Finite Horizon Portfolio Selection with Transaction Costs

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Abstract

We study the problem of maximising expected utility of terminal wealth over a finite horizon, with one risky and one riskless asset available, and with trades in the risky asset subject to proportional transaction costs. In a discrete time setting, using a utility function with hyperbolic risk aversion, we prove that the optimal trading strategy is characterised by a function of time $\zeta(t)$, which represents the ratio of wealth held in the risky asset to that held in the riskless asset. There is a time varying no transaction region with boundaries $\zeta_b(t) < \zeta_s(t)$, such that the portfolio is only rebalanced when $\zeta(t)$ is outside this region. The results are consistent with similar studies of the infinite horizon problem with intermediate consumption, where the no transaction region has a similar, but time independent, characterisation. We solve the problem numerically and compute the boundaries of the no transaction region for typical model parameters. We show how the results can be used to implement option pricing models with transaction costs based on utility maximisation over a finite horizon.

1 Optimal Portfolios and Transaction Costs

The impact of transaction costs on the trading decisions of investors has been studied intensively in recent years. The earliest papers looked at the optimal investment and consumption decisions of an agent seeking to maximise expected utility of consumption over an infinite horizon with just two investment instruments: a riskless bank account B and a risky stock S whose price is usually taken to be a geometric Brownian motion. This problem was first tackled in the absence of transaction costs by Merton (1969, 1971), who was able, in this ideal setting, to derive a closed-form solution to the stochastic control problem faced by the agent. Remarkably, this is one of the few nonlinear stochastic control problems that can be explicitly solved, and it turns out that for utility functions in the HARA (hyperbolic absolute risk aversion) class the optimal investment strategy is to keep a constant fraction of total wealth in the risky asset and to consume at a rate proportional to total wealth (the "Merton strategy").

The introduction of proportional transaction costs to Merton's model was first accomplished by Magill and Constantinides (1976). This yielded the fundamental insight that any attempt to apply the Merton strategy in the face of transaction costs would be infinitely costly since it involves incessant trading, so there must be some no transaction (NT) region inside which the portfolio is not rebalanced. Further insights were provided by Constantinides (1979, 1986), who showed that proportional transaction costs have only a second-order effect on the liquidity premium (the amount of increase in the rate of return of the stock which would be required to compensate the investor for the presence of the transaction costs), essentially because investors deflect the impact of even large transaction costs by drastically reducing the frequency and volume of trading. Then, in a landmark paper Davis and Norman (1990) showed (using the tools of singular stochastic control that were unavailable to Magill and Constantinides in 1976) that in continuous time the NT region is a wedge in (x, y) space, where x, y represent the wealth in the bond and stock respectively. More recently Shreve and Soner (1994) have studied this problem using a viscosity solution approach to Hamilton-Jacobi-Bellman equations of dynamic programming, allowing some restrictive assumptions of Davis and Norman to be removed.

The work cited above deals with the portfolio choice problem over an infinite horizon with intermediate consumption. The problem of maximising the expected utility of final wealth over a finite horizon, without consumption, has received much less attention. It has been studied by Hodges and Neuberger (1989), Davis, Panas and Zariphopoulou (1993), Davis and Panas (1994) and Barles and Soner (1998), for the exponential utility function, in the context of utility maximisation approaches to option pricing with transaction costs. Cvitanić and Karatzas (1996) employ a martingale methodology to the finite horizon problem with a general utility function, and draw some conclusions on the link between utility maximisation and the hedging of a contingent claim. They prove the existence of an optimal trading policy, but they do not analyse the numerical solution of the problem.

In this paper we study the problem of maximising expected utility of wealth over a finite horizon, T, in the presence of proportional transaction costs, for logarithmic and power utility functions. We obtain numerical results for these utility functions and also for exponential utility, using a Markov chain approximation technique pioneered by Kushner (1990). We provide a proof that, for logarithmic and power utility, the optimal trading strategies imply a time-varying no transaction region which is a wedge in (x, y) space, as opposed to the fixed wedge that one obtains in the infinite horizon case. We are not aware of any previous demonstration of this fact. We also indicate how the methods in this paper can be applied to utility based approaches to option pricing with transaction costs, a subject explored in more depth in Monoyios (1998).

Other work on optimal portfolio selection with transaction costs has been presented by Duffie and Sun (1990), Dumas and Luciano (1991), Taksar, Klass and Assaf (1988), Morton and Pliska (1995), Akian, Menaldi and Sulem (1996) and Korn (1998). These models tend to differ in one or other of their specifications. For example, Duffie and Sun (1990) consider the case of fixed plus proportional transaction costs. Morton and Pliska (1995) and Korn (1998) analyse a problem involving fixed costs using impulse control techniques. Taksar, Klass and Assaf (1988) focus on the problem of maximising the long run growth rate of the total wealth.

The present paper is organised as follows. In Section 2 we summarise the results of the no transaction cost problem, for logarithmic, power and exponential utility functions, so as to give insight into the nature of the optimal policies in each case. In Section 3 we introduce transaction costs into the model and give informal arguments concerning the nature of the optimal policies, as well as the dynamic programming eqautions (actually a variational inequality) satisfied by the value function. In Section 4 we use a discrete time setting to show that, for logarithmic and power utility, the optimal trading strategy is characterised by a time varying no transaction region with boundaries $\zeta_b(t) < \zeta_s(t)$, where $\zeta(t)$ represents the ratio of wealth held in the stock to that held in the bond at time t. Then, by discretising the portfolio state space we are able to numerically solve the utility maximisation problem using a Markov chain approximation. Numerical results are presented in Section 5, and in Section 6 we indicate how the numerical techniques of the paper can be applied to various option pricing models with transaction costs. In Section 7 we present our conclusions. In an Appendix we give details of the proof of the nature of the optimal policies for power and logarithmic utility.

2 Finite Horizon Utility Maximisation in a Frictionless Market

In this section we study the classical problem of choosing a trading strategy to maximise utility of wealth over a finite horizon $[0, T]$ in a market that is free from transaction costs, or frictionless. We shall employ results due to Karatzas (1989).

Consider a market consisting of a bond and a stock whose prices $B(t)$ and $S(t)$ at time t satisfy

$$\begin{aligned} dB(t) &= rB(t)dt, \quad B(0) = 1, \\ dS(t) &= S(t)[bdt + \sigma dW(t)], \quad S(0) = S. \end{aligned} \tag{1}$$

Here $W(t)$ is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with natural filtration $\mathbf{F} = \{\mathcal{F}(t), 0 \le t \le T\}$. The coefficients r, b, σ will be taken to be constants in this paper, though for the problem without transaction costs treated in this section they could just as well be taken to be processes $r(t), b(t), \sigma(t)$, which are bounded and progressively measurable with respect to $\mathbf{F}$.

