# Six-dimensional space-time from quaternionic quantum mechanics

Dorje C. Brody[1] and Eva-Maria Graefe[2]

[1]*Mathematical Sciences, Brunel University, Uxbridge UB8 3PH, United Kingdom*

[2]*Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom*

(Received 26 May 2011; revised manuscript received 18 August 2011; published 14 December 2011)

Quaternionic quantum Hamiltonians describing nonrelativistic spin particles require the ambient physical space to have five dimensions. The quantum dynamics of a spin-$\frac{1}{2}$ particle system characterized by a generic Hamiltonian is worked out in detail. It is shown that there exists, within the structure of quaternionic quantum mechanics, a canonical reduction to three spatial dimensions upon which standard quantum theory is retrieved. In this dimensional reduction, three of the five dynamical variables are shown to oscillate around a cylinder, thus behaving in a quasi-one-dimensional manner at large distances. An analogous mechanism is shown to exist in the case of octavic Hamiltonians, where the ambient physical space has nine dimensions. Possible experimental tests in search for the signature of extra dimensions at low energies are briefly discussed.

DOI: 10.1103/PhysRevD.84.125016 PACS numbers: 03.65.Ta, 03.65.Aa, 04.50.Cd, 11.10.Kk

In many models that attempt to reconcile quantum theory with gravity, the notion of extra dimensions is introduced. If we take seriously the hypothesis that these extra dimensions may be relevant to physical reality, then we should likewise take seriously the quantum theory underlying these models. Yet, surprisingly little attention has been paid to foundational investigations into measurable effects of higher-dimensional quantum mechanics. It is well known that extra dimensions can change spectral properties of particles, but the standard argument is that if the sizes of extra dimensions are sufficiently small, then low-energy spectra are typically unaltered, and indications of the existence of extra dimensions may be revealed only at inaccessibly large energies [1]. Quantized energy spectra of particles, however, are not the only quantum effect measured in laboratories. It appears that other quantum effects arising, e.g., from geometric phase, interference, or entanglement, that may be used to probe extra dimensions at low energies, have not been fully explored. Furthermore, the following issue concerning higher-dimensional quantum theory is often overlooked: The spin-orbit interaction in standard quantum mechanics naturally singles out four-dimensional space-time. There seems to be no structure, within the complex framework, that allows for higher-dimensional extensions.

Here we take the first step towards addressing these fundamental issues and exploring the possibility of detecting higher-dimensional quantum effects at low energies by investigating certain quaternionic extensions of quantum mechanics that naturally lead to six-dimensional space-time structures. Specifically, we analyze the dynamical aspects of a two-level system in quaternionic quantum mechanics. Two-level systems are of great importance in many physical applications, both as approximations in cases where only two states are of relevance to the dynamics, and in the description of the internal degrees of freedom for spin-$\frac{1}{2}$ particles. We show that there is an intrinsic mechanism for dimensional reduction such that observed phenomena in three spatial dimensions can be restored. Similarly, octavic quantum mechanics is shown to lead to nine spatial dimensions—a dimensionality often considered in string theory models—within which three-dimensional space is naturally embedded. By determining dynamical aspects of quaternionic and octavic quantum states of a spin particle, we point the way towards the possible detection of extra dimensions at low energies.

We begin by remarking that there are two fundamental ways in which the use of quaternions in physics is related to the notion of six-dimensional space-time. The first is the representation of space-time points in terms of quaternionic spinors: The points of four-dimensional Minkowski space correspond to two-by-two Hermitian matrices $\{x^{AA'}\}_{A,A'=1,2}$. Lorentz transformations are given by conjugating $x^{AA'}$ by elements of $\mathrm{SL}(2,\mathbb{C})$, and the Minkowski metric for the interval between two points is given by the determinant of their difference [2,3]. In a standard basis this correspondence reads

$$x^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \leftrightarrow (t,x,y,z), \qquad (1)$$

