and calibrated to obtain predictions for out of sample data. Eight models, with increasing complexity and number of parameters have been explained here. The method of Kalman calibration has been explained and applied to the Eight models to attach values to the parameters of the different models for several time steps. Application of some of the interest rate models for scenario generation for optimization will be described in subsequent chapters. In particular, Vasicek type models (i.e. models with rate-independent volatility) can be approximated well with re-combining trees, which leads to a major computational advantage in a stochastic optimization set-up; this will be discussed later in chapter 4. Parts of this chapter will be also of use for multi-factor simulations in chapter 7.

Chapter 4

Scenario generation for interest rates

Some of the more common ways of modelling the evolution of the interest rates were described in the previous chapter. These models can be adapted to generate discrete scenarios of future bond prices. There are several methods for generating scenarios. We will look at two different popular methods. The need to generate scenarios stems from the need to forecast several possible interest rates or other macroeconomic variables as needed. In this chapter, we will explain the methodology used in the subsequent chapters to generate possible scenarios of interest rates. These scenarios will later be used for the decision models described in the next couple of chapters, as well as for simulation in chapter 7. In the case of decision models, the decisions made will turn out to be effective in practice only if the back-tested generated scenarios adequately reflect reality. The methods described in this chapter are focused on the use of back-testing scenario generation with the decision models. In this context, we model a set of scenarios to be used with a stochastic programming problem. The scenarios are generated using a re-combining tree or Monte Carlo simulations to carry the short rate and respective bond prices to the mixed integer models described in chapter 6.

Monte Carlo [1] can be used here to generate data used in our scenarios. For any given underlying process, we can always generate M sample paths by Monte Carlo simulation [57] (or M possible yield vectors), at each time step. These values which the uncertain variable can take are called nodes and the computational complexity of a decision model under uncertainty depends on the number of nodes (or the number of possible values of the

[1] For the purpose of this chapter, 'Monte Carlo' refers to the use of random number generator to generate scenarios, while 'trees' refers to generation of scenarios using deterministic rules based on statistical properties (such as moments) of the underlying distribution. We will use the terminology 'tree' or 'lattice' interchangeably to describe re-combining trees, which are discussed in the next section.

uncertain variable on which the decision will be based). We will look at Monte Carlo method later in section 4.2. First, we look at back-tested generating scenarios with re-combining sample paths which typically leads to a much smaller number of nodes than a more general Monte Carlo implementation.

4.1 Polynomial lattice method

Several standard terminology definitions are going to be explored briefly. We will start with nodes:

Definition 18. *A node is a set of data characterized by its inheritance of potential prior and later data sets. A node that inherits a state from a prior node is called a child node. A node that has given a basic state to a child node, is called a parent node. A root node is a node with no parent nodes.*

We can also define scenarios:

Definition 19. *A scenario is defined here as a set of descendant nodes starting from the root node that describe a possible outcome according to a model.*

A tree is a set of scenarios. When the descendant nodes recombine, it forms a recombining lattice tree. We will use a polynomial recombining lattice here as our principal means of generating scenarios of future interest rates. There are several added benefits to using a lattice (or a *tree*) over using traditional Monte Carlo method with non-recombining sample paths:

- The construction of a polynomial lattice for a given stochastic process is deterministic and can still capture a fairly wide array of possible values.
- In addition, using a re-combining lattice make the nodes of the tree grow linearly in time. This is important in stochastic optimization models when the number of decision variables and the number of constraints are determined by the number of nodes. The linear growth of data only involves the generation of input data and not the size of the mixed integer problem solved in subsequent chapters which remains exponential in time.

We use trinomial trees for capturing the interest rates to price bonds, although higher order polynomial trees (e.g. pentanomial trees) can also be used. As an example, pentanomial trees have been used for option pricing in [90].

Let $t = 1, 2, \cdots, n$ be the steps in the tree, or the i^{th} generation out of the n descendants from the root node. Let r_i^j be the short rate at the i^{th} time step and j^{th} scenario. For single

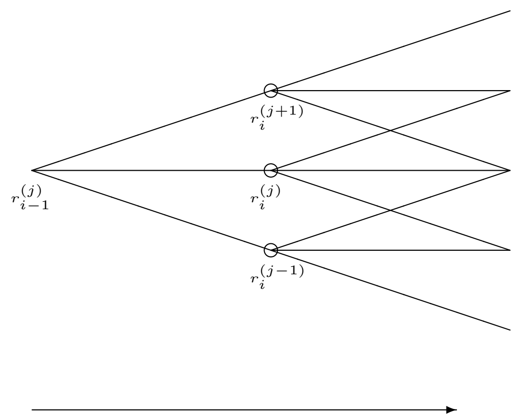

Figure 4.1: Example of a trinomial recombining lattice tree centered at the i^{th} time and j^{th} scenario

factor models, we only need a one dimensional [2] tree to model the evolution of the uncertain interest rates and the number of nodes in the final stage grows linearly, and the number of total nodes grows quadratically over time, see figure (4.1).

4.1.1 Trinomial tree with a single factor Vasicek model

Let $r_i^{(j)}$ denote the interest rate at time step i and node j. In the case of a single factor Vasicek model described in section 3.1.1, we will describe the evolution of the tree as in [21]:

$$\begin{cases} r_i^{(j+1)} = r_{i-1}^{(j)} u, & \text{for the upper branch of the lattice} \\ r_i^{(j)} = r_{i-1}^{(j)}, & \text{for the middle branch of the lattice} \\ r_i^{(j-1)} = r_{i-1}^{(j)} d & \text{for for the lower branch of the lattice.} \end{cases} \tag{4.1}$$

where $u = e^{\frac{\sigma^2(1-exp(-\kappa\delta_t))}{2\kappa}}$ and $d = e^{\frac{-\sigma^2(1-exp(-\kappa\delta_t))}{2\kappa}}$. The above choice of u, d is a common choice in finance provided they satisfy $u = 1/d > 0$, as they remain positive. The advantage of using a single factor tree in a multi-stage optimization set-up is its computational simplicity; as mentioned above, it leads to number of scenarios which grow linearly with the number of time steps. Reducing the size of the time steps, which is equivalent to increasing the number of time steps for the same time horizon makes our model more accurate, as it captures more scenarios at the expense of increasing computational complexity. A more detailed example of a trinomial tree with the Hull-White one factor model can be found in [58]. The probability of each individual scenario is not considered to be important in this

[2] Dimension of a tree here refers to the number of sources of uncertainty used.

chapter and will be assumed in 6.4.2 to be uniform over all scenarios, i.e. $p_j = 1/J$ where p_j is the probability of scenario j and J the total of scenarios considered.

Remark. It should be noted that as the volatility of the CIR model is dependent on the short rate r_t, it can't be modeled using a re-combining tree. Therefore only Monte Carlo simulation with a Gaussian random number generator can be used with the CIR model for generating bond price scenarios.

4.1.2 Multi-dimensional trees

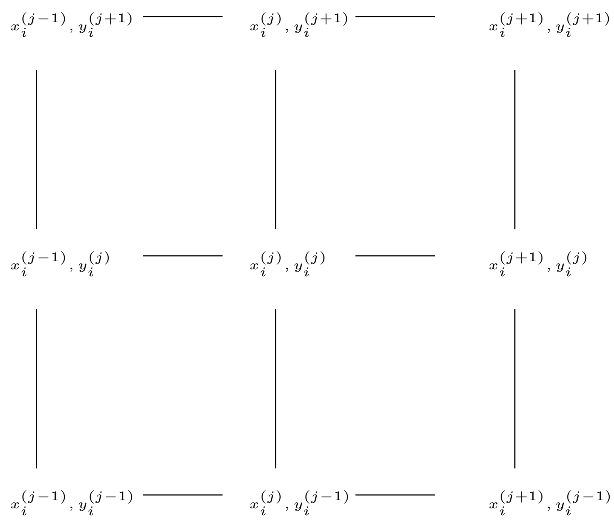

Figure 4.2: A cut from a 2-dimensional trinomial tree centered at the i^{th} and j^{th} scenario at time t.

Multidimensional trees are necessary to model multiple factors of interest rate models or modelling inflation as well as interest rates. For instance, a two dimensional tree can model two factors of randomness and hence can be used to model an interest rate and an inflation rate. An example of two dimensional tree is shown in figure 4.2. While this is not the only way to generate a scenario tree, it is the way employed in the optimisation model in this thesis.

4.2 Monte Carlo scenario generation

Given a mathematical description of a stochastic process, back-tested Monte Carlo simulations enables us to predict a very large number of possible outcomes, allowing for better decision making under uncertainty as long as we are able to sample from a distribution at any future time. An early reference of Monte Carlo simulations was published in 1949 [77]. Its use is widespread both in academia and in the industry, and it is used for solving a wide variety of problems [48].

4.2.1 Back-tested Monte Carlo method

Monte Carlo simulation samples a large number of random possible paths of a given process. If the process is an Ito type process described earlier in chapter 2, the random paths are picked from the model with the addition of a Brownian Motion component. Monte Carlo simulation can also be used with jump processes. The expected destination of the random paths and its variance are computed in the end to provide information as to a most likely destination paths. The steps in the Monte Carlo simulation can be outlined as follows:

- Sample a set of random inputs Z_i from a specific distribution,
- Evaluate how a specific model performed under Z_i and let S^i denote the performance measure,
- Repeat until enough paths have been obtained,
- Analyze the results and compute the required functions of the sample path, e.g. the expected value of $\mathbb{E}[S^i]$.

For a detailed treatment of the Monte Carlo method and its applications in financial mathematics, please refer to [48]. The Monte Carlo method can also be applied to multi dimensional problems by using jump processes or the standard d-dimensional Brownian motion described in chapter 2. A Brownian Motion process $W_t = (W_t^1, W_t^2, \ldots, W_t^d)^\top$ for $0 \leq t \leq T$ is a standard d-dimensional Brownian motion if $W_0 = 0$, is a continuous paths with independent increments and:

$$W_t - W_s \sim N(0, (t-s)I), \tag{4.2}$$

for all $0 \leq s \leq t \leq T$ and I the identity square matrix of dimension $d \times d$. Each individual W_t^i behaves like a standard Brownian motion and for all $j \neq i$ we get W_t^i and W_t^j to be independent.

This methodology has been modified and applied to a wide variety of stochastic processes including jump diffusions and pure jump processes; see [48]. Alternative classes of methods to evaluate integrals are quasi-Monte Carlo methods or low discrepancy methods, which are based on sequences of pseudo-random numbers.

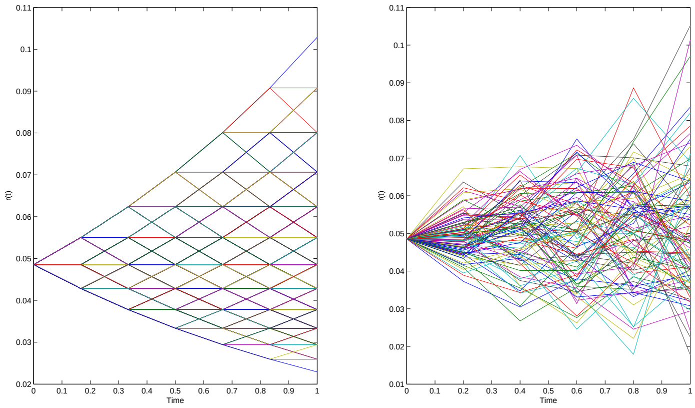

Figure 4.3: A trinomial tree with 6 time steps and a Monte Carlo fan with 1000 paths at $t = 0$

4.2.2 Comparison between the polynomial tree and Monte Carlo method

Both the methods are very different. The lattices usually require a much smaller number of nodes for the same time horizon and are hence very useful in decision models which are computationally intensive and are often non-convex, such as problems with integer constraints for each scenario. The Monte Carlo method, on the other hand, is extremely flexible and can be used for simulating a wide variety of processes including jump process, see [23]. However, the traditional Monte Carlo method requires a lot of memory as each scenario needs to be stored in memory. Hence it is less suited for optimization, especially when multi-stage optimization is considered (please see chapter 5) As the debt management problem leads naturally to a multi-stage decision problem, we will use lattice-based scenario generator in our model, see figure 4.3.

4.3 Macroeconomic models

Generating scenarios with macroeconomic factors, other than the spot rate of a market or the inflation index rate, is often used in actuarial sciences [103] and finance [15]. By incorporating several key macroeconomic factors such as the GDP, the output gap (the difference between the actual GDP and the highest level of GDP that can be sustained over a long term when the economy's resources are fully employed), the unemployment levels, it is theoretically easier to model the financial requirement for the year and render the scenarios generated more credible. Certain models were created specifically for simulation

[14] only and others for optimization [16]. However, some of the factors, such as the market sentiment are not directly quantifiable and are replaced by proxies in macroeconomic models (e.g. return of equity, risk premium, unemployment or real estate returns). These *soft factors* make the models highly prone to errors from a decision modelling point of view. [36] reports that the presence of soft factors doesn't always improve the analysis of data compared to a standard multifactor short-rate model.

4.4 Summary

In this chapter we have covered several methodologies for scenario generation. Tree based scenario generation were covered to some extent and compared to Monte Carlo scenario generation. This will be of use in chapter 6.

Chapter 5

Multi-stage stochastic programming

5.1 Background on multi-stage stochastic programming

In this section, we will provide several definitions to help define the debt management problem in the next section. This section will be more about multi-stage programming in general, and not specifically about debt management problem. To begin, several symbols and relevant notations need to be defined:

5.1.1 Notation for this chapter

1. x: a vector of real numbers, $x \in \mathbb{R}^n$.

2. y: a vector of integer numbers, $y \in \mathbb{N}^m$.

3. U: a vector of random real variables, generally not independent and identically distributed.

4. $f(x, y) : \mathbb{R}^n \times \mathbb{N}^m \mapsto \mathbb{R}$ will represent objective functions in future definitions.

5. $g_i(x, y) : \mathbb{R}^n \times \mathbb{N}^m \mapsto \mathbb{R}$ for $i = 1, 2, \cdots, M$. Where M is a specified integer constant such as $M \geq 1$. The set of functions $g_i(x, y)$ will be used to satisfy equality constraints.

6. $h_j(x, y) : \mathbb{R}^n \times \mathbb{N}^m \mapsto \mathbb{R}$ for $i = 1, 2, \cdots, M$ and M is defined as previously. The set $h_i(x, y)$ will be used to satisfy inequality constraints.

7. $\mathbb{E}[f_i(x, u)]$ be the expected value of the function $f_i(x, u)$.

Stochastic programming methodology was developed in the late 60s ([99], [101], [66]) and have had a widespread influence in the world since then. These are a practical set of techniques to model problems where some parameters of the problem are uncertain and are only described by a probability distribution. In our case, these techniques will be used to model the short rate and other macroeconomical values. By using the scenarios generated in the previous chapter 4, we can model certain stochastic variables. In the recent years, several commercial and non-commercial solvers have appeared, such as FortSP [41] or FuncDesigner [67].

A stochastic program is a mathematical program where one or more variables are random as defined in chapter 2. We will define a simple recourse stochastic programming problem first:

Definition 20. *A stochastic program (or SP) is a mathematical optimization problem where some of the data is uncertain. Uncertainty is defined in terms of a probability distribution on the parameters. The problem can be written in the following form:*

$$\textit{minimize } f_1(x) + \mathbb{E}\left[f_2(x,u)\right] \tag{5.1}$$

$$\textit{subject to } g_i(x) = 0, \quad i = 1, ..., m, \tag{5.2}$$

$$h_j(x) \leq 0, \quad j = m+1, \dots, M, \tag{5.3}$$

$$l_r(x,u) <= 0, \quad r = 1, \dots, K, \forall u \in U, \tag{5.4}$$

$$x \in X \subseteq \mathbb{R}^n \tag{5.5}$$

$$u \in U, \textit{is a random variable that takes values in } \mathbb{R}^n. \tag{5.6}$$

where $\mathbb{E} f_2(x,u)$ is the expected value of $f_2(x,u)$ with respect to the random variable $u \in U$, where U is the set of possible values within $\mathbb{R}^n$. X is a subset of possible values within $\mathbb{R}^n$. The set of functions $l_r(x,u)$ are required to hold for each constraint with probability 1 and for each $u \in U$ and are the link between the first stage decisions x and the second stage decisions u.

The above definition is for a two stage program as it involves only one set of decisions and a second set of decisions that are stochastic to minimize the expected value of the objective function. To be solved a deterministic equivalent problem is often defined.

Definition 21. *A two stage stochastic problem can be reformulated as a deterministic equivalent linear program. It is a large linear programming problem over a finite number of scenarios, where the optimal first-stage decision are computed and attach a probability p_k to* *each scenario.*

$$\textit{minimize } f_1(x) + \sum_{r=1}^{K} f_{2,r}(x,u) \tag{5.7}$$

$$\textit{subject to } g_i(x) = 0, \quad i = 1, ..., m, \tag{5.8}$$

$$h_j(x) \leq 0, \quad j = m+1, \ldots, M, \tag{5.9}$$

$$l_r(x,u) <= 0, \quad r = 1, \ldots, K, \forall u \in U, \tag{5.10}$$

$$x \in X \subseteq \mathbb{R}^n, \tag{5.11}$$

$$u \in U, \textit{is a random variable that takes values in } \mathbb{R}^n. \tag{5.12}$$

where the notation is identical to definition 20.

A two stage stochastic problem can be extended to a multi-stage model. A multi-stage model, or recourse model, is a model where decisions must be made having an incomplete set of information over deterministic and/or stochastic parameters. If the information becomes available before a later stage than the model should be able to alter the set of decisions which were already made for the subsequent stages. If the parameters with incomplete information are modeled with random variables with a known probability distribution then it becomes a multi-stage stochastic model:

Definition 22. *A multi-stage stochastic program (or MSP) is a mathematical optimization problem in the form:*

$$\textit{minimize } \sum\nolimits_{k=1}^{n} \left[p^k f(x^k, u^k)\right] \tag{5.13}$$

$$\textit{subject to } g_i(x_t^k, u_t^k) = 0, \quad i = 1, ..., m, \ \forall k, \forall t \in [1,T], \tag{5.14}$$

$$h_j(x_t^k, u_t^k) \leq 0, \quad j = m+1, \ldots, n, \ \forall k, \forall t \in [1,T], \tag{5.15}$$

$$x_1^k = x_1^l, \quad \forall k, \forall l, \tag{5.16}$$

$$x_t^k = x_t^l, \quad \textit{if} \quad (u_1^k, \cdots, u_{t-1}^k) \equiv (u_1^l, \cdots, u_{t-1}^l), \quad \forall k, \forall l, \forall t \in [1,T], \tag{5.17}$$

$$x_t^k \in X_t(u^k), \quad \forall k, \forall t \in [1,T], \tag{5.18}$$

$$u_t^k \in U_t^k, \quad \forall k, \forall t \in [1,T]. \tag{5.19}$$

where $X_t(u^k)$ is the subset of possible values within $\mathbb{R}^n$ at time t and scenario k. U_t^k is the set of values within $\mathbb{R}^n$ at time t and scenario k. p^k is the probability of scenario k occurring within the n scenarios considered, $x^k = (x_1^k, x_2^k, \cdots, x_T^k)$ and $u^k = (u_1^k, u_2^k, \cdots, u_T^k)$ are the decision and random variables at time t and scenario k. The functions $f(x,u)$, $g_i(x,u)$ and $h_j(x,u)$ remain defined as above.

The 3^{th} constraint makes sure that all scenarios and decisions are identical at the start of the optimization. The 4^{th} constraint exists to ensure that under the same random variables, the same decision is taken. See [32] for a more detailed multi-stage stochastic methodology using scenario trees. For a detailed explanation of mathematical programming see [17].