We have an investor who can decide at each instant $t \in [0, T]$ how much money $\pi(t)$ to invest in the stock. If we denote by $X_x^\pi(t) \equiv X(t)$ the wealth of the agent at time t corresponding to a trading strategy π, when starting at time zero with initial wealth x, then $X(t) - \pi(t)$ is the amount invested in the bond and the wealth process satisfies

$$dX(t) \equiv dX_x^\pi(t) = (rX(t) + \pi(t)(b - r))dt + \sigma\pi(t)dW(t). \tag{2}$$

The investor's objective is to maximise expected utility of wealth at a fixed final time T. That is, to find a trading strategy π which acieves the supremum

$$\mathcal{V}(x) = \sup_\pi \mathbf{E}[U(X_x^\pi(T))], \tag{3}$$

where U is an increasing, concave utility function. In this paper we shall be mainly concerned with the functions $U(x) = \log x$, $U(x) = x^\gamma/\gamma$, $\gamma < 1$, $\gamma \neq 0$, but we shall also give some results in the case of the negative exponential utility function $U(x) = -\exp(-\gamma x)$, with constant risk aversion index γ.

If we introduce the martingale

$$Z_0(t) \equiv \exp\left(-\theta W(t) - \frac{1}{2}\theta^2 t\right), \tag{4}$$

where $\theta = (b-r)/\sigma$. Then we can define the (equivalent to $\mathbf{P}$) martingale probability measure $\mathbf{P_0}$ by

$$Z_0(t) = \mathbf{E}\left[\frac{d\mathbf{P_0}}{d\mathbf{P}} \mid \mathcal{F}(t)\right]. \tag{5}$$

The solution to the optimisation problem (3) is given by Karatzas (1989) as follows. There exists an $\mathcal{F}(T)$-measurable random variable (i.e. a contingent claim) Ψ which is "attained" by the optimal trading strategy π^* in the sense that

$$X_x^{\pi^*}(t) = \frac{1}{H_0(t)}\mathbf{E}\left[H_0(T)\Psi|\mathcal{F}(t)\right], \tag{6}$$

where we have defined the positive semimartingale $H_0(t) \equiv e^{-rt}Z_0(t)$. Furthermore, if we define the function $I = (U')^{-1}$ as the inverse of the gradient of U then Ψ has the representation

$$\Psi = I(\mathcal{Y}(x)H_0(T)), \tag{7}$$

where $\mathcal{Y}(x)$ is the inverse of the function

$$\mathcal{X}(x) = \mathbf{E}\left[H_0(T)I(xH_0(T))\right]. \tag{8}$$

Applying the above methodology to our optimisation problem for different utility functions, we obtain the following characterisation of the optimal trading strategy in a frictionless market. We will make use of these values in constructing the numerical solution to the problem with costs later in the paper.

Logarithmic utility, $U(x) = \log x$ **:** The optimal strategy is to keep the ratio $\zeta(t)$ of wealth in the stock to wealth in the bond equal to the constant value $\theta/(\sigma-\theta)$, where $\theta = (b-r)/\sigma$.

Power utlity, $U(x) = x^\gamma/\gamma$, $\gamma < 1$, $\gamma \neq 0$ **:** The optimal strategy is to keep $\zeta(t)$ equal to the constant value $\theta/[(\sigma-\theta)(1-\gamma)]$.

Expnential utility $U(x) = -\exp(-\gamma x)$ **:** The optimal strategy is to keep the *wealth invested in the stock* $\pi(t)$ equal to $e^{-r(T-t)}\theta/(\sigma\gamma)$.

We note the fundamental distinction between the solution for the exponential utility function compared with that for other forms of utility. Namely that the exponential utility function has a constant index of risk aversion so that the relevant variable is the amount of money invested in the stock, and the amount invested in the bond ceases to matter, whilst it is the *ratio* of these two quantities that determines the optimal strategy for logarithmic or power utility functions.

3 Dynamic Portfolios with Transaction Costs

We now introduce proportional transaction costs into the model of the previous section. Then the wealth process becomes inherently two dimensional, in that we consider separately the wealth held in the stock and in the bond.

The investor has wealth $X_x(t)$ dollars invested in the bond and $Y_y(t)$ invested in the stock at time t, with initial values x, y respectively. We define a pair of right-continuous, non-decreasing processes $(L(t), M(t))$ such that $L(t)$ is the cumulative wealth transferred into the stock account up to time t and $M(t)$ is the cumulative wealth transferred out of the stock account, with $L(0) = M(0) = 0$. Then the wealth held in the stock is the following stochastic process:

$$Y(t) \equiv Y_y(t) = y + \int_0^t bY(s)ds + \int_0^t \sigma Y(s)dW(s) + L(t) - M(t). \tag{9}$$

We assume that transfers of wealth between stock and bond incur transaction costs which are proportional to the dollar value of wealth transferred. Thus the cumulative transfer $L(t)$ of wealth into the stock reduces the wealth in the bond by $(1+\lambda)L(t)$, where λ $(0 \leq \lambda < 1)$ represents the proportional transaction cost rate associated with buying stock. Similarly the cumulative transfer $M(t)$ of wealth out of the stock increases the wealth in the bond by $(1-\mu)M(t)$, where μ $(0 \leq \mu < 1)$ represents the proportional transaction cost rate associated with selling stock. Then the wealth held in the bond is the process:

$$X(t) \equiv X_x(t) = x + \int_0^t rX(s)ds - (1+\lambda)L(t) + (1-\mu)M(t). \tag{10}$$

The investor's holdings $(X(t), Y(t))$ are constrained to lie in the closed solvency region

$$\mathcal{S} = \{X(t), Y(t)) : X(t) + Y(t) \geq 0\}. \tag{11}$$

A *trading policy* is a choice of $(L(t), M(t))$ such that the investor's holdings remain within $\mathcal{S}$. We denote the set of admissable trading strategies by $\mathcal{A}(x, y)$ and consider an investor who derives utility $U(X(T) + Y(T))$ from his terminal wealth. The investor's optimisation problem is to find a pair $(L, M) \in \mathcal{A}(x, y)$ that maximizes expected utility from terminal wealth. That is, a policy which attains the supremum

$$V(x, y) \equiv \sup_{(L,M)\in\mathcal{A}(x,y)} \mathbf{E}U(X(T) + Y(T)). \tag{12}$$

An alternative expression for the terminal wealth in (12) can be used if it assumed that the portfolio is converted to cash at the final time. Then the terminal wealth is given by the expression

$$\begin{aligned} &X(T) + (1+\lambda)Y(T), \quad Y(T) \leq 0 \\ &X(T) + (1-\mu)Y(T), \quad Y(T) \geq 0, \end{aligned} \tag{13}$$

and a similar alternative characterisation of the solvency region $\mathcal{S}$ in (11) can also be defined.