and we have the relation $2\det(x^{AA'}) = t^2 - x^2 - y^2 - z^2$. Similarly, points of six-dimensional Minkowski space correspond to two-by-two quaternionic Hermitian matrices of the form (1) with $i$ replaced by $\boldsymbol{i} = (iy_1 + jy_2 + ky_3)/y$, where $y^2 = y_1^2 + y_2^2 + y_3^2$. Here, $i$, $j$, and $k$ denote three imaginary units of a quaternion, satisfying $i^2 = j^2 = k^2 = ijk = -1$ and the cyclic relations $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. In this case we have the relation $2\det(x^{AA'}) = t^2 - x^2 - y_1^2 - y_2^2 - y_3^2 - z^2$. These two correspondences are related to the facts that the universal covering group of SO(3,1) is isomorphic to $\mathrm{SL}(2,\mathbb{C})$, and that of SO(5,1) is isomorphic to $\mathrm{SL}(2,\mathbb{H})$, where $\mathbb{H}$ denotes the field of quaternions [4].

 

Perhaps what is less appreciated is the second connection between quaternions and six dimensions arising in the context of quantum mechanics. Complex Hermitian matrices represent physical observables in conventional quantum mechanics. A trace-free two-by-two complex Hermitian matrix, for instance, represents the energy of a spin-$\frac{1}{2}$ particle. The spin-orbit interaction of elementary quantum mechanics then requires that the (Euclidean) space-time dimension is four. Mathematically, this can be seen from the fact that the state space $\mathbb{CP}^1 \simeq S^2$ of a spin-$\frac{1}{2}$ particle system, obtained by the identification $|\Psi\rangle \sim \lambda|\Psi\rangle$, $\lambda \in \mathbb{C} - \{0\}$, admits a natural embedding in $\mathbb{R}^3$, and this allows us to make the so-called Pauli correspondence whereby we can speak of "spin in such and such direction." The group isomorphism that underlies this identification is that between the universal covering group Spin(3) of SO(3) and the two-by-two complex unitary matrices $SU(2) \simeq Sp(1)$.

Similarly, we can regard a trace-free two-by-two quaternionic Hermitian matrix representing the energy of a spin-$\frac{1}{2}$ particle in quaternionic quantum mechanics. Then the spin-orbit interaction demands that the (Euclidean) space-time dimension is six [5] (see also [6]). Here the Pauli correspondence is characterized by the fact that the state space $\mathbb{HP}^1 \simeq S^4$ of a spin-$\frac{1}{2}$ particle system, obtained by the identification $|\Psi\rangle \sim |\Psi\rangle\lambda$, $\lambda \in \mathbb{H} - \{0\}$, admits a natural embedding in $\mathbb{R}^5$. Alternatively stated, there is an isomorphism between the universal covering group Spin(5) of SO(5) and the group of two-by-two quaternionic unitary matrices Sp(2). (A third connection between quaternions and six-dimensional cosmology has been noted by Dirac [7].) We thus see that, be it Euclidean or Lorenzian, complex Hermitian form naturally leads to the notion of four-dimensional space-time, and quaternionic Hermitian form naturally leads to the notion of six-dimensional space-time. Evidently, octavic Hermitian forms lead to dimensionality 10.

The quaternionic Schrödinger equation

$$|\dot{\Psi}\rangle = -\boldsymbol{i}\hat{H}|\Psi\rangle, \tag{2}$$

with $\hat{H}$ Hermitian and $\boldsymbol{i}$ skew-Hermitian unitary, generates a unitary time evolution if both $\hat{H}$ and $\boldsymbol{i}$ commute with $\hat{U}_t = \exp(-\boldsymbol{i}\hat{H}t)$. One standard approach is to regard $\boldsymbol{i}\hat{H}$ as a generic skew-Hermitian operator [8]. Another approach, which we shall follow here, is to impose a superselection rule that fixes $\boldsymbol{i}$ and restrict $\hat{H}$ to the ones that commute with $\boldsymbol{i}$ [9]. The condition $[\boldsymbol{i}, \hat{H}] = 0$ thus implies that the specification of the Hamiltonian *a fortiori* determines the superselection rule dynamically.

