# Dequantisation of the Dirac Monopole

By Dorje C. Brody

*Department of Mathematics, Imperial College London, London SW7 2BZ, UK*

Using a sheaf-theoretic extension of conventional principal bundle theory, the Dirac monopole is formulated as a spherically symmetric model free of singularities outside the origin such that the charge may assume arbitrary real values. For integral charges, the construction effectively coincides with the usual model. Spin structures and Dirac operators are also generalised by the same technique.

**Keywords: magnetic monopole; sheaf cohomology; topological quantisation; spin structure; Dirac operator**

## 1. Introduction

In his classical paper on quantisation of magnetic poles Dirac (1931) remarked that "Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation." In accordance with this principle, the present paper further extends the work of Dirac by exploring the 'dequantisation' of magnetic poles.

Diverse and numerous versions of the magnetic pole construction and the associated charge quantisation condition of Dirac (1931, 1948) have appeared in the literature, but the basic model can be most concisely and accurately formulated in terms of the Hopf fibration (Wu & Yang 1975; Trautman 1977; Yang 1977). From a mathematical viewpoint, the rationale is as follows. The monopole potential is modelled as a connection $A$ on a nontrivial principal $U(1)$-bundle $P$ over a subset of Minkowski space. A charged matter field interacting with the monopole is accordingly modelled as a section of a vector bundle $E_n$ associated with $P$ via the representation $\rho(e^{i\theta})\psi = e^{in\theta}\psi$ of $U(1)$ on a complex vector space. Since $A$ then induces the connection $nA$ on $E_n$, the integer $n$ can be identified with the electric charge of the matter field (see Urbantke 2003 for a detailed account of the description of a magnetic monopole in the language of the Hopf fibration).

However, following the spirit of Dirac, if we describe the monopole by a more general mathematical scheme, then the interaction of matter fields and magnetic poles *with arbitrary real charges* can also be modelled in a consistent manner. The present paper introduces a new sheaf-theoretic framework permitting an explicit construction of arbitrarily charged magnetic monopoles. This framework is likewise applied to generic $U(1)$-bundles and also yields, as a by-product, the notion of a quasispin structure defined on arbitrary space-times. The results suggest that topological quantisation in general can be viewed from a more flexible perspective.

Although magnetic monopoles have not been observed experimentally, one important physical consequence of the present model is that their detection would not necessarily imply the quantisation of electric charge. Likewise, an observed violation of charge quantisation would not necessarily imply the nonexistence of magnetic monopoles. Furthermore, grand unified theories require the existence of magnetic monopoles, and, according to conventional field theory described in terms of manifolds this necessarily implies charge quantisation, hence, the detection of nonintegral electric charge would indicate that the present scheme, based on sheaves rather than manifolds, may be physically more realistic. Also, a sheaf construction permits the global description of fermion fields on Lorentzian manifolds possessing no conventional spin structure, as an alternative to the cumbersome machinery of Kähler fermions.

The paper is organised as follows. In §2 we present a very brief sketch of sheaf theory for the benefit of readers less acquainted with the subject. This is intended to provide the bare minimum of information necessary for following the ensuing discussion; for further details on sheaf theory, see, e.g., Bredon (1997), Kultze (1970), or Wells (2008). In §3 we develop the basic mathematical machinery used in later sections. The key idea here is the construction of a principal $\mathcal{G}$-sheaf bundle which generalises the conventional notion of a principal bundle. In §4 we prove that equivalence classes of principal $G$-bundles can, under certain hypotheses, be mapped injectively into equivalence classes of $\mathcal{G}$-sheaf bundles. This implies that the Dirac sheaf bundle constructed in §5 does indeed constitute a generalisation of the conventional Dirac monopole. The spherically symmetrical connection and curvature of the Dirac sheaf bundle are constructed in §6, demonstrating that the magnetic charge of this model can assume arbitrary real values. The interaction of the generalised monopole with a charged matter field is considered in §7. Dequantisation of more general $U(1)$ bundles is considered in §8, with particular attention to gravitational and electromagnetic instantons. In §9 the basic machinery developed earlier is applied to spin structures and Dirac operators. In §10 the paper concludes with a brief discussion of possible implications in diverse areas of physics.

## 2. Elements of sheaf theory

The concept of a sheaf over a manifold $X$ provides a way of interpolating local data and global data on X. We begin with the definition of a presheaf. A *presheaf* $\mathfrak{F}$ on $X$ is a functor assigning, to each open $U \subset X$, a group $F(U)$, abelian or otherwise, such that for each $V \subset U$ the restriction map $r_V^U : F(U) \to F(V)$, $r_U^U = 1$, defines a homomorphism and such that $r_W^V r_V^U = r_W^U$ for $W \subset V \subset U$. An element $\sigma \in F(U)$ is referred to as a *section* of $F(U)$ over $U$. The restriction of $\sigma \in F(U)$ on $V \subset U$ is thus given by $\sigma|_V = r_V^U(\sigma)$.

The sections $\sigma \in F(U)$ and $\tau \in F(V)$ are said to be *equivalent* at $x \in U \cap V$ if there is a neighbourhood $W$ of $x$ such that $r_W^U(\sigma) = r_W^V(\tau)$. The equivalence class containing $\sigma \in F(U)$ is called the *germ* of $\sigma$ at $x$, and the set of all such germs for any fixed $x$ is denoted by $\mathcal{F}_x$.

The disjoint union $\mathcal{F}$ of all the sets $\mathcal{F}_x$ provides local information about the structure of $\mathfrak{F}$. However, information concerning global structure has been lost, since we have discarded relations between the $\mathcal{F}_x$ for varying $x$. To retrieve some global structure, we introduce a topology in the following manner. For a fixed

$\sigma \in F(U)$ the set of all germs $\sigma_x \in \mathcal{F}_x$ for $x \in U$ is taken to be an open set in $\mathcal{F}$, and the topology of $\mathcal{F}$ is defined as that generated by these open sets.

The projection $\pi : \mathcal{F} \to X$ mapping $\mathcal{F}_x$ into $x$ has the property that for any point $t \in \mathcal{F}$ with $\pi(t) = y$ there is a neighbourhood $N = \{\sigma_x | x \in U\}$ for $\sigma \in F(U)$ and $\sigma_y = t$ such that the restriction $\pi|_N$ is a homeomorphism onto a neighbourhood of $y$. These ideas can be summarised as follows:

**Definition 1.** A *sheaf* of groups on $X$ is a pair $(\mathcal{F}, \pi)$ such that

  i) $\mathcal{F}$ is a topological space (in general, not Hausdorff);

 ii) $\pi$ is a local homeomorphism of $\mathcal{F}$ onto $X$;

iii) each $\mathcal{F}_x = \pi^{-1}(x)$, $x \in X$, is a group called the *stalk* of $\mathcal{F}$ at $x$; and

 iv) the group operations are continuous with respect to the relative topology on the subset $\mathcal{F} \bigtriangleup \mathcal{F} = \{(f, f') \in \mathcal{F} \times \mathcal{F} | \pi(f) = \pi(f')\}$ of $\mathcal{F} \times \mathcal{F}$.

For $U \subset X$, a continuous map $\sigma : U \to \mathcal{F}$ such that $\pi\sigma(x) = x$ is called a *section* of $\mathcal{F}$ over $U$. The totality of such $\sigma$ will be denoted $\mathbf{\Gamma}(U, \mathcal{F})$. Note that every element of the group $F(U)$ specified by the presheaf functor naturally determines an element of $\mathbf{\Gamma}(U, \mathcal{F})$, but the converse is only true locally.

Some of the key properties of a sheaf are as follows. If $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open covering of an open set $U \subset X$, and if $\sigma, \sigma' \in F(U)$ are such that $\sigma|_{U_\alpha} = \sigma'|_{U_\alpha}$ for all $\alpha$, then $\sigma = \sigma'$. Furthermore, if $\sigma_\alpha \in F(U_\alpha)$ are such that $\sigma_\alpha|_{U_\alpha \cap U_\beta} = \sigma_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta$, then there exists an element $\sigma \in F(U)$ with $\sigma|_{U_\alpha} = \sigma_\alpha$ for all $\alpha$. The first property implies that if the restrictions of a pair of sections always agree, then the two sections are identical—thus a section over $U$ is determined by the totality of its restrictions to subsets of $U$. The second property, somewhat complementary to the first, implies that if pairs of sections always agree on their overlapping regions then global section can be constructed from the local data—thus a section over $U$ may be assembled from consistent local sections on subsets of $U$.

A sheaf $\mathcal{F}$ contains localised information concerning the topological space $X$. Global information about $X$ can then be extracted from $\mathcal{F}$ by consideration of exact sequences, quotients, and so on. Given two sheaves $\mathcal{F}$ and $\mathcal{F}'$ over $X$, a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{F}'$ is a continuous map such that the stalk map $\phi_x := \phi|_{\mathcal{F}_x}$ is a homomorphism of $\mathcal{F}_x$ into $\mathcal{F}'_x$ for each $x \in X$. A sequence of sheaf homomorphisms of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

such that the corresponding sequence of stalk maps is exact for all $x \in X$ is called a short exact sequence of sheaves. Evidently, exactness is a local property. Given a short exact sequence of sheaves, the induced sequence

$$0 \longrightarrow \mathbf{\Gamma}(X, \mathcal{F}) \longrightarrow \mathbf{\Gamma}(X, \mathcal{G}) \longrightarrow \mathbf{\Gamma}(X, \mathcal{H}) \longrightarrow 0$$

is exact at $\mathbf{\Gamma}(X, \mathcal{F})$ and $\mathbf{\Gamma}(X, \mathcal{G})$, but in general not at $\mathbf{\Gamma}(X, \mathcal{H})$. That is to say, the local exactness of a sequence does not imply exactness with respect to the global sections over $X$. The measure of inexactness at $\mathbf{\Gamma}(X, \mathcal{H})$ can then be characterised by cohomology.

Recall that in the cohomology theories of $X$ one computes $H^i(X, G)$ where $G$ is an abelian group. In sheaf cohomology the coefficients are not elements of a fixed group $G$ but are, rather, local sections of some sheaf $\mathcal{F}$ over $X$. More precisely, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of $X$. For any $U = (U_1, \ldots, U_{q+1})$ such that $V_U = U_1 \cap \cdots \cap U_{q+1} \neq \varnothing$, we define the set of $q$-cochains by $C^q(\mathcal{U}, \mathcal{F}) = \Pi_U \mathbf{\Gamma}(V_U, \mathcal{F})$. For any $f \in C^q(\mathcal{U}, \mathcal{F})$ define the coboundary operator $\delta$ by

$$\delta f(V) = \sum_{i=1}^{q+2} (-1)^i r_V^{V_i} f(V_i), \tag{2.1}$$

where $U_i = (U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{q+1})$ and $r_V^{V_i}$ is the sheaf restriction map. These coboundary operators define a complex

$$\cdots \longrightarrow C^{q-1} \xrightarrow{\delta^{q-1}} C^q \xrightarrow{\delta^q} C^{q+1} \longrightarrow \cdots$$

and the cohomology groups of this complex are then defined in the usual manner:

$$H^q(\mathcal{U}, \mathcal{F}) = \frac{\operatorname{Ker} \delta^q}{\operatorname{Im} \delta^{q-1}}. \tag{2.2}$$

Passing to a direct limit over progressively finer coverings, we obtain the sheaf cohomology groups $H^q(X, \mathcal{F})$.