3.1 Dynamic Programming Equations

The fundamental insight into portfolio selection problems in the presence of transaction costs was first provided by Magill and Constantinides (1976), who realised that there must be a "no transaction" region in the state space such that the portfolio is not rebalanced if its holdings reside in this region. This was proved more rigorously by Davis and Norman (1990) and Shreve and Soner (1994) in the context of the infinite horizon problem, and Davis, Panas and Zariphopoulou (1993) gave arguments to support this notion in the finite horizon case. Using their insight we can give a sketch of the derivation of the PDE (which turns out to be a variational inequality with gradient constraints) satisfied by the value function of (12). Although we shall not solve the optimisation problem via the PDE, it is useful in describing the nature of the optimal policies and for motivating the Markov chain approximation for the portfolio process $(X(t), Y(t))$ that will be used to compute the optimal trading policy in the next section.

We define $V(t, x, y)$ as the maximum expected utility of wealth at T, when starting at time $t \in [0, T]$ with holdings (x, y). The state space is divided into three distinct regions - the BUY, SELL and no transaction (NT) regions, from which it is optimal to buy stock, sell stock and not to trade, respectively. We denote the boundaries between the NT region and the BUY (SELL) regions by ∂B (∂S).

In the BUY region the value function satisfies

$$V(t, x, y) = V(t, x - (1+\lambda)\delta y_b, y + \delta y_b) \tag{14}$$

where δy_b, the wealth transferred into the stock, can take any positive value up to the one required to take the state to ∂B. Allowing $\delta y_b \downarrow 0$ we have

$$\frac{\partial V}{\partial y} - (1+\lambda)\frac{\partial V}{\partial x} = 0. \tag{15}$$

Similarly, in the SELL region, the value function satisfies the equations

$$V(t, x, y) = V(t, x + (1-\mu)\delta y_s, y - \delta y_s) \tag{16}$$

and

$$\frac{\partial V}{\partial y} - (1-\mu)\frac{\partial V}{\partial x} = 0. \tag{17}$$

In the NT region the process $(X(t), Y(t))$ becomes an uncontrolled diffusion, drifting under the influence of the stock process only, and the value function satisfies

$$\frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + by\frac{\partial V}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 V}{\partial y^2} = 0. \tag{18}$$

and the left hand sides of equations (15) and (17) are non-positive and non-negative respectively.

The above equations can be condensed into the PDE

$$\begin{aligned} \max \quad & \left[\frac{\partial V}{\partial y} - (1+\lambda)\frac{\partial V}{\partial x}, -\left(\frac{\partial V}{\partial y} - (1-\mu)\frac{\partial V}{\partial x}\right),\right. \\ & \left.\frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + by\frac{\partial V}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 V}{\partial y^2}\right] = 0. \end{aligned} \tag{19}$$

For the case of the negative exponential utility function $U(x) = -\exp(-\gamma x)$ the above problem was studied by Davis, Panas and Zariphopoulou (1993) and also by Whalley and Wilmott (1997), who gave an analytic formula for the boundaries of the NT region in the limiting case of small transaction costs. The choice of exponential utility renders the amount of money in the bond irrelevant and reduces the dimensionality of the problem. For other choices of utility function, $U(x) = \log x$ and $U(x) = x^\gamma/\gamma$, the relevant quantity is the ratio $Y(t)/X(t)$ of wealth held in the stock to wealth held in the bond, just as in the no transaction cost case of the previous section. This has been demonstrated for the infinite horizon problem with intermediate consumption by a number of authors (Constantinides (1979, 1986), Davis and Norman (1990), Shreve and Soner (1994)). For the finite horizon case a proof is given in a discrete time setting in the next section and in the Appendix. We shall assume that the boundaries ∂B and ∂S of the NT region are smooth functions of t, denoted by $\zeta_b(t)$ and $\zeta_s(t)$, where $\zeta_b(t) < \zeta_s(t)$, and these boundaries collapse to a single constant ζ^*, independent of time, in the frictionless market case, as we saw in Section 2.

In continuous time one can prove the following properties of the value function $V(t,x,y)$ for utility functions with hyperbolic risk aversion. We shall prove these and more properties of the value function in discrete time, later in the paper.

Proposition 1 *For $U(x) = \log x$ and $U(x) = x^\gamma/\gamma$, $\gamma \in (0,1)$,*

1. *$V(t,x,y)$ is increasing and concave in x and y;*
2. *$V(t,x,y)$ has the homotheticity property: for $\rho > 0$*

$$\begin{aligned} V(t,\rho x,\rho y) &= V(t,x,y) + \log\rho \quad [U(x) = \log x]; \\ V(t,\rho x,\rho y) &= \rho^\gamma V(t,x,y) \quad [U(x) = x^\gamma/\gamma]. \end{aligned}$$

Proof

The proof follows Davis and Norman (1990).

1. This is established by considering convex combinations of initial states (x,y) and control policy (L,M) and using the linearity of equations (9) and (10) and concavity of the utility function. This approach appears in Karatzas and Shreve (1986).

2. From (9) and (10) we see that for any $\rho > 0$,

$$\mathcal{A}(\rho x,\rho y) = \{(\rho L,\rho M) : (L,M) \in \mathcal{A}(x,y)\}. \tag{20}$$

   Therefore, defining the optimal portfolio holdings $(X_x^*(t), Y_y^*(t))$ at time $t \in [0,T]$ by

$$\begin{aligned} V(x,y) \equiv V(0,x,y) &= \sup_{(L,M)\in\mathcal{A}(x,y)} \mathbf{E}U(X_x(T) + Y_y(T)) \\ &\equiv \mathbf{E}U(X_x^*(T) + Y_y^*(T)), \end{aligned}$$

we have from (20) that

$$(X^*_{\rho x}(t), Y^*_{\rho y}(t)) = \rho(X^*_x(t), Y^*_y(t)) \quad \forall t \in [0,T]. \tag{21}$$

In other words the value function satisfies

$$\begin{aligned} V(\rho x, \rho y) = &\quad \sup\nolimits_{\mathcal{A}(\rho x,\rho y)} \mathbf{E} U(X_{\rho x}(T) + Y_{\rho y}(T)) = \quad \mathbf{E} U(X^*_{\rho x}(T) + Y^*_{\rho y}(T)) \\ = &\quad \sup\nolimits_{\mathcal{A}(x,y)} \mathbf{E} U(\rho X_x(T) + \rho Y_y(T)) = \quad \mathbf{E} U(\rho X^*_x(T) + \rho Y^*_y(T)), \end{aligned}$$

from which the homotheticity of $V(x,y)$ follows. Then since $V(t,x,y)$ inherits the same properties of $V(x,y)$, the proof is complete.