For a two-level system, a generic quaternionic Hermitian Hamiltonian can be expressed in the form

$$\hat{H} = u_0 \mathbb{1} + \sum_{l=1}^{5} u_l \hat{\sigma}_l, \tag{3}$$

where $\{u_l\}_{l=0\ldots5} \in \mathbb{R}$, and

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\hat{\sigma}_4 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_5 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \tag{4}$$

are the quaternionic Pauli matrices. This follows from the fact that elements of a quaternionic Hermitian matrix satisfy $H_{mn} = \bar{H}_{nm}$. Then the right eigenvalues $E_\pm$ of $\hat{H}$ in (3), determined by $\hat{H}|\phi_\pm\rangle = |\phi_\pm\rangle E_\pm$, are real. Having specified the Hamiltonian (3) we must select a unit imaginary quaternion such that the evolution operator $\hat{U}_t = \exp(-\boldsymbol{i}\hat{H}t)$ is unitary. This is given by

$$\boldsymbol{i} = (iu_2 + ju_4 + ku_5)/\nu, \tag{5}$$

where $\nu = \sqrt{u_2^2 + u_4^2 + u_5^2}$. Then the Schrödinger Eq. (2) can be expressed more explicitly in terms of the components $(\psi_1, \psi_2)$ of the state vector $|\Psi\rangle$ as follows:

$$\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \begin{pmatrix} -(u_0 + u_3)\boldsymbol{i}\psi_1 - u_1\boldsymbol{i}\psi_2 - \nu\psi_2 \\ -(u_0 - u_3)\boldsymbol{i}\psi_2 - u_1\boldsymbol{i}\psi_1 + \nu\psi_1 \end{pmatrix}. \tag{6}$$

We can think of the Hamiltonian (3) as representing the interaction of a "spin vector" $\vec{\sigma}$ with an external field $\vec{B} = (u_1, u_2, u_3, u_4, u_5)$ in five dimensions. The quaternionic Pauli matrices are related to the 10 generators of the rotation group SO(5), in a way similar to the relation between the three Pauli matrices and the group SO(3). The 10 skew-Hermitian generators $\hat{\Sigma}_{mn} = \frac{1}{2}[\hat{\sigma}_m, \hat{\sigma}_n]$ of the dynamics, each inducing a rotation that mixes $\hat{\sigma}_m$ and $\hat{\sigma}_n$, fulfil the algebraic relation $[\hat{\Sigma}_{mn}, \hat{\Sigma}_{m'n'}] = \delta_{mm'}\hat{\Sigma}_{nn'} + \delta_{nn'}\hat{\Sigma}_{mm'} - \delta_{mn'}\hat{\Sigma}_{nm'} - \delta_{nm'}\hat{\Sigma}_{mn'}$. The spin vector can be seen to fulfil formally the "superspin" algebra of Zhang [10]: $[\hat{\Sigma}_{lm}, \hat{\sigma}_n] = \delta_{mn}\hat{\sigma}_l - \delta_{ln}\hat{\sigma}_m$. The generator of the evolution operator is then expressed as

$$\begin{aligned} \boldsymbol{i}\hat{H} = {} & \nu\hat{\Sigma}_{31} + u_0(u_2\hat{\Sigma}_{54} + u_4\hat{\Sigma}_{25} + u_5\hat{\Sigma}_{42})/\nu \\ & + u_1(u_2\hat{\Sigma}_{23} + u_4\hat{\Sigma}_{43} + u_5\hat{\Sigma}_{53})/\nu \\ & + u_3(u_2\hat{\Sigma}_{12} + u_4\hat{\Sigma}_{14} + u_5\hat{\Sigma}_{15})/\nu. \end{aligned} \tag{7}$$

We see that while each of the 10 generators of pairwise-mixing rotations appear once, there are only 6 degrees of freedom. This follows from the Hermiticity condition imposed on $\hat{H}$. The time evolution thus gives rise to certain rotations in five-dimensional space.