## 3. Basic machinery

The reader will notice that the definitions in this section run parallel, *mutatis mutandis*, to the conventional definitions in the theory of fibre bundles. The $\mathcal{G}$-sheaf plays the role of a trivial and the $\mathcal{G}$-sheaf bundle that of a generally nontrivial fibre bundle.

**Definition 2.** Let $X$ and $F$ be topological spaces and $\mathcal{G} = (G, \tilde{\pi}, X)$ a sheaf of groups over $X$. A $\mathcal{G}$-*sheaf* over $X$ is a triple $\mathcal{F} = (F, \pi, X)$ such that

  i)  $\pi$ is a local homeomorphism of $F$ onto $X$;

  ii) for each $x \in X$, the stalk $\mathcal{G}_x$ operates (by left action) upon $\mathcal{F}_x = \pi^{-1}(x)$; and

  iii) if $F \bigtriangleup G = \{(f, g) \in F \times G \,|\, \pi(f) = \tilde{\pi}(g)\}$ is equipped with the relative topology in $F \times G$, then the mapping $k: F \bigtriangleup G \to F$ defined by $k(f, g) = gf$ is continuous.

For any subset $A \subset X$, the obvious restriction maps define a $\mathcal{G}|_A$-sheaf $\mathcal{F}|_A$ over $A$. In the sequel, when $A$ is clearly understood, we shall, for brevity, use the term $\mathcal{G}$-sheaf in place of $\mathcal{G}|_A$-sheaf. We call $\mathcal{F}$ a *principal $\mathcal{G}$-sheaf* if $\mathcal{F} = \mathcal{G}$ and $\mathcal{G}$ operates by left translation.

**Definition 3.** Given two $\mathcal{G}$-sheaves $\mathcal{F} = (F, \pi, X)$ and $\mathcal{F}' = (F', \pi', X)$ over $X$, a continuous map $\phi: F \to F'$ will be called a $\mathcal{G}$-*sheaf map* of $\mathcal{F}$ into $\mathcal{F}'$ provided

  i)  $\pi' \phi = \pi$; and

ii) for each $x \in X$, the induced map $\phi_x : \mathcal{F}_x \to \mathcal{F}'_x$ satisfies

$$\phi_x(g_x f_x) = g_x \phi_x(f_x) \tag{3.1}$$

for all $f_x \in \mathcal{F}_x$ and $g_x \in \mathcal{G}_x$.

Again, if $A \subset X$, then a $\mathcal{G}|_A$-sheaf map will, for brevity, be called a $\mathcal{G}$-sheaf map when the restriction is clearly understood. A *$\mathcal{G}$-sheaf isomorphism* is a bijective $\mathcal{G}$-sheaf map.

**Definition 4.** A *$\mathcal{G}$-sheaf bundle* $\mathcal{B}$ over a topological space $X$ is defined by the following data:

i) an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of $X$;

ii) for each $\alpha \in \Lambda$, a $\mathcal{G}$-sheaf $\mathcal{F}_\alpha$ over $U_\alpha$; and

iii) for each nonempty intersection $U_\alpha \cap U_\beta \neq \varnothing$, a $\mathcal{G}$-sheaf isomorphism

$$T_{\alpha\beta} : \ \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \tag{3.2}$$

such that the cocycle condition

$$T_{\alpha\beta} T_{\beta\gamma} = T_{\alpha\gamma} \tag{3.3}$$

is satisfied on $U_\alpha \cap U_\beta \cap U_\gamma$.

We call $\mathcal{B}$ a *principal $\mathcal{G}$-sheaf bundle* if each $\mathcal{F}_\alpha$ is a principal $\mathcal{G}$-sheaf. Two $\mathcal{G}$-sheaf bundles $\mathcal{B}$ and $\mathcal{B}'$ over $X$, defined in terms of the respective open coverings $\mathcal{U}$ and $\mathcal{U}'$, are *equivalent* provided that the $\mathcal{G}$-sheaf bundles $\bar{\mathcal{B}}$ and $\bar{\mathcal{B}}'$ induced by the respective restriction maps on some common refinement $\bar{\mathcal{U}} = \{U_\alpha\}_{\alpha \in \Lambda}$ of $\mathcal{U}$ and $\mathcal{U}'$ are such that there exist $\mathcal{G}$-sheaf isomorphisms $T_\alpha : \mathcal{F}_\alpha \to \mathcal{F}'_\alpha$ for each $\alpha \in \Lambda$ satisfying

$$\bar{T}_{\alpha\beta} = T_\alpha^{-1} \bar{T}'_{\alpha\beta} T_\beta \tag{3.4}$$

for all $\alpha, \beta \in \Lambda$.

Remark: Of academic interest is the fact that, given a $\mathcal{G}$-sheaf bundle $\mathcal{B}$, one can, from the presheaf defined by local sections of $\mathcal{B}$, define a $\mathcal{G}$-sheaf $\mathcal{F}_\mathcal{B}$ which in turn uniquely determines the structure of $\mathcal{B}$ up to equivalence (Kashiwara & Schapira 2006). However, this fact will not be required in the sequel.

**Definition 5.** Let $X$ be a smooth manifold, $G$ a Lie group, and $H$ a closed central subgroup of $G$. For each open subset $U$ of $X$, let $\Gamma(U, G)$ denote the totality of smooth maps $U \to G$, which forms a group under pointwise multiplication, i.e. $(\sigma\sigma')(x) := \sigma(x)\sigma'(x)$. Let $\Gamma_c(U, H)$ denote the totality of constant maps $U \to H$, which forms a central subgroup of $\Gamma(U, G)$. Assign to each open $U \subset X$ the quotient group

$$\mathfrak{G}_H^G(U) := \Gamma(U, G)/\Gamma_c(U, H). \tag{3.5}$$

With the obvious restriction maps, this defines a presheaf; denote by $\mathcal{G}_H^G(X)$ the corresponding sheaf of groups over $X$, regarded as a principal $\mathcal{G}(X)$-sheaf, and its restriction to a subset $A$ of $X$ by $\mathcal{G}_H^G(A)$. A principal $\mathcal{G}$-sheaf bundle over $X$ such that $\mathcal{F}_\alpha = \mathcal{G}_H^G(U_\alpha)$ for some open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of $X$ will, for brevity, be called a *$G_H$-bundle*.

*Example 1.* With the notation of Definition 4, let $(P, \pi, X)$ be a principal $G$-bundle, and $\{U_\alpha\}_{\alpha \in \Lambda}$ an open covering of $X$ by trivialising neighbourhoods. On each nonempty intersection $U_\alpha \cap U_\beta$ the bundle transition function $g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, G)$ defines an element of $\mathfrak{G}_H^G(U_\alpha \cap U_\beta)$ and hence a continuous section $\tilde{g}_{\alpha\beta}$ of the sheaf $\mathcal{G}_H^G(U_\alpha \cap U_\beta)$. The germs of $\tilde{g}_{\alpha\beta}$, acting by right multiplication, provide a $\mathcal{G}$-sheaf map $T_{\alpha\beta} : \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta}$, which is obviously bijective, i.e. is a $\mathcal{G}$-sheaf isomorphism. Moreover, the cocycle condition (3.3) follows immediately from the corresponding cocycle condition for the bundle transition functions. Thus, we obtain a $G_H$-bundle $\mathcal{F}_P(H)$ over $X$.

## 4. Classification of $G_H$-bundles

The following result shows that under a certain simple hypothesis the conventional principal $G$-bundles may be identified with a subset of the $G_H$-bundles over $X$. Throughout the sequel, we shall assume that $X$ is paracompact, i.e. that every open covering of $X$ has a locally finite subcovering.

**Proposition 4.1.** *For fixed $H$, the correspondence $P \to \mathcal{F}_P(H)$ induces an injective map of equivalence classes of principal $G$-bundles into equivalence classes of $\mathcal{G}$-sheaf bundles over $X$, provided that the Čech cohomology group $\check{H}^1(X, H) = 0$.*

We first prove three lemmata.

**Lemma 4.2.** *Let $G$ be an arbitrary group. A mapping $\phi : G \to G$ which commutes with all left translations is a right translation.*

Proof. Given that $\phi(g_1 g_2) = g_1 \phi(g_2)$ for all $(g_1, g_2) \in G \times G$, let $g_2 = e$. Then $\phi(g_1) = g_1 \phi(e)$ for all $g_1 \in G$. $\qquad\square$

Recall that $\mathbf{\Gamma}(V, \mathcal{F})$ denotes the totality of continuous sections of a sheaf $\mathcal{F}$ over a subset $V \subset X$.

**Lemma 4.3.** *Let $U$ be an open subset of the topological space $X$, $G$ a group, $\mathcal{F}$ a principal $\mathcal{G}$-sheaf over $X$ and $T : \mathcal{F}|_U \to \mathcal{F}|_U$ a $\mathcal{G}$-sheaf isomorphism. Then, for each point $x \in U$, there exists a neighbourhood $V_x$ of $x$ in $U$ and a $g \in \mathbf{\Gamma}(V_x, \mathcal{F})$ such that $T\sigma(y) = \sigma(y)g(y)$ for every section $\sigma \in \mathbf{\Gamma}(V_x, \mathcal{F})$ and all $y \in V_x$.*

Proof. By Lemma 4.2 and condition ii) of Definition 3, for each $x \in U$ there exists a $g_0(x) \in \mathcal{F}_x$ such that $T_x f_x = f_x g_0(x)$ for all $f_x \in \mathcal{F}_x$. Now, fix $x_0 \in U$, and choose any $f_{x_0} \in \mathcal{F}_{x_0}$ and any section $\sigma \in \mathbf{\Gamma}(V_{x_0}, \mathcal{F})$ with $V_{x_0}$ open in $U$, $x_0 \in V_{x_0}$ and $\sigma(x_0) = f_{x_0}$. Since $T$ is continuous and stalk-preserving, $T\sigma(y) = \sigma(y)g_0(y)$ $(y \in V_{x_0})$ defines a continuous section of $\mathcal{F}$, and by the continuity of the group operations $g_0(y) = \sigma(y)^{-1}T\sigma(y)$ $(y \in V_{x_0})$ is also a continuous section. For any other section $\sigma'(y) \in \mathbf{\Gamma}(V_{x_0}, \mathcal{F})$, writing $\sigma'(y) = f_y'$ we have $T\sigma'(y) = T_y f_y' = f_y' g_0(y) = \sigma'(y)g_0(y)$. Hence, $g_0(y) \in \mathbf{\Gamma}(V_{x_0}, \mathcal{F})$ has the desired property. $\qquad\square$

**Lemma 4.4.** *Let $\mathcal{F} = (F, \pi, X)$ and $\mathcal{F}' = (F, \pi', X)$ be sheaves, $\phi : \mathcal{F} \to \mathcal{F}'$ an epimorphic sheaf map and $\sigma' \in \mathbf{\Gamma}(X, \mathcal{F}')$. Then, for any $x \in X$, there exists a neighbourhood $V_x$ of $x$ and a $\sigma \in \mathbf{\Gamma}(V_x, \mathcal{F})$ such that $\phi\sigma = \sigma'|_{V_x}$.*