In this chapter we will look at a generalization of the above problem with some integer valued variables as follows:

Definition 23. *A mixed integer optimization program (or MIP) is a mathematical optimization problem in the form:*

$$\textit{minimize } f(x,y) \tag{5.20}$$

$$\textit{subject to } g_i(x,y) = 0, \quad i = 1,\dots,m, \tag{5.21}$$

$$h_j(x,y) \le 0, \quad j = m+1,\dots,n, \tag{5.22}$$

$$x \in X \subseteq \mathbb{R}^{n_1}. \tag{5.23}$$

$$y \in Y \subseteq \mathbb{N}^{n_2}. \tag{5.24}$$

where x is a vector of size n_1 of real numbers: $x \in \mathbb{R}^{n_1}$ and y a vector of n_2 integer numbers: $y \in \mathbb{N}^{n_2}$. f is the objective function, g_i are the equality constraints, h_i are the inequality constraints. The subset $X \subseteq \mathbb{R}^{n_1}$ is the subset of feasible solutions of x,and $Y \subseteq \mathbb{N}^{n_2}$ is the subset of feasible solutions of y.

A number of commercial and non-commercial solvers exist to solve mixed integer problems such as Gurobi [86] and CPLEX [60]. For more information regarding integer programming see [112].

5.2 Applications of multi-stage stochastic programming

Several real life problems can be posed as a multi-stage stochastic optimization problem. An example is a traveling salesman problem where every node is a new stage, and the distance is fixed but the time to travel varies with a stochastic variable depending on traffic [63]. A more detailed explanation of the traveling salesman problem can be found in [69] and its many applications in [92]. The scheduling problem where each time slot is a stage and the potential attendance or rating can be stochastic is also a multi-stage stochastic problem [75]. Specific scheduling applications featuring multi-stage stochastic optimization include power distribution [62] and economic lot allocation [42]. The knapsack problem is a problem of space allocation [53], it can be made as a multi-stage stochastic problem in the case where the space required or the elements needed to be allocated are changing. The main use in this thesis will be the public debt issuance problem described in chapter 6.

5.2.1 Receding horizon method

The receding horizon method used on a stochastic program differs from a multi-stage stochastic program. In a standard multi-stage stochastic program, all decisions must be re-evaluated at every decision node or time node. In a receding horizon framework, all decisions are re-evaluated at every interval of time where more information becomes available, similar in fashion to a weekly, monthly or quarterly budget review. In the current context, the application of the receding horizon method can be further explained as follows. We assume that there are n different points in time $t_0, t_1, \ldots, t_n$ when the optimization will be carried out or repeated, and $T > t_n$ is the end of planning horizon. We also assume that the objective function, the constraints and the variables for the optimization model at each t_i is known a priori. The problem parameters or data for future times t_i is not known a priori.

- At t_0, a multi-stage stochastic programming problem is set up to minimize a particular objective function for a given set of constraints and and solved. The first stage decisions are then implemented.
- At t_i, i = 1,2 or 3, a new multi-stage stochastic programming problem is set up, once the data for the problem becomes available. Note that, in general, this data depends on the decisions taken at time t_{i-1} (in the debt cost minimization example in chapter 6, this data will be prior debt issuance).

The main advantage of the receding horizon method is that the computational difficulty of each stage is independent to the amount of decisions needed to be done. This idea of using multiple optimizations over the trajectory of an uncertain variable is similar to the receding horizon approach used in predictive process control, see *e.g.* [73].

5.2.2 Re-combining lattice for interest rates

As in the previous section, after each (re-)calibration we build a re-combining trinomial lattice using a procedure in [59] and use it for setting up an optimization problem at each auction. This idea of solving multiple, possibly multi-stage optimization problems during the financial year is realistic as the sovereign debt issuing authority can dynamically adjust its decisions during the year as the economic environment evolves. We build a Q step lattice at the beginning of each quarter using the parameters of recent calibration. A construction for $Q = 3$ is shown in figures 5.2.2 to explain the idea of a receding horizon.

We will now visit the debt management problem using the contents of this chapter to describe it in the next chapter, and using the contents of all previous chapters to solve it.

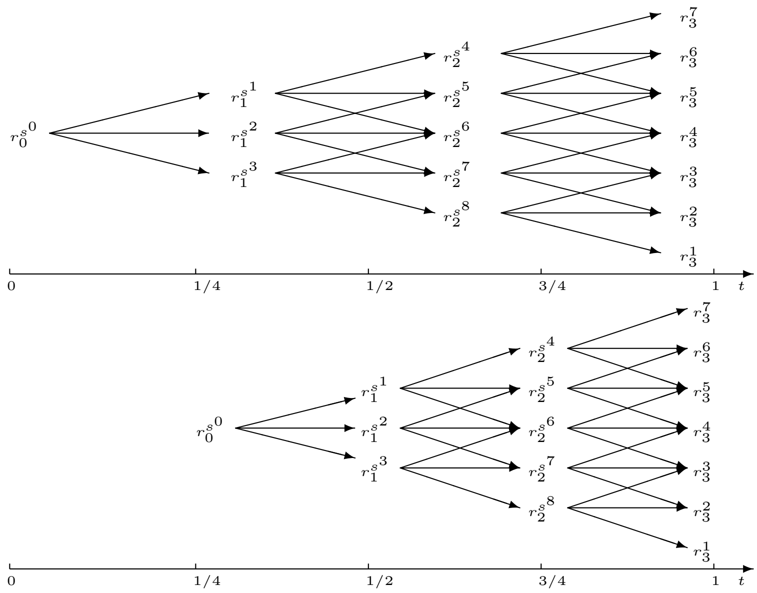

Figure 5.1: Lattice at the beginning of the 1^{st} and 2^{nd} quarter.

Chapter 6

SP based optimization model for debt issuance

Governments make use of debt instruments in order to finance two major components of the national accounts among other needs of financing:

1. the government net cash requirement, which is essentially the difference between government's income and expenditure in cash returns;

2. the redemption of maturing government bonds. This is the amount needed to finance the annual repayment of maturing debt.

In managing the government debt, several governments have as their stated debt strategy objective the minimization of long-term financing cost while maintaining a low downside risk around those costs. In the UK, for example, the government explicitly states in [107] that "the primary objective of debt management policy shall be to minimize, over the long term, the cost of meeting the Government's financing needs whilst:

- taking account of risk and
- so far as possible, to avoid conflict with monetary policy".

Although phrased in many different ways, similar statements relating the objective of government debt management are found in most of the Ministry of Finance code of practices around Europe and it is explicitly mentioned in the IMF guidelines for public debt management; see [61].

The trade-off between cost and risk is a familiar concept in the asset-pricing literature where investors attempt to optimally select the proportion of risky and riskless assets that maximize their expected utility functions subject to appropriate wealth constraints. This suggests that the government might be able to apply corporate finance theory in determining its debt issuance strategy. However, asset-liability management can not be applied to sovereign debt management in a straightforward manner. First, the objective and horizon of government debt management differ from those of private institutions and the types of risks actively managed at sovereign level also differ from private sector. In particular, while asset portfolio managers try to maximize asset returns over holding period subject to upper limit on risk, sovereign debt managers try to minimize the debt-service cost over a longer horizon subject to an implicit or explicit constraint on the volatility of debt-service cost(as a proxy for risk). Second, government debt managers are concerned with maintaining a liquid and well-functioning government security market. Sovereign fixed-income market often serve as a benchmark for corporate issuers, thus implying that small alterations of the government portfolio often have large impacts on the entire bond market. Therefore, the objective of minimizing the cost of debt servicing is subject to the constraint that a minimum level of bonds has to be issued at each maturity bracket. Finally, the implementation and transmission of monetary policy interventions occur through financial markets. According to the liquidity preference theory, debt management has a clear influence on the term structure of interest rates. Therefore, some constraints are imposed on debt management by the need to consider consistency with monetary policy.

The purpose of this work is to integrate corporate portfolio optimization theory in a general framework which can be used by government debt managers to inform the issuance policy. In doing so we assume that one of the main sources of risk in sovereign debt portfolio management is the uncertainty about future short term interest rates. Other important sources of uncertainty such as the exposure to currency risk or fluctuations of macroeconomic variables (e.g., the rate of inflation) are not inserted directly into our cost minimization problem and are assumed to be closely correlated with the single source of uncertainty used. To model the evolution of interest rates, we use an affine term structure model introduced in [109] and discussed in chapter 3, and calibrate to multivariate time series data on government bond yields using a Kalman filter. This filtering-based calibration approach allows us to use the short term rate as an unobservable variable rather than using a proxy for it and to use potentially noisy yield data from which to estimate the short rate. Similar approaches have been previously employed in [5], [98], [50] and [30] among others. [28] provides a review of using Kalman filtering in financial time series models.

To generate scenarios of uncertain future interest rates (and hence the yields, which are affine functions of short rate for the chosen short term rate model) evolving through time, we use a trinomial recombining lattice. Using a recombining lattice is an industry standard way of modelling asset price or interest rate evolution for pricing purposes. In the present context, using a recombining lattice means that the number of possible values the yield vector can take grows linearly with time steps. In a non-recombining lattice, the number of steps can grow exponentially or combinatorially. The use of a recombining lattice keeps the mixed integer linear programming (MILP) based multi-stage stochastic programming problem numerically tractable, even on a desktop with modest hardware specifications. An alternative approach would be to use a non-recombining lattice followed by scenario generation heuristics, as proposed in [39] and [55].

The use of scenario based stochastic optimization in bond portfolio management is not new, although most of the applications are demand-side applications (i.e. the optimization problem as seen from the bond purchaser's point of view). A two stage stochastic program was formulated in [49] to address fixed income portfolio management under interest rate and cash flow uncertainty, while a similar formulation was used in [114] to illustrate management of portfolios containing mortgage backed securities. In [38], bond portfolio management is formulated as a multiperiod stochastic program in a dynamic setting. Like our paper, [38] also uses a recombining lattice as a temporal model for uncertain interest rates.

On the supply side (i.e., for a sovereign issuance problem), a linear programming based model is presented in [1] for minimization of the total cost of issuance under regulatory constraints. This model is illustrated using debt issuance data of the Italian government. Other notable work in this area includes [14], which provides results on the multivariate simulation of interest rates using observable (ECB) rates as well as analysis of principal components. The research reported in [22] and [7] is the closest in spirit to the work reported here, in the sense that, both these papers also develop multi-stage stochastic programming models for sovereign debt issuance.

6.1 Assumptions

We start with the following assumptions about the process of raising debt by a sovereign government, as described in [27]:

1. The sovereign body raises debt through a series of auctions. At the beginning of the financial year, the dates of debt auctions are fixed. There are three separate auction calenders; one each for short, medium and long dated bonds. Imperatives other than purely financial ones play a significant role in deciding this calendar and we consider these calendars as input data.

2. At each auction, a single bond is issued, either from the existing (or pre-issued) stock or a bond with new maturity.

3. The *average* price of any bond at the auction is its arbitrage free price as determined by the yield curve. The yield of any new bond issued is also determined by the yield curve on the auction date.

4. Fiscal policy is responsible for the government net cash requirement. Thus, the total amount to be raised over the financial year is dictated by the government's borrowing requirements and is assumed to be an exogenous constant. Further, the *total* amounts to be raised through each set of auctions *viz.* auctions of short, medium, long dated bonds, are fixed. These are dictated by the government's need to maintain liquidity in markets of bonds with different maturities.

5. The government does not engage in opportunistic borrowing. As a consequence financial strategies that attempt to take advantage of the market conditions for issuance of various debt instruments are ruled out. This operational principle together with the need of pursuing an issuance policy that is open and transparent are often described in the code of practice of the debt managing agencies.

Under these assumptions, the optimization model uses a *receding horizon* approach as mentioned earlier; see *e.g.* [73] for applications in control engineering. This approach in the present context may be explained as follows. Suppose that there are N auction dates indexed from t_1 through t_N. At each auction date t_i, $i > 1$, data of previous auctions $t_1, \cdots t_{i-1}$ are already available. Also, the multivariate time series data are available until time t_i. It is assumed that, at each t_i, a new recombining interest rate lattice is established and a multi-state stochastic optimization problem is solved to generate the choice of bond and the amount of debt to be auctioned from t_i onwards, i.e. at $t_i, t_{i+1}, \cdots, t_N$. If $\mathcal{T}_i$ is the number of stages in the stochastic optimization problem solved at t_i, then $\mathcal{T}_i - \mathcal{T}_j = i - j$ for $1 \leq i \leq j \leq N$. In the numerical experiments reported in 6.6, this receding horizon approach is followed based on real UK data and the results are compared with the actual issuance during the same period.

6.2 Notation for the debt management problem

As outlined later in section 5.2.2, we use a re-combining tree (or lattice) to model the evolution of interest rates. For a re-combining tree, the number of nodes (and hence decision variables) grows linearly with time and the problem remains tractable even for a long time horizon. Given an interest rate lattice and hence a set of scenarios for future bond yields, we outline here the notation used in our development of the optimization model. The scenario generation will be discussed separately in latter subsections.

1. $\mathcal{N} = \{1, 2, \cdots, N\}$, $\mathcal{J} = \{1, 2, \cdots, J\}$ and $\mathcal{K} = \{1, 2, \cdots, K\}$ are the index sets for auctions over the budget year, interest rate scenarios and bonds to be auctioned respectively.

2. $X_{i.j}^{(k)}$ is a binary decision variable which has a value 1 if k^{th} gilt is auctioned at i^{th} auction, in j^{th} scenario; $X_{ij}^{(k)}$ is 0 otherwise.

3. $u_{i.j}^{(k)}$ is a real valued decision variable which gives the number of units of bond k sold at auction i in j^{th} scenario. The unit is enforced to be a multiple of a constant ϖ in the model with the use of $\omega_{i.j}$ as an integer variable.

4. $P_{i.j}^{(k)}$ is the forecasted price of the k^{th} bond at the i^{th} auction in the j^{th} scenario; this is explained in more detail later in section 5.2.2.

5. The amount raised at a single auction is bounded from above by $\overline{D}$ and from below by $\underline{D}$, both of which are specified constants.

6. $D0_{i.j}^{(k)}$ is the prior total holding in the secondary market of the k^{th} bond at the i^{th} auction in the j^{th} scenario. The parameter $D0_{i.j}^{(k)}$ keeps track of the total amount of a particular bond existing in circulation and is updated after every iteration of the receding horizon. It also keeps track of maturing debt of the time period in question. $\overline{\psi}$ is the upper bound for the liquidity constraint this is a constant for a specific problem to make sure that to many already existing bonds don't exist.

7. $\tau_i^{(k)}$ is time to maturity of bond k starting from time t_i. It needs to satisfy the maturity constraint, i.e. $\underline{\tau} \leq \tau_i^{(k)} \leq \overline{\tau}$, where $\underline{\tau}, \overline{\tau}$ are given constants.

8. $B \geq 1$ is a constant integer which limits the number of times a specific bond can be used in the considered financial year. The choice of integer B is a trade-off between flexibility in choosing the lowest cost issuance and ensuring enough liquidity across all maturities.

9. L is the principal of each bond. I_j represents the total cost of issuance over the lifetime of debt in scenario j:

$$I_j = \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} u_{i.j}^{(k)} L(1 + \chi_i^{(k)}),$$

   where $\chi_i^{(k)}$ represents the total amount of coupons over the remaining life of bond k from time t_i onwards. This cost function is calculated in accordance with the European System of Accounts (ESA95).

Now we will provide some clarification to certain constants used in our problem.

6.3 Requirements of a debt management office

A debt office, as written previously, has certain requirements:

1. An amount D must be raised in cash,

2. This amount must be issued over the financial year over different maturities in the N auctions available,
3. This amount must be issued in conventional and index-linked bonds,
4. The bonds issued must be grown to a benchmark amount for 5 year and 10 year maturity,
5. The debt management office must announce a specific auction calendar at the beginning of the year,
6. and the coupon yield of the specific bond issued at a closer date to the auction,
7. The amount raised must be done in increments of ϖ,
8. The debt management office must pick the best coupon yield according to the yield curve to minimize costs while making sure to raise the full amount of money required subject to certain risk measures,
9. The debt management office must avoid influencing the operations of the central bank or the validity of the currency.

These requirements exists as a way to keep the bond market (sovereign and corporate) liquid enough and provide enough information on their operations as not to have a sudden impact.

6.4 Optimal debt issuance models

Using the scenarios created in chapter 5, the forecasted short rate used for future auctions is then linearly interpolated from the tree if the auction date does not coincide with a tree node; see appendix C for a small example of how this is done. Linear interpolation here means:

$$r_t = r_i^j + (r_{i+1}^j - r_i^j)\frac{t - t_i}{t_{i+1} - t_i}, \tag{6.1}$$

Using the Vasicek pricing formula, we can obtain the price of a bond with maturity T_k, at time t_i and corresponding to a short rate $r_i^{(j)}$ by summing over all coupons:

$$P_{i.j}^{(k)} = \sum_{t_i < t_c \leq T_k} \chi^{(k)} A(t_i, t_c) e^{-B(t_i,t_c) r_i^{(j)}} + L A(t_i, T_k) e^{-B(t_i,T_k) r_i^{(j)}}, \tag{6.2}$$

where L is the principal of each bond, $\chi^{(k)}$ is the coupon of the k^{th} bond , t_c belongs to the the set of maturities of all the remaining coupons for the bond considered and $A(t,T), B(t,T)$ are as defined in section 3.1.1.

6.4.1 A simplified optimization model for the debt issuance problem

For the set-up outlined above, it is worth considering a deterministic optimization problem first. Let us assume that future prices are "known" as $P_i^{(k)}$ for the auction date at t_i and for the unit of bond k, the auctions will sell out. Thus the second subscript for prices P, which indexes the scenarios, is not used and the overall notation is simplified.

Let $X_i^{(k)}$ be the binary variable that represents which bond k to issue at the i^{th} auction and $u_i^{(k)}$ be a real variable that estimates the amount of bonds to issue for k bond at i^{th} auction date. The total cost is simply the un-discounted total cash flow from the issuance of the year. The simplified version of optimization model without uncertainty can then be expressed as follows.

$$\text{minimize} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} u_i^{(k)} L(1 + \chi_i^{(k)}) \text{ subject to} \tag{6.3}$$

$$\sum_{(i,k) \in (\mathcal{N},\mathcal{K})} u_i^{(k)} P_i^{(k)} \geq D, \tag{6.4}$$

$$u_i^{(k)} P_i^{(k)} + D0_i^{(k)} \leq \overline{\psi} - \sum_{pt=1}^{i} u_{pt}^{(k)} P_{pt}^{(k)} \; \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{6.5}$$

$$\underline{D} X_i^{(k)} \leq u_i^{(k)} P_i^{(k)} \leq \overline{D} X_i^{(k)} \; \forall\, i \in \mathcal{N}, k \in \mathcal{K}, \tag{6.6}$$

$$\sum_{(i,k) \in (\mathcal{N},\mathcal{K})} X_i^{(k)} = N, \tag{6.7}$$

$$\sum_{k \in \mathcal{K}} X_i^{(k)} = 1, \forall i \in \mathcal{N}, \tag{6.8}$$

$$\sum_{i \in \mathcal{N}} X_i^{(k)} \leq B, \forall k \in \mathcal{K}. \tag{6.9}$$

The equations in this model can be explained as follows.

- Inequality (6.4) guarantees that the minimum required amount of debt is raised over through the specified series of auctions.
- Inequality (6.5) ensures that the total issuance for a particular bond (or a particular maturity) remains under a specified constant $\overline{\psi}$.
- Inequality (6.6) constrains the minimum and the maximum issuance size at each auction.

- Equations (6.7)-(6.8) ensure that all auctions are used and only one bond is issued at each auction.
- Finally, equation (6.9) is a constraint to ensure that one bond is used at most B times in the series of auctions.

Analytically, this model can be solved using a deterministic mixed integer linear program, with the amounts auctioned and the issuance choice (binary) variables as the decision variables. Although the model constitutes a useful exercise, it is overly simplified to illustrate the issues involved in public debt issuance. The assumption that prices are known and a lack of measure to control the issuance risk make the problem highly unrealistic. In the subsequent sections, we will introduce the necessary risk measures and will also introduce a mechanism to generate scenarios for different possible future prices for bonds.

6.4.2 Risk Measures for stochastic programming

Risk measures provide information about the uncertainty of future debt-service cost, therefore the value at risk plays a central role in the management of government debt. An increase in the value of the debt portfolio reflects an increase in the future burden for taxpayers or it may boost the cost of other debt instruments often used by debt managers such as swaps or buybacks.