To determine the dynamics we introduce a quaternionic Bloch vector $\vec{\sigma}$, whose components are given by

$$\sigma_l = \langle\Psi|\hat{\sigma}_l|\Psi\rangle/\langle\Psi|\Psi\rangle, \qquad l = 1, \ldots, 5. \tag{8}$$

Then for each component we work out the dynamics by making use of the Schrödinger Eq. (6). After rearrangements we deduce that

$$\begin{aligned}
\frac{1}{2}\dot{\sigma}_1 &= \nu\sigma_3 - u_3(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5)/\nu \\
\frac{1}{2}\dot{\sigma}_2 &= (u_2u_3\sigma_1 - u_1u_2\sigma_3 + u_0u_5\sigma_4 - u_0u_4\sigma_5)/\nu \\
\frac{1}{2}\dot{\sigma}_3 &= -\nu\sigma_1 + u_1(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5)/\nu \\
\frac{1}{2}\dot{\sigma}_4 &= (u_3u_4\sigma_1 - u_0u_5\sigma_2 - u_1u_4\sigma_3 + u_0u_2\sigma_5)/\nu \\
\frac{1}{2}\dot{\sigma}_5 &= (u_3u_5\sigma_1 + u_0u_4\sigma_2 - u_1u_5\sigma_3 - u_0u_2\sigma_4)/\nu.
\end{aligned} \tag{9}$$

These equations constitute the general quaternionic Bloch equations. The special case of (9) for which $u_1 = \cdots = u_5 = 0$, i.e., when $\hat{H} = u_0\mathbb{1}$, has previously been obtained by Wolff [11]. These evolution equations preserve the normalization condition:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 = 1, \tag{10}$$

which can be interpreted as the defining equation for the state space $S^4$.

As in any physical theory modeled on a higher-dimensional space-time, it is important to identify a dimensional reduction leading to a theory consistent with observed phenomena perceived in three spatial dimensions. In the present context, this amounts to finding a reduction of the dynamics on $S^4$ to the conventional Bloch sphere $S^2$. For this purpose, let us define the three spin variables according to

$$\sigma_x = \sigma_1, \quad \sigma_y = (u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5)/\nu, \quad \sigma_z = \sigma_3. \tag{11}$$

Then it follows from (9) that

$$\frac{1}{2}\dot{\sigma}_x = \nu\sigma_z - u_3\sigma_y \qquad \frac{1}{2}\dot{\sigma}_y = u_3\sigma_x - u_1\sigma_z$$
$$\frac{1}{2}\dot{\sigma}_z = u_1\sigma_y - \nu\sigma_x. \tag{12}$$

These equations are, indeed, the standard Bloch equations for a spin-$\frac{1}{2}$ particle immersed in a magnetic field with strength $\vec{B} = (u_1, \nu, u_3)$. The reduced spin dynamics is thus confined to the state space

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = r^2, \tag{13}$$

where $r \leq 1$ is time independent. The dynamical Eqs. (12) thus generate Rabi oscillations on the reduced state space $S^2$ about the axis $(u_1, \nu, u_3)$, with angular frequency $\omega$, where $\omega^2 = 4(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2)$.