Proof. This is an elementary fact of sheaf theory (Kultze 1970, Hilfsatz 3.4). $\square$

Proof of Proposition 4.1. Clearly, equivalent principal bundles give rise to equivalent sheaf bundles. Conversely, let $(P, \pi, X)$ and $(P', \pi', X)$ be principal $G$-bundles defined in terms of trivialising open coverings $\mathcal{U}$ and $\mathcal{U}'$, respectively. Choosing, if necessary, a common refinement, we may suppose that $\mathcal{U} = \mathcal{U}' = \{U_\alpha\}_{\alpha \in \Lambda}$. Then, with the foregoing notation, $\mathcal{F}_P(H) \sim \mathcal{F}'_P(H)$ iff there exist $\mathcal{G}$-sheaf isomorphisms $T_\alpha : \mathcal{F}_\alpha \to \mathcal{F}'_\alpha$, $\alpha \in \Lambda$, such that

$$T'_{\alpha\beta} = T_\alpha T_{\alpha\beta} T_\beta^{-1} \tag{4.1}$$

on $U_\alpha \cap U_\beta$ $(\alpha, \beta \in \Lambda)$. The $\mathcal{G}$-sheaves $\mathcal{F}_\alpha$ and $\mathcal{F}'_\alpha$ are identical, both arising from the presheaf functor $\mathfrak{F}_\alpha(V) = \Gamma(V, G)/\Gamma_c(V, H)$, $V$ open in $U_\alpha$. Let $\mathcal{G}_\alpha$ denote the sheaf over $U_\alpha$ defined by the presheaf functor $V \to \Gamma(V, G)$. Then, the presheaf epimorphism $\Gamma(V, G) \to \mathfrak{F}_\alpha(V)$ induces an epimorphic sheaf map $\phi : \mathcal{G}_\alpha \to \mathcal{F}_\alpha$, hence, by Lemmata 4.3 and 4.4, for each $x \in U_\alpha$ there exists a neighbourhood $V_x \subset U_\alpha$ and a section $g_\alpha^x \in \Gamma(V_x, G)$ such that $T_\alpha[\sigma_\alpha] = [\sigma_\alpha][g_\alpha^x] = [\sigma_\alpha g_\alpha^x]$ for all $\sigma_\alpha \in \Gamma(V_x, G)$, where the square brackets denote cosets in $\mathfrak{G}_H^G(V_x)$, regarded as continuous sections in $\mathbf{\Gamma}(V_x, \mathcal{F}_\alpha)$. Then, choosing the covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ sufficiently fine, we may, by paracompactness, omit the superscript $x$ and simply write $T_\alpha[\sigma_\alpha] = [\sigma_\alpha g_\alpha]$ for all $\sigma_\alpha \in \Gamma(U_\alpha, G)$, with square brackets denoting cosets in $\mathfrak{G}_H^G(U_\alpha)$, again regarded as elements of $\mathbf{\Gamma}(U_\alpha, \mathcal{F}_\alpha)$. Hence, recalling the definition of the $T_{\alpha\beta}$ in Example 1, relation (4.1) implies that

$$[\sigma_\alpha g'_{\alpha\beta}] = [\sigma_\alpha g_\beta^{-1} g_{\alpha\beta} g_\alpha] \tag{4.2}$$

on $U_\alpha \cap U_\beta$. Since $\sigma_\alpha$ is arbitrary, this clearly means that

$$g'_{\alpha\beta}(x) = g_\beta^{-1}(x) g_{\alpha\beta}(x) g_\alpha(x) h_{\alpha\beta} \tag{4.3}$$

for certain constant functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to H. \tag{4.4}$$

Applying the cocycle condition for principal bundles to both members of (4.3), we deduce that the constants $h_{\alpha\beta}$ also satisfy the cocycle condition, and therefore define an element of $\check{H}^1(X, H)$. Hence, if the covering $\{U_\alpha\}_{\alpha \in \Lambda}$ is chosen sufficiently fine, then, by hypothesis, there exists a 0-cochain $\{h_\alpha\}_{\alpha \in \Lambda}$ such that $h_{\alpha\beta} = h_\beta^{-1} h_\alpha$. Defining $\bar{g}_\alpha(x) = h_\alpha g_\alpha(x)$ we find that (4.3) becomes

$$g'_{\alpha\beta}(x) = \bar{g}_\beta^{-1}(x) g_{\alpha\beta}(x) \bar{g}_\alpha(x), \tag{4.5}$$

which is the condition for equivalence of $P$ and $P'$. $\qquad\square$

As used in the foregoing proof, Lemmata 4.3 and 4.4 show that for a sufficiently fine covering $\mathcal{U}$ the transition sheaf isomorphisms of a $G_H$-bundle are of the form

$$T_{\alpha\beta} \bar{\sigma} = \bar{\sigma} \bar{g}_{\alpha\beta} \tag{4.6}$$

for fixed $\bar{g}_{\alpha\beta}$ and arbitrary $\bar{\sigma} \in \mathbf{\Gamma}(U_\alpha \cap U_\beta, \mathcal{F}_\beta)$. Likewise, the $\mathcal{G}$-sheaf isomorphisms $T_\alpha$ in the definition of $\mathcal{G}$-sheaf bundle equivalence are of the form

$$T_\alpha \bar{\sigma} = \bar{\sigma} \bar{g}_\alpha \tag{4.7}$$

for fixed $\bar{g}_\alpha$ and arbitrary $\bar{\sigma} \in \mathbf{\Gamma}(U_\alpha, \mathcal{F}_\alpha)$.

A standard theorem states that equivalence classes of principal $G$-bundles over $X$ correspond biuniquely to elements of the sheaf cohomology set $H^1(X, \mathcal{G})$. In view of (4.6), the argument used in proving this theorem (see, e.g., Lawson & Michelsohn 1989, Appendix A) can also be applied in a straightforward manner to yield the following result.

**Corollary 4.5.** *The equivalence classes of $G_H$-bundles over $X$ correspond biuniquely to elements of the sheaf cohomology set $H^1(X, \mathcal{F})$. For $H = \{e\}$, this correspondence reduces to the standard theorem cited above.*

If $G$ is abelian, then Proposition 4.1 follows more directly from standard results as follows. Let $\mathcal{H}_c$ denote the sheaf arising from the constant presheaf defined by $H_c(U) = \Gamma_c(U, H) \sim H$. The short exact sequence

$$0 \longrightarrow \Gamma_c(U, H) \overset{i}{\longrightarrow} \Gamma(U, G) \overset{j}{\longrightarrow} \Gamma(U, G)/\Gamma_c(U, H) \longrightarrow 0$$

induces a sequence of sheaf maps

$$0 \longrightarrow \mathcal{H}_c \overset{\boldsymbol{i}}{\longrightarrow} \mathcal{G} \overset{\boldsymbol{j}}{\longrightarrow} \mathcal{F} \longrightarrow 0$$

which is easily seen to be exact. This in turn induces a long exact cohomology sequence which includes, in particular, the segment

$$\cdots \longrightarrow H^1(X, \mathcal{H}_c) \overset{\boldsymbol{i}^*}{\longrightarrow} H^1(X, \mathcal{G}) \overset{\boldsymbol{j}^*}{\longrightarrow} H^1(X, \mathcal{F})$$
$$\overset{\boldsymbol{\delta}^*}{\longrightarrow} H^2(X, \mathcal{H}_c) \longrightarrow \quad \cdots$$

For a paracompact $G$-manifold $X$ the sheaf cohomology group $H^p(X, \mathcal{H}_c)$ is known to be isomorphic to the Čech cohomology group $\check{H}^p(X, H)$ as well as the singular cohomology group $H^p(X, H)$ with coefficients in $H$ (Spanier 1994, Chapter 6). Hence, if $\check{H}^1(X, H) = 0$, then the exactness of the above sequence implies that $\boldsymbol{j}^*$ is injective. Denoting the totality of equivalence classes of principal $G$-bundles (resp. $G_H$-bundles) by $P_G(X)$ (resp. $\mathcal{P}_{G_H}(X)$), we note that the diagram

$$
\begin{array}{ccc}
P_G(X) & \longrightarrow & \mathcal{P}_{G_H}(X) \\
\downarrow & & \downarrow \\
H^1(X, \mathcal{G}) & \overset{\boldsymbol{j}^*}{\longrightarrow} & H^1(X, \mathcal{F})
\end{array}
$$

where the upper arrow represents the correspondence $P \to \mathcal{F}_P(X)$ and the vertical arrows the biunique correspondence of Corollary 4.5, is commutative. The result follows.

Relations (4.6) and (4.7) also imply:

**Corollary 4.6.** *For the special case $H = \{e\}$ (the identity of $G$), the $G_H$-bundles can be identified with conventional principal $G$-bundles, and $\mathcal{G}$-sheaf equivalence is merely conventional bundle equivalence.*

Thus, for $H = \{e\}$ the present theory yields nothing new. However, for $H \neq \{e\}$ and $\check{H}^1(X, H) = 0$ we obtain a nontrivial extension of the conventional theory

of principal bundles, as we shall see, in particular, from the example in the next section, wherein $H = G$.

Note that the set of data $\mathcal{T} = \{U_\alpha, \mathcal{F}_\alpha, \bar{g}_{\alpha\beta}\}$ (see Definition 5 and (4.6)), subsequently referred to as a *presentation* of the $G_H$-bundle under consideration, plays a role analogous to that of a system of local trivialisations in the conventional theory of principal bundles.

If $\mathcal{U}'$ is a refinement of $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$, then $\mathcal{T}$ induces, in the obvious manner, a presentation $\mathcal{T}'$ associated with $\mathcal{U}'$. Recalling the definition of $\mathcal{G}$-sheaf equivalence (see (3.4)), relations (4.6) and (4.7) imply that two presentations $\mathcal{T}$ and $\mathcal{T}'$ associated with the same covering $\{U_\alpha\}_{\alpha\in\Lambda}$ define equivalent $G_H$-bundles iff there exist elements $\bar{g}_\alpha \in \mathbf{\Gamma}(U_\alpha, \mathcal{F}_\alpha)$ such that the respective transition elements $\bar{g}_{\alpha\beta}$ and $\bar{g}'_{\alpha\beta}$ satisfy

$$\bar{g}'_{\alpha\beta}(x) = \bar{g}_\beta^{-1}(x)\bar{g}_{\alpha\beta}(x)\bar{g}_\alpha(x)h_{\alpha\beta} \tag{4.8}$$

for all $\alpha, \beta \in \Lambda$ and $x \in U_\alpha \cap U_\beta$. If the associated open coverings $\mathcal{U}$ and $\mathcal{U}'$ are different, then the condition for equivalence is given by (4.8) with respect to some common refinement $\mathcal{U}''$ of $\mathcal{U}$ and $\mathcal{U}'$.