As a measure of risk, we use two different measures: Conditional Value at Risk (CVaR) and a quantile based supply-side measure called Cost at Risk (CaR), as discussed in [94]. They both measure the potential extra cost incurred by the DMO with respect to

The CVaR risk measure is the weighted average of the Value at Risk (VaR) and the losses exceeding VaR. CVaR due to its very definition, is always an upper bound of VaR and therefor provides a good control of risk within the optimization model. It is defined as a system of linear constraints in [96]. The CVaR constraint is also bounded to control the maximum amount of conditional risk tolerated. A similar bound is explained in [22]:

$$CVaR := \frac{\sum_{j \in \mathcal{J}} p_j \phi_j}{1 - \beta} + \zeta, \tag{6.10}$$

with

$$\phi_j := \max\left(I_j - \frac{\sum_{j \in \mathcal{J}} I_j}{J} - \zeta, 0\right), \tag{6.11}$$

where $\mathcal{J}$ is an index set as defined in section 5.1.1, $\zeta \in \mathbb{R}$, p_j is the probability of the j^{th} scenario, or a branch of the tree to occur and β corresponds to the confidence rate between 0 and 1. In the proposed model, the value of CVaR will be bounded from above by a constant ρ as done in [113] and [22]. We also consider that each branch will have an equal probability to occur, so that we can take p_j to be $\frac{1}{J}$. Now the CVaR constraint becomes:

$$CVaR := \frac{\sum_{j\in\mathcal{J}} \phi_j}{J(1-\beta)} + \zeta$$

As the CVaR upper bound is reduced, the difference between costs of different scenarios is reduced as well. As theory suggests, this will raise the expected cost in general.

The CaR measure is defined in [52] as:

$$CaR := \mathbb{E}(I_j) + 1.645\varsigma,$$

where $\mathbb{E}(I_j) = \sum_{j\in\mathcal{J}} I_j/J$ and ς is the standard deviation of the achieved cost. This supply side measure is similar to the popular Value at Risk (VaR) measure on the demand side, under the assumption of normally distributed scenarios. In our case, the standard deviation is computed *a posteriori* as the sample standard deviation over all the scenarios. As such, there is no linear constraint to model CaR, it merely provides extra information to estimate the cost of issuance risk.

6.4.3 A stochastic MILP model for public debt issuance

The mixed integer linear programming model for the optimal debt issuance problem is defined as follows.

$$\text{minimize } \frac{1}{J}\sum_{j\in\mathcal{J}} I_j \text{ subject to} \tag{6.12}$$

$$\sum_{(i,k)\in(\mathcal{N},\mathcal{K})} u_{i.j}^{(k)} P_{i.j}^{(k)} \geq D, \forall j \in \mathcal{J}, \tag{6.13}$$

$$u_{i.j}^{(k)} P_{i.j}^{(k)} = \varpi \omega_{i.j}, \forall\, i \in \mathcal{N}, k \in \mathcal{K}, j \in \mathcal{J}, \tag{6.14}$$

$$\phi_j = I_j - \frac{1}{J} \sum_{j\in\mathcal{J}} I_j - \zeta, \forall\, j \in \mathcal{J}, \tag{6.15}$$

$$\frac{1}{J(1-\beta)} \sum_{j\in\mathcal{J}} \phi_j + \zeta \leq \rho, \tag{6.16}$$

$$u_{i.j}^{(k)} P_{i.j}^{(k)} + D0_{i.j}^{(k)} \leq \overline{\psi} - \sum_{pt=1}^{i} u_{pt.j}^{(k)} P_{pt.j}^{(k)}\ \forall i \in \mathcal{N}, k \in \mathcal{K}, j \in \mathcal{J}, \tag{6.17}$$

$$\underline{D} X_{i.j}^{(k)} \leq u_{i.j}^{(k)} P_{i.j}^{(k)} \leq \overline{D} X_{i.j}^{(k)}\ \forall\, i \in \mathcal{N}, k \in \mathcal{K}, j \in \mathcal{J}, \tag{6.18}$$

$$\sum_{(i,k)\in(\mathcal{N},\mathcal{K})} X_{i.j}^{(k)} = N, \forall j \in \mathcal{J}, \tag{6.19}$$

$$\sum_{k\in\mathcal{K}} X_{i.j}^{(k)} = 1, \forall i \in \mathcal{N}, j \in \mathcal{J}, \tag{6.20}$$

$$\sum_{i\in\mathcal{N}} X_{i.j}^{(k)} \leq B, \forall j \in \mathcal{J}, k \in \mathcal{K}, \tag{6.21}$$

$$if\, \underline{\tau} \geq \tau_i^k\, or\, \overline{\tau} \leq \tau_i^k\, then\, X_{i.j}^k = 0\ \forall\, i \in \mathcal{N}, j \in \mathcal{J}. \tag{6.22}$$

The optimization procedure is schematically illustrated in figure 1. The goal is to minimize the average cost of debt servicing as defined by ESA95 [43] over all interest rate scenarios $\mathcal{J}$ and the set of auctions $\mathcal{N}$. The rest of the notation in the above model is as defined in section 5.1.1. The set of equations is an expanded version of the model presented earlier in section 6.4.1 and can be explained as follows.

- Inequality (6.13) is a constraint to make sure the amount raised is at least the fixed objective D over the year.
- Equation (6.14) exists to ensure that the auctions are done in increments of ϖ, $\omega_{i.j}$ being an integer variable to ensure the increments are respected.
- The systems of inequalities (6.15)-(6.16) corresponds to the CVaR risk measure bounded from above by ρ with confidence β.
- Equation (6.17) is a liquidity constraint and ensures that the total amount of a specific

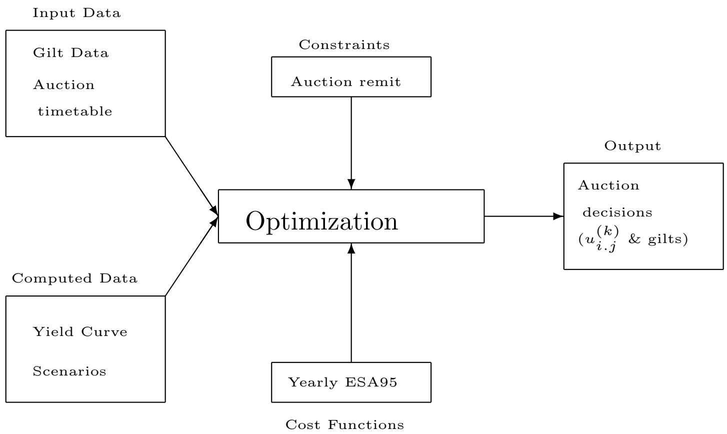

Figure 6.1: Optimization procedure

bond in issuance doesn't exceed an upper bound $\overline{\psi}$.

- Equation (6.18) ensures that each auction will raise funds within the boundaries set by a government.
- Equations (6.19)-(6.20) impose constraints that all auctions are used and only one bond is auctioned on each auction date.
- The inequality (6.21) ensures that a single bond is used no more than B times.
- Finally, the last constraint (6.22) ensures that if the maturity of a particular bond does not match the maturity constraint of a problem at the i^{th} auction it may not be auctioned by the model.

The optimization model discussed so far assumes that a mechanism is available for generating scenarios of bond prices. These scenarios need to be arbitrage-free, since we are assuming that the auction prices are determined by the secondary market.

Remark. There are no non-anticipitivaty constraints defined explicitly in the model. Allowing the model access to all of the data means the solutions for every scenario is far more diverse and informative. As the receding horizon occurs, the prior issuance, of the scenario closest to the actual short rate, is added to the portfolio of issued bonds $D0_{i.j}^{(k)}$ and a new set of forecasts are made. This makes the non-anticipitivaty of our deterministic equivalent problem (as defined in 21) implicit.

6.5 A stochastic MILP model with liquidity and interest rate risk measurements

In addition to minimizing the average cost of issuance, we can also try to minimize the average loss of value of gilts due to issuance, with respect to potential change of short rate. We will start by looking at an interest rate measure which will allow us to do this.

We will use the definitions from section 3.2.2. The risk measure defined in that section is more involved than the Cost at Risk (CaR) measure; however, it can provide useful information on the potential portfolio movements related to interest rates. CaR measure depends only on the mean and the standard deviation; however, optimizing it would be a linear programming problem with quadratic constraints. As the values of those interest rate risk measures are pre-computed, they can be used as parameters and optimized linearly by taking the values of the derivatives from the forecasted yield curves. By inputting the values into the data files, we can evaluate an interest risk measure at each time step and each scenario that would represent the potential gain or loss of a zero-coupon bond value if the short rate were to change by 1%. As interest rate risk is one of the main risks when issuing fixed income debt, it is important to monitor and control it. It also represents an extra penalty on the issuance of bonds with a coupon that diverges too much from the yield curve at each time and scenario. Let the variable $IRM_{i,j}$ denote the interest rate measure just defined.

Another issue is with the bond devaluation due to an excess of liquidity which can occur when issuing in tens of millions of bonds per auction.

6.5.1 Modified prices of bonds depending on existing liquidity

Let us consider a modified bond price, where the new price is the existing price discounted to account for the impact of the already issued quantity. The denominator can represent several things. Usually it is taken to be the discount factor between fair or mathematical value and market value. The following equations shows the approach used here:

$$\tilde{P}_{i.j}^{(k)} = \frac{P_{i.j}^{(k)}}{1 + \varepsilon(D0_{i.j}^{(k)} + \sum_{i=0}^{t-1} P_{i.j}^{(k)} u_{i.j}^{(k)})}, \tag{6.23}$$

where $P_{i.j}^{(k)}$ corresponds to the price of k^{th} bond at time i and scenario j. $\tilde{P}$ corresponds to the new modified price and ε is a constant. As previously, $D0_{i.j}^{(k)}$ will represent the total prior issuance of the specific bond at time i and scenario j, to build the bond to a benchmark, while making it less worthwhile to issue further. Let $\mu_{i.j}^{(k)}$ be $1+\varepsilon(D0_{i.j}^{(k)})$, then:

$$\tilde{P}_{i.j}^{(k)} := \frac{P_{i.j}^{(k)}}{\mu_{i.j}^{(k)} + \varepsilon \sum_{l=1}^{t-1} u_{l.j}^{(k)} P_{l.j}^{(k)}}, \tag{6.24}$$

and let us denote:

$$z_{i.j}^{(k)} := \mu_{i.j}^{(k)} + \varepsilon \sum_{l=1}^{t-1} u_{l.j}^{(k)} P_{l.j}^{(k)}, \tag{6.25}$$

Now as $z_{i.j}^{(k)}$ depend on the prior issuance $x_{i.j}^{(k)}$, it is a decision variable and is always greater than 1. Lets take the quadratic constraint:

$$\sum_{(i,k)\in(\mathcal{N},\mathcal{K})} \frac{u_{i.j}^{(k)} P_{i.j}^{(k)}}{z_{i,j}^{(k)}} \geq D \; \forall j \in \mathcal{J}, \tag{6.26}$$

We can change the value of the bond everywhere or just modify the amount to be raised to accommodate for the difference. It can be rewritten as:

$$\sum_{(i,k)\in(\mathcal{N},\mathcal{K})} u_{i.j}^{(k)} P_{i.j}^{(k)} \geq D z_{i.j}^{(k)}, \; \forall j \in \mathcal{J}, \tag{6.27}$$

as $z_{i.j}^{(k)}$ by definition will never be negative. This particular constraint is used to make sure that the total amount raised still meets the requirement of the budget with the modified prices.

The model can take such a liquidity variable into account while remaining linear, as explained in [17]. It can be rewritten as:

$$\text{minimize } \frac{1}{J} \sum_{j\in\mathcal{J}} I_j \text{ subject to:} \tag{6.28}$$

$$\sum_{(i,k)\in(\mathcal{N},\mathcal{K})} u_{i.j}^{(k)} P_{i.j}^{(k)} \geq D z_{i,j}^{(k)}, \forall j \in \mathcal{J}, \tag{6.29}$$

$$u_{i.j}^{(k)} P_{i.j}^{(k)} = \varpi \omega_{i.j}, \forall\, i \in \mathcal{N}, k \in \mathcal{K}, j \in \mathcal{J}, \tag{6.30}$$

$$\phi_j = I_j - \frac{1}{J} \sum_{j\in\mathcal{J}} I_j - \zeta, \forall\, j \in \mathcal{J}, \tag{6.31}$$

$$\frac{1}{J(1-\beta)} \sum_{j\in\mathcal{J}} \phi_j + \zeta \leq \rho, \tag{6.32}$$

$$(u_{i.j}^{(k)} + h_{i.j}^{(k)}) P_{i.j}^{(k)} + D0_j \leq \overline{\psi}_{i.j}^{(k)} \; \forall i \in \mathcal{N}, k \in \mathcal{K}, j \in \mathcal{J}, \tag{6.33}$$

$$IRM_{i.j}^{(k)} = (D0_{i.j}^{(k)} + \sum_{ti=1}^{i} x_{ti.j}^{(k)} \frac{\partial P_{i.j}^{(k)}}{\partial r_i^{(k)}}, \tag{6.34}$$

$$\sum_{k \in \mathcal{K}, j \in \mathcal{J}} IRM_{i.j}^{(k)} < \tilde{\rho}, \tag{6.35}$$

with the additional auction specific constraints :

$$\underline{D}X_{i.j}^{(k)} \leq u_{i.j}^{(k)} P_{i.j}^{(k)} \leq \overline{D}X_{i.j}^{(k)} \; \forall\, i \in \mathcal{N}, k \in \mathcal{K}, j \in \mathcal{J}, \tag{6.36}$$

$$\sum_{(i,k) \in (\mathcal{N},\mathcal{K})} X_{i.j}^{(k)} = N, \forall j \in \mathcal{J}, \tag{6.37}$$

$$\sum_{k \in \mathcal{K}} X_{i.j}^{(k)} = 1, \forall i \in \mathcal{N}, j \in \mathcal{J}, \tag{6.38}$$

$$\sum_{i \in \mathcal{N}} X_{i.j}^{(k)} \leq B, \forall j \in \mathcal{J}, k \in \mathcal{K}, \tag{6.39}$$

$$if\; \underline{\tau} \geq \tau_i^k \; or\; \overline{\tau} \leq \tau_i^k \; then\; X_{i.j}^k = 0 \; \forall\, i \in \mathcal{N}, j \in \mathcal{J}. \tag{6.40}$$

where $\varpi, \omega_{i.j}, \phi_j, \zeta, \beta$ are as defined in the previous model 6.4.3. $IRM_{i.j}^{(k)}$ corresponds to the interest rate measure defined earlier in this section, and $\tilde{\rho}$ is the upper bound set over the portfolio of all bonds and all scenarios. Equation (6.29) is slightly modified to accommodate the extra issuance of bonds, whereas equations (6.30)-(6.35) are new linear constraints that have been added to the model.

6.6 Numerical results

We apply the optimization model defined in 6.4.3 to the UK government debt problem for the $2007-08$ year. The model parameters N, K and D for the debt problem (auctions, bonds and amounts to be raised), as defined in section 6.2 are:

Subproblem	N	K	D (in bn)
short $(1-7\,years)$	4	16	10
medium $(7-15\,years)$	4	8	10
long $(>15\,years)$	11	10	23.4

Table 6.1: Parameters used for optimization.

As well as using the real bonds that were available during that financial year, some of the parameters of the optimization models are chosen based on the government remit as follows with those defined in table 6.6.

- $\gamma = 250$ is the amount in million pound sterling to increment the amount raised at an auction.

- $\underline{D} = 1,500$ million and $\overline{D} = 4,000$ million, these are set in the remit.
- $B = 2$ is the maximum amount of times we choose to issue a particular bond in the set of auction considered for short dated and medium dated bonds. $B = 3$ for the long dated bonds issuance problem.

We will refer to the problem of issuance of short dated bonds as the short subproblem. Similarly the medium dated bonds and long dated bonds correspond to the medium subproblem and the long subproblem respectively. The CVaR measure of risk will be compared to the traditional VaR measure and the CaR measure introduced in [52]. However unlike CVaR which is evaluated in our optimization model, VaR and CaR are computed out of sample for a posteriori analysis. $\overline{\psi}$ is 20 billion for the short subproblem, 22 billion for the medium subproblem and 40 billion for the long subproblem. The use of longer term debt doesn't decrease the expected cost, as the cost function takes into account all coupon and principal repayments, not discounted by the effect of inflation. The effect of inflation reduces quite considerably the actual cost of the debt.

In this section several tables and plots with key results from the different subproblems will be shown. Let us begin with the results for the short, medium and long subproblem with no upper limit constraint of CVaR (ρ). The solutions from (6.2 - 6.4) [1] were obtained on AMD Phenom X6 1055T processor with 4GB of RAM using the Gurobi 5.0.1 solver.

Q	CaR in MN	VaR in MN	CVaR in MN	S.D. in MN	$\mathbb{E}[I]$ in MN	time in s
1	13570.951	13235.678	12487.400	603.345	12604.396	4.667
2	13236.067	13130.600	11791.049	441.366	11909.629	3.215
3	13080.330	13084.245	11411.655	362.734	11771.610	5.790
4	12988.885	13058.574	11187.301	315.572	11828.529	37.602
5	12925.182	13048.864	11103.565	280.070	11567.305	206.276
6	12877.629	13041.224	11037.916	253.707	11628.767	327.514

Table 6.2: Results for the short subproblem with no ρ constraint.

Q	CaR in MN	VaR in MN	CVaR in MN	S.D. in MN	$\mathbb{E}[I]$ in MN	time in s
1	24938.271	23208.025	70547.039	3308.965	15034.343	1.149
2	23190.877	22563.763	66334.643	2503.593	14160.075	0.591
3	22334.971	22264.818	63924.727	2087.910	14801.552	0.852
4	21823.823	22231.914	63741.804	1791.331	14794.298	2.737
5	21465.394	22093.170	62553.862	1619.776	14767.257	4.521

[1] MN in the tables is a common abbreviation for million

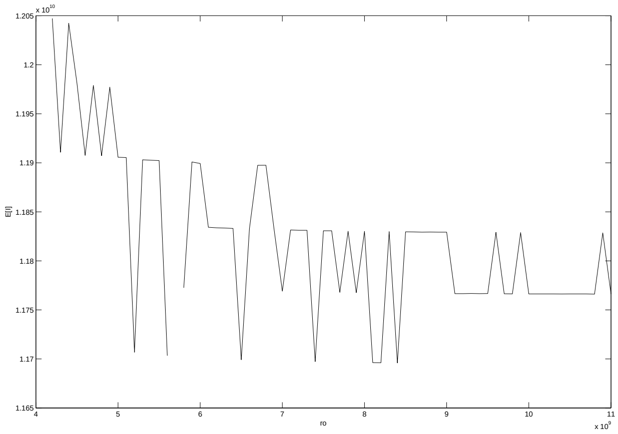

Figure 6.2: Efficient frontier for short subproblem.

Table 6.3: Results for the medium subproblem with no ρ constraint.

Q	CaR in MN	VaR in MN	CVaR in MN	S.D. in MN	$\mathbb{E}[I]$ in MN	time in s
1	117307.585	105333.152	481727.596	22692.107	49679.162	1.188
2	104380.241	99787.568	439813.347	16863.673	48472.099	1.224
3	98320.306	98024.002	424732.222	13769.380	50866.865	2.116
4	94822.590	97072.583	416614.623	11961.758	50710.848	6.484
5	92492.104	96767.387	414004.704	10647.074	22119.416	22.114

Table 6.4: Results for the long subproblem with no ρ constraint.

By restricting the values of the upper limit of the $CVaR$ constraint, ρ, we are able to produce results which are given in the Appendix C. Those results are used to produce efficient frontiers for all three problems. These frontiers are shown in figures (6.2)-(6.4). As can be seen, integer constraints make the efficient frontiers highly discontinuous. The variation in $\mathbb{E}[I]$ for the short subproblem with changes in ρ is much higher than the corresponding variation for the medium subproblem or the large subproblem. This can be explained by the larger choice in short term maturity, coupons and smaller availability to issue within the liquidity constraints (all maturities roll over eventually towards the short term subproblems). This means that as ρ is reduced, the model will pick bonds with higher coupons and available in a larger quantity.