To identify the structure characterizing the evolution of the "internal" dynamical variables of $\sigma_y$: $\sigma_2$, $\sigma_4$, and $\sigma_5$, let us subtract (13) from (10) to eliminate $\sigma_1$ and $\sigma_3$. Then we deduce that the motion lies on a cylinder in $\mathbb{R}^3$:

$$(u_2\sigma_4 - u_4\sigma_2)^2 + (u_4\sigma_5 - u_5\sigma_4)^2 + (u_5\sigma_2 - u_2\sigma_5)^2 = \nu^2c^2, \tag{14}$$

that is, $|(u_2, u_4, u_5) \times (\sigma_2, \sigma_4, \sigma_5)| = \nu c$, where $c^2 = 1 - r^2$ is the squared radius of the cylinder, whose axis points in the $y$ direction. In Fig. 1 we plot typical motions of the variables $\sigma_2$, $\sigma_4$, $\sigma_5$ on the cylinder.

The time evolution of these dynamical variables can also be represented in the form of Bloch equations if we transform to the auxiliary variables $\sigma_{y_1} = u_4\sigma_5 - u_5\sigma_4$, $\sigma_{y_2} = u_5\sigma_2 - u_2\sigma_5$, and $\sigma_{y_3} = u_2\sigma_4 - u_4\sigma_2$. Then we have $\dot{\sigma}_{y_1} = 2u_0(u_5\sigma_{y_2} - u_4\sigma_{y_3})/\nu$, $\dot{\sigma}_{y_2} = 2u_0(u_2\sigma_{y_3} - u_5\sigma_{y_1})/\nu$, and $\dot{\sigma}_{y_3} = 2u_0(u_4\sigma_{y_1} - u_2\sigma_{y_2})/\nu$. These variables are useful in understanding the dynamics in five dimensions: We let $\hat{\sigma}_{x,y,z}$ be the operators for $\sigma_{x,y,z}$, and $\hat{\sigma}_{y_1,y_2,y_3}$ be the operators for $\sigma_{y_1,y_2,y_3}$. Additionally, define a new set of rotation generators by $\hat{\Sigma}_x = \frac{1}{2}[\hat{\sigma}_y, \hat{\sigma}_z]$, $\hat{\Sigma}_y = \frac{1}{2}[\hat{\sigma}_z, \hat{\sigma}_x]$, $\hat{\Sigma}_z = \frac{1}{2}[\hat{\sigma}_x, \hat{\sigma}_y]$, $\hat{\Sigma}_{y_1} = \hat{\Sigma}_{54}$, $\hat{\Sigma}_{y_2} = \hat{\Sigma}_{25}$, and $\hat{\Sigma}_{y_3} = \hat{\Sigma}_{42}$. These operators fulfil a pair of closed algebraic relations $\frac{1}{2}[\hat{\Sigma}_a, \hat{\Sigma}_b] = -\epsilon_{abc}\hat{\Sigma}_c$ for $a, b, c$ ranging over $x, y, z$; and $\frac{1}{2}[\hat{\Sigma}_{y_l}, \hat{\Sigma}_{y_m}] = \epsilon_{lmn}\hat{\Sigma}_{y_n}$ for $l, m, n$ ranging over 1, 2, 3. Then (7) can be expressed in the concise form

$$\boldsymbol{i}\hat{H} = u_1\hat{\Sigma}_x + \nu\hat{\Sigma}_y + u_3\hat{\Sigma}_z + u_0\hat{\Sigma}_\perp, \tag{15}$$

where $\hat{\Sigma}_\perp = (u_2\hat{\Sigma}_{y_1} + u_4\hat{\Sigma}_{y_2} + u_5\hat{\Sigma}_{y_3})/\nu$ is the generator of the planar rotation about the three-space spanned by $x, y, z$. In this manner we see how the subgroup SO(3) × U(1) of SO(5) emerges naturally, on account of the fact that $[\hat{\Sigma}_{x,y,z}, \hat{\Sigma}_\perp] = 0$. In particular, if $\mathrm{tr}\,\boldsymbol{i}\hat{H} = 0$, i.e., if $u_0 = 0$, then it is not possible to detect extra dimensions dynamically.