Another immediate consequence of (4.6) is the fact that $G_H$-bundles, like conventional $G$-bundles, are functorial, i.e. a $G_H$-bundle $\mathcal{P}$ over $X$ and a continuous map $f : Y \to X$ naturally induce a $G_H$-bundle $f^*(\mathcal{P})$ over $Y$, since the transition sections $\bar{g}_{\alpha\beta}$ on $X$ pull back to sections on $Y$ which obviously satisfy the cocycle condition. Moreover, one can prove that if $X$ and $Y$ are compact Hausdorff spaces and the maps $f$ and $f' : Y \to X$ are homotopic, then $f^*(\mathcal{P})$ and $f'^*(\mathcal{P})$ are equivalent (cf. Lawson & Michelsohn 1990, Appendix A).

## 5. Dirac sheaf bundles

Let $X = S^2$ and $G = H = U(1)$. We represent $S^2$ by the unit sphere in $\mathbb{R}^3$, with spherical polar coordinates $(\theta, \phi)$, $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$, and $U(1)$ by $S^1 = \{e^{i\gamma} | 0 \leq \gamma < 2\pi\}$. For the open covering

$$U_1 = \{(\theta, \phi) | 0 \leq \theta < \tfrac{3}{2}\pi\}, \quad U_2 = \{(\theta, \phi) | \tfrac{1}{2}\pi < \theta \leq \pi\} \tag{5.1}$$

of $S^2$, we have $U_1 \cap U_2 = \{(\theta, \phi) | \tfrac{1}{2}\pi < \theta < \tfrac{3}{2}\pi\}$. Let $\mathcal{G}$ be the sheaf of smooth $U(1)$-valued functions over $S^2$. Let $\mathfrak{F}$ denote the presheaf over $S^2$ defined for each open $V$ by

$$\mathfrak{F}(V) = \Gamma(V, U(1))/\Gamma_c(V, U(1)), \tag{5.2}$$

$\mathcal{F}$ the associated $\mathcal{G}$-sheaf, and $\mathcal{F}_\alpha = \mathcal{F}|_{U_\alpha}$ $(\alpha = 1, 2)$. If $\nu$ is an arbitrary real number, define a $\mathcal{G}$-sheaf isomorphism

$$T_{12} : \mathcal{F}_2|_{U_1\cap U_2} \to \mathcal{F}_1|_{U_1\cap U_2} \tag{5.3}$$

by

$$T_{12}(x)f_x = f_x\bar{g}^\nu_{12}(x) \tag{5.4}$$

for $x = (\theta_0, \phi_0) \subset U_1 \cap U_2$, where $\bar{g}^\nu_{12}$ is the section of $\mathcal{F}|_{U_1\cap U_2}$ defined locally by

$$g^\nu_{12}(\theta, \phi) = \left[e^{-i\nu\phi}\right]. \tag{5.5}$$

Here the square brackets indicate the equivalence class modulo constant sections $c_h(\theta, \phi) = e^{ih}$. Denote the $G_H$-bundle so defined by $\mathcal{D}_\nu$. If $\nu = n$ is an integer, then $\mathcal{D}_n$ is just the principal $\mathcal{G}$-sheaf bundle $\mathcal{F}_{P_n}(U(1))$ arising from the conventional $U(1)$-bundle $P_n$ of charge $\frac{1}{2}n$ (in the appropriate units), in accordance with the construction of Example 1. In general, for an arbitrary abelian group $H$, the Čech cohomology groups are given by

$$\check{H}^q(S^n, H) = \left\{ \begin{array}{ll} H & \text{if } q = 0 \text{ or } n, \\ 0 & \text{otherwise} \end{array} \right. \tag{5.6}$$

(see, e.g., Spanier 1994). In particular, we have $\check{H}^1(S^2, U(1)) = 0$. Hence, by Proposition 4.1, the sheaf bundles $\mathcal{D}_n$ are mutually inequivalent. That $\nu \neq \nu'$ implies $\mathcal{D}_\nu \nsim \mathcal{D}_{\nu'}$ for arbitrary real $\nu$ and $\nu'$ will be proved in §6 below.

From the foregoing construction, the truth of the following proposition should be evident.

**Proposition 5.1.** *Let $g_{\alpha\beta}$ be the transition functions for some trivialising atlas of a principal $U(1)$-bundle $P$ over the base space $X$. Then, for any real number $\nu$, the powers $(g_{\alpha\beta})^\nu$ define the transition isomorphisms of a $U(1)_{U(1)}$-bundle $\mathcal{P}_\nu$ over $X$. If the Čech cohomology group $\check{H}^1(X, U(1)) = 0$, then the principal $U(1)$-bundles $P_n$ defined by the transition functions $(g_{\alpha\beta})^n$, if mutually inequivalent, correspond bi-uniquely with mutually inequivalent $U(1)_{U(1)}$-bundles $\mathcal{P}_n$.*

The final assertion is merely an application of Proposition 4.1.

## 6. Connections on $G_H$-bundles

The Lie algebra of $G$ will be denoted by $\mathfrak{g}$ and the algebra of smooth $\mathfrak{g}$-valued differential forms on a manifold $M$ by $\Lambda(M, \mathfrak{g})$.

**Definition 6.** A *connection* $\mathcal{A}$ on a $G_H$-bundle $\mathcal{F}$ given in terms of a presentation $(U_\alpha, \mathcal{F}_\alpha, \bar{g}_{\alpha\beta})$ is specified by a family of $\mathfrak{g}$-valued 1-forms $A_\alpha \in \Lambda^1(U_\alpha, \mathfrak{g})$ satisfying†

$$A_\alpha(x) = \bar{g}_{\alpha\beta}(x) A_\beta(x) \bar{g}_{\beta\alpha}(x) + \bar{g}_{\alpha\beta}(x) \mathrm{d}\bar{g}_{\beta\alpha}(x) \tag{6.1}$$

for all $\alpha, \beta \in \Lambda$ and $x \in U_\alpha \cap U_\beta$.

Both terms on the right side of (6.1) are unambiguously defined as follows. The value of $\bar{g}_{\alpha\beta}$ at $x \in U_\alpha \cap U_\beta$ is an element of $\mathcal{F}_{\beta x}$, represented by $f_{\alpha\beta} \in \mathfrak{F}(V_x) = \mathfrak{O}_H^G(V_x)$ for some neighbourhood $V_x$ of $x$, and $f_{\alpha\beta}$ is in turn represented by a section $g_{\alpha\beta} \in \Gamma(V_x, G)$. The first term on the right side of (6.1) is then defined by $g_{\alpha\beta}(x) A_\beta(x) g_{\beta\alpha}(x)$, and the second by $g_{\alpha\beta}(x) \mathrm{d} g_{\beta\alpha}(x)$. Any two possible choices $g_{\alpha\beta}$ differ only by a constant factor in $H$, which affects neither term. Two such connections $\mathcal{A}$ and $\mathcal{A}'$ defined in terms of the respective presentations $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent*, provided that, for some common refinement $\{U_\alpha\}_{\alpha \in \Lambda}$ of the two respectively associated open coverings $\mathcal{U}$ and $\mathcal{U}'$ with (cf. (4.8))

$$\bar{g}'_{\alpha\beta}(x) = \bar{g}_\beta^{-1}(x) \bar{g}_{\alpha\beta}(x) \bar{g}_\alpha(x) h_{\alpha\beta}, \tag{6.2}$$

† For simplicity of notation we assume that $G$ is a matrix group.

the relation

$$A'_\alpha(x) = \bar{g}_\alpha^{-1}(x)A_\alpha(x)\bar{g}_\alpha(x) + \bar{g}_\alpha^{-1}(x)\mathrm{d}\bar{g}_\alpha(x) \tag{6.3}$$

holds for all $\alpha \in \Lambda$ and $x \in U_\alpha$. Here the right side of (6.3) is interpreted in the same manner as that of (6.1). By choosing a sufficiently fine covering, the above-mentioned neighbourhoods $V_x$ may, by paracompactness, be identified with the trivialising neighbourhoods $U_\alpha$, so that the representative sections $g_{\alpha\beta}$ and $g_\alpha$ may be regarded as elements of $\Gamma(U_\alpha \cap U_\beta, G)$ and $\Gamma(U_\alpha, G)$, respectively. The $g_{\alpha\beta}$ then satisfy the cocycle condition modulo locally constant sections in $H$, which suffices to ensure the mutual consistency of the relations (6.1) as the indices $\alpha$ and $\beta$ vary over $\Lambda$.

The *curvature* of such a connection is defined in the conventional manner, i.e. for each $U_\alpha$ we have

$$F_\alpha = \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha = \mathrm{d}A_\alpha + \tfrac{1}{2}[A_\alpha, A_\alpha]. \tag{6.4}$$

By the same calculation as that used for conventional principal bundles, one finds, using (6.1), that

$$F_\alpha(x) = \bar{g}_{\alpha\beta}(x)F_\beta(x)\bar{g}_{\beta\alpha}(x) \tag{6.5}$$

for $x \in U_\alpha \cap U_\beta$. Furthermore, if $A'_\alpha$ and $A_\alpha$ are related by (6.3), then

$$F'_\alpha(x) = \bar{g}_\alpha^{-1}(x)F_\alpha(x)\bar{g}_\alpha(x). \tag{6.6}$$

If $(P, \pi, X)$ is a principal $G$-bundle and $H$ a central closed subgroup of $G$, then an ordinary connection $A$ on $P$ clearly gives rise to a connection $\mathcal{A}$ on $\mathcal{F}_P(H)$ in the sense of Definition 6, and the conventional local curvature forms then coincide with those of $\mathcal{A}$ as defined in (6.4). By virtue of the relations (6.5) and (6.6), characteristic classes of $G_H$-bundles can be defined and shown to be independent of the choice of the presentation and the choice of the connection $\mathcal{A}$, as for conventional principal bundles.

We now define a connection on the Dirac sheaf bundle $\mathcal{D}_\nu$ ($\nu \in \mathbb{R}$). Using the notation of §5, let $\mathcal{T}^\nu$ denote the presentation $\{U_1, U_2, \mathcal{F}_1, \mathcal{F}_2, \bar{g}_{12}^\nu\}$, and let

$$A_\alpha = \tfrac{1}{2}\mathrm{i}\nu\left[(-1)^{\alpha+1} - \cos\theta\right]\mathrm{d}\phi \in \Lambda^1(U_\alpha, \mathfrak{u}(1)) \tag{6.7}$$

be the corresponding local 1-forms. On $U_1 \cap U_2$ we have

$$A_1 = A_2 + \mathrm{i}\nu = A_2 + g_{12}^\nu \mathrm{d}g_{21}^\nu, \tag{6.8}$$

in accordance with (6.1). The curvature is given by

$$F = \mathrm{d}A_\alpha = \tfrac{1}{2}\mathrm{i}\nu\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{6.9}$$

and hence the first Chen† class is $-\nu\mathrm{d}S$, where $\mathrm{d}S$ is the area form on the unit 2-sphere. This shows that the sheaf bundles $\mathcal{D}_\nu$ for distinct $\nu$ are inequivalent, as claimed above.

† I use the current officially valid romanisation in place of the still prevalent "Chern", which was officially discarded about sixty years ago.