Remark. The discontinuity is also seen in tables (C.1)-(C.3), as empty lines. The empty lines are unfeasible solutions which are due to the limitations of the Gurobi 5.0.1 solver.

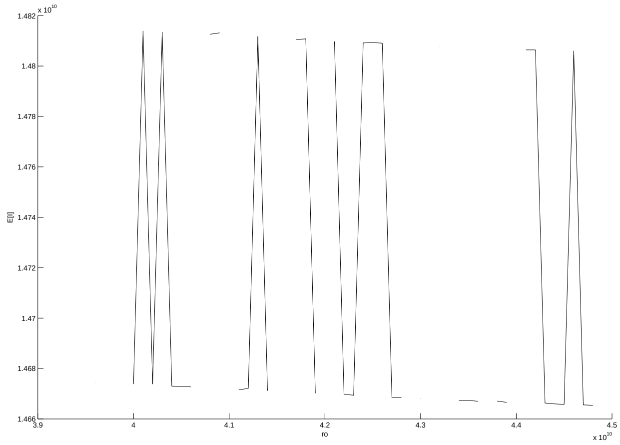

Figure 6.3: Efficient frontier for medium subproblem.

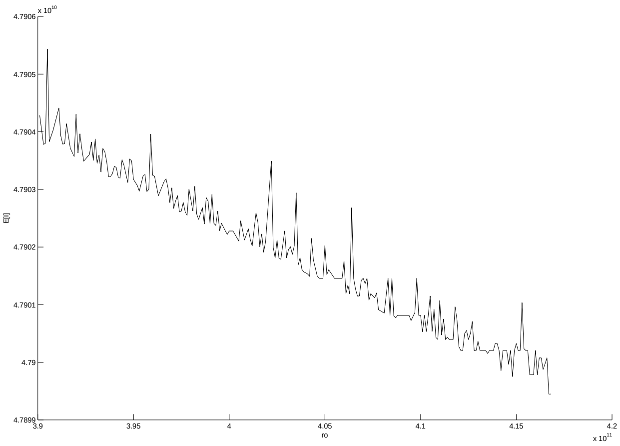

Figure 6.4: Efficient frontier for long subproblem.

The table C.4 represents the solutions for the short subproblem using the CPLEX 12.2.0.0 solver. It can be found in Appendix C. As the number of unfeasible solutions with the CPLEX solver are greater than with the Gurobi solver, they will be ignored in this thesis.

6.6.1 Comparison with the DMO debt issuance

The above model proposes an optimization based approach to debt issuance. However, the issuance is often driven by exogenous factors of uncertainty, such as a change in political sentiment or macroeconomic shocks. Not all the sources of uncertainty can be adequately represented in an optimization model. From table 6.5 it appears that the implementation of our model would have resulted in a significant cost reduction for the UK government in the period considered. What the model does not tell is whether the implementation of the proposed cost minimization procedure leads to a maturity structure which is radically different from the one adopted in the real world. It is therefore of interest to compare our model with the actual issuance by the UK government. An important exogenous factor which the government takes into account when issuing debt is the net debt to GDP ratio. The differences in the amounts issued from table 6.5 are rather small. However, they have an important impact on the total cost of the issuance and can be seen in the debt to GDP ratios. [2]

Subproblem	Actual cost	Model cost with no ρ	Model cost with ρ
Short problem	12.587500	11.628767	12.047241
Medium problem	15.375000	14.767257	14.674645
Long problem	55.353750	50.710848	47.904285
Total cost	83.316250	77.106872	74.626171

Table 6.5: Comparison against real debt issuance in billions of pounds

Remark. Note that the interest rate model used for scenario generation is calibrated on one data set and the optimization is carried out on a different (out-of-sample) data set throughout this exercise and the actual issuance decisions are not used as inputs to the model.

[2] This work was first reported by the author in [27].

Chapter 7

Multifactor simulation models

Optimization modelling is useful in decision making when we have decision variables which can influence the future outcomes, the size of the system is modest enough to make optimization tractable and we have the dynamics of the system can be forecast with a reasonable accuracy over a time horizon of interest. When one of these conditions is not satisfied, one resorts to simulation models instead as an aid in decision making. We focus in this chapter on macroeconomic simulation models which can be of use in decision making for public debt issuance by providing useful insight and information into possible future outcomes; see [2] and [87]. The next section introduces some notation common to both the models. The two subsequent sections introduce the two models, followed by a discussion on their comparative advantages. Simulations can be used for medium to long term forecasts and use several more factors because it does not involve decision making. Both models examined are very different, the first one is from an actuarial point of view and the second from a government point of view, although they both look at similar aspects of the economy such as inflation or the short and long rate. After simulating, the probable medium to long term scenario, we will apply an optimization model to assess how a specific strategy of issuance fares, e.g. only long term bond issuance or a mixed maturities issuance of fixed income debt. The choice of issuance policy or strategy is described in [24] and [91]. [51] describes the calibration and testing of the policies on macroeconomic models. [64] gives examples of stochastic simulations for dynamic economic models, whereas [46] propose a set of Bayesian econometric models.

7.1 Notations for simulation models

In the next section, two models are going to be presented where the following notations will be used:

1. r_t be the short term rate at time t,
2. l_t be the long term rate at time t. It corresponds to the r_∞ previously used and can be defined in certain arbitrage free models,
3. q_t be the inflation rate (Consumer Price Index or CPI rate) at time t,
4. p_t be the inflation rate (Retail Price Index or RPI rate) at time t,
5. u_t be the reversion level at time t,
6. μ_t be the mean reversion level of inflation at time t,
7. λ_t be the risk premium or excess equity return attributable to capital appreciation,
8. s_t be the return of equity defined as $q_t + r_t + \lambda_t$ as seen in [54],
9. y_t be the equity dividend yields, it is assumed that the natural logarithm of y_t follows an autoregressive process like ([56], [110] [111]),
10. re_t be the real estate returns,
11. ue_t be the unemployment rate,
12. og_t be the output gap at time t,
13. f_t be the financial requirement of the government at time t,
14. d_t be the debt to nominal GDP ratio at time t,
15. ζ be the inflation target,
16. dW_i be the Brownian motion for each factor i; as defined in section 2.2;

7.2 Macroeconomic models

With the above notation, let us demonstrate a short example applied to the UK from [2], which uses several Ornstein Uhlenbeck processes:

$$dr_t = \kappa_r(l_t - r_t)dt + \sigma_r dW_r, \tag{7.1}$$

$$dq_t = \kappa_q(\mu_q - q_t)dt + \sigma_q dW_q, \tag{7.2}$$

$$dl_t = \kappa_l(\mu_r - l_t)dt + \sigma_l dW_l, \tag{7.3}$$

$$d(ln(y_t)) = \kappa_y(\mu_y - ln(y_t))dt + \sigma_y dW_y, \tag{7.4}$$

$$d(re)_t = \kappa_{re}(\mu_{re} - (re)_t)dt + \delta_{re} q_t + \sigma_{re} dW_{re}, \tag{7.5}$$

$$d(ue)_t = \kappa_{ue}(\mu_{ue} - ue_t)dt + \delta_{ue} dq_t + \sigma_{ue} dW_{ue}. \tag{7.6}$$

where δ_{re} are δ_{ue} are constants between 0 and 1. This model is used to give a possible outcome depending on the values chosen to begin, and decision making can occur afterwards assuming the decisions don't impact greatly on the prediction. The equations can be discretized to:

$$r_{t+1}(0) = r_t(0) + (1-\kappa_r)(l_t - r_t(0)) + \sigma_r \varepsilon_{r,t}, \tag{7.7}$$

$$q_{t+1} = q_t + (1-\kappa_q)(\mu_q - q_t) + \sigma_q \varepsilon_{q,t}, \tag{7.8}$$

$$l_{t+1} = l_t + (1-\kappa_l)(\mu_r - l_t) + \sigma_l \varepsilon_{l,t}, \tag{7.9}$$

$$ln(y_{t+1}) = ln(y_t) + (1-\kappa_y)(\mu_y - ln(y_t)) + \sigma_y \varepsilon_{y,t}, \tag{7.10}$$

$$re_{t+1} = re_t + (1-\kappa_{re})(\mu_{re} - re_t) + \delta_{re} q_t + \sigma_{re} \varepsilon_{re,t}, \tag{7.11}$$

$$ue_{t+1} = ue_t + (1-\kappa_{ue})(\mu_{ue} - ue_t) + \sigma_{ue} \varepsilon_{ue,t} \tag{7.12}$$

$$+ \delta_{ue}((1-\kappa_q)(\mu_q - q_{t-1}) + \sigma_q \varepsilon_{q,t-1}). \tag{7.13}$$

The use of the natural logarithm permits us to work with a positive and negative equity dividend yields, to resemble more closely the equity markets appetite for risk.

Another simulation model used by the UK debt management office can be found in [87] and is defined as:

$$r_t(0) = \phi + \omega q_{t-1} + \chi og_{t-1} + \varepsilon_{r,t}, \tag{7.14}$$

$$q_t = \zeta(1-\xi) + \xi q_{t-1} + \psi og_{t-1} + \varepsilon_{q,t}, \tag{7.15}$$

$$p_t = \kappa + q_{t-1} + \iota r_t + \varepsilon_{p,t}, \tag{7.16}$$

$$f_t = \mu + \nu f_{t-1} - \pi og_{t-1} - \theta(d_{t-1} - d^*) + \varepsilon_{f,t}, \tag{7.17}$$

$$og_t = \alpha_t + \rho og_{t-1} - \beta(r_{t-1}(0) - q_{t-1}) + \varepsilon_{og,t}. \tag{7.18}$$

with α_t a Markov switching intercept with two states, ρ measures the degree to which the output gap is affected by its previous value and β is the short rate from the previous period. ν indicates the extent to which the primary net financing requirement is influenced by its previous value and θ indicated the extent to which the government has to change its fiscal policy in order to ensure that the debt to GDP ratio does not deviate too far from the long-run average ratio. ξ measure the strength with which the CPI inflation is influenced by its previous value. κ is a constant and ι indicate the extent to which the short rate affects the RPI inflation. ϕ is a constant, ω indicate the degree to which the previous period's value of the CPI inflation is affected. d^* is the long-run average debt to nominal GDP. π ψ and χ are the lagged value of the output gap. The ε variables correspond to the errors and are defined as such:

- $\varepsilon_{og,t} \sim N(0, \sigma_{og}^2)$,

- $\varepsilon_{f,t} \sim N(0, \sigma_f^2)$,
- $\varepsilon_{q,t} \sim N(0, \sigma_q^2)$,
- $\varepsilon_{p,t} \sim N(0, \sigma_p^2)$,
- $\varepsilon_{r,t} \sim N(0, \sigma_r^2)$,
- $\varepsilon_{l,t} \sim N(0, \sigma_l^2)$,
- $\varepsilon_{y,t} \sim N(0, \sigma_y^2)$,
- $\varepsilon_{re,t} \sim N(0, \sigma_{re}^2)$,
- $\varepsilon_{ue,t} \sim N(0, \sigma_{ue}^2)$,

Given a medium to long term simulation, we can use a debt issuance strategy to forecast the costs and risks associated to such a strategy for that particular simulation. The simulation is done on a quarterly basis over a long period of time. Several assumptions will be taken for simplicity:

- the amount needed to be raised will be the financial requirement f_t,
- only new bonds will be issued and there will not be any re issuance of existing bonds,
- all new bonds will be issued at face value,
- the initial debt to GDP ratio is set to a constant,
- the cost of the debt at any given time t is the sum of all remaining fixed and inflation linked coupons and a realized inflation compensation effect on maturing inflation linked bonds,
- bonds will be issued to a 5,10 and 20 years maturity for fixed coupons and 30 years for fixed or inflation linked bonds.

The models are calibrated and used to estimate yield curves evaluated with a Dynamic Nelson-Siegel model. The yield curves are assumed to depend on the short rate, the CPI inflation and the output gap at time t, in the following manner according to [87]:

$$\begin{pmatrix} l_t \\ s_t \\ c_t \end{pmatrix} = \begin{pmatrix} \mathbb{E}(r_t(0)) \\ -\mathbb{E}(r_t(0)) \\ 0.03 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0.7 & -0.6 & -2.2 \end{pmatrix} \begin{pmatrix} og_t \\ r_t(0) \\ q_t \end{pmatrix} + \begin{pmatrix} \eta_l(t) \\ \eta_s(t) \\ \eta_c(t) \end{pmatrix} \tag{7.19}$$

with the yield at time t for a zero-coupon bond maturing τ:

$$r_t(\tau) = l_t + s_t \left(\frac{1 - exp(-\tau/\lambda)}{\tau/\lambda} \right) + c_t \left(\frac{1 - exp(-\tau/\lambda)}{\tau/\lambda} - exp(-\tau/\lambda) \right), \tag{7.20}$$

where λ is associated a constant value, in [87] the value $\lambda = 1.45$.

Remark. Note that the authors of [87] do not explain the calibration procedure used to arrive at these values.

Several values have been allocated constant values:

$$y_{t=0} = 0, \tag{7.21}$$

$$f_{t=0} = \frac{\mu}{\nu - 1}, \tag{7.22}$$

$$q_{t=0} = \zeta, \tag{7.23}$$

$$p_{t=0} = \kappa + \iota\phi + (1 + \iota\omega)\zeta, \tag{7.24}$$

$$r_{t=0}(0) = \phi + \omega\zeta. \tag{7.25}$$

Each scenario generated will allow a yield curve to be constructed at each quarter of the timescale considered and will allow the cost of a specific strategy to be evaluated.

7.3 Public debt strategy testing

The goal of testing a strategy is to assess how a particular strategy of issuance will perform over a prolonged period of time, it's cost, risk attached to it, and it's result on debt to GDP ratio dynamics. In [87] they test a fixed strategy over a period of 125 years, and keep issuing the required financial requirement with a fixed ratio of short, medium, long and/or inflation linked bonds at each quarter. Assuming a scenario with a short rate and an inflation factor as well as a financial requirement: a yield curve can be computed. The inflation linked bonds are easy to price once, the inflation is assumed. Once a yield curve can be evaluated at each quarter and the amounts to be raised is given by the financial requirements, the model only needs to issue the bonds.

7.4 Comparison of macroeconomic models

The second model is used by the UK debt management office. It models financial requirement and the evolution of the GDP with the output gap as well as other key macroeconomic factors, and create yield curves along the year to create new bonds. However their assumptions appear to be unreasonable and unrealistic, as it is not possible to issue a new bond per quarter for short, medium and long term bonds as well as an inflation linked bond. There are no restrictions on the sizes of auctions or financial requirements, the coupon payments are done quarterly instead of semiannually and issuance occurs at face value. Figures 7.1-7.2 are an example of a single scenario, using the data obtained from [102]. We can see the consequence of having a constant drift with a relatively large volatility. We can see in figure

7.1 that the unemployment rate can move by more than 2% within a single year quite regularly. Clearly several macroeconomic indicators do not vary by over 2% per year regularly, this helps the notion that the assumptions appear to be unreasonable and unrealistic.

The two methods are very different but they can both be used for decision making. More information about macroeconomic scenarios and forecasting can be found in [102]. The first model (7.7)-(7.13) is an actuary model and therefore was never intended to be used for public debt issuance, there is no financial requirement modeled or the GDP growth, making it useful for macroeconomic simulations but not for public debt issuance.

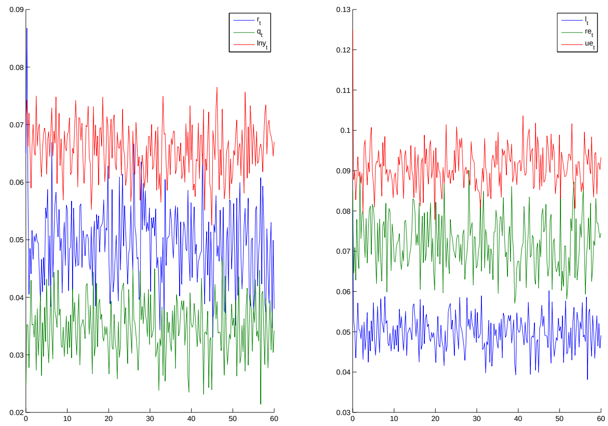

Figure 7.1: A single scenario of the 1^{st} model macroeconomical evolution over 60 years.

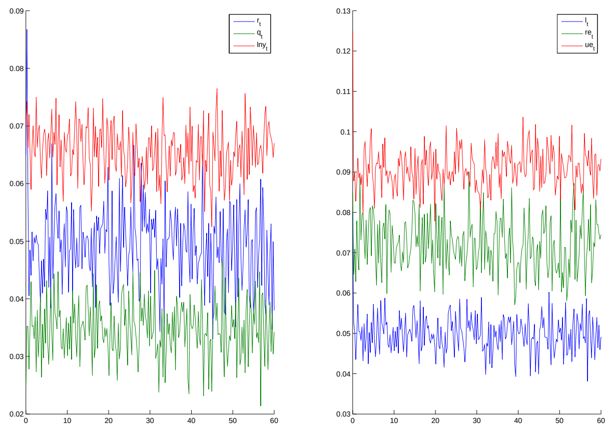

Figure 7.2: A single scenario of the 2^{nd} model macroeconomical evolution over 60 years.

Chapter 8

Conclusion

8.1 Contributions

The main contributions to knowledge of this thesis are contained in chapter 3 and 6. They are summarized as below:

1. **Chapter 3**
   Several calibrations and comprehensive numerical experiments with real financial UK data are given to compare different one, two and three factor interest rate models in terms of their explanatory and predictive power. Comparisons of different one, two and three factor interest rate models, show that the arbitrage free dynamic Nelson-Siegel model outperforms all other models in-sample and the two-factor Vasicek model outperforms out-of-sample. The two-factor Vasicek model is shown to be more accurate out-of-sample despite being less complex and having considerably faster computation times as compared to three factor models.

2. **Chapter 6**

   - We calibrate a one factor, linear Gaussian interest rate model using a Kalman filter and noisy yield measurements and use this to create bond price scenarios for the optimization model. Arguably, this reflects better market expectation of the bond prices obtainable through auctions than using primary economic variables. Using a filter based interest rate model also allows for easy re-calibration and hence allows for generating interest rate scenarios which are tuned to more recent market data. For demand-side optimization, a similar approach was taken in [83] where a two factor interest rate model is used along with a multi-factor stochastic program to manage mortgage-backed securities. Filtering-based model is also used in a simulation framework in the report by [34]. The authors are not aware of the use of filtering based framework to generate scenarios in a supply-side optimization.

- We use a recombining lattice-based stochastic programming model [27] as opposed to a non-recombining scenario tree used in [22] and [7] while discussing the sovereign debt issuance. This makes the problem computationally significantly simpler, as the number of scenarios is reduced significantly, while retaining consistency with the underlying theoretical interest rate model.
- We use a *receding horizon* approach to carry out multiple, multistage stochastic programs over a period of time to optimize debt issuance cost over a given horizon; see [73] for control engineering applications of the receding horizon approach. Once a stochastic programming exercise is carried out, one need not stick to the full sequence of optimal decisions with passage of time, as the uncertainty progressively resolves itself. We propose re-calibrating the scenario generation (i.e. interest rate) model periodically and use it to re-optimize the issuance over the remaining period, using the issuance data up to that time. To our knowledge, the use of receding horizon strategy in an optimization model is new.
- We carry out out-of-sample back-testing to compare the performance of our strategy against the actual debt issuance by the UK government in the budgetary years 2006-2008. Our results show that a significant debt-service cost reduction can be achieved by carrying out a rigorous optimization exercise.

3. A significant amount of reusable software was created in Matlab (for filtering based calibration and prediction) and in Matlab and AMPL (for supply side optimization). The software was used to create the numerical experiments in chapter 3 and 6. It is fairly easy to use and documentation has been provided in Appendix A.

Further research

This thesis tries to cover all aspects of public debt issuance from an operational research point of view. The topic has received more interest from academia in recent years. However as DMOs have become independent fairly recently, very little has been published or shared. This leaves many facets of the topic unexplored by operational research. This work can be extended in several areas:

1. A dynamic interest rate model to price bonds, taking into account existing liquidity and secondary market liquidity would be ideal for optimized decision making of public debt issuance.