The homothetic property implies that $V(t,x,y)$ can be re-expressed as a function of the ratio y/x. Specifically, if we define the function $\psi(t,z)$ by $\psi(t,z) \equiv V(t,1,z)$ then

$$\begin{aligned} V(t,x,y) = &\quad \psi(t, y/x) + \log x \quad [U(x) = \log x] \\ V(t,x,y) = &\quad x^\gamma \psi(t, y/x) \quad [U(x) = x^\gamma/\gamma]. \end{aligned}$$

The above reduction in dimensionality is very suggestive. It strengthens the assertion made earlier that that the optimal trading strategy can be parametrised in terms of the ratio $\zeta(t) \equiv Y(t)/X(t)$. It also suggests that one possible way of solving for the value function $V(t,x,y)$ is to rewrite the variational inequality (19) in terms of $\psi(t,z)$ and exploit the resulting reduction in dimensionality. This is essentially what is done in Davis and Norman (1990) and Shreve and Soner (1994), in the infinite horizon (and hence time independent) problem. The residual time dependence in our case will make this approach perhaps less powerful, but there remains the possibility of using a technique such as Laplace transformation to turn the resulting two variable PDE in (t,z) into an ODE in z. We are currently investigating this method of solution.

The above remarks suggest (and indeed we shall show this in the next section) that the optimal trading strategy $(X^*(t), Y^*(t))$ is a reflected diffusion in the NT region, such that $\zeta_b(t) \le Y^*(t)/X^*(t) \le \zeta_s(t), \forall t \in [0,T]$ (except perhaps at the initial time when the portfolio holdings might lie outside the NT region). If the state is in NT it drifts under the influence of the stock price diffusion on a surface defined by the number of shares, $Y(t)/S(t) =$ constant. If the state is in the BUY or SELL regions an immediate transaction occurs taking the state to ∂B or ∂S. Therefore, the optimal trading strategy $(L^*(t), M^*(t))$ consists of a pair of "local time" type processes which are non-decreasing, adapted, right-continuous processes. Moreover, if we can compute the value function in the NT region along with the boundaries of this region, then we can calculate its value in the BUY and SELL regions using equations (14) and (16).

4 Discretisation and Numerical Solution of the Model

In this section we go to a discrete time setting and formally prove various properties of the value function and of the optimal trading strategy. We shall then construct a discrete state space of portfolio holdings (X,Y) so that the process $(X(t),Y(t))$ is represented by a discrete time discrete space Markov chain. This allows for numerical implementation of a dynamic programming algorithm in which we represent the function $V(t, x, y)$ by a number for each possible value of the initial holdings (x, y) in the discrete state space.

4.1 Time Discretisation and Optimal Solution Properties

We approximate the bond and stock processes in (1) by discrete time processes which generate correspnding discrete time processes for the portfolio holdings $(X(t), Y(t))$. In this setting we can prove rigorously that the optimal trading strategy for logarithmic and power utility is characterised by a NT region whose boundaries are two distinct values of $Y(t)/X(t)$.

We discretise the time interval $[0, T]$ into N steps, each of size δt. The asset prices $B(t), S(t)$ then evolve according to

$$\begin{aligned} B(t+\delta t) &= e^{r\delta t} B(t) \\ S(t+\delta t) &= \varepsilon S(t), \end{aligned} \tag{22}$$

where $\varepsilon = \exp\left[(b - \sigma^2/2)\delta t \pm \sigma\sqrt{\delta t}\right]$, each value occurring with probability one half. If we now take $(X(t), Y(t))$ to be the discrete time portfolio holdings *prior* to a possible transaction at time t, then we have that

$$\begin{aligned} X(t+\delta t) &= e^{r\delta t}(X(t) - v(t) - \theta|v(t)|) \\ Y(t+\delta t) &= \varepsilon(Y(t) + v(t)), \end{aligned} \tag{23}$$

where $v(t)$ represents the amount of wealth (if any) transferred into the stock at time t, and can be positive, negative or zero, and θ represents the appropriate transaction cost parameter. In the notation of the previous section we have the correspondence

$$\begin{array}{rcl} v(t) = & \delta L(t) > 0, & \text{if } \delta L(t) \text{ worth of stock is bought } (\theta = \lambda), \\ v(t) = & -\delta M(t) < 0, & \text{if } \delta M(t) \text{ worth of stock is sold } (\theta = \mu), \\ v(t) = & 0, & \text{if no transaction takes place.} \end{array} \tag{24}$$

The discrete time dynamic programming equation for $V(t, x, y)$ is, in the notation of (23)

$$V(t, x, y) = \max_{v(t)} \mathbf{E}_\varepsilon V(t + \delta t, e^{r\delta t}(x - v(t) - \theta|v(t)|), \varepsilon(y + v(t))), \tag{25}$$

where $\mathbf{E}_\varepsilon$ denotes expectation over the random variable ε. The above form of the dynamic programming algorithm for the portfolio selection problem allows for a proof of the nature of the optimal policies (although it is not the most useful for numerical solution of the problem, which will be done using a so-called Markov chain approximation below). In the Appendix we prove the following.

Theorem 1 *1. For utility functions with hyperbolic risk aversion, $U(x) = \log x$ and $U(x) = x^\gamma/\gamma$ there is a no transaction region at time t, denoted by $NT(t)$ and defined by the set of portfolios $(X(t), Y(t))$ satisfying*

$$NT(t) = \{(X(t), Y(t)) : \zeta_b(t) \le Y(t)/X(t) \le \zeta_s(t)\}, \tag{26}$$

and $\zeta_b(t) < \zeta_s(t)$ are the boundaries of the NT region at time t.

2. *If* $Y(t)/X(t) < \zeta_b(t)$ *then the portfolio lies in the BUY region and shares are bought so as to take the portfolio to the boundary* $\zeta_b(t)$*. The optimal wealth transferred into the stock,* $v^*(t) = \delta L(t) > 0$*, then satisfies*

$$\frac{Y(t) + \delta L(t)}{X(t) - (1+\lambda)\delta L(t)} = \zeta_b(t). \tag{27}$$

3. *Similarly, if* $Y(t)/X(t) > \zeta_s(t)$ *then the portfolio lies in the SELL and the amount of wealth transferred into the stock is* $v^\dagger(t) = -\delta M(t) < 0$ *and satisfies*

$$\frac{Y(t) - \delta M(t)}{X(t) + (1-\mu)\delta M(t)} = \zeta_s(t). \tag{28}$$

Following Constantinides (1986) we refer to such trading policies as *simple*.

Proof

See the Appendix

4.2 A Markov Chain Approximation

We now set out to numerically compute the optimal trading strategies, and in particular the boundaries of the NT region. At first sight one is tempted to try and directly solve the Bellman equation (25), but this presents many difficulties. The equation cannot be solved analytically, and the presence of transaction costs makes the discrete time processes of (23) path dependent. This means that the binomial process for the stock price will generate an exponentially growing number of paths for the portfolio $(X(t), Y(t))$ as the number $N = T/\delta t$ of time steps is increased. This makes a direct numerical solution of the Bellman equation computationally intensive.