For simplicity of exposition and to emphasise the analogy with the conventional Hopf fibration $S^1 \to S^3 \to S^2$, the Dirac sheaf bundle and the corresponding monopole connection have been constructed over $X = S^2$ as base space. However, with a view to physical applications, the same construction applies verbatim if the base space is the subset of Minkowski space defined by $\tilde{\mathbb{R}}^4 := \{(x^0, x^1, x^2, x^3) | (x^1)^2 + (x^2)^2 + (x^3)^2 > 0\}$. Again, expressing $(x^1, x^2, x^3)$ by spherical polar coordinates, formulae (5.1), (5.2) and (5.6) remain unchanged. Thus, for arbitrary real $\nu$, one obtains a $U(1)_{U(1)}$-bundle over $\tilde{\mathbb{R}}^4$ with field strength (6.9).

Remark: According to the conventional bundle picture, there are no singularity-free gauge potentials $\{A_\alpha\}_{\alpha=1,2}$ with the properties that (i) their curls are equal to the field, and that (ii) they are related by gauge transformations on the overlapping region $U_1 \cap U_2$, unless electromagnetic charges are quantised (Wu & Yang 1975, Theorem 3). The above result demonstrates that this conclusion is not valid in the present generalised model.

## 7. Particle fields

Let $G$ be a Lie group, $H$ a closed subgroup of $G$, and $\rho : G \to Aut(\mathbb{V})$ a representation of $G$ on a (real or complex) vector space $\mathbb{V}$. Let $X$ be a smooth paracompact manifold, and for each open $U \subset X$, let $\Gamma(U, \mathbb{V})$ denote the totality of smooth maps $\tau : U \to \mathbb{V}$. If $\tau \in \Gamma(U, \mathbb{V})$ and $h \in H$, define $h\tau$ by $(h\tau)(x) = h(\tau(x))$. Denote by $\Gamma_H(U, \mathbb{V})$ the quotient space of $\Gamma(U, \mathbb{V})$ under this action of $H$. If $U' \subset U$, then the restriction of $\Gamma(U, \mathbb{V})$ to $\Gamma(U', \mathbb{V})$ obviously commutes with the action of $H$, and thus one obtains a restriction map $r_{U'}^U$. If $U'' \subset U' \subset U$, then $r_{U''}^{U'} r_{U'}^U = r_{U''}^U$, hence the system $\{\Gamma_H(U, \mathbb{V}), r_{U'}^U\}$ defines a presheaf $\mathfrak{V}$, and a corresponding sheaf $\mathcal{V}$.†

Next, for $x \in X$, define a left action of the stalk $(\mathcal{G}_H^G)_x$ of the sheaf $\mathcal{G}_H^G$ (see Definition 5) upon $\mathcal{V}_x$ as follows. For some neighbourhood $U$ of $x$, the germ $\bar{g}_x \in (\mathcal{G}_H^G)_x$ is represented by an element $\bar{g} \in \mathfrak{G}_H^G(U)$ and $\bar{g}$ in turn by a section $g \in \Gamma(U, G)$, while $\bar{v}_x$ is represented by an element $\bar{v}_x$ of $\Gamma_H(U, \mathbb{V})$ and $\bar{v}_x$ in turn by a section $v \in \Gamma(U, \mathbb{V})$. Define $\bar{g}_x \bar{v}_x$ as the element of $\mathcal{V}_x$ represented by the section $(gv)(x) = \rho(g(x))v(x)$. One can readily check that this is independent of the choices of $g$ and $v$.

**Definition 7.** Let $(\mathcal{P}, \pi, X)$ be a $G_H$-bundle defined by a presentation $\mathcal{T} = \{U_\alpha, \mathcal{F}_\alpha, \bar{g}_{\alpha\beta}\}_{\alpha \in \Lambda}$. A *particle field* is a system of sections $\{\bar{v}_\alpha \in \mathbf{\Gamma}(U_\alpha, \mathcal{V})\}_{\alpha \in \Lambda}$ satisfying the condition

$$\bar{v}_\alpha(x) = \bar{g}_{\alpha\beta}(x)\bar{v}_\beta(x) \tag{7.1}$$

for every $\alpha, \beta \in \Lambda$ and $x \in U_\alpha \cap U_\beta$. The totality of such particle fields will be denoted by $\mathcal{V}_\rho(\mathcal{P})$.

Inspection of the foregoing construction clearly shows that for $H = \{e\}$ this definition effectively reduces to that of a conventional particle field (cf. Corollary 4.6), i.e. a section of the associated bundle $P \times_G \mathbb{V}$ defined by a principal $G$-bundle and a representation $\rho : G \to Aut(\mathbb{V})$. Also, note that if $H \neq \{e\}$, then $\mathcal{V}_\rho(\mathcal{P})$ is *not*

† In this context, all the definitions in §3 are to be interpreted with groups replaced by vector spaces and homomorphisms by linear transformations.

a vector space. Nevertheless, one can define covariant derivatives on $\mathcal{V}_\rho(\mathcal{P})$, as is shown by the ensuing Examples 2 and 3.

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be the local potentials of a connection $\mathcal{A}$ with respect to some presentation $\mathcal{T}$, as in Definition 6. Consider a point $x \in U_\alpha$, and choose local coordinates $\{x^\mu\}$ in a neighbourhood $U_x \subset U_\alpha$ of $x$ so that $A_\alpha = A_{\alpha\mu}\mathrm{d}x^\mu$ with $A_{\alpha\mu} \in \Gamma(U_x, \mathfrak{g})$. Let $\bar{v}_\alpha \in \mathcal{V}(U_\alpha)$ and choose a section $v_\alpha = v_\alpha(y)$ representing $\bar{v}_\alpha$ in some neighbourhood $W_x$ of $x$, with $W_x \subset U_x$. Then define

$$[(\partial_\mu - A_{\alpha\mu})\bar{v}_\alpha](x) = ((\partial_\mu - A_{\alpha\mu})v_\alpha)_x, \tag{7.2}$$

where $A_{\alpha\mu}$ acts upon $v_\alpha$ in accordance with the Lie algebra representation induced by $\rho$. This germ is clearly independent of the choice of $v$, and thus one obtains a well-defined section of $(\partial_\mu - A_{\alpha\mu})\bar{v}_\alpha \in \boldsymbol{\Gamma}(W_x, \mathcal{V})$. Moreover, for $x \in U_\alpha \cap U_\beta$,

$$(\partial_\mu - A_{\alpha\mu}(x))\bar{v}_\alpha(x) = \bar{g}_{\alpha\beta}(x)(\partial_\mu - A_{\beta\mu}(x))\bar{v}_\beta(x), \tag{7.3}$$

that is, if $\bar{v}_\alpha$ and $\bar{v}_\beta$ are related by (7.1), then so are their covariant derivatives. To establish (7.3) note that both sides of the equation are well-defined local sections of $\mathcal{V}(U_\alpha \cap U_\beta)$. Hence we may choose arbitrary representatives $v_\alpha$, $v_\beta$, and $g_{\alpha\beta}$ of $\bar{v}_\alpha$, $\bar{v}_\beta$, and $\bar{g}_{\alpha\beta}$, respectively, to perform the calculation. We choose representatives satisfying the relation $v_\alpha(x) = \rho(g_{\alpha\beta}(x))v_\beta(x)$, and by a routine computation, as for conventional minimal coupling, we verify that

$$(\partial_\mu - A_{\alpha\mu}(x))v_\alpha(x) = g_{\alpha\beta}(x)(\partial_\mu - A_{\beta\mu}(x))v_\beta(x), \tag{7.4}$$

which implies (7.3). Choosing a sufficiently fine open covering, we may, by paracompactness, identify the aforesaid open sets $W_x$ with the $U_\alpha$, and hence assume that $(\partial_\mu - A_{\alpha\mu})\bar{v}_\alpha \in \boldsymbol{\Gamma}(U_\alpha, \mathcal{V})$. Accordingly, we let $(\partial_\mu - A_{\alpha\mu})\bar{v}_\alpha$ denote the element of $\mathcal{V}(\mathcal{P})$ determined by (7.3). For each $\alpha$ one can define a 1-form

$$D_\alpha^A := (\partial_\mu - A_{\alpha\mu})\bar{v}_\alpha \mathrm{d}x^\mu \tag{7.5}$$

on $U_\alpha$ assuming values in the sheaf $\mathcal{V}$, and (7.3) shows that these combine to provide a global $\mathcal{V}$-valued form $D^A\bar{v}$ on $X$.

The quantisation of the gauge and particle fields described above will be investigated elsewhere. Accordingly, we shall not attempt to define Lagrangians in the present paper. Rather, we shall merely postulate that the equations of motion derived in conventional gauge field theory are also valid in the present context.

*Example 2. Generalised interaction of Dirac monopole with charged scalar field.* Consider the fundamental representation of $U(1)$, that is, $\rho(\mathrm{e}^{\mathrm{i}\theta})z = \mathrm{e}^{\mathrm{i}\theta}z$ for $z \in \mathbb{C}$. We consider the Dirac sheaf bundle $\mathcal{D}_\nu$ and connection† $A_\nu$ defined over the subset $\tilde{\mathbb{R}}^4$ of Minkowski space, as described in §6. Let $\bar{v} \in \mathcal{V}(\mathcal{D}_\nu)$ be a particle field associated with $\mathcal{D}_\nu$ by the representation $\rho$, as in Definition 7. The classical equation of motion for a spin-0 charged particle interacting with an electromagnetic potential $A$ on Minkowski space is

$$(\partial_\mu - A_\mu)(\partial^\mu - A^\mu)\phi + m^2\phi = 0, \tag{7.6}$$

† As explained in §8 below, $A_\nu = \nu A$, where $A$ is the connection form for the conventional Dirac monopole.

where the charge factor i$e$ has been absorbed into $A$. If $\phi$ is replaced by $\bar{v} \in \mathcal{V}(\mathcal{D}_\nu)$, then, by virtue of relation (7.3), both sides of the equation

$$(\partial_\mu - A_\mu)(\partial^\mu - A^\mu)\bar{v} = -m^2\bar{v} \tag{7.7}$$

are well-defined elements of $\mathcal{V}(\mathcal{D}_\nu)$.

Remark: A solution of (7.7) might be physically described as a 'wave function' determined only up to a locally constant phase factor. Current dogma holds that a globally constant phase factor is undetectable, but the detectability of a locally constant phase factor using a physical measuring apparatus also seems problematical. Hence, the model described in the present example appears to be physically plausible.