2. The optimization models can be extended to take into account the inflation linked bond subproblem. It is a long term problem, where modeling inflation is key for pricing and for selecting the maturities of the bonds to be issued.

3. Very little work has been done with respect to macroeconomic simulations for long term planning of public debt issuance. The issuance strategies require too many unrealistic assumptions in the current literature, as mentioned in chapter 7. An optimization model based on more realistic macroeconomic dynamics could be potentially a very useful contribution. The issuance of debt can be made to mimic the real conditions of debt issuance.

4. Long term debt issuance strategies have only been done with static strategies. In a simulation environment, dynamic strategies have more potential to model macroeconomic events with greater accuracy.

5. Currently scenarios are considered independent to public debt issuance strategies. The issuance of debt doesn't impact the scenarios in any way, which is another topic that can be explored in much more detail.

Appendix A

Function Documentation

In this part of the thesis, we will give a short user interface documentation for the matlab code developed during this thesis and provided in the accompanying CD-ROM.

We will begin with the scripts:

`kalmanscript.m`

Call `kalmanscript` from the `Matlab terminal`. Within `kalmanscript`, a user can choose the calibration model used:

1. One factor Vasicek model,
2. One factor CoxIngersollRoss model,
3. Two factor Vasicek model,
4. Two factor CoxIngersollRoss model,
5. Basic dynamic Nelson-Siegel model, where the curvature is omitted,
6. Non mean reverting dynamic Nelson-Siegel model,
7. Mean reverting dynamic Nelson-Siegel model,
8. Arbitrage free dynamic Nelson-Siegel model with independent factors,
9. Arbitrage free dynamic Nelson-Siegel model with corrolated factors,
10. Macroeconomic dynamic Nelson-Siegel model.

The script will call other scripts to load the data for the time period considered and load a large amount of data related to the data. It will then perform a Kalman calibration to obtain the parameters as explained in chapter 3. It will be followed by an attempted to adjust the factors of the model to the real data. The script depends on `initialization` and `kalmandataload.m` or `kalmanweeklydataload.m`.

`kalmandataload.m`

Is called by `kalmanscript` to load the daily bond prices from an excel file called `Allconvyields0607.xlsx`.

`kalmanweeklydataload.m`

Is called by `kalmanscript` to load the weekly bond prices from an excel file called `Allconvyields0607weekly.xlsx`.

`initialization.m`

Is called by `kalmanscript` to process the loaded data and load additional information about the bond data such as the model parameters for each model for a specific time step, the timescale of the problem and the evaluate time points, the bond maturity and the next dividend dates, the coupon yields, number of coupons left. The script also selects the number of bonds to consider for the calibration problem. All the data is processed into matrices for Matlab to process.

`scriptshort.m`

Is the script used to solve the set of four short term bond problems. The solvers are chosen with their options within the script then each problem is solved and sends the decisions taken between receding horizons to the next problem. There is a data file for each problem needed containing the prices, yields for each scenario at each auction data making some of those data files quite substantial.

`scriptmedium.m`

Is the script used to solve the set of four medium term bond problems, similarly to the `scriptshort.m` except the data and problems are adjusted to the medium term bond problem specifications.

`scriptlong.m`

Is the script used to solve the set of four long term bond problems, similarly to the `scriptshort.m` except the data and problems are adjusted to the long term bond problem specifications.

`bsgmodel.m`

is a script that is used to create the left hand side A matrix and right hand side b vector of the optimisation problems based on the AMPL model. It helps to by-pass the AMPL language all together and just call the solver withing Matlab.

And now for the functions:

`mymle.m`

The function `mymle` is in the form: `[L x] = mymle(param)`. Where the input *param* is a vector containing the parameters for the time specific problem and the length of the vector will depend on the model used. The outputs are L, the maximum likelihood estimator and x the vector or matrix of interest rate factors evolving along the timescale of the problem. The maximum likelihood estimator L is estimated from the Kalman filtration process adjusted to the different models that can be chosen from `kalmanscript`.

`price.m`

The function `price` is in the form: `p = price(y,L,C,mat,divid,annual)`. Where the input y corresponds to yield of the bond, L is the face value of the bond (the value at 1^{st} issuance). C is the coupon of the bond, *mat* is the time to maturity from the beginning of the time horizon, *divid* is the time until the next dividend is payed and *annual* specifies how often the bonds pay dividends per year. *annual* is set to be two, i.e. the bond pays a dividend every 6 months from 1^{st} issuance. The output p of the function is the price of the bond corresponding to the yield y.

`yield.m`

The function `yield` is in the form: `Y = yield(x,param,t,tau)`. Where x is a vector or matrix of the factors evolving in time for the duration of the problems timescale, *param* is the vector of parameters obtained from Kalman filtration for the model referenced by the number t. *tau* corresponds to the time to maturity from the current time point. The output Y is the corresponding yield obtained using a specific interest rate model.

`yieldbisection.m`

The function `yieldbisection` is in the form:
`ymt = yieldbissection(L,P,C,mat,divid,annual)`. The input are as usual:

1. L is the face value of the bond,
2. P is the price of the bond,
3. C is the coupon of the bond,
4. *mat* is the time to maturity of the bond,
5. *divid* is the time until the next dividend occurs,
6. and *annual* is the frequency at which a bond pays coupons.

The output of the function is yield *ymt* obtained by using a bisection algorithme.

`yieldsecant.m`

The function `yieldsecant` is in the form:
`ymt = yieldsecant(L,P,C,mat,divid,annual)`. The input are as defined earlier:

1. L is the face value of the bond,
2. P is the price of the bond,
3. C is the coupon of the bond,
4. *mat* is the time to maturity of the bond,
5. *divid* is the time until the next dividend occurs,
6. *annual* is the frequency at which a bond pays coupons.

The output of the function is yield *ymt* obtained by using a secant algorithm. It is faster for long term bonds however doesn't behave well when a singularity occurs, so when called if an error message occurs it should switch to the slower but more stable `yieldbisection`.

`Vtreepricing.m`

The function `Vtreepricing` is in the form:
`[P bt br r] = Vtreepricing(Q, N, k, L, aucd, divid, coupon, nbcl , param, objt, MOD)`. The input argument Q is the number of steps the tree is allowed to take (e.g. 4 step tree means $3^4 = 81$ nodes).

1. N is the number of auctions,
2. k the number of bonds to consider,
3. L is the face value of the bond,
4. *aucd* the time to auction dates,
5. *divid* the time until the next dividends,
6. *coupon* the bond coupon yields,
7. *nbcl* the number of coupon left,
8. *objt* the timescale of the problem,
9. *param* is the vector of parameters for the model number MOD.

The output of `Vtreepricing` is a set of 4 matrices:

1. *bt* is the basic tree created by `treesetup.m`, it is a tree with integer value for nodes that grow by 1 or -1,
2. *br* is the basic interest tree build over the basic tree *bt* and is obtained from `interesttree.m`, it replaces the integer value of nodes with possible values of interest rates for the model MOD,
3. r is a matrix with linearly interpolated values of interest rates at auction dates obtained from `makelinearinterR.m`, upon which the set of bonds are priced at the interest rate values of auction dates to obtain the P matrix.

`treesetup.m`

The function `treesetup` is in the form:

`basicT = treesetup(Q,cstraint,branches)`. It is called by `Vtreepricing` and is used to created a general basic tree *basicT* with recombining lattices with Q tree steps and *branches* amount of nodes per node, i.e. if $branches = 3$ then it will create a trinomial tree with recombining lattice. If *cstraint* is 0 then the tree will grow unconstrained. If *cstraint* is 1 then the tree will grow until it reaches a maximum or minimum value and will not be able to grow above it.

`interesttree.m`

The function `interesttree` is in the form:

`basicR = interesttree(Q,branches,param,MOD,objt,basicT)`.

It is called by `Vtreepricing` after having called `treesetup`. The input arguments are:

1. Q is the number of steps in the tree,
2. *branches* is the number of branch nodes after each time step,
3. *param* is the vector of parameters obtained from the Kalman filtration with the interest rate model number MOD,
4. *objt* is the timescale of the tree,
5. and *basicT* is the basic tree where nodes have integer values.

The output *basicR* is a basic tree with *branches* branches for Q time steps where each node has a possible interest rate value instead.

`makelinearinterR.m`

The function `makelinearinterR` is in the form:

`liniR = makelinearinterR(Q,branches,N,objT,basicR,aucd)`.

It is called by `Vtreepricing` after having called `interesttree`. The input arguments are:

1. Q is the number of steps in the tree,
2. *branches* is the number of branch nodes after each time step,
3. N is the number of auctions in the problem,
4. *objt* is the timescale of the problem considered,
5. *basicR* is the basic interest tree obtained from `interesttree`,
6. *aucd* is the vector of length N with the auction dates.

The function will perfom a linear interpolation on the basic interest tree to get the output matrix *liniR*. It is made of the linearly interpolated values of interest rates at the auction dates.

`adddaynoise.m`

The function `adddaynoise` is in the form:

`str = adddaynoise(timestr,delta,basis)`. Its input arguments are:

1. *timestr* is a date in string format of Matlab,
2. *delta* is a integer to specify by how many days,
3. *basis* is the type of string time template used.

The output is a string with a new date, it can basically move the yields forward of backward to obtain different yields for the same days while using the same interest rate model.

`Bondpricing.m`

The function `Bondpricing` is in the form:

`P = Bondprice1(L,k,t,aucd,dividd,coupon,nbcl,param,MOD)`. The function has several input arguments:

1. L is the face value of the bond,
2. k is the number of bonds in the problem,
3. t is the time at which the bond needs to be priced,
4. *aucd* is the vector with the auction dates,
5. *dividd* is the vector with the next dividend dates,
6. *coupon* is the vector with the coupon yields,
7. *nbcl* is the vector with the number of remaining coupons,
8. *param* is the vector of parameters for the interest rate model numbered MOD.

The output is the matrix P with the prices of the bonds at each auction date, using the interest rate model MOD.

`savedn.m`

The function `savedn` is in the form:

`n = savedn(N,endt,aucd)`. Is a simple function which takes as input arguments:

1. N is the number of auctions in the problem,
2. *endt* is the amount of time to consider in the time scale in years,
3. *aucd* is the vector with the auction dates.

and it returns the number of coupons that will occur in the time considered. It is important to separate data that needs to be moved to the next problem.

`VPmntCrlo.m`

The function `VPmntCrlo` is in the form:

`[P MC MCr] = VPmntCrlo(N, param, MOD, fact, objt, size, L , aucd, mat, dividd, coupon, nbcl).`

The function has a number of input arguments:

1. N is the number of auctions in the problem,
2. *param* is the vector of parameters for the interest rate model numbered MOD,
3. $fact$ is the number of factors in the interest rate model,
4. *objt* is the timescale of the problem considered,
5. *size* is the number of Monte Carlo scenarios to create,
6. L is the face value of the bond,
7. *aucd* is the vector with the auction dates,
8. *mat* is the vector with the maturity dates of the bonds,
9. *dividd* is the vector with the next dividend dates of the bonds,
10. *coupon* is the vector with the coupon yields of the bonds,
11. *nbcl* is the vector with the number of coupons left for each bond.

`VPmntCrlo` is a function which calls `MCscengenerator.m` and `MCinterest.m` to create a Monte Carlo tree of interest rates. The output arguments are:

1. MC is a matrix with all the values of the interest rates along the the time scale of the problem,
2. MCr is a matrix of the interest rates at the auction dates,
3. P is a matrix of bond prices at each auction date using the interest rate model MOD.

`MCscengenerator.m`

The function `MCscengenerator` is in the form:

`mntCrlo = MCscengenerator(N, param, MOD, fact, objt, size).`

The function takes as input arguments:

1. N is the number of auctions in the problem,
2. *param* is the vector of parameters for the interest rate model numbered MOD,
3. $fact$ is the number of factors in the interest rate model,
4. *objt* is the timescale of the problem considered,
5. *size* is the number of Monte Carlo scenarios to create.

The function outputs a matrix MC with *size* scenarios using the interest rate model MOD.

`MCinterest.m`

The function `MCinterest` is in the form:

`MCr = MCinterest(MC, aucd, objt, size).`

The function takes as input arguments:

1. MC is the matrix from `MCscengenerator.m` which contains the interest rate values over the time scale of the problem,
2. *aucd* is the vector with the auction dates,
3. *objt* is the timescale of the problem considered,
4. *size* is the number of Monte Carlo scenarios to create.

The function outputs a matrix MCr with *size* scenarios using the interest rate model MOD at each auction date. MCr is used in `VPmntCrlo` to create a matrix P with the prices of the bonds with the interest rates from MCr using model MOD.

A set of functions also exist for the short, medium and long problems:

`short`*i*`.m`

`medium`*i*`.m`

`long`*i*`.m`

where i takes the values $1, 2, 3$ and 4. The set of functions takes all the data compiled by Matlab and exports the necessary data into `.dat` files for AMPL to load.

Appendix B

Extra calibrations using the Kalman filter

In this appendix, several plots have been added. They are the evolution of actual yields to maturity, after each calibration experiment for the different models discussed in chapters 2 and 3 using a Kalman Filter. The calibrations for the 1^{st} quarter were added to chapter 3. The plots for following quarters for the single and two factor Vasicek and Cox Ingersoll Ross models can be found below, as well as the plots for three different Nelson Siegel models discussed in chapter 2 and calibrated in chapter 3.