A more promising approach is to construct a discrete grid in (X, Y) space to represent possible values that the portfolio $(X(t), Y(t))$ might reach, and to approximate the processes $X(t)$, $Y(t)$ by discrete time, discrete state Markov chains, along the lines pioneered by Kushner (see, for example, Kushner and Dupuis (1992)). The stochastic control problem is then solved for the discrete Markov chain, and the solution of the discrete problem can be shown to converge to the solution of the continuous time model. The success of this approach ultimately lies in the fact that, although the discrete state space may contain merely tens of thousands of points (much less than the billions of paths of the portfolio process), many of the paths will pass through (or close to) the same values. We exploit this fact along with our knowledge of the nature of the optimal policies to make the numerical solution tractable.

A similar technique has been employed by Davis and Panas (1994) for exponential utillty. We employ a different state space to them and describe a modified algorithm below. It should be noted that there are, in general, many Markov chain approximations that will work for a particular problem, and we construct one which adequately reflects the structure and supposed properties of the original problem. The optimisation problem that we are facing here is one of singular control, in which the control processes are not continuous (see Kushner and Martins (1991) for the original application of the Markov chain technique to a singular control problem).

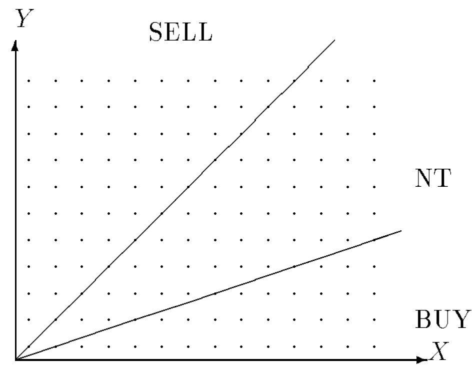

Figure 1: Discrete state space for Markov chain approximation

We discretise the time interval $[0,T]$ into N steps, each of size δt. We also discretise the portfolio state space (X,Y) into discretisation steps h_X, h_Y, as shown schematically in Figure 1. The quantities change in steps, indexed by an integer n, which are either "diffusion" steps (no trading occurs) or "control" steps, where shares are either bought or sold. We define a number of increments, reflecting changes in t, X, Y. The time increment is $\delta t(n) = \delta t$ for a diffusion step and zero for a control step, which takes place instantaneously.

The main requirement of the Markov chain approximation method is to construct chains which "respect" (to order δt) the original features of the problem, in the sense that the changes $\delta X(n)$, $\delta Y(n)$ should have first and second moments which approximate those of the continuous time processes of (10) and (9), which constitute a reflected diffusion in the NT region. This "local consistency" requirement for the process $Y(n)$, for example, is that in a diffusion step

$$\mathbf{E}_n\left[\delta Y(n)\right] = bY(n)\delta t(n) \tag{29}$$

and

$$\mathbf{E}_n\left[\left[\delta Y(n) - \mathbf{E}(\delta Y(n))\right]^2\right] = \sigma^2 Y(n)\delta t(n), \tag{30}$$

where $\mathbf{E}_n$ denotes expectation at n over the next time step. In satisfying local consistency, however, we must also ensure that in a transition the Markov chain $(X(n), Y(n))$ moves from one point in the grid of Figure 1 to another. We shall be solving the Bellman equation for the value function $V(t,x,y)$ by approximating it by a numerical value for each point (x,y) in the state space grid.

The increment describing the change in Y over a diffusion step is

$$\delta Y(n) = (\varepsilon - 1)Y(n) + \delta\tilde{Y}(n) \tag{31}$$

where $\delta\tilde{Y}(n)$ is a random process constructed to ensure that the increment in Y is an integral multiple of h_Y, taking the state to another one of the points on our discrete grid. This process is given by

$$\delta\tilde{Y}(n) = \begin{cases} -qh_Y & \text{with probability } 1-q \\ (1-q)h_Y & \text{with probability } q, \end{cases} \tag{32}$$

where $q = R((\varepsilon-1)Y(n), h_Y)$, and the function $R(a,b)$ is the remainder of a/b.

We also define two increments $\delta L(n)$, $\delta M(n)$, which describe the changes in Y when shares are bought or sold, and are affected only by control steps. They are given by

$$\begin{aligned} \delta L(n) &= h_Y, \quad \text{if } h_Y \text{ worth of stock is bought} \\ \delta M(n) &= h_Y, \quad \text{if } h_Y \text{ worth of stock is sold} \end{aligned} \tag{33}$$

with $\delta L(n) = \delta M(n) = 0$ in a diffusion step. The above increments allow us to write the increment in Y during a control step as

$$\delta Y(n) = \delta L(n) - \delta M(n), \qquad n \text{ a control step.} \tag{34}$$

Finally, the increment describing the change in X is given by

$$\delta X(n) = (e^{r\delta t} - 1)X(n) + \delta\tilde{X}(n), \qquad n \text{ a diffusion step,} \tag{35}$$

where $\delta\tilde{X}(n)$ is constructed in a similar manner to $\delta\tilde{Y}(n)$ and

$$\begin{aligned} \delta X(n) = &\ -(1+\lambda)\delta L(n), \quad n \text{ a "buy" step,} \\ \delta X(n) = &\ (1-\mu)\delta M(n), \quad n \text{ a "sell" step.} \end{aligned} \tag{36}$$

The increments can be used to create discrete time processes $\xi(n)$ defined by

$$\xi(n) = \sum_{i=0}^{n-1} \delta\xi(i) \tag{37}$$

where $\xi(n)$ is any of $t(n)$, $X(n)$, $Y(n)$, $L(n)$, $M(n)$, $\delta\tilde{X}(n)$, $\delta\tilde{Y}(n)$. The value function $V(t,x,y)$ that we are interested in has a discrete analogue, which we also denote by $V(t,x,y)$, when starting at time $t = t(n)$ with initial holdings $X(n) = x$ and $Y(n) = y$. In the next section we shall give an algorithm for computing this value function and the boundaries of the NT region.