This result shows that the superselection rule for $\boldsymbol{i}$ emerges from symmetry breaking. In complex quantum mechanics, given a state one can always unitarily transform it to another arbitrary state by a suitable choice of Hamiltonian. In quaternionic quantum mechanics with the superselection rule (5), the ratio $u_2$:$u_4$:$u_5$ is fixed so that the only parametric degrees of freedom in the Hamiltonian are

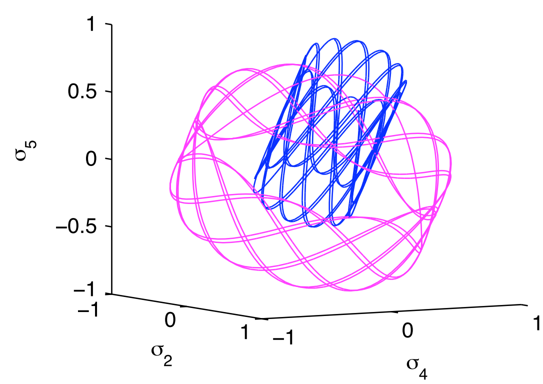


FIG. 1 (color online). Examples of dynamical trajectories traced by the variables $(\sigma_2(t), \sigma_4(t), \sigma_5(t))$ for two different initial conditions. For each choice of $c$ the orbits form cylindrical Rabi oscillations. The axis of the cylinder is determined by the vector $(u_2, u_4, u_5)$.

those appearing in (15). It follows that a state with a given value of $r$ in (13) cannot unitarily evolve into another state with a different value of $r$.

The superselection rule resulting from the symmetry breaking circumvents a difficulty associated with combined systems in quaternionic quantum mechanics (cf. [8,12,13]). If all systems share the same $\boldsymbol{i}$, then one is working with a commuting subalgebra of quaternions, thus circumventing the issues associated with the construction of tensor products for combined systems. While the standard choice of complex quantum mechanics $\boldsymbol{i} = i$ can be regarded as a special case of this formalism, the embedding into the quaternionic space nevertheless accommodates extra dimensions. These extra dimensions are not introduced "by hand"; rather, they emerge from the requirement of unitary time evolution generated by a Hermitian quaternionic Hamiltonian of a two-level system. Furthermore, the resulting dynamics naturally factorizes into a motion in a three-space and a motion for the remaining "hidden coordinates."

It is worth remarking that the structure revealed in the foregoing analysis carries through to an octavic representation of a spin-$\frac{1}{2}$ system. In this case, the spin vector $\vec{\sigma}$ lies on an eight sphere $S^8 \subset \mathbb{R}^9$. If we define $\sigma_y$ in a manner analogous to (11) involving the seven spin components $\sigma_2, \sigma_4, \ldots, \sigma_9$, then a calculation shows that the dynamical equations satisfied by the reduced spin variables are given by (12), with $\nu^2 = u_2^2 + u_4^2 + \cdots + u_9^2$. To characterize the surface upon which the remaining degrees of freedom are confined, let us write $[l, m, n] = |(u_l, u_m, u_n) \times (\sigma_l, \sigma_m, \sigma_n)|^2$. Hence the left side of (14), for instance, becomes $[2, 4, 5]$. Then in the octavic case these dynamical variables are confined to a real six-dimensional manifold determined by the relation

$$[2,4,5] + [2,6,7] + [2,8,9] + [4,6,8] + [4,7,9] + [5,6,9] + [5,7,8] = \nu^2 c^2. \tag{16}$$

This manifold, which is the octavic generalization of (14), has the structure of a cylinder $S^5 \times \mathbb{R}^1$ in the direction of the vector $(u_2, u_4, u_5, u_6, u_7, u_8, u_9)$, with radius $c$.

It is important to note that here we consider dynamics in the angular momentum space, and that the "thickness" $c$ of the $y$-axis is not related to the size of extra dimensions in coordinate space. The higher-dimensional angular momentum discussed here can be related to a higher-dimensional coordinate space in the usual manner: $L_{mn} = x_m p_n - p_n x_m$, with $p_n = \boldsymbol{i}\partial_n$. The size of the $x_n$ does not affect the size of $c$.