The use of induced bundles to describe conventional interactions between gauge fields and particle fields initially defined in terms of different principal bundles (see Bleecker 1981) can be extended to $G_H$-bundles in a straightforward manner. If $\mathcal{F}$ and $\mathcal{F}'$ are, respectively, $G_H$- and $G'_{H'}$-bundles defined over the same base space $X$, then, passing to a common refinement if necessary, we may assume that $\mathcal{F}$ and $\mathcal{F}'$ are given in terms of presentations $\mathcal{T}$ and $\mathcal{T}'$ over the same system of trivialising neighbourhoods $\{U_\alpha\}_{\alpha \in \Lambda}$ with transition isomorphisms determined by $\bar{g}_{\alpha\beta}$ and $\bar{g}'_{\alpha\beta}$, respectively. Then, the pairs $(\bar{g}_{\alpha\beta}, \bar{g}'_{\alpha\beta})$ define a $(G \times G')_{H \times H'}$-bundle $\mathcal{F} \times \mathcal{F}'$ over $X$, with a presentation $\mathcal{T} \times \mathcal{T}' = \{U_\alpha, \mathcal{F}_\alpha \times \mathcal{F}', \bar{g}_{\alpha\beta} \times \bar{g}'_{\alpha\beta}\}$. Furthermore, if $A_\alpha \in \Lambda^1(U_\alpha, \mathfrak{g})$ and $A'_\alpha \in \Lambda'^1(U_\alpha, \mathfrak{g})$ are the local 1-forms of connections $\mathcal{A}$ and $\mathcal{A}'$ relative to the presentations $\mathcal{T}$ and $\mathcal{T}'$, respectively, then the local 1-forms $A_\alpha \oplus A'_\alpha \in \Lambda^1(U_\alpha, \mathfrak{g} \oplus \mathfrak{g}')$ determine a connection on $\mathcal{F} \times \mathcal{F}'$ relative to the presentation $\mathcal{T} \times \mathcal{T}'$. Now, let $\rho : G \to Aut(\mathbb{V})$ and $\rho' : G' \to Aut(\mathbb{V})$ be linear representations on a vector space $\mathbb{V}$ such that $\rho(g)\rho'(g') = \rho'(g')\rho(g)$ for all $(g, g') \in G \times G'$. Then $(g, g') \mapsto \rho(g)\rho'(g')$ defines a linear representation $\rho \times \rho' : G \times G' \to Aut(\mathbb{V})$. The elements of $\mathcal{V}_{\rho \times \rho'}(\mathbb{V})$ represent particle fields which interact with potentials defined on $\mathcal{F}$ as well as those defined on $\mathcal{F}'$.

*Example 3. Generalised interaction of Dirac monopole and spinor field.* Let $\rho$ denote the representation of $U(1)$ on $\mathbb{C}^4$ defined by scalar multiplication $\rho(e^{i\theta})\psi = e^{i\theta}\psi$, and let $\rho' = D^{1/2,0} \oplus D^{0,1/2} : SL(2, \mathbb{C}) \to Aut(\mathbb{C}^2 \times \mathbb{C}^2) = Aut(\mathbb{C}^4)$, where $D^{1/2,0}(g) = g$ and $D^{0,1/2}(g) = (g^\dagger)^{-1}$. Then $\rho$ and $\rho'$ obviously commute in the foregoing sense, hence $(\rho \times \rho')(e^{i\theta}, g)\psi = e^{i\theta}\rho'(g)\psi$ defines a representation of $U(1) \times SL(2, \mathbb{C})$ on $\mathbb{C}^4$. Let $\mathcal{D}_\nu$ be the Dirac $U(1)_{U(1)}$-bundle over $\tilde{\mathbb{R}}^4$, with connection forms $A_{\nu\alpha}$ ($\alpha = 1, 2$) relative to the presentation $\mathcal{T}^\nu$, as in (6.7), extended from $S^2$ to $\tilde{\mathbb{R}}^4$ in the obvious manner, as described in §6. Let $\mathcal{F}'$ be the trivial $SL(2, \mathbb{C})_{\{e\}}$-bundle over $\tilde{\mathbb{R}}^4$, or essentially $\mathcal{F}' = \mathcal{G}^{SL(2,\mathbb{C})}_{\{e\}}(\tilde{\mathbb{R}}^4)$, with trivial connection forms $A'_\alpha = 0$ corresponding to the trivial presentation $\mathcal{T}' = \{\tilde{\mathbb{R}}^4, \mathcal{F}'\}$. Thus, in this case the local 1-forms $A_\alpha \oplus A'_\alpha = A_{\nu\alpha} \oplus 0$ so the action of $A_\alpha \oplus A'_\alpha$ upon the local section $\bar{v}_\alpha \in \mathcal{V}(U_\alpha)$ is simply pointwise multiplication by the imaginary number $A_{\nu\alpha}(x)$. The conventional Dirac equation in the presence of an electromagnetic potential, transcribed in the present context, becomes

$$\gamma^\mu(\partial_\mu - A_{\nu\mu})\bar{v} = -im\bar{v}. \tag{7.8}$$

Again, by virtue of (7.3), both sides of (7.8) represent well-defined elements of $\mathcal{V}_{\rho \times \rho'}(\mathcal{D}_\nu \times \mathcal{F}')$.

## 8. Sheaf bundles for electromagnetic instantons

As indicated by the remarks in §1, the mechanism of charge quantisation *a la* Dirac is not an exclusive feature of the classical Dirac monopole alone. The essential ingredient is a *nontrivial $U(1)$ bundle over some spacetime*.† Therefore, consideration of various other examples could be theoretically instructive as well as suggesting possible examples of applications to physical phenomena.

The foregoing Ansatz for construction of the Dirac sheaf bundle can likewise be applied to cases where the underlying manifold $X$ is not a two-sphere. Recall that given a principal bundle $P$ with abelian structure group $G$ and transition functions $g_{\alpha\beta}$ for some trivialising atlas $\mathcal{U}$, the powers $g_{\alpha\beta}^n = (g_{\alpha\beta})^n$ for integral $n$ are the transition functions of a principal $G$-bundle $P_n$ with respect to the same atlas. Moreover, if $\{A_\alpha\}$ are the local 1-forms of a connection $A$ on $P$, then one can easily check that $A_{n\alpha} := nA_\alpha$ provide the local 1-forms of a connection $A_n$ on $P_n$. Furthermore, if $F = \mathrm{d}A$, then $F_n = \mathrm{d}A_n = nF$ for the curvature (field strength) of $A_n$. This means that if $I(F^j)$ is a characteristic class of $P$, then the corresponding class of $P_n$ is just $I(F_n^j) = n^j I(F^j)$. In particular, this applies to the Chen classes.

Now, in accordance with the discussion in §6, we can proceed similarly in the context of $U(1)_{U(1)}$-bundles, replacing the integer $n$ by the arbitrary real number $\nu$, and deduce that if $c_j$ is the $j$th Chen class of the circle bundle $P$, then $\nu^j c_j$ is the corresponding Chen class of the $U(1)_{U(1)}$-bundle $\mathcal{P}_\nu$. To illustrate, consider the canonical connection on a circle bundle over the complex projective plane, which defines a gravitational and electromagnetic instanton. As local minima of the Riemannian Hilbert-Maxwell action, such instantons provide significant contributions to the partition function in the joint path integral quantisation of the gravitational and electromagnetic fields, thus playing a role analogous to that of the usual instantons in the pure Yang-Mills theory (cf. Gibbons & Hawking 1979; Eguchi & Hanson 1979).

*Example 4. Gravitational and electromagnetic instanton.* Let $X = \mathbb{CP}^2$ and consider the canonical $U(1)$ bundle over $X$. The complex projective plane $X = \mathbb{CP}^2$ with coordinates $\{z_i\}_{i=1,2,3}$ satisfying $\bar{z}_j z^j = 1$ is the quotient space of the five-sphere $S^5$ by the circle action $z \to \mathrm{e}^{\mathrm{i}\phi}z$. We regard $S^5$ as a subspace of $\mathbb{C}^3$:

$$S^5 = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \,\middle|\, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \right\}. \tag{8.1}$$

The Hopf map $\pi : S^5 \to \mathbb{CP}^2$ is defined by $\pi(z_1, z_2, z_3) = (\bar{z}_3 z_1 + \bar{z}_1 z_3, \mathrm{i}\bar{z}_3 z_1 - \mathrm{i}\bar{z}_1 z_3, \bar{z}_2 z_3 + \bar{z}_3 z_2, -\mathrm{i}\bar{z}_2 z_3 + \mathrm{i}\bar{z}_3 z_2, \bar{z}_3 z_3 - \bar{z}_2 z_2 - \bar{z}_1 z_1)$. Parameterising $S^5$ by

$$z_1 = \sin\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 \mathrm{e}^{\mathrm{i}\phi_1}, \quad z_2 = \sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2 \mathrm{e}^{\mathrm{i}\phi_2}, \quad z_3 = \cos\tfrac{1}{2}\theta_1 \mathrm{e}^{\mathrm{i}\phi_3}, \tag{8.2}$$

where $\theta_1, \theta_2 \in [0, \pi]$, $\phi_i \in \mathbb{R}$, formula (8.2) becomes

$$\begin{aligned} \pi(z_1, z_2, z_3) \quad = \quad & \left( \sin\theta_1 \cos\tfrac{1}{2}\theta_2 \cos(\phi_3 - \phi_1), \sin\theta_1 \cos\tfrac{1}{2}\theta_2 \sin(\phi_3 - \phi_1), \right. \\ & \left. \sin\theta_1 \sin\tfrac{1}{2}\theta_2 \cos(\phi_3 - \phi_2), \sin\theta_1 \sin\tfrac{1}{2}\theta_2 \sin(\phi_3 - \phi_2), \cos\theta_1 \right). \end{aligned} \tag{8.3}$$

As trivialising neighbourhoods, let $U_\alpha = \mathbb{CP}^2 - \{z_\alpha = 0\}$. Writing $(\zeta^1, \zeta^2) = (z_1/z_3, z_2/z_3)$ for the inhomogeneous coordinates on $U_3$, a section $\sigma_3$ over $U_3$ is

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† For a trivial bundle, one need not satisfy any consistency (gauge invariance) relations between local connection forms, hence, the connection on an associated vector bundle could be any real multiple $eA$.

given by

$$\sigma_3 = \frac{1}{\sqrt{1 + |\zeta^1|^2 + |\zeta^2|^2}} \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2 e^{i\varphi_1} \\ \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2 e^{-i\varphi_2} \\ \cos\frac{1}{2}\theta_1 \end{pmatrix}. \tag{8.4}$$