4.3 The Discrete Dynamic Programming Algorithm

We apply a dynamic programming algorithm which relates the value function $V(t,x,y)$ to its counterpart at the next time step. This expresses $V(t,x,y)$ as the maximum of the choices available to the investor at each time - namely, buy some shares, sell some shares or do not trade. The discrete Bellman equation that we shall use is

$$\begin{aligned} V(t,x,y) = \ & \max\{\mathbf{E}_n[V(t+\delta t, X_B(t+\delta t), Y_B(t+\delta t))], \\ & \mathbf{E}_n[V(t+\delta t, X_S(t+\delta t), Y_S(t+\delta t))], \\ & \mathbf{E}_n[V(t+\delta t, X_{NT}(t+\delta t), Y_{NT}(t+\delta t))]\}. \end{aligned} \tag{38}$$

where

$$\begin{aligned} X_B(t+\delta t) &= e^{r\delta t}(x - (1+\lambda)\delta L(n)) + \delta\tilde{X}_L(n) \\ Y_B(t+\delta t) &= \varepsilon(y + \delta L(n)) + \delta\tilde{Y}_L(n) \\ X_S(t+\delta t) &= e^{r\delta t}(x + (1-\mu)\delta M(n)) + \delta\tilde{X}_M(n) \\ Y_S(t+\delta t) &= \varepsilon(y - \delta M(n)) + \delta\tilde{Y}_M(n) \\ X_{NT}(t+\delta t) &= e^{r\delta t}x + \delta\tilde{X}_{NT}(n) \\ Y_{NT}(n+1) &= \varepsilon y + \delta\tilde{Y}_{NT}(n), \end{aligned} \tag{39}$$

and the processes $\delta\tilde{X}_L(n)$, $\delta\tilde{Y}_L(n)$, $\delta\tilde{X}_M(n)$, $\delta\tilde{Y}_M(n)$, $\delta\tilde{X}_{NT}(n)$, $\delta\tilde{Y}_{NT}(n)$ are constructed in the same manner as (32). For example

$$\delta\tilde{X}_L(n) = \begin{cases} -qh_X & \text{with probability } 1-q \\ (1-q)h_X & \text{with probability } q, \end{cases} \tag{40}$$

with $q = R(e^{r\delta t}(x - (1+\lambda)\delta L(n)) - x, h_X)$.

The boundary condition at the final time T for the above value function is $V(T, x, y) = U(x+y)$.

The above equations thus determine the value function by comparing: (i) buying h_Y worth of shares and allowing the stock to diffuse or (ii) selling h_Y worth of shares and allowing the stock to diffuse or (iii) allowing the stock to diffuse only. To implement the above algorithm the following sequence of steps is performed. Suppose we know the value function $V(t+\delta t, x, y)$ for all points x, y in the discrete state space. Starting from values x, y which are in the optimal proportions for the problem without transaction costs (e.g. $y/x = \zeta^*$) as at the end of Section 2), we compare the second and third terms in the maximisation operator of (38) for increasing values of y in steps of h_Y, until the former is greater than the latter, at say y_s, which we assume satisfies $y_s = \zeta_s(t)x$, marking the boundary between the NT and SELL regions at time n. We repeat this procedure in decreasing steps of h_Y to locate the boundary $\zeta_b(t)$. Having located the boundaries of the NT region at time n, the value function at all points outside this region is determined by assuming the investor transacts to its boundaries (i.e. applying equations (14) and (16)), whilst the function in the NT region is found by assuming the investor does not transact, and applying

$$V(t, x, y) = \mathbf{E}_n[V(t+\delta t, X_{NT}(t+\delta t), Y_{NT}(t+\delta t))], \tag{41}$$

for all (x, y) in the NT region at time t. Of course, we recognise the right hand side of (41) as the third term in the maximisation of (38).

5 Numerical Results

The algorithm described in the previous section was implemented to compute the optimal trading strategies and boundaries of the NT region for typical model parameters. The main results we can report are:

- The NT region becomes wider as the time difference $T-t$ becomes smaller, as the transaction costs outweigh the benefits of rebalancing the portfolio.
- The NT region becomes wider as the transaction costs increase, with the lower boundary $\zeta_b(t)$ being more sensitive than $\zeta_s(t)$.

We shall report detailed numerical results in a future version of this paper.

6 Application to Option Pricing with Transaction Costs

The utility maximisation problem analysed in this paper can be used to implement option pricing models with transaction costs, as described in Monoyios

(1998). With transaction costs the perfect replication policy of Black and Scholes (1973) becomes infinitely costly, and a number of authors have suggested ways around this problem, with none of them being totally satisfactory. One promising technique is to examine the effect of buying or selling an option on an investor's achievable utility, when it is assumed that the investor is trading to maximise utility at the option expiration time T. Hodges and Neuberger (1989) pioneered this approach, and they defined a value for an option as one which results in the investor achieving the same utility as when not trading the option. Given the utility maximisation problem (3), we ask the question of whether this maximum utility can be increased by the purchase (or short-selling) of a European option whose cash value at time T is some non-negative random variable Γ, the purchase price at time zero being p. To be precise, if an amount of cash δ is diverted into options, we define

$$Q(\delta, p, x) = \sup_{\pi} \mathbf{E}\left[U(X^{\pi}_{x-\delta}(T) + \frac{\delta}{p}\Gamma)\right]. \tag{42}$$

Clearly $Q(0, p, x)$ coincides with $\mathcal{V}(x)$.

Hodges and Neuberger (1989) define the *reservation selling price* P_s of an option with transaction costs as the solution to the equation

$$\mathcal{V}(0) = Q(-P_s, P_s, 0), \tag{43}$$

with the same notation as in (3) and (42). The reasoning behind (43) is clearly that P_s is the money received for the option by an investor with zero initial endowment, who then trades optimally and sets the terminal wealth against the option liability, and which results in the investor achieving the same utility as when not trading the option. (A similar definition of the reservation buying price can also be made.) Because the terminal wealth cannot, in general, replicate the option payoff, we have that $X^{\pi}_{P_s}(T) + \Gamma$ can be negative. This precluded Hodges and Neuberger from using common utility functions such as $U(x) = \log x$, so they specialised their model to the negative exponential function.

An alternative approach, first suggested by Davis (1997) and examined in detail in Monoyios (2000) is to use a "marginal rate of substitution" argument: p is defined as a fair price for the option if diverting an infinitesimal amount of the initial wealth into it at time zero has a neutral effect on the investor's achievable utility. Thus the fair option price is defined as the solution (if one exists) $\hat{p}$ of the equation

$$\frac{\partial Q}{\partial \delta}(0, p, x) = 0. \tag{44}$$

This results in the pricing formula

$$\hat{p} = \frac{\mathbf{E}[U'(X^{\pi^*}_x(T))\Gamma]}{\mathcal{V}'(x)}, \tag{45}$$

where the prime denotes differentiation and π^* denotes the trading strategy which maximises the expected utility in (3). We note that this is the trading strategy which optimises a portfolio *without* options.

The methods described above show promise in that they yield approximate hedging strategies in which the hedging portfolio is only rebalanced at finite intervals, which are optimally chosen by embedding the pricing problem in a utility maximisation framework.