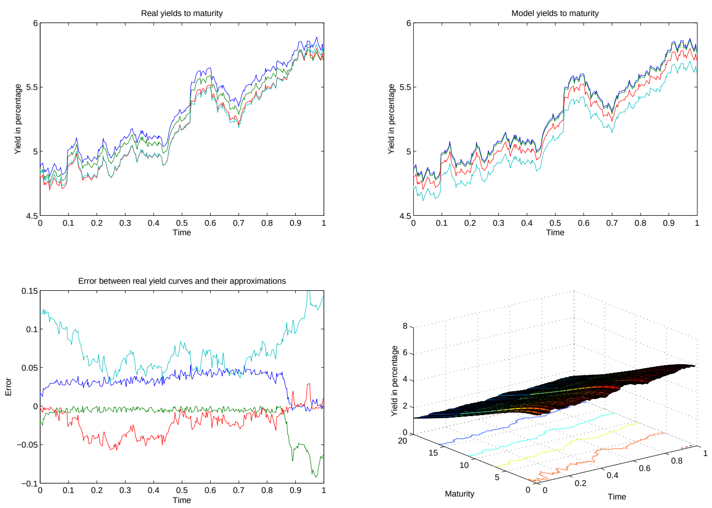

Figure B.1: Vasicek Model calibrated with Kalman filtration at $t = 06/2007$

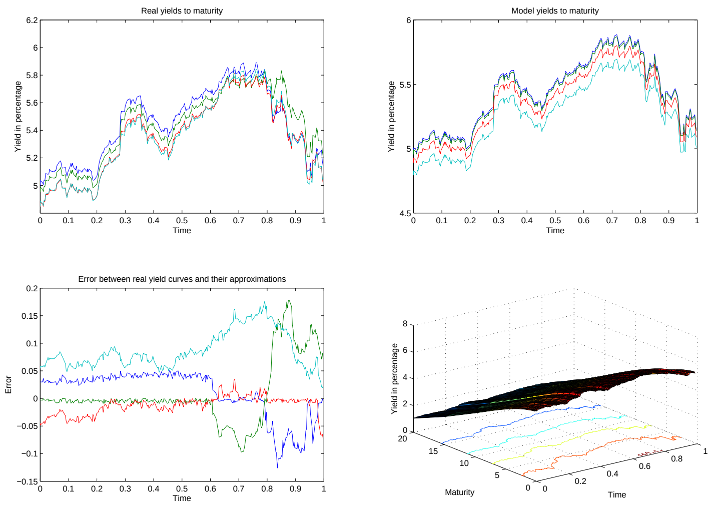

Figure B.2: Vasicek Model calibrated with Kalman filtration at $t = 09/2007$

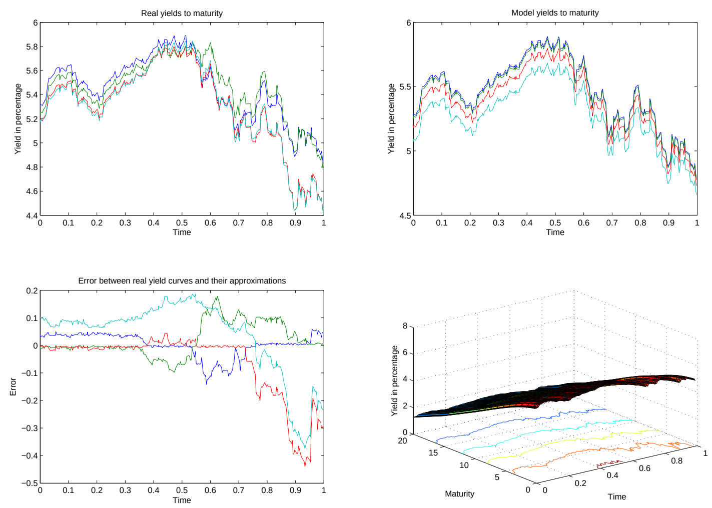

Figure B.3: Vasicek Model calibrated with Kalman filtration at $t = 12/2007$

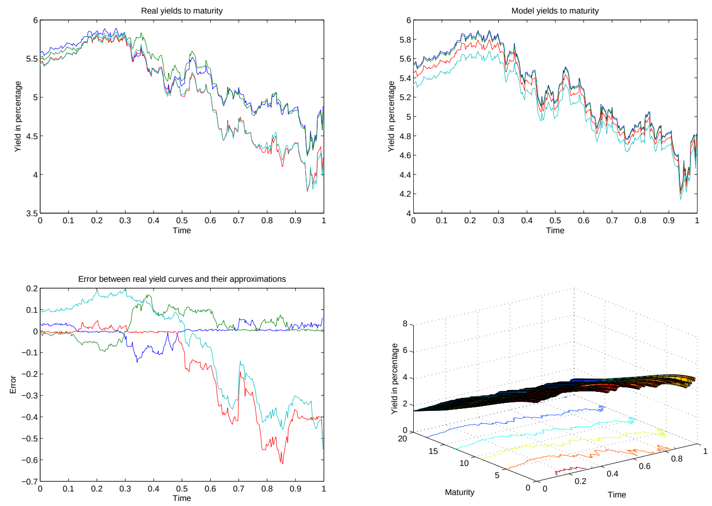

Figure B.4: Vasicek Model calibrated with Kalman filtration at $t = 03/2008$

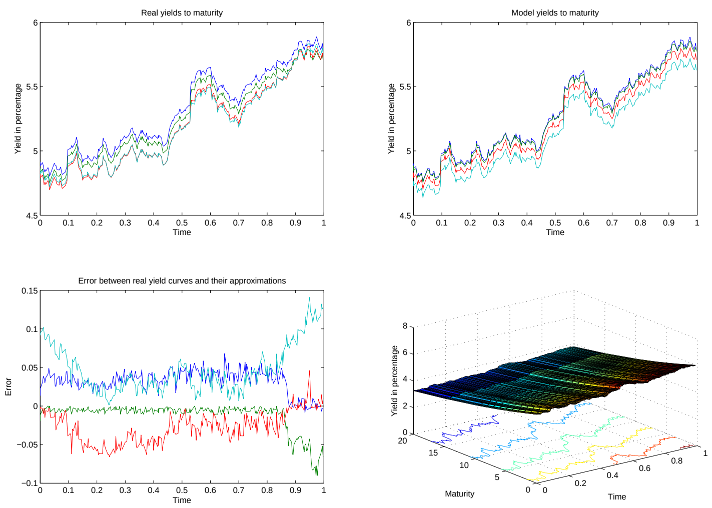

Figure B.5: CIR Model calibrated with Kalman filtration at $t = 06/2007$

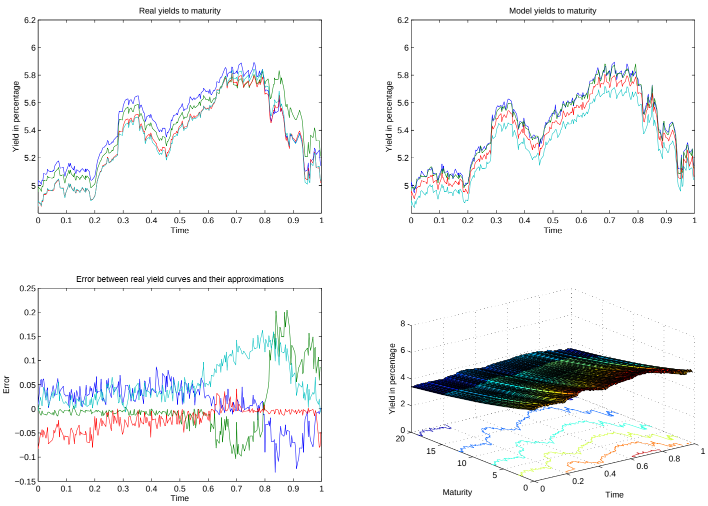

Figure B.6: CIR Model calibrated with Kalman filtration at $t = 09/2007$

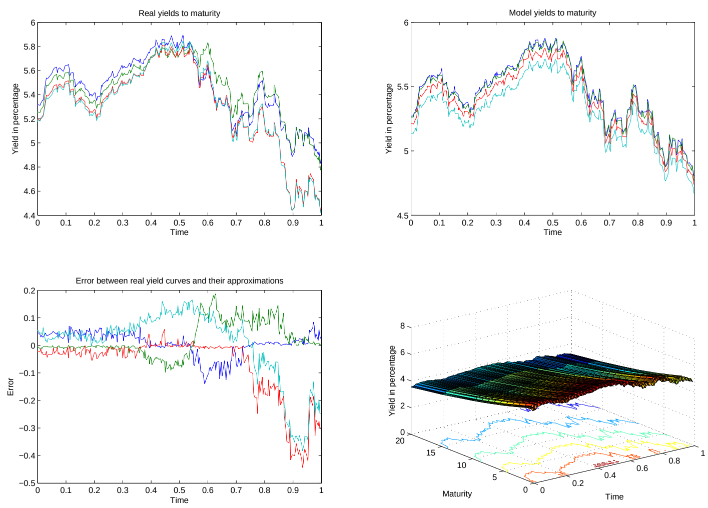

Figure B.7: CIR Model calibrated with Kalman filtration at $t = 12/2007$

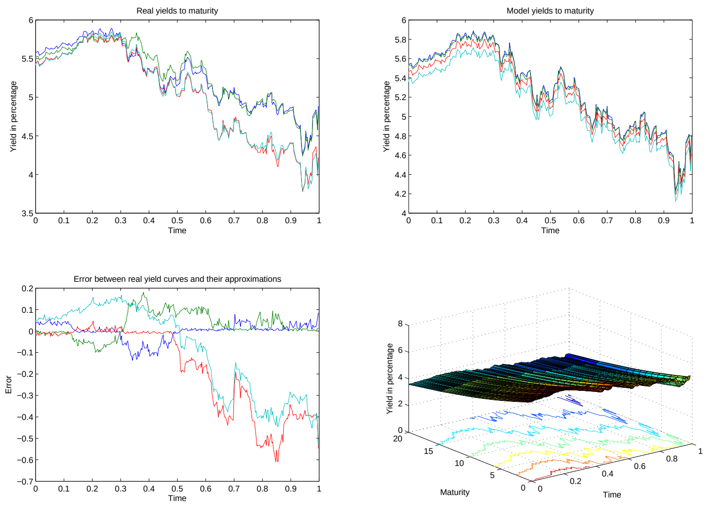

Figure B.8: CIR Model calibrated with Kalman filtration at $t = 03/2008$

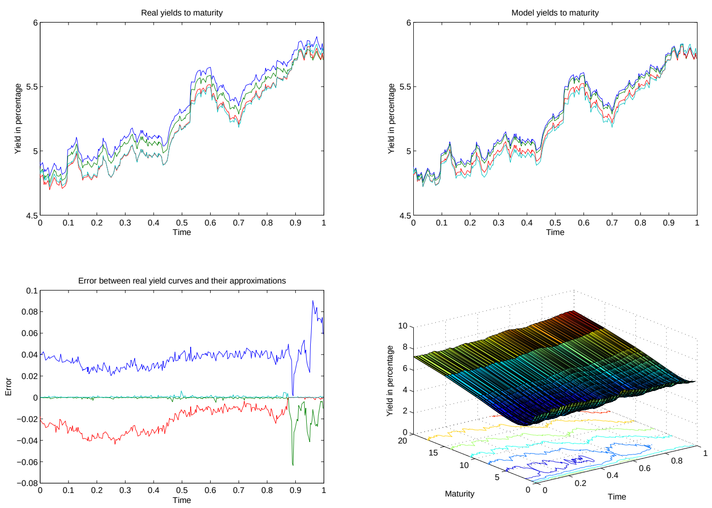

Figure B.9: 2 factor Vasicek Model calibrated with Kalman filtration at $t = 06/2007$

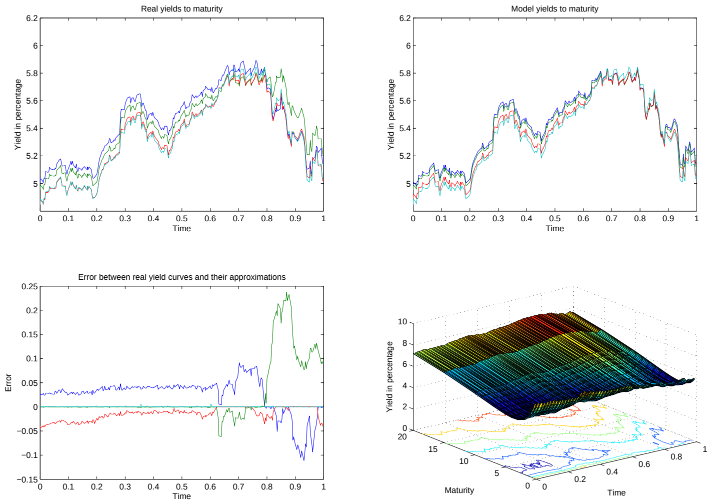

Figure B.10: 2 factor Vasicek Model calibrated with Kalman filtration at $t = 09/2007$

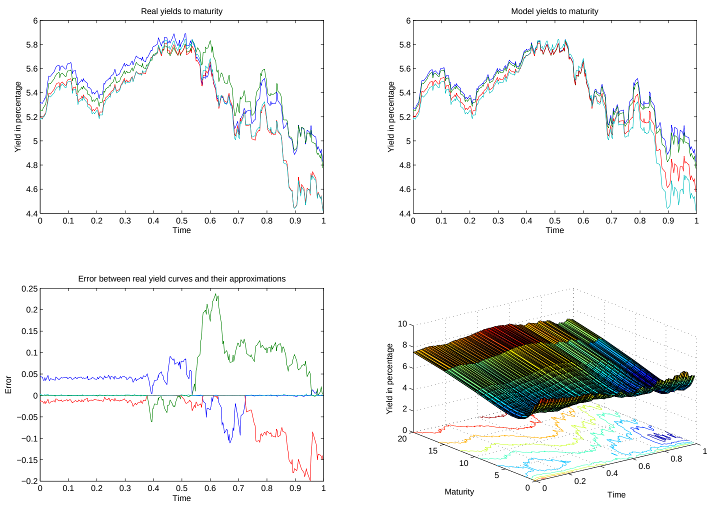

Figure B.11: 2 factor Vasicek Model calibrated with Kalman filtration at $t = 12/2007$

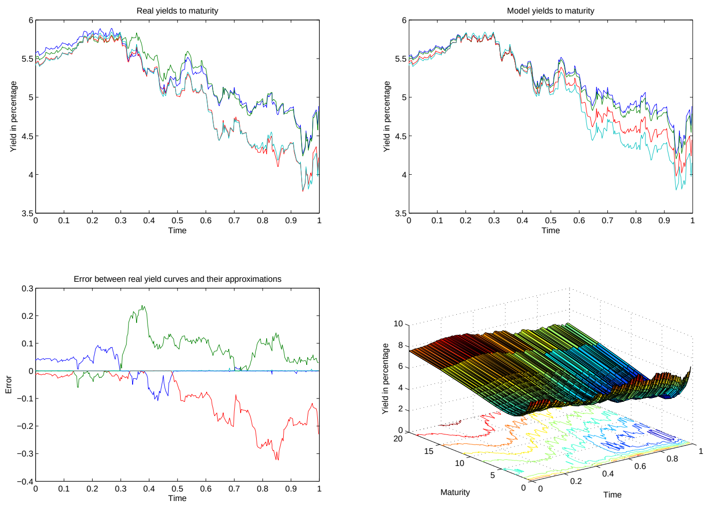

Figure B.12: 2 factor Vasicek Model calibrated with Kalman filtration at $t = 03/2008$

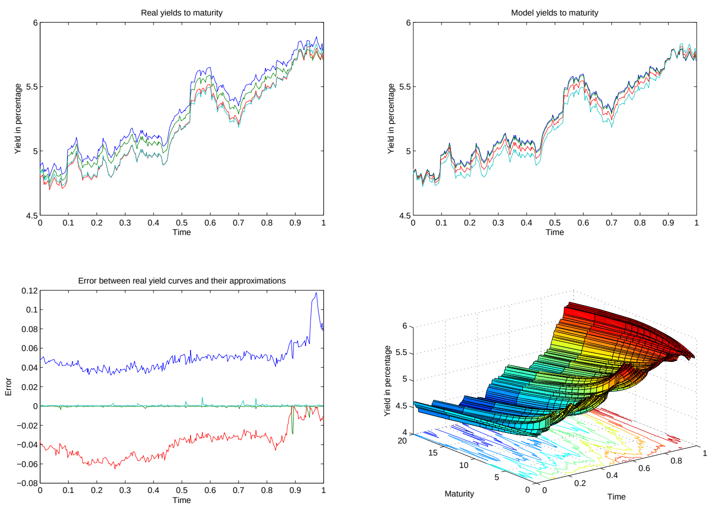

Figure B.13: 2 factor CIR Model calibrated with Kalman filtration at $t = 06/2007$

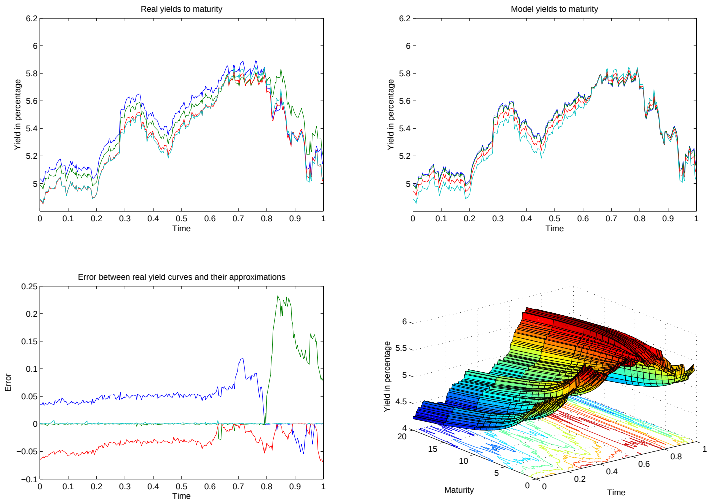

Figure B.14: 2 factor CIR Model calibrated with Kalman filtration at $t = 09/2007$

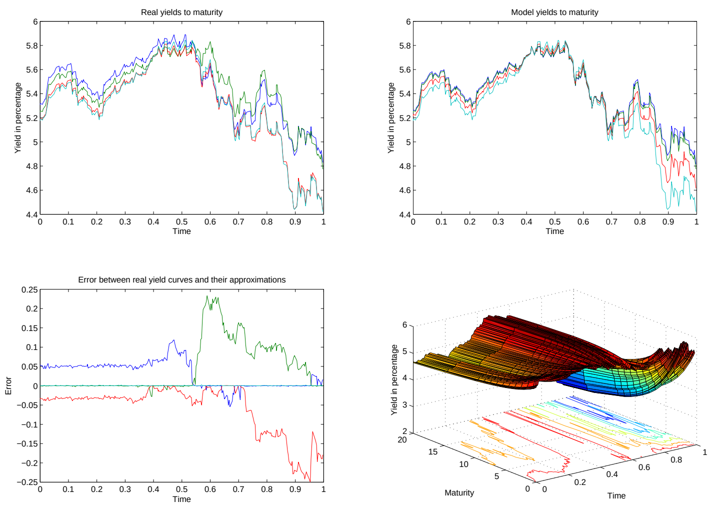

Figure B.15: 2 factor CIR Model calibrated with Kalman filtration at $t = 12/2007$

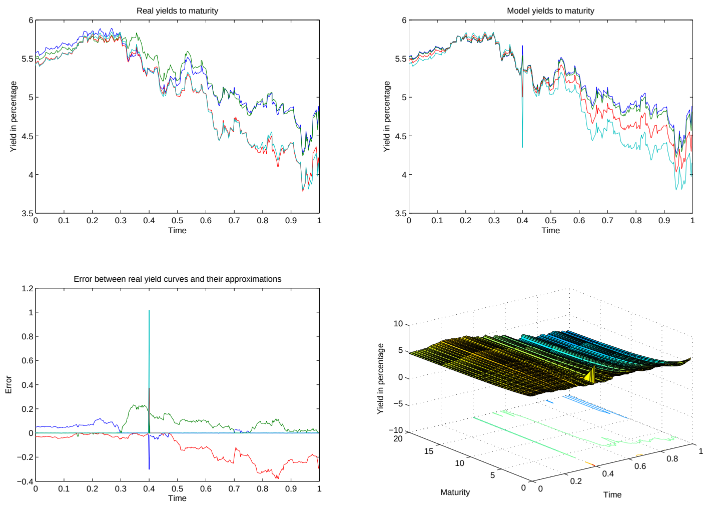

Figure B.16: 2 factor CIR Model calibrated with Kalman filtration at $t = 03/2008$