Similarly, we have

$$\sigma_1 = \begin{pmatrix} \sin\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2 \\ \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2 e^{i\varphi_3} \\ \cos\frac{1}{2}\theta_1 e^{-i\varphi_1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \sin\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2 e^{-i\varphi_3} \\ \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2 \\ \cos\frac{1}{2}\theta_1 e^{i\varphi_2} \end{pmatrix}, \tag{8.5}$$

for sections over $U_1$ and $U_2$, respectively. By virtue of the relation $\varphi_1 + \varphi_2 + \varphi_3 = 0$ the transition function of the $U(1)$ bundle on, say, $U_1 \cap U_3 \subset \mathbb{CP}^2$ is given by $\sigma_3 = e^{i\varphi_1}\sigma_1$. Similarly, we have $\sigma_2 = e^{i\varphi_2}\sigma_3$ and $\sigma_1 = e^{i\varphi_3}\sigma_2$. Thus the generic transition function is simply $g_{\alpha\beta} = e^{i\varphi}$. The canonical connection on the bundle is given by the Hermitian inner product $\omega = \langle \bar{z}, dz \rangle$. In terms of the local coordinates on $U_3$, given by $\zeta^1 = \tan\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2 e^{i\varphi_1}$ and $\zeta^2 = \tan\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2 e^{-i\varphi_2}$ the local connection form is $\omega_3 = i[\sin^2\frac{1}{2}\theta_1 \cos^2\frac{1}{2}\theta_2 d\varphi_1 - \sin^2\frac{1}{2}\theta_1 \sin^2\frac{1}{2}\theta_2 d\varphi_2]$. A similar calculation shows that the local connection form on $\{U_\alpha\}_{\alpha=1,2}$ reads $\omega_1 = i[\sin^2\frac{1}{2}\theta_1 \sin^2\frac{1}{2}\theta_2 d\varphi_3 - \cos^2\frac{1}{2}\theta_1 d\varphi_1]$ and $\omega_2 = i[\cos^2\frac{1}{2}\theta_1 d\varphi_2 - \sin^2\frac{1}{2}\theta_1 \cos^2\frac{1}{2}\theta_2 d\varphi_3]$. Using the relation $\varphi_1 + \varphi_2 + \varphi_3 = 0$, we see at once that the local connection forms are related by the gauge transformation $\omega_1 = \omega_2 + g_{12}^{-1}dg_{12}$, where $g_{12} = e^{i\varphi_3}$. Similarly we have $\omega_2 = \omega_3 + g_{23}^{-1}dg_{23}$ with $g_{23} = e^{i\varphi_2}$ and $\omega_3 = \omega_1 + g_{31}^{-1}dg_{31}$ with $g_{31} = e^{i\varphi_1}$. Calculating the field strength $F = d\omega$ on $U_3$, say, we obtain

$$\begin{aligned} F &= i \left[ \sin\theta_1 \, d\theta_1 \wedge \left( \cos^2\tfrac{1}{2}\theta_2 \, d\varphi_1 - \sin^2\tfrac{1}{2}\theta_2 \, d\varphi_2 \right) \right. \\ &\quad \left. - \sin^2\tfrac{1}{2}\theta_1 \sin\theta_2 \, d\theta_2 \wedge (d\varphi_1 + d\varphi_2) \right]. \end{aligned} \tag{8.6}$$

This expression for $F$ in (8.6) agrees with the one obtained by Trautman (1977); an alternative expression is given by Gibbons & Pope (1978) using different local coordinates. The field $F$ is self-dual with vanishing energy-momentum tensor, and thus, along with the Fubini-Study metric on $\mathbb{CP}^2$, solves the Einstein-Maxwell equation with cosmological constant. At this point, one can merely speculate upon the possible incorporation of the corresponding $U(1)_{U(1)}$-bundles $\mathcal{P}_\nu$ into the above-mentioned path integral formalism.

*Example 5.* The field $F$ defined on $\mathbb{CP}^n$ induces solutions to the Maxwell equation on analytic submanifolds of $\mathbb{CP}^n$. Trautman (1977) considered an example given by the Veronese embedding (cf. Brody & Hughston 2001) of $\mathbb{CP}^1$ in $\mathbb{CP}^n$. For $n = 2$ this is the embedding $(z_1, z_2) \hookrightarrow (z_1^2, \sqrt{2}z_1 z_2, z_2^2)$, which defines a conic $\mathcal{C}$ in $\mathbb{CP}^2$. A short calculation shows that, in terms of the spherical polar coordinates $(\theta, \phi)$ of $\mathbb{CP}^1 \simeq S^2$, the local connection forms of the bundle on the two hemispheres $\{U_\alpha\}_{\alpha=1,2}$ are given by $\omega_\alpha = i((-1)^{\alpha+1} - \cos\theta)d\phi$. Comparison with (6.7) for $\nu = 1$ shows that $\omega_\alpha = 2A_\alpha$. Thus, the electromagnetic field induced on $S^2$ corresponds to a magnetic pole of unit strength, which might appropriately be called the *Trautman monopole*. Another elementary example is the solution to the Maxwell equation arising from the Segré embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $\mathbb{CP}^3$; this defines a quadric $\mathcal{Q}$ in $\mathbb{CP}^3$. In terms of the spherical polar coordinates $(\theta_1, \theta_2, \phi_1, \phi_2)$ of $\mathcal{Q}$ one finds at once that the local connection forms on $\mathcal{Q}$

are given by $\omega_\alpha = \frac{1}{2}\mathrm{i}\left[((-1)^{\alpha+1} - \cos\theta_1)\mathrm{d}\phi_1 + ((-1)^{\alpha+1} - \cos\theta_2)\mathrm{d}\phi_2\right]$ in the respective trivialising neighbourhoods $\{U_\alpha\}_{\alpha=1,2}$. Thus, we obtain a pair of disjoint Dirac monopoles. The potential physical significance of the extensions to nonintegral charges $\nu$ would, of course, be similar to that of the simple Dirac sheaf bundles described in §5.

Remark: A nontrivial example of a new solution to the Maxwell equation on a torus $T^2$ can be constructed by pulling the field $F$ back to $T^2$ via the elliptic curve $(z) \hookrightarrow (1, \wp(z), \wp'(z))$ in $\mathbb{CP}^2$. Here, $z$ is the complex coordinate of the torus and $\wp(z)$ denotes the Weierstraß $\wp$-function.

Remark: Since the equivalence classes of principal $U(1)$-bundles over $X$ are indexed by $H^2(X, \mathbb{Z})$ (see Lawson & Michelsohn 1990, Appendix A), any realisation of the usual hypothesis of charge quantisation through a topological mechanism is excluded in cosmological models over a contractible base, such as the Schwarzschild universe (see also Trautman 1979). This constitutes an apparent contradiction with certain grand unified theories which require the existence of monopoles ('t Hooft 1974). The author is not aware of any Ansatz whereby this apparent contradiction could be resolved.

## 9. Quasispin structures and Dirac operators

Consider the short exact sequence of Lie groups

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\ \phi\ } K \longrightarrow 1$$

with $H$ closed and central in $G$. As described in Example 1, a principal $G$-bundle $(P, \pi, X)$ canonically determines a $G_H$-bundle $\mathcal{F}_H(P)$ over $X$. In the special case where $H$ is discrete, a principal $K$-bundle $(Q, \pi, X)$ canonically determines a $G_H$-bundle over $X$ as follows. Let $\{k_{\alpha\beta}\}$ denote the transition functions of $Q$ relative to a trivialising open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of $X$. Since $H$ is discrete, the above epimorphism $\phi : G \to K$ is a local homeomorphism, hence, for any $x \in U_\alpha \cap U_\beta$, there exists a neighbourhood $V_x$ of $x$, with $V_x \subset U_\alpha \cap U_\beta$, and a smooth mapping $g_{\alpha\beta}^x : V_x \to G$ such that $\phi \circ g_{\alpha\beta}^x = k_{\alpha\beta}|_{V_x}$, and any two such $g_{\alpha\beta}^x$ and $g_{\alpha\beta}'^x$ are related by $g_{\alpha\beta}^x = hg_{\alpha\beta}'^x$ for some constant $h \in H$. Therefore, $g_{\alpha\beta}^x$ uniquely defines an element $\tilde{g}_{\alpha\beta}^x$ of $\mathfrak{G}_H^G(V_x)$ (see (3.5)) and hence a continuous section $\bar{g}_{\alpha\beta}^x$ of the sheaf $\mathcal{G}_H^G(V_x)$. For $y \in V_x \cap U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$\phi(g_{\alpha\beta}^x(y)g_{\beta\gamma}^x(y)g_{\gamma\alpha}^x(y)) = k_{\alpha\beta}(y)k_{\beta\gamma}(y)k_{\gamma\alpha}(y) = 1, \tag{9.1}$$

hence, $g_{\alpha\beta}^x(y)g_{\beta\gamma}^x(y)g_{\gamma\alpha}^x(y) \in \mathrm{Ker}(\phi) = H$, and since $H$ is discrete, this product must be constant in some neighbourhood $W_x \subset V_x$ of $x$. Thus,

$$\tilde{g}_{\alpha\beta}^x\tilde{g}_{\beta\gamma}^x\tilde{g}_{\gamma\alpha}^x = 1 \tag{9.2}$$

in $G_H(W_x)$, and we replace $V_x$ by $W_x$. Passing to a suitable refinement of $\mathcal{U}$, we may, by paracompactness, identify the $W_x$ with the intersections $U_\alpha \cap U_\beta$, discard the superscripts $x$ from the $\tilde{g}_{\alpha\beta}^x$, and thereby obtain, as in Example 1, transition sheaf isomorphisms

$$T_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}, \tag{9.3}$$

where $\mathcal{F}_\alpha := \mathcal{G}_H(U_\alpha)$, which satisfy the cocycle condition by virtue of (9.2). We denote the resulting $G_H$-bundle by $\mathcal{S}_Q(H)$.

In particular, if $P$ is a principal $G$-bundle covering $Q$ in the sense that the transition functions $g_{\alpha\beta}$ of $P$ with respect to some common trivialising open covering for $P$ and $Q$ satisfy the relation

$$\phi(g_{\alpha\beta})(x) = k_{\alpha\beta}(x), \tag{9.4}$$

then we may select $g_{\alpha\beta}^x = g_{\alpha\beta}(x)$ in the foregoing construction, and thus $\mathcal{S}_Q(H)$ coincides with the $G_H$-bundle $\mathcal{F}_P(H)$ described in Example 1.

Let $\mathfrak{k}$ denote the Lie algebra of $K$, and let $A$ be a conventional connection on $Q$. Relative to a given system of local trivialisations over the open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$, with transition functions $k_{\alpha\beta}$, $A$ is determined by a family of $\mathfrak{k}$-valued 1-forms $A_\alpha \in \Lambda^1(U_\alpha, \mathfrak{k})$ satisfying

$$A_\alpha(x) = k_{\alpha\beta}(x)A_\beta(x)k_{\beta\alpha}(x) + k_{\alpha\beta}(x)\mathrm{d}k_{\beta\alpha}(x) \tag{9.5}$$

for all $x \in U_\alpha \cap U_\beta$. Relative to another system of local trivialisations over $\mathcal{U}$, with transition functions $k'_{\alpha\beta}(x) = k_\alpha^{-1}(x)k_{\alpha\beta}(x)k_\beta(x)$, the same connection $A$ is represented by the 1-forms

$$A'_\alpha(x) = k_\alpha^{-1}(x)A_\alpha(x)k_\alpha(x) + k_\alpha^{-1}(x)\mathrm{d}k_\alpha(x). \tag{9.6}$$

For a sufficiently fine covering, we can choose sections $g_\alpha \in \Gamma(U_\alpha, G)$ and $g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, G)$ such that $\phi(g_\alpha(x)) = k_\alpha(x)$ and, as above, $\phi(g_{\alpha\beta}(x)) = k_{\alpha\beta}(x)$. These choices are unique modulo constant multiplicative factors in $\Gamma_c(U_\alpha, H)$ and $\Gamma_c(U_\alpha \cap U_\beta, H)$, respectively. Furthermore, the covering map $\phi$ induces a Lie algebra isomophism $\phi_* : \mathfrak{g} \to \mathfrak{k}$. Applying $\phi_*^{-1}$ to (9.5) and (9.6), we obtain

$$\phi_*^{-1}A_\alpha(x) = g_{\alpha\beta}(x)(\phi_*^{-1}A_\beta(x))g_{\beta\alpha}(x) + g_{\alpha\beta}(x)\mathrm{d}g_{\beta\alpha}(x) \tag{9.7}$$

and

$$\phi_*^{-1}A'_\alpha(x) = g_\alpha^{-1}(x)(\phi_*^{-1}A_\alpha(x))g_\alpha(x) + g_\alpha^{-1}(x)\mathrm{d}g_\alpha(x). \tag{9.8}$$