7 Conclusions and Extensions

The main conclusion of this paper is that stochastic control theory provides a promising framework in which to study transaction costs, and that the Markov chain approximation is a powerful technique for solving the computationally challenging portfolio optimisation problem. Transaction costs are a major impediment to the implementation of dynamic hedging strategies and their study is therefore crucial to determining the true nature of risk management policies. At the moment, these are often based on the notions of frictionless markets, where any risk can, by suitable trading, be covered. It is hoped that the study in this paper shows that a more careful analysis of risk management strategies is required, and that this might lead to a reduction in the highly geard positions that investors take in primary assets as well as in derivatives.

There are a number of directions in which this work could be extended, for example, the inclusion of different transaction cost structures and different stock price processes. These would render analytic methods even more obsolete, and numerical methods such as the ones described in this paper would become even more necessary.

Appendix

We prove that the optimal trading strategy is simple for $U(x) = x^\gamma/\gamma$ (the proof for $U(x) = \log x$ follows similar reasoning).

We write the Bellman equation (25) as

$$V(t,x,y) = \max_{v(t)} \phi(t, v(t), x, y), \tag{46}$$

where

$$\begin{aligned} \phi(t, v(t), x, y) &= \mathbf{E}_\varepsilon V(t+\delta t, e^{r\delta t}(x - v(t) - \theta|v(t)|), \varepsilon(y + v(t))), \\ &\equiv \mathbf{E}_\varepsilon V_{t+\delta t}(x_{t+\delta t}, y_{t+\delta t}), \end{aligned}$$

and we have written $V(t+\delta t, \cdot, \cdot) = V_{t+\delta t}(\cdot, \cdot)$, as well as defining $x_{t+\delta t}, y_{t+\delta t}$ according to (23):

$$\begin{aligned} x_{t+\delta t} &= e^{r\delta t}(x - v(t) - \theta|v(t)|), \\ y_{t+\delta t} &= \varepsilon(y + v(t)). \end{aligned} \tag{47}$$

Then the no transaction region (26) is the set of portfolios $(X(t), Y(t)) = (x, y)$ for which $\phi(t, v(t), x, y) \leq \phi(t, 0, x, y)$.

To prove Theorem 1 we analyse the function $\phi(t, v(t), x, y)$, via a series of lemmas. We restrict the analysis to non-negative values of x and y, but the generalisation to the case where borrowing and short-selling is allowed is straightforward (see, for example, Constantinides (1979,1986)).

Lemma 1 *$\phi(t, v(t), x, y)$ is a homogeneous function of degree γ with respect to $v(t)$, x and y.*

Proof

This follows easily from the definition provided that $V(t+\delta t, x, y)$ is homogeneous, and this itself is guaranteed since at the final time $V(T,x,y) = (x+y)^\gamma/\gamma$, so that the value function at all earlier times is homogeneous.

Lemma 2 *$\phi(t, v(t), x, y)$ is concave with respect to $v(t)$.*

Proof

We first show that $\phi(t, v(t), x, y)$ is concave with respect to $v(t)$ provided that $V_{t+\delta t}(x, y)$ is also concave with respect to x and y. In that case $\phi(t, v(t), x, y)$ is clearly concave for $v(t) \neq 0$. For $v(t) = 0$ we have that the right derivative $\partial^{(r)}\phi/\partial v(t)$ is given by

$$\frac{\partial^{(r)}\phi}{\partial v(t)}(t, 0, x, y) = \lim_{v(t)\downarrow 0} \mathbf{E}_\varepsilon \left[\frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}} \frac{\partial x_{t+\delta t}}{\partial v(t)} + \frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}} \frac{\partial y_{t+\delta t}}{\partial v(t)} \right]. \tag{48}$$

Now, for $v(t) > 0$, we have that $\partial x_{t+\delta t}/\partial v(t) = -e^{r\delta t}(1+\lambda)$ and $\partial y_{t+\delta t}/\partial v(t) = \varepsilon$. Hence

$$\begin{aligned} \frac{\partial^{(r)}\phi}{\partial v(t)}(t, 0, x, y) &= -e^{r\delta t}(1+\lambda)\mathbf{E}_\varepsilon \frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}}(e^{r\delta t}x, \varepsilon y) \\ &+ \mathbf{E}_\varepsilon \varepsilon \cdot \frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}}(e^{r\delta t}x, \varepsilon y), \end{aligned} \tag{49}$$

where the dot $(\cdot)$ denotes the inner product in the two-dimensional Euclidean space of the random variable ε.

Similarly the left derivative $\partial^{(\ell)}\phi/\partial v(t)$ at $v(t) = 0$ is given by

$$\begin{aligned} \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t, 0, x, y) &= -e^{r\delta t}(1-\mu)\mathbf{E}_\varepsilon \frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}}(e^{r\delta t}x, \varepsilon y) \\ &+ \mathbf{E}_\varepsilon \varepsilon \cdot \frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}}(e^{r\delta t}x, \varepsilon y), \end{aligned} \tag{50}$$

from which we see that

$$\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t, 0, x, y) \geq \frac{\partial^{(r)}\phi}{\partial v(t)}(t, 0, x, y). \tag{51}$$

Therefore, $\phi(t, v(t), x, y)$ is concave with respect to $v(t)$ everywhere, provided $V(t+\delta t, x, y)$ is concave in x and y. Moreover, since $V(T, x, y) = U(x+y)$ is certainly concave, we have that $V(t, x, y)$ is concave for all $t < T$, and the proof is complete.

The concavity with respect to $v(t)$ implies that any local maximum of $\phi(t, v(t), x, y)$ with respect to $v(t)$ will also be a global maximum. Hence an initial portfolio $(X(t), Y(t)) \equiv (x, y)$ at time t will lie in the NT region if and only if $v(t) = 0$ is a maximum of $\phi(t, v(t), x, y)$ with respect to $v(t)$. Equivalently,

$$(x, y) \in NT(t) \quad \text{iff} \quad \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t, 0, x, y) \geq 0 \quad \text{and} \quad \frac{\partial^{(r)}\phi}{\partial v(t)}(t, 0, x, y) \leq 0. \tag{52}$$

Lemma 3

$$NT(t) = \left\{(x,y) : \zeta_b(t) \le \frac{y}{x} \le \zeta_s(t)\right\},$$

where $\zeta_b(t)$ *and* $\zeta_s(t)$ *are defined by*

$$\begin{aligned} \zeta_b(t) &\equiv \min\left\{y : \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,y) \ge 0, \quad \frac{\partial^{(r)}\phi}{\partial v(t)}(t,0,1,y) \le 0\right\}, \\ \zeta_s(t) &\equiv \max\left\{y : \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,y) \ge 0, \quad \frac{\partial^{(r)}\phi}{\partial v(t)}(t,0,1,y) \le 0\right\}. \end{aligned}$$

Proof

Suppose we are given a portfolio $(x,y) \in NT(t)$. Then from (52) and the homogeneity of $\phi(t,v(t),x,y)$ we obtain

$$\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,y/x) = x^{1-\gamma}\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,x,y) \ge 0.$$

Similarly we can show that

$$\frac{\partial^{(r)}\phi}{\partial v(t)}(t,0,1,y/x) \le 0.$$

Hence we see that the portfolio $(x,y) \in NT(t)$ satisfies $\zeta_b(t) \le y/x \le \zeta_s(t)$.