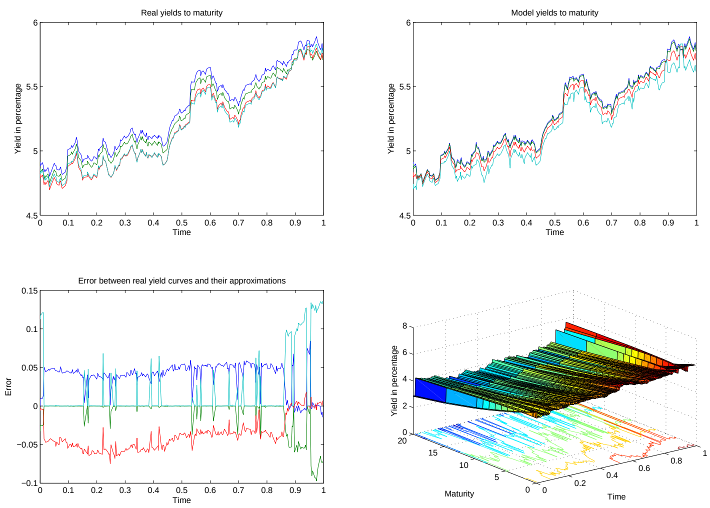

Figure B.17: DNS model calibrated with Kalman filtration at $t = 06/2007$

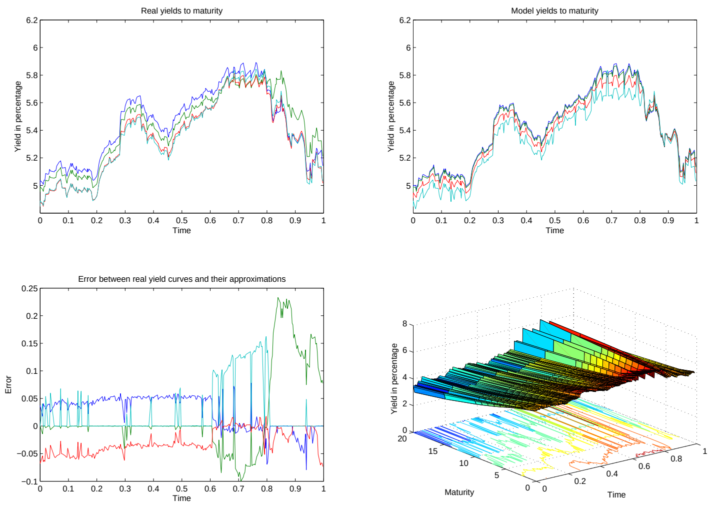

Figure B.18: DNS model calibrated with Kalman filtration at $t = 09/2007$

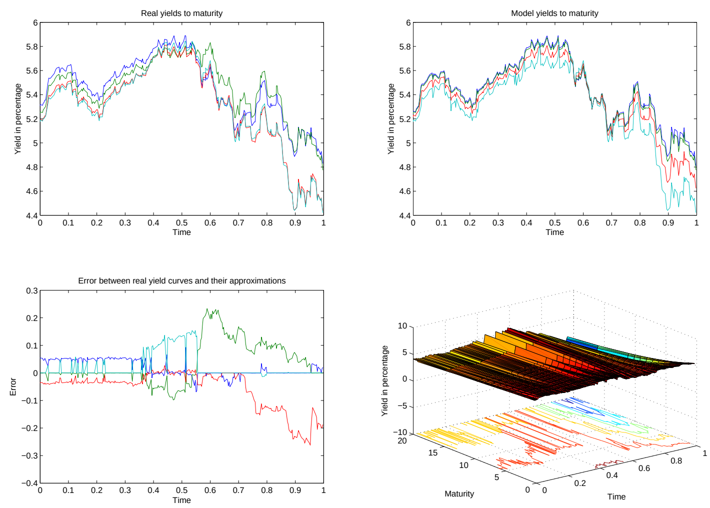

Figure B.19: DNS model calibrated with Kalman filtration at $t = 12/2007$

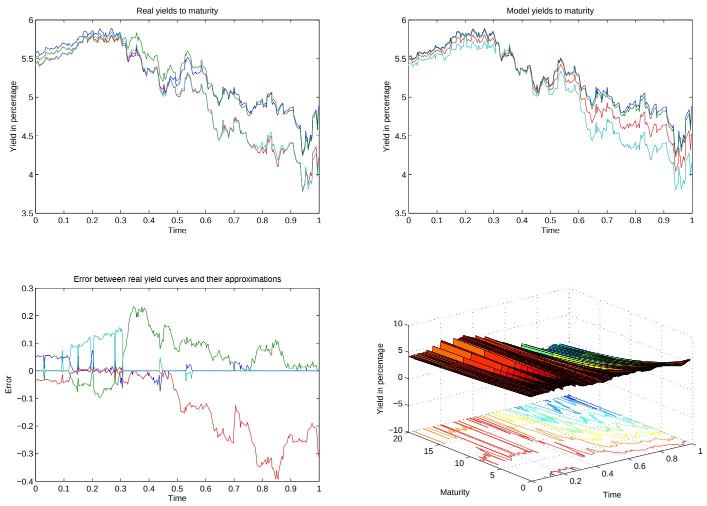

Figure B.20: DNS model calibrated with Kalman filtration at $t = 03/2008$

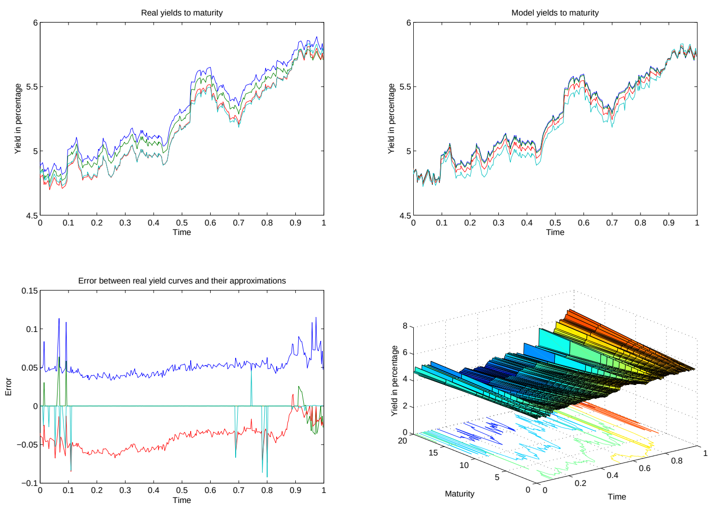

Figure B.21: AFDNSi model calibrated with Kalman filtration at $t = 06/2007$

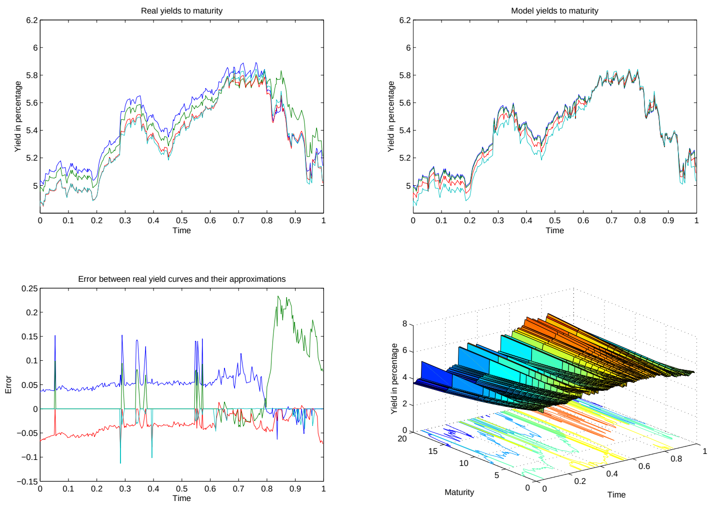

Figure B.22: AFDNSi model calibrated with Kalman filtration at $t = 09/2007$

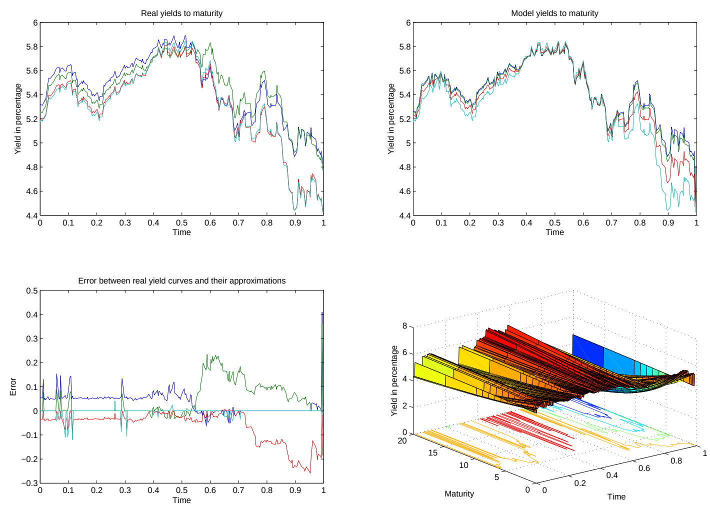

Figure B.23: AFDNSi model calibrated with Kalman filtration at $t = 12/2007$

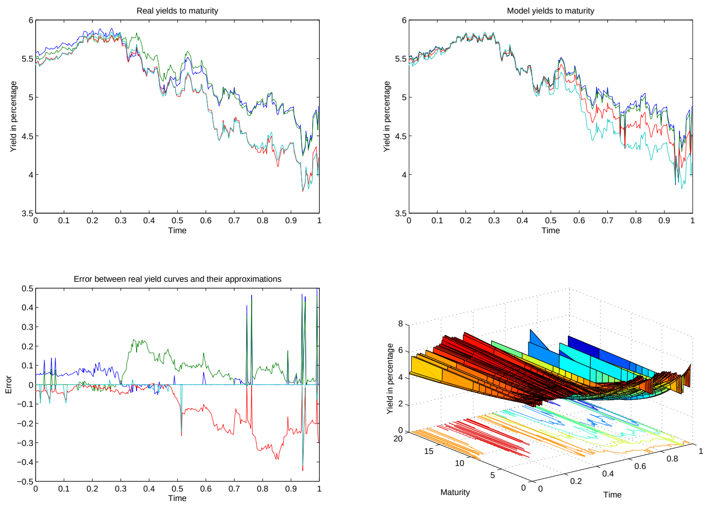

Figure B.24: AFDNSi model calibrated with Kalman filtration at $t = 03/2008$

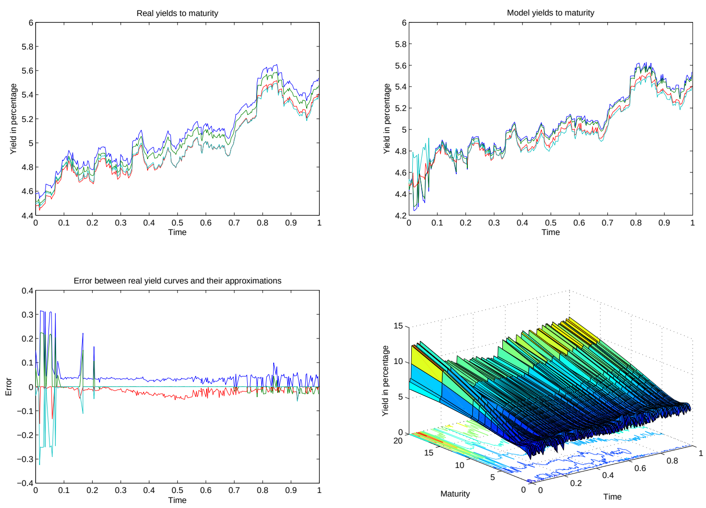

Figure B.25: AFDNSc model calibrated with Kalman filtration at $t = 06/2007$

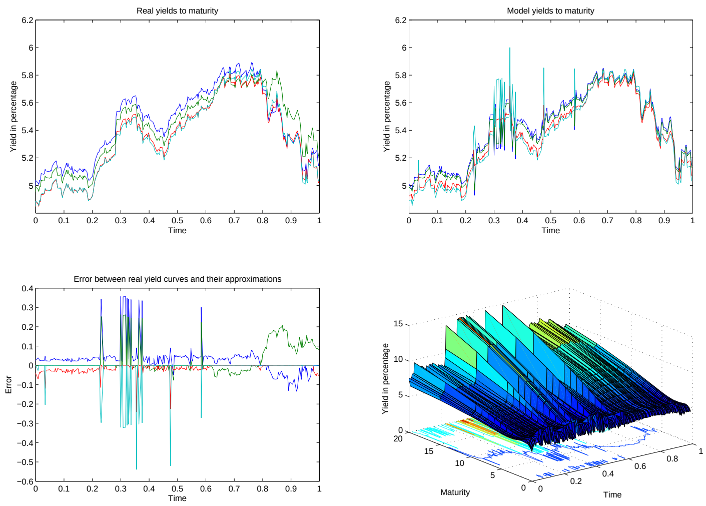

Figure B.26: AFDNSc model calibrated with Kalman filtration at $t = 09/2007$

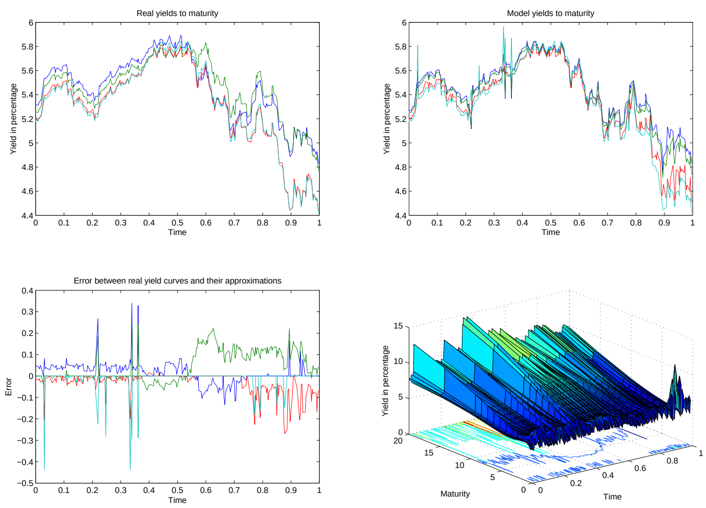

Figure B.27: AFDNSc model calibrated with Kalman filtration at $t = 12/2007$

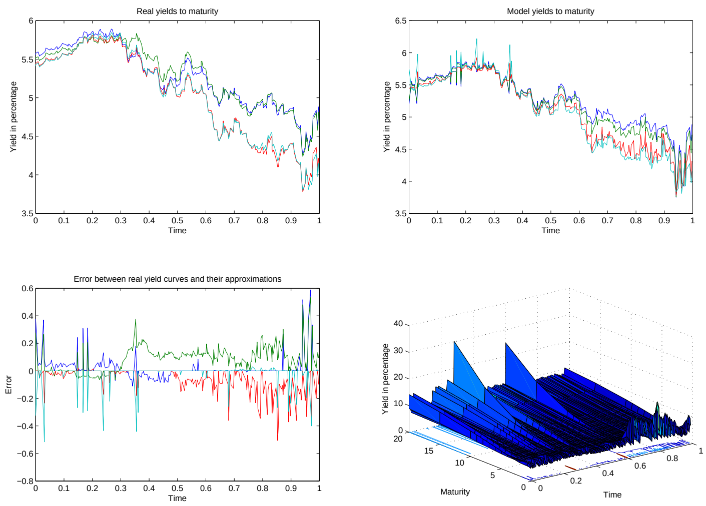

Figure B.28: AFDNSc model calibrated with Kalman filtration at $t = 03/2008$

Appendix C

Detailed numerical results from optimisation of subproblems in chapter 6

The following tables correspond to the numerical results obtained from the optimizations of the short, medium and long subproblems from chapter 6. The results were obtained using the single factor Vasicek model and a recombining trinomial tree with $Q = 4$. For the short subproblem, the maximum amount of a specific bond issued is going to be 22 billions and for the medium and long subproblem 25 and 36 billions respectively. The results were obtained using AMPL, Gurobi 5.0.1 on an AMD Phenom X6 1055T processor with 4GB of RAM.

Remark. An empty row indicates that the problem was unsolvable for a particular value of ρ.

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
11000.000	12988.614	13048.849	10899.515	315.326	11766.062
10900.000	12988.500	13043.616	10899.515	315.223	11828.576
10800.000	12988.248	13033.480	10704.489	314.991	11766.108
10700.000	12987.103	12998.059	10019.506	313.912	11766.266
10600.000	12987.103	12998.059	10019.506	313.912	11766.266
10500.000	12987.103	12998.059	10019.506	313.912	11766.266
10400.000	12987.666	13009.043	10234.307	314.450	11766.185
10300.000	12987.180	12997.969	10018.648	313.986	11766.254
10200.000	12987.103	12998.059	10019.506	313.912	11766.266

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Table C.1 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
10100.000	12987.103	12998.059	10019.506	313.912	11766.266
10000.000	12986.959	12997.114	10000.000	313.775	11766.286
9900.000	12986.716	12987.600	9816.490	313.539	11828.822
9800.000	12986.796	12984.226	9753.293	313.618	11766.310
9700.000	12985.820	12972.326	9515.743	312.658	11766.461
9600.000	12985.015	12954.260	9162.550	311.850	11829.092
9500.000	12984.440	12955.105	9170.577	311.244	11766.697
9400.000	12985.227	12953.975	9159.844	312.066	11766.556
9300.000	12984.315	12955.333	9172.740	311.098	11766.726
9200.000	12985.213	12953.996	9160.040	312.051	11766.559
9100.000	12985.249	12950.809	9100.000	312.089	11766.552
9000.000	12984.026	12946.374	8998.913	310.807	11829.273
8900.000	12983.908	12937.579	8831.188	310.715	11829.282
8800.000	12983.196	12931.252	8701.299	309.973	11829.409
8700.000	12983.773	12930.548	8695.876	310.579	11829.304
8600.000	12982.638	12919.406	8468.508	309.387	11829.510
8500.000	12981.961	12921.600	8500.000	308.650	11829.644
8400.000	12981.545	12913.620	8342.239	308.200	11695.914
8300.000	12980.915	12911.721	8295.653	307.481	11829.863
8200.000	12980.553	12906.736	8196.599	307.122	11696.110
8100.000	12979.892	12899.983	8058.023	306.392	11696.245
8000.000	12979.582	12893.368	7927.675	306.054	11830.117
7900.000	12979.576	12891.791	7897.545	306.045	11767.619
7800.000	12978.814	12883.733	7732.312	305.194	11830.279
7700.000	12978.043	12881.433	7675.209	304.297	11767.955
7600.000	12977.079	12877.946	7592.260	303.176	11830.675
7500.000	12977.055	12873.070	7500.000	303.174	11830.670
7400.000	12975.424	12868.320	7380.496	301.246	11697.244
7300.000	12974.709	12864.736	7300.000	300.415	11831.218
7200.000	12974.823	12859.350	7200.000	300.559	11831.187
7100.000	12973.588	12852.502	7047.083	299.079	11831.487
7000.000	12973.017	12850.594	7000.000	298.385	11769.130
6900.000	12971.856	12845.249	6875.914	296.959	11831.961
6800.000	12969.394	12843.970	6800.000	293.810	11897.516

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Table C.1 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
6700.000	12969.551	12838.540	6700.000	294.007	11897.475
6600.000	12968.631	12834.216	6600.000	292.877	11832.834
6500.000	12967.899	12828.862	6483.077	291.946	11699.188
6400.000	12967.491	12824.943	6400.000	291.422	11833.112
6300.000	12966.199	12821.193	6300.000	289.717	11833.490
6200.000	12965.341	12816.954	6200.000	288.573	11833.746
6100.000	12963.845	12813.568	6100.000	286.522	11834.251
6000.000	12963.132	12809.160	6000.000	285.569	11899.340
5900.000	12960.692	12810.039	5900.000	280.352	11900.876
5800.000	12961.158	12801.209	5800.000	282.803	11772.573
5700.000					
5600.000	12957.294	12798.778	5600.000	275.533	11703.286
5500.000	12956.844	12794.596	5500.000	274.602	11902.278
5400.000	12955.144	12790.848	5400.000	272.648	11902.657
5300.000	12953.407	12787.306	5300.000	270.546	11903.087
5200.000	12951.668	12791.888	5200.000	263.503	11706.827
5100.000	12950.865	12785.908	5100.000	263.452	11905.369
5000.000	12950.594	12781.922	5000.000	262.510	11905.689
4900.000	12949.584	12777.218	4899.042	261.525	11977.152
4800.000	12947.479	12777.676	4800.000	256.799	11907.259
4700.000	12946.836	12774.054	4700.000	255.410	11978.980
4600.000	12944.852	12768.531	4600.000	254.362	11907.604
4500.000	12942.508	12772.187	4500.000	247.515	11981.144
4400.000	12942.873	12760.648	4381.123	250.948	12042.324
4300.000	12942.667	12765.330	4300.000	245.381	11910.771
4200.000	12939.534	12770.171	4200.000	237.334	12047.241

Table C.1: Results for short subproblem with ρ constraint

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
44800.000	21756.727	21279.342	44800.000	1723.574	14665.381
44700.000	21755.750	21274.909	44700.000	1722.475	14665.588

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Table C.2 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
44600.000	21753.859	21265.694	44496.320	1720.411	14805.973
44500.000	21754.799	21265.122	44500.000	1721.447	14665.773
44400.000	21753.826	21260.650	44400.000	1720.375	14665.971
44300.000	21752.200	21251.717	44204.670	1718.567	14666.308
44200.000	21751.950	21243.112	44037.430	1718.296	14806.365
44100.000	21751.827	21246.505	44100.000	1718.161	14806.390
44000.000					
43900.000	21750.920	21236.730	43900.000	1717.152	14666.570
43800.000	21748.456	21232.527	43779.969	1714.369	14667.099
43700.000					
43600.000	21748.894	21222.104	43589.771	1714.8860	14666.995
43500.000	21747.071	21218.187	43486.284	1712.848	14667.378
43400.000	21747.000	21213.708	43400.000	1712.767	14667.393
43300.000					
43200.000	21744.648	21203.448	43167.066	1710.122	14807.902
43100.000					
43000.000	21743.703	21193.099	42954.785	1709.046	14668.099
42900.000					
42800.000	21742.120	21184.311	42762.173	1707.263	14668.437
42700.000	21741.848	21181.270	42700.000	1706.958	14668.495
42600.000	21739.264	21173.528	42510.090	1704.017	14809.066
42500.000	21738.229	21173.663	42495.352	1702.834	14809.294
42400.000	21738.847	21168.100	42400.000	1703.542	14809.157
42300.000	21737.697	21163.845	42300.000	1702.229	14669.401
42200.000	21735.705	21159.604	42185.909	1699.946	14669.842
42100.000	21736.321	21154.530	42100.000	1700.657	14809.712
42000.000					
41900.000	21734.091	21145.991	41900.000	1698.093	14670.201
41800.000	21731.566	21141.689	41775.072	1695.176	14810.777
41700.000	21732.867	21136.555	41700.000	1696.685	14810.482
41600.000					
41500.000					
41400.000	21729.772	21121.180	41354.754	1693.105	14671.173
41300.000	21727.553	21119.258	41278.017	1690.469	14811.710

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Table C.2 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
41200.000	21725.571	21116.915	41200.000	1688.192	14672.143
41100.000	21728.243	21107.989	41077.599	1691.327	14671.521
41000.000					
40900.000	21721.793	21104.066	40880.173	1683.474	14813.147
40800.000	21723.619	21095.508	40758.888	1685.905	14812.603
40700.000					
40600.000	21723.098	21086.553	40579.748	1685.301	14672.713
40500.000	21722.063	21081.000	40456.069	1684.090	14672.952
40400.000	21721.828	21078.267	40400.000	1683.815	14673.007
40300.000	21720.186	21074.555	40300.000	1681.875	14813.403
40200.000	21718.183	21070.608	40188.628	1679.492	14673.873
40100.000	21718.376	21065.394	40093.140	1679.725	14813.834
40000.000	21718.407	21060.450	40000.000	1679.769	14673.816
39900.000					
39800.000					
39700.000					
39600.000	21714.966	21042.715	39600.000	1675.660	14674.645

Table C.2: Results for medium subproblem with ρ constraint

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
416800.000	94822.590	97082.339	416800.000	11961.758	47899.448
416700.000	94822.590	97077.076	416700.000	11961.758	47899.448
416600.000	94817.561	97074.325	416600.000	11957.174	47900.076
416500.000	94818.396	97068.642	416500.000	11957.937	47899.971
416400.000	94819.148	97063.002	416400.000	11958.622	47899.877
416300.000	94817.561	97058.535	416300.000	11957.174	47900.076
416200.000	94817.561	97053.272	416200.000	11957.174	47900.076
416100.000	94819.896	97046.839	416100.000	11959.304	47899.783
416000.000	94816.555	97043.252	416000.000	11956.254	47900.203
415900.000	94819.896	97036.313	415900.000	11959.304	47899.783
415800.000	94819.896	97031.049	415800.000	11959.304	47899.783

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Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
415700.000	94819.896	97025.786	415700.000	11959.304	47899.783
415600.000	94816.555	97022.200	415600.000	11956.254	47900.203
415500.000	94816.555	97016.937	415500.000	11956.254	47900.203
415400.000	94816.315	97011.794	415400.000	11956.035	47900.233
415300.000	94810.022	97009.733	415300.000	11950.263	47901.033
415200.000	94816.555	97001.147	415200.000	11956.254	47900.203
415100.000	94816.555	96995.884	415100.000	11956.254	47900.203
415000.000	94815.579	96991.113	415000.000	11955.362	47900.326
414900.000	94816.555	96985.358	414900.000	11956.254	47900.203
414800.000	94820.143	96978.294	414800.000	11959.530	47899.753
414700.000	94816.555	96974.831	414700.000	11956.254	47900.203
414600.000	94818.448	96968.616	414600.000	11957.984	47899.965
414500.000	94816.555	96964.305	414500.000	11956.254	47900.203
414400.000	94816.555	96959.042	414400.000	11956.254	47900.203
414300.000	94816.555	96953.779	414300.000	11956.254	47900.203
414200.000	94819.322	96947.126	414200.000	11958.781	47899.855
414100.000	94816.555	96943.252	414100.000	11956.254	47900.203
414000.000	94815.579	96938.482	414000.000	11955.362	47900.326
413900.000	94815.579	96933.218	413900.000	11955.362	47900.326
413800.000	94816.555	96927.463	413800.000	11956.254	47900.203
413700.000	94816.555	96922.200	413700.000	11956.254	47900.203
413600.000	94816.555	96916.937	413600.000	11956.254	47900.203
413500.000	94816.949	96911.475	413500.000	11956.615	47900.153
413400.000	94816.555	96906.410	413400.000	11956.254	47900.203
413300.000	94816.555	96901.147	413300.000	11956.254	47900.203
413200.000	94816.555	96895.884	413200.000	11956.254	47900.203
413100.000	94816.555	96890.621	413100.000	11956.254	47900.203
413000.000	94815.280	96886.001	413000.000	11955.088	47900.363
412900.000	94816.555	96880.095	412900.000	11956.254	47900.203
412800.000	94816.555	96874.831	412800.000	11956.254	47900.203
412700.000	94812.586	96871.580	412700.000	11952.619	47900.705
412600.000	94814.221	96865.486	412600.000	11954.118	47900.498
412500.000	94815.041	96859.806	412500.000	11954.870	47900.394
412400.000	94813.795	96855.175	412400.000	11953.728	47900.552

Continued on Next Page. . .

Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
412300.000	94814.221	96849.696	412300.000	11954.118	47900.498
412200.000	94816.555	96843.252	412200.000	11956.254	47900.203
412100.000	94816.555	96837.989	412100.000	11956.254	47900.203
412000.000	94816.007	96833.002	412000.000	11955.754	47900.272
411900.000	94812.293	96829.624	411900.000	11952.350	47900.743
411800.000	94810.587	96825.233	411800.000	11950.783	47900.961
411700.000	94815.041	96817.701	411700.000	11954.870	47900.394
411600.000	94815.041	96812.438	411600.000	11954.870	47900.394
411500.000	94815.041	96807.175	411500.000	11954.870	47900.394
411400.000	94814.703	96802.083	411400.000	11954.560	47900.436
411300.000	94815.041	96796.648	411300.000	11954.870	47900.394
411200.000	94812.248	96792.805	411200.000	11952.308	47900.749
411100.000	94814.420	96786.437	411100.000	11954.301	47900.472
411000.000	94809.725	96783.570	411000.000	11949.990	47901.071
410900.000	94815.011	96775.611	410900.000	11954.842	47900.398
410800.000	94814.757	96770.477	410800.000	11954.610	47900.430
410700.000	94810.931	96767.162	410700.000	11951.099	47900.917
410600.000	94813.915	96760.377	410600.000	11953.838	47900.536
410500.000	94809.118	96757.566	410500.000	11949.431	47901.149
410400.000	94811.743	96750.957	410400.000	11951.845	47900.813
410300.000	94813.925	96744.583	410300.000	11953.847	47900.535
410200.000	94811.743	96740.431	410200.000	11951.845	47900.813
410100.000	94813.964	96734.037	410100.000	11953.882	47900.530
410000.000	94811.743	96729.905	410000.000	11951.845	47900.813
409900.000	94811.743	96724.642	409900.000	11951.845	47900.813
409800.000	94806.738	96721.953	409800.000	11947.238	47901.457
409700.000	94811.363	96714.310	409700.000	11951.496	47900.862
409600.000					
409500.000	94812.448	96703.229	409500.000	11952.492	47900.723
409400.000	94811.743	96698.326	409400.000	11951.845	47900.813
409300.000	94811.743	96693.063	409300.000	11951.845	47900.813
409200.000	94811.743	96687.799	409200.000	11951.845	47900.813
409100.000	94811.743	96682.536	409100.000	11951.845	47900.813
409000.000	94811.743	96677.273	409000.000	11951.845	47900.813

Continued on Next Page. . .

Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
408900.000	94811.743	96672.010	408900.000	11951.845	47900.813
408800.000	94811.743	96666.747	408800.000	11951.845	47900.813
408700.000	94812.065	96661.319	408700.000	11952.140	47900.772
408600.000	94811.804	96656.189	408600.000	11951.901	47900.805
408500.000	94806.738	96653.531	408500.000	11947.238	47901.457
408400.000	94811.743	96645.694	408400.000	11951.845	47900.813
408300.000	94806.738	96643.005	408300.000	11947.238	47901.457
408200.000					
408100.000	94811.433	96630.063	408100.000	11951.560	47900.853
408000.000					
407900.000					
407800.000	94810.964	96614.514	407800.000	11951.129	47900.913
407700.000	94808.711	96610.407	407700.000	11949.056	47901.202
407600.000	94809.372	96604.804	407600.000	11949.665	47901.117
407500.000					
407400.000	94808.794	96594.575	407400.000	11949.133	47901.191
407300.000	94809.680	96588.856	407300.000	11949.948	47901.077
407200.000	94806.738	96585.110	407200.000	11947.238	47901.457
407100.000	94807.448	96579.480	407100.000	11947.892	47901.365
407000.000	94806.738	96574.584	407000.000	11947.238	47901.457
406900.000	94807.017	96569.177	406900.000	11947.495	47901.420
406800.000	94809.112	96562.832	406800.000	11949.426	47901.150
406700.000	94809.125	96557.562	406700.000	11949.438	47901.148
406600.000	94808.184	96552.784	406600.000	11948.571	47901.270
406500.000	94806.738	96548.268	406500.000	11947.238	47901.457
406400.000	94797.415	96547.900	406400.000	11938.595	47902.680
406300.000	94808.824	96536.665	406300.000	11949.160	47901.187
406200.000	94807.661	96532.002	406200.000	11948.089	47901.337
406100.000	94808.794	96526.154	406100.000	11949.133	47901.191
406000.000	94804.430	96523.152	406000.000	11945.105	47901.756
405900.000	94806.738	96516.689	405900.000	11947.238	47901.457
405800.000					
405700.000					
405600.000					

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Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
405500.000	94806.738	96495.637	405500.000	11947.238	47901.457
405400.000					
405300.000					
405200.000	94805.590	96480.443	405200.000	11946.177	47901.605
405100.000	94806.240	96474.842	405100.000	11946.778	47901.521
405000.000	94802.379	96471.593	405000.000	11943.206	47902.025
404900.000	94806.738	96464.058	404900.000	11947.238	47901.457
404800.000	94806.738	96458.795	404800.000	11947.238	47901.457
404700.000	94806.738	96453.531	404700.000	11947.238	47901.457
404600.000	94806.430	96448.428	404600.000	11946.954	47901.496
404500.000					
404400.000	94804.297	96439.011	404400.000	11944.982	47901.774
404300.000	94801.440	96435.244	404300.000	11942.336	47902.148
404200.000	94806.462	96427.359	404200.000	11946.983	47901.492
404100.000	94806.165	96422.250	404100.000	11946.708	47901.531
404000.000	94806.008	96417.068	404000.000	11946.564	47901.551
403900.000	94805.881	96411.870	403900.000	11946.447	47901.568
403800.000	94805.552	96406.779	403800.000	11946.142	47901.610
403700.000	94803.996	96402.326	403700.000	11944.704	47901.813
403600.000	94804.967	96396.556	403600.000	11945.602	47901.686
403500.000	94795.464	96396.310	403500.000	11936.776	47902.941
403400.000	94802.398	96387.372	403400.000	11943.224	47902.022
403300.000	94803.527	96381.518	403300.000	11944.270	47901.874
403200.000	94802.535	96376.775	403200.000	11943.351	47902.004
403100.000	94802.898	96371.321	403100.000	11943.688	47901.956
403000.000	94803.996	96365.484	403000.000	11944.704	47901.813
402900.000	94800.459	96362.076	402900.000	11941.426	47902.277
402800.000					
402700.000	94804.180	96349.598	402700.000	11944.874	47901.789
402600.000	94804.084	96344.385	402600.000	11944.785	47901.801
402500.000	94801.648	96340.398	402500.000	11942.529	47902.120
402400.000	94803.996	96333.905	402400.000	11944.704	47901.813
402300.000	94802.636	96329.353	402300.000	11943.444	47901.991
402200.000	94791.388	96330.085	402200.000	11932.962	47903.490

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Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
402100.000					
402000.000					
401900.000	94801.886	96308.694	401900.0000	11942.749	47902.089
401800.000	94803.249	96302.716	401800.000	11944.013	47901.911
401700.000	94800.847	96298.714	401700.000	11941.785	47902.226
401600.000	94802.535	96292.564	401600.000	11943.351	47902.004
401500.000	94799.301	96289.004	401500.000	11940.350	47902.430
401400.000	94798.083	96284.387	401400.000	11939.216	47902.591
401300.000					
401200.000	94802.435	96271.564	401200.000	11943.259	47902.017
401100.000	94801.557	96266.762	401100.000	11942.444	47902.132
401000.000	94800.173	96262.227	401000.000	11941.160	47902.315
400900.000					
400800.000	94801.639	96250.929	400800.000	11942.520	47902.122
400700.000					
400600.000	94799.120	96241.732	400600.000	11940.181	47902.454
400500.000	94801.776	96235.067	400500.000	11942.648	47902.104
400400.000					
400300.000					
400200.000	94800.459	96219.971	400200.000	11941.426	47902.277
400100.000					
400000.000	94800.459	96209.445	400000.000	11941.426	47902.277
399900.000	94800.895	96203.952	399900.000	11941.830	47902.220
399800.000	94800.459	96198.919	399800.000	11941.426	47902.277
399700.000					
399600.000	94799.447	96188.927	399600.000	11940.485	47902.411
399500.000	94800.409	96183.156	399500.000	11941.379	47902.284
399400.000	94797.853	96179.246	399400.000	11939.002	47902.622
399300.000	94799.711	96172.998	399300.000	11940.731	47902.376
399200.000	94799.455	96167.870	399200.000	11940.493	47902.410
399100.000	94795.671	96164.620	399100.000	11936.968	47902.913
399000.000	94799.447	96157.348	399000.000	11940.485	47902.411
398900.000	94796.566	96153.615	398900.000	11937.804	47902.793
398800.000	94796.098	96148.602	398800.000	11937.367	47902.856

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Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
398700.000	94799.537	96141.511	398700.000	11940.569	47902.399
398600.000	94797.411	96137.376	398600.000	11938.590	47902.681
398500.000	94798.170	96131.709	398500.000	11939.297	47902.580
398400.000	94798.927	96126.044	398400.000	11940.002	47902.479
398300.000	94798.220	96121.156	398300.000	11939.344	47902.573
398200.000	94794.634	96117.807	398200.000	11936.000	47903.052
398100.000	94797.819	96110.843	398100.000	11938.971	47902.627
398000.000					
397900.000	94794.978	96101.833	397900.000	11936.322	47903.006
397800.000	94798.411	96094.739	397800.000	11939.521	47902.548
397700.000	94797.928	96089.732	397700.000	11939.073	47902.612
397600.000	94796.714	96085.115	397600.000	11937.942	47902.774
397500.000	94797.837	96079.254	397500.000	11938.987	47902.624
397400.000	94797.947	96073.933	397400.000	11939.090	47902.609
397300.000	94795.842	96069.792	397300.000	11937.128	47902.890
397200.000	94796.515	96064.169	397200.000	11937.756	47902.800
397100.000	94797.488	96058.387	397100.000	11938.662	47902.671
397000.000	94794.830	96054.544	397000.000	11936.183	47903.026
396900.000	94796.732	96048.264	396900.000	11937.958	47902.771
396800.000	94794.805	96044.032	396800.000	11936.160	47903.029
396700.000	94793.656	96039.386	396700.000	11935.087	47903.183
396600.000	94794.003	96033.936	396600.000	11935.411	47903.137
396500.000					
396400.000					
396300.000	94795.839	96017.162	396300.000	11937.125	47902.890
396200.000					
396100.000	94793.340	96007.977	396100.000	11934.790	47903.226
396000.000	94793.195	96002.792	396000.000	11934.655	47903.245
395900.000	94787.951	96000.379	395900.000	11929.734	47903.958
395800.000	94795.011	95991.290	395800.000	11936.352	47903.001
395700.000	94795.305	95985.869	395700.000	11936.627	47902.962
395600.000	94793.107	95981.787	395600.000	11934.573	47903.257
395500.000	94793.296	95976.422	395500.000	11934.749	47903.232
395400.000					

Continued on Next Page…

Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
395300.000	94795.253	95964.844	395300.000	11936.578	47902.969
395200.000	94794.521	95959.973	395200.000	11935.895	47903.067
395100.000	94794.131	95954.920	395100.000	11935.531	47903.119
395000.000	94793.741	95949.866	395000.000	11935.166	47903.172
394900.000	94791.348	95945.896	394900.000	11932.925	47903.495
394800.000	94791.107	95940.764	394800.000	11932.699	47903.528
394700.000	94794.121	95933.872	394700.000	11935.521	47903.121
394600.000					
394500.000	94791.893	95924.548	394500.000	11933.436	47903.421
394400.000	94791.212	95919.654	394400.000	11932.798	47903.513
394300.000	94793.562	95913.121	394300.000	11934.998	47903.196
394200.000	94793.440	95907.923	394200.000	11934.884	47903.212
394100.000	94792.199	95903.330	394100.000	11933.723	47903.380
394000.000	94792.048	95898.148	394000.000	11933.581	47903.400
393900.000	94792.982	95892.381	393900.000	11934.455	47903.274
393800.000	94793.358	95886.915	393800.000	11934.808	47903.223
393700.000	94793.376	95881.642	393700.000	11934.824	47903.221
393600.000	94791.472	95877.408	393600.000	11933.042	47903.478
393500.000	94790.174	95872.849	393500.000	11931.824	47903.654
393400.000	94789.778	95867.802	393400.000	11931.452	47903.708
393300.000	94792.778	95860.912	393300.000	11934.265	47903.302
393200.000	94790.598	95856.830	393200.000	11932.221	47903.597
393100.000	94791.649	95850.996	393100.000	11933.207	47903.454
393000.000	94788.565	95847.412	393000.000	11930.312	47903.874
392900.000	94791.278	95840.671	392900.000	11932.859	47903.505
392800.000	94788.947	95836.677	392800.000	11930.671	47903.822
392700.000	94790.525	95830.553	392700.000	11932.153	47903.607
392600.000					
392500.000					
392400.000	94791.391	95814.294	392400.000	11932.965	47903.489
392300.000	94789.778	95809.907	392300.000	11931.452	47903.708
392200.000	94787.912	95805.664	392200.000	11929.697	47903.963
392100.000	94790.337	95799.076	392100.000	11931.977	47903.632
392000.000	94785.442	95796.497	392000.000	11927.370	47904.303

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Table C.3 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
391900.000	94790.800	95788.299	391900.000	11932.411	47903.569
391800.000					
391700.000	94789.737	95778.351	391700.000	11931.413	47903.714
391600.000					
391500.000	94786.632	95769.525	391500.000	11928.492	47904.139
391400.000	94789.155	95762.879	391400.000	11930.867	47903.793
391300.000	94789.209	95757.586	391300.000	11930.917	47903.786
391200.000	94788.146	95752.904	391200.000	11929.918	47903.931
391100.000	94784.656	95749.564	391100.000	11926.627	47904.412
391000.000					
390900.000					
390800.000	94787.445	95732.236	390800.000	11929.258	47904.027
390700.000					
390600.000	94788.902	95720.912	390600.000	11930.629	47903.828
390500.000	94777.367	95722.072	390500.000	11919.712	47905.434
390400.000	94789.111	95710.271	390400.000	11930.825	47903.799
390300.000	94789.248	95704.933	390300.000	11930.954	47903.781
390200.000					
390100.000	94785.576	95696.423	390100.000	11927.496	47904.285

Table C.3: Results for long subproblem with ρ constraint

To compare results, a set of short subproblems were evaluated with CPLEX 12.2.0.0 in the table (C.4). As the results are very similar to the Gurobi 5.0.1 solver and more solutions are found to be unfeasible, only the results from the Gurobi solver will be considered in this thesis.

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
11000.000	12987.739	13049.303	11000.000	314.518	11766.175
10900.000	12988.430	13043.677	10900.000	315.158	11766.085
10800.000	12988.048	13038.611	10800.000	314.807	11766.134
10700.000	12988.155	13033.292	10700.000	314.906	11766.120

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Table C.4 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
10600.000	12987.991	13028.115	10600.000	314.754	11766.142
10500.000	12987.905	13022.897	10500.000	314.674	11766.153
10400.000	12987.805	13017.688	10400.000	314.580	11766.166
10300.000	12987.580	13012.548	10300.000	314.369	11766.197
10200.000	12987.368	13007.405	10200.000	314.167	11766.227
10100.000	12986.401	13002.735	10100.000	313.218	11766.375
10000.000	12986.920	12997.137	10000.000	313.737	11766.292
9900.000	12986.914	12991.877	9900.000	313.732	11766.293
9800.000	12986.802	12986.680	9800.000	313.624	11766.309
9700.000	12985.210	12982.434	9700.000	312.038	11766.563
9600.000	12985.122	12977.236	9600.000	311.944	11766.580
9500.000	12985.752	12971.539	9500.000	312.591	11766.471
9400.000	12984.230	12967.311	9400.000	311.036	11766.730
9300.000	12985.779	12960.992	9300.000	312.620	11766.466
9200.000	12984.862	12956.339	9200.000	311.691	11766.619
9100.000	12983.777	12951.871	9100.000	310.549	11766.817
9000.000	12984.923	12945.763	9000.000	311.759	11766.606
8900.000	12983.292	12941.663	8900.000	310.061	11766.897
8800.000	12983.256	12936.407	8800.000	310.034	11766.899
8700.000	12983.607	12930.884	8700.000	310.405	11766.834
8600.000	12982.760	12926.239	8600.000	309.515	11766.988
8500.000	12982.069	12921.500	8500.000	308.776	11767.119
8400.000	12982.625	12915.808	8400.000	309.374	11767.012
8300.000	12981.569	12911.372	8300.000	308.230	11767.219
8200.000	12981.390	12906.227	8200.000	308.049	11695.938
8100.000	12980.850	12901.392	8100.000	307.461	11767.356
8000.000	12980.419	12896.477	8000.000	306.987	11767.443
7900.000	12979.001	12892.531	7900.000	305.324	11767.772
7800.000	12978.631	12887.495	7800.000	304.962	11767.829
7700.000	12978.405	12882.422	7700.000	304.708	11767.876
7600.000	12977.703	12877.787	7600.000	303.900	11768.033
7500.000	12977.120	12873.041	7500.000	303.231	11768.162
7400.000	12976.453	12868.347	7400.000	302.479	11768.305
7300.000	12975.262	12864.207	7300.000	301.073	11768.586

Continued on Next Page...

Table C.4 – Continued

ρ in MN	CaR in MN	VaR in MN	CVaR in MN	S.D in MN	$\mathbb{E}[I]$ in MN
7200.000	12975.019	12859.159	7200.000	300.794	11896.050
7100.000	12974.131	12854.752	7100.000	299.734	11896.264
7000.000	12973.021	12850.603	7000.000	298.382	11896.543
6900.000	12972.007	12846.359	6900.000	297.146	11896.798
6800.000	12971.377	12841.750	6800.000	296.365	11896.961
6700.000	12970.560	12837.336	6700.000	295.352	11897.174
6600.000	12969.232	12833.515	6600.000	293.668	11897.534
6500.000	12968.315	12829.304	6500.000	292.472	11897.797
6400.000	12967.733	12824.660	6400.000	291.741	11897.952
6300.000					
6200.000	12965.657	12816.567	6200.000	289.000	11969.871
6100.000					
6000.000					
5900.000					
5800.000					
5700.000	12960.119	12797.356	5700.000	281.315	11971.647

Table C.4: CPLEX results for short subproblem with ρ constraint

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Acknowledgements

I would like to thank my supervisor, Dr. Paresh Date, for which I have no words to express how grateful I am for all he has done for me. I would also like to thank my father, Dr. Mahmoud Abdel-Jawad, for his unconditional financial support throughout my PhD at Brunel University.