Let $\bar{g}_{\alpha\beta}$ and $\bar{g}_\alpha$ denote the sections of $\mathcal{G}_H^G(U_\alpha \cap U_\beta)$ and $\mathcal{G}_H^G(U_\alpha)$ determined by $g_{\alpha\beta}$ and $g_\alpha$, respectively, and write $\phi_*^{-1}A_\alpha = \tilde{A}_\alpha$, $\phi_*^{-1}A'_\alpha = \tilde{A}'_\alpha$. Then (9.7) and (9.8) imply the relations

$$\tilde{A}_\alpha(x) = \bar{g}_{\alpha\beta}(x)\tilde{A}_\beta(x)\bar{g}_{\beta\alpha}(x) + \bar{g}_{\alpha\beta}(x)\mathrm{d}\bar{g}_{\beta\alpha}(x) \tag{9.9}$$

and

$$\tilde{A}'_\alpha(x) = \bar{g}_\alpha^{-1}(x)\tilde{A}_\alpha(x)\bar{g}_\alpha(x) + \bar{g}_\alpha(x)\mathrm{d}\bar{g}_\alpha(x), \tag{9.10}$$

in accordance with the explanatory remarks following (6.1) and (6.3). Thus, the family of 1-forms $\{\tilde{A}_\alpha\}_{\alpha\in\Lambda}$, corresponding to the presentation $\mathcal{T} = \{U_\alpha, \mathcal{F}_\alpha, \bar{g}_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$, where, as usual, $\mathcal{F}_\alpha = \mathcal{G}_H^G(U_\alpha)$, determines a connection $\mathcal{A}$ on the $G_H$-bundle $\mathcal{S}_Q(H)$ in the sense of Definition 6.

Of particular physical interest is the case where $G = Spin^e(r,s)$, $K = SO^e(r,s)$ (the superscript $e$ denoting the component of the identity) and $\phi$ is the canonical

homomorphism with kernel $H = \mathbb{Z}_2$. Given a principal $K$-bundle $Q$ over $X$, a well-known theorem (Lawson & Michelsohn 1990, Friedrich 2000) states that $Q$ is covered by some principal $G$-bundle $P$ in the above sense iff the second Stiefel class $w_2(Q) \in H^2(X, \mathbb{Z}_2)$ vanishes, and then the equivalence classes (i.e. bundle equivalence compatible with the covering map onto $Q$) of such coverings $P$ (known as *spin structures*) are parameterised by the elements of $H^1(X, \mathbb{Z}_2)$. Inequivalent spin structures may (or may not) be inequivalent as abstract $G$-bundles over $X$. Nevertheless, this can entail no inconsistency with Proposition 4.1, since, if the hypothesis of that proposition is satisfied, then, by the above-cited theorem, there exists at most one spin structure $P$ over $Q$, and hence at most one $\mathcal{F}_P(H)$, which, if existent, coincides with $\mathcal{S}_Q(H)$. However, the *quasispin structure* $\mathcal{S}_Q(H)$ exists in any case, even if $\mathcal{F}_P(H)$ does not.

Specifically, let $X$ be a smooth Lorentzian manifold and $Q$ the orthonormal frame bundle of $X$. The Levi-Civita connection $A_{LC}$ on $Q$ then determines, as above, a connection $\mathcal{A}_{LC}$ on $\mathcal{S}_Q(H)$, and the spin representation $\rho : G \to Aut(\mathbb{S})$, where $\mathbb{S}$ denotes the spin module, gives rise to a space $\mathcal{V} := \mathcal{V}_\rho(\mathcal{S}_Q(H))$ of particle fields (see Definition 7). Covariant differentiation in $\mathcal{V}$ is then well-defined, just as was indicated, using local coordinates, in Examples 2 and 3.

Relative to a given trivialisation over $U_\alpha$ in a sufficiently fine open covering, a quasispinor field $\psi \in \mathcal{V}$ is represented by a smooth section $\psi_\alpha \in \Gamma(U_\alpha, \mathbb{S})$, determined up to a constant factor $h \in H$, that is, up to a sign $\pm 1$. Using the local 1-form $(A_{LC})_\alpha$ relative to the given trivialisation, one obtains a conventional Dirac operator $\slashed{D}_\alpha$ over each $U_\alpha$, which, applied to $\psi_\alpha$, yields a section $\slashed{D}_\alpha \psi_\alpha$, likewise defined up to a sign. Let $\bar{\psi}_\alpha$ denote the element of $\mathcal{V}(U_\alpha)$ determined by $\psi_\alpha$. Since the Dirac operator behaves covariantly with respect to gauge transformations $g \in G$, the relations (7.1), with $\bar{v}_\alpha = \bar{\psi}_\alpha$, imply corresponding relations for the sections $\slashed{D}_\alpha \psi_\alpha$, which thus combine to yield an element $\slashed{D}_\alpha \psi \in \mathcal{V}_\rho(\mathcal{S}_Q(H))$.

According to a well-known theorem of Geroch (1968), the orthonormal frame bundle of a noncompact Lorentzian manifold $X$ possesses a spin structure iff $X$ is parallelisable. A specific example of a noncompact Lorentzian manifold not admitting a spin structure is described, e.g., in Clarke (1971). The present formalism permits the global description of fermionic fields on such spacetimes, and the local properties of quasispinor fields and quasi-Dirac operators in this context are clearly identical with those of their conventional counterparts. The foregoing considerations are likewise applicable to the construction of quasispinor fields and quasi-Dirac operators on oriented Riemannian manifolds, with $G = Spin(n)$, $K = SO(n)$ and $H = \mathbb{Z}_2$. In the conventional theory, the index of the Dirac operator on a 'spinnable' manifold $X$ is expressible, by the Atiyah-Singer index formula, in terms of the Chen classes of $X$ (Lawson & Michelsohn 1990; Berline *et al.* 1992). Even if $X$ is not spinnable, this expression is still well-defined. On the other hand, since $\mathcal{V}$ is not a vector space, the index of the quasi-Dirac operator $\slashed{D}$ cannot be defined in the conventional manner. Of course, one could, by fiat, simply define the index of the quasi-Dirac operator as the value of the Atiyah-Singer expression. However, a more interesting approach would consist in providing a general geometrical definition of the index for differential operators on $\mathcal{V}$, and proving that its value for the quasi-Dirac operator is given by the Atiyah-Singer formula. This appears to pose a not entirely trivial problem.

In the pseudoriemannian case, if $X$ is not space and time orientable, then one

deals with pin structures (see Chamblin and Gibbons 1997 for a discussion of physical applications). Since the homomorphism $Pin(r,s) \to O(r,s)$ is also a double covering, the foregoing construction is directly applicable, and one thus obtains *quasipin structures*, even if true pin structures fail to exist.

$Spin^c$-structures can also be generalised in a similar vein, but since in this case the central subgroup $H = U(1)$ is not discrete, the situation is somewhat more complicated and will be dealt with elsewhere.

## 10.  Discussion

Mathematically sophisticated expositions of gauge theory tend to define connections and curvature *ab initio* on the total space of a principal bundle before pulling these quantities down to the base space via local trivialisations. Some readers may, perhaps, inquire why we have not adopted this more elegant approach. The reason is that our sheaf bundles are not Hausdorff spaces, hence not manifolds, so the tangent spaces are undefined. Consequently, we must adopt the orthodox physical practice of working in terms of local sections.

In the relevant literature one occasionally observes heuristic statements to the effect that topological solitons already display certain quantal effects at the classical level. Since charge renormalisation and running coupling constants can only result from field quantisation in the conventional theory, one could, in a similar vein, assert that the generalised solitons described above already display quantum field theoretic effects at the classical level. The coupling constants can walk even before they begin to run.

We have constructed a family of generalised Dirac monopoles $\mathcal{D}_\nu$ which include the conventional $\mathcal{D}_n$ as special cases. The $\mathcal{D}_n$ interact with conventional wave functions $\psi$, i.e. sections of the associated vector bundles, and likewise, mathematical consistency in the description of interactions involving the $\mathcal{D}_\nu$ requires the introduction of generalised wave functions $\Psi$ which are sections of certain sheaves. Every conventional $\psi$ defines a $\Psi$, but there also exist $\Psi$ that do not arise from any $\psi$. For example, consider normalised wave functions defined over the circle $X = S^1$. In the conventional model, these include, e.g., $\psi(x) = \exp(inx)$ for integral $n$, but not $\exp(i\nu x)$ for arbitrary real $\nu$. In our model, however, the $\exp(i\nu x)$ (and more generally, $\exp(i\nu x)f(x)$, where $f(x)$ is an ordinary normalised complex-valued function on $S^1$) represent well-defined sections of the sheaf $\mathcal{G}_H^G(S^1)$, where $G = H = U(1)$. Thus, rather than demanding knowledge of a global function on $X$, determined only up to a globally constant phase factor, we merely demand, for any point $x \in X$, the knowledge of a function on some neighbourhood $U$ of $x$, and determined only up to a constant phase factor on $U$. The probabilistic interpretation of such a generalised wave function remains the same as in the conventional theory. Moreover, differentiation with respect to $x$ and multiplication by ordinary functions $V(x)$ are well-defined, so we can apply Hamiltonian operators such as $H = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$.

Magnetic monopoles, of either integral charge or otherwise, have not yet been encountered in experimental studies (see, e.g., Milton 2006 for the current status of the experimental limits). The discovery of either nonintegral magnetic monopoles or nonintegral electric charges would tend to indicate that the above-described model is not merely more general mathematically but also more realistic physically than the conventional principal bundle model. Such a discovery might therefore cast

doubt upon the long-standing assumption that the venues for physical phenomena are necessarily manifolds, and suggest that sheaves, which provide a more local description, constitute the appropriate arenas for physical models.

The sheaf-theoretic dequantisation of solitons and instantons for nonabelian gauge groups constitutes a challenging topic for research, and might have implications for grand unified theories. The above treatment of the Dirac sheaf bundle can be adapted to cases where the Dirac monopole is embedded within a nonabelian principal bundle, such as the 't Hooft monopole.

The validity of the model constructed above would have certain obvious physical consequences. Firstly, the detection of a magnetic monopole would not necessarily imply the quantisation of electromagnetic charges. Conversely, an observed violation of charge quantisation would not necessarily imply the nonexistence of magnetic monopoles.

A definite conclusion concerning the actual physical quantisation of electric charges remains elusive. For example, explosions of electron bubbles in liquid helium have been observed in the laboratory, suggesting that fractionally charged particles can possibly exist in isolation (Konstantinov & Maris 2003), although the interpretation of such experiments is controversial (Jackiw *et al.* 2001; Bender *et al.* 2005). Other physical observations apparently consistent with nonintegral charges include the fractional quantum Hall effect (Laughlin 1983) and geometric phase measurements in anisotropic spin systems (Bruno 2004). The possible variability of the fine structure constant over the cosmological timescale (Bekenstein 1982) and the postulated fractional charges of the quarks in the deconfinement phase could also be relevant to this issue. A sheaf-theoretic formulation of the underlying quantum theory along the foregoing lines might conceivably clarify some of these diverse phenomena.

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