We conclude by discussing the possibility of detecting extra dimensions in a laboratory. An experimental test for quaternionic quantum mechanics has previously been proposed by Peres [14], which has subsequently been shown to yield null outcome by Adler [15]. Given the analysis presented here of a quaternionic spin system, another obvious proposal arises from relation (13), since the left side involves quantities that can be estimated directly from experimental data, whereas the value of the right side, according to complex quantum mechanics, is unity. However, in the quaternionic case there are states for which $c > 0$, and we have $r^2 = 1 - c^2 < 1$. To perform an experiment, one prepares a large number of spin-$\frac{1}{2}$ particles in a pure state and measures the spin in three orthogonal directions to estimate $\sigma_x^2 + \sigma_y^2 + \sigma_z^2$. If the result is less than one, then this gives a strong indication that there can be extra dimensions.

Although such a basic experiment is easily performed, it need not constitute a useful test for the following two reasons: (i) the prepared states must be pure; and (ii) the measurements have to be performed along three strictly orthogonal directions. Any impurity or deviation from orthogonality will lead to a number less than one even in three dimensions. Hence it may be difficult to extract useful insights from this simple experiment. Nevertheless, this example illustrates the important point that in principle it is possible to probe extra dimensions at low energies. Viable experiments may be constructed by making use of interference effects arising from, for instance, geometric phases (cf. [16–18]). Alternatively, the existence of an SO (5) symmetry between antiferromagnetic and superconducting phases that can be described by a five-dimensional superspin [10] might provide a clue along this line of investigation; and, conversely, an extension of Zhang's SO(5) representation to SO(9) might lead to new predictions in superconductor physics. The identifications made here of the structures of "commutative" quaternionic and octavic state spaces will undoubtedly help in making progress towards these directions.

E. M. G. is supported by the Imperial College JRF scheme.

[1] B. Zwiebach, *A First Course in String Theory* (Cambridge University Press, Cambridge, 2004).

[2] R. Penrose and W. Rindler, *Spinors and Space-Time* (Cambridge University Press, Cambridge, 1984), Vol. 1.

[3] D. C. Brody and L. P. Hughston, Proc. R. Soc. A **461**, 2679 (2005).

[4] T. Kugo and P. Townsend, Nucl. Phys. **B221**, 357 (1983).

[5] D. C. Brody and E. M. Graefe, J. Phys. A **44**, 072001 (2011).

[6] V. I. Arnold, in *The Arnoldfest: Proceedings of a Conference in Honour of V. I. Arnold for his Sixtieth Birthday*, edited by E. Bierstone *et al.*. (AMS, Providence, Rhode Island, 1999).
[7] P. A. M. Dirac, Proc. R. Irish Acad., Sect. A **50**, 261 (1945), http://www.jstor.org/stable/20520646.
[8] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford University Press, Oxford, 1995).
[9] D. Finkelstein, J. M. Jaueh, S. Schiminovieh, and D. Speiser, J. Math. Phys. (N.Y.) **3**, 207 (1962).
[10] S.-C. Zhang, Science **275**, 1089 (1997).
[11] U. Wolff, Phys. Lett. A **84**, 89 (1981).
[12] A. Razon and L. P. Horwitz, Acta Appl. Math. **24**, 141 (1991).
[13] J. Baez, Found. Phys., 1-37 (2011).
[14] A. Peres, Phys. Rev. Lett. **42**, 683 (1979).
[15] S. L. Adler, Phys. Rev. D **37**, 3654 (1988).
[16] S. L. Adler and J. Anandan, Found. Phys. **26**, 1579 (1996).
[17] E. Demler and S.-C. Zhang, Ann. Phys. (N.Y.) **271**, 83 (1999).
[18] K. Hasebe, Phys. Rev. Lett. **94**, 206802 (2005).