Conversely, suppose that we have a portfolio (x,y) which is such that $\zeta_b(t) \le \zeta = y/x \le \zeta_s(t)$. Then we can show that this portfolio lies in the NT region. Define

$$\tilde{v}(t) = \frac{\zeta - \zeta_s(t)}{1 + (1-\mu)\zeta} < 0 \tag{53}$$

and

$$\alpha = \frac{\zeta(1 + (1-\mu)\zeta_s(t))}{y(1+(1-\mu)\zeta)} > 0. \tag{54}$$

Since the derivative of $\phi(t,v(t),x,y)$ with respect to $v(t)$ is homogeneous of degree $\gamma - 1$, we have that

$$\alpha^{\gamma-1}\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,x,y) = \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,\alpha x,\alpha y). \tag{55}$$

Using (55) and (50) we find that

$$\begin{aligned} & \alpha^{\gamma-1}\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,x,y) = -e^{r\delta t}(1-\mu)\mathbf{E}_\varepsilon \frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}}(\alpha e^{r\delta t}x, \alpha\varepsilon y) \\ + \quad & \mathbf{E}_\varepsilon \varepsilon \cdot \frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}}(\alpha e^{r\delta t}x, \alpha\varepsilon y) \\ = \quad & -e^{r\delta t}(1-\mu)\mathbf{E}_\varepsilon \frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}}(e^{r\delta t}(1-(1-\mu)\tilde{v}(t)), \varepsilon(\zeta_s(t)+\tilde{v}(t))) \\ + \quad & \mathbf{E}_\varepsilon \varepsilon \cdot \frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}}(e^{r\delta t}(1-(1-\mu)\tilde{v}(t)), \varepsilon(\zeta_s(t)+\tilde{v}(t))) \\ = \quad & \frac{\partial\phi}{\partial v(t)}(t,\tilde{v}(t),1,\zeta_s(t)) \ge \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,\zeta_s(t)) \ge 0. \end{aligned}$$

Hence

$$\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,x,y) \geq 0,$$

and by a similar suitable choice of $\tilde{v}(t)$ and α we can show that, also

$$\frac{\partial^{(r)}\phi}{\partial v(t)}(t,0,x,y) \leq 0,$$

so that $(x,y) \in NT(t)$ and the proof is complete.

Lemma 4 *The boundaries of the NT region satisfy*

$$\frac{\partial^{(r)}\phi}{\partial v(t)}(t,0,1,\zeta_b(t)) = 0, \qquad \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,\zeta_s(t)) = 0.$$

Proof

Consider the first statement, which we shall prove by contradiction. Suppose that $(\partial^{(r)}\phi/\partial v(t))(t,0,1,\zeta_b(t)) < 0$ (we know it cannot be positive from the definition of $\zeta_b(t)$). Then since $\phi(t,v(t),x,y)$ is continuous there exists $y < \zeta_b(t)$ such that $(\partial^{(r)}\phi/\partial v(t))(t,0,1,y) \leq 0$. But from the proof of Lemma 3 we know that there exist $\tilde{v}(t) < 0$ and $\alpha > 0$ (obtained by replacing ζ and $\zeta_s(t)$ in (53) and (54) by y and $\zeta_b(t)$), such that

$$\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,y) = \alpha^{1-\gamma}\frac{\partial\phi}{\partial v(t)}(t,\tilde{v(t)},1,\zeta_b(t)) \geq \alpha^{1-\gamma}\frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,\zeta_b(t)) \geq 0.$$

Thus we have that

$$\frac{\partial^{(r)}\phi}{\partial v(t)}(t,0,1,y) \leq 0, \qquad \frac{\partial^{(\ell)}\phi}{\partial v(t)}(t,0,1,y) \geq 0, \tag{56}$$

so that $\zeta_b(t)$ is not the minimum value of the stock wealth for which (56) holds, which is a contradiction, so the first part of the lemma is proved. By a similar argument the second part of the lemma can also be shown to be true.

Lemma 5 *If $(y/x) < \zeta_b(t)$ then*

$$V(t,x,y) = \max_{v(t)}\phi(t,v(t),x,y) = \phi(t,v^*(t),x,y) = \phi(t,0,\bar{x},\bar{y}), \tag{57}$$

where

$$\begin{aligned} v^*(t) &= \frac{\zeta_b(t)x - y}{1+(1+\lambda)\zeta_b(t)} \\ \bar{x} &= x - (1+\lambda)v^*(t) \\ \bar{y} &= y + v^*(t), \end{aligned} \tag{58}$$

with $(\bar{x},\bar{y}) \in NT(t)$ and $\bar{y}/\bar{x} = \zeta_b(t)$.

If $(y/x) > \zeta_s(t)$ then

$$V(t,x,y) = \max_{v(t)}\phi(t,v(t),x,y) = \phi(t,v^\dagger(t),x,y) = \phi(t,0,\tilde{x},\tilde{y}), \tag{59}$$

where

$$\begin{aligned} v^{\dagger}(t) &= \frac{\zeta_s(t)x - y}{1+(1-\mu)\zeta_s(t)} \\ \tilde{x} &= x-(1-\mu)v^{\dagger}(t) \\ \tilde{y} &= y+v^{\dagger}(t), \end{aligned} \tag{60}$$

with $(\tilde{x},\tilde{y}) \in NT(t)$ *and* $\tilde{y}/\tilde{x} = \zeta_s(t)$.

Proof

We shall prove the first half of the lemma. The proof of the second half follows exactly the same reasoning.

It is easy to see that $v^*(t) > 0$, $\bar{y}/\bar{x} = \zeta_b(t)$, so that $(\bar{x},\bar{y}) \in NT(t)$ and $\phi(t, v^*(t), x, y) = \phi(t, 0, \bar{x}, \bar{y})$, by definition. Then we only need to show that $v^*(t)$ achieves a maximum of $\phi(t, v(t), x, y)$. To do this it suffices to show that

$$\frac{\partial\phi}{\partial v(t)}(t, v^*(t), x, y) = 0,$$

since $v^*(t) > 0$. Now,

$$\begin{aligned} & \frac{\partial\phi}{\partial v(t)}(t, v^*(t), x, y) \\ = \;& -e^{r\delta t}(1+\lambda)\mathbf{E}_{\varepsilon}\frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}}(e^{r\delta t}(x-(1+\lambda)v^*(t)), \varepsilon(y+v^*(t)) \\ + \;& \mathbf{E}_{\varepsilon}\varepsilon \cdot \frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}}(e^{r\delta t}(x-(1+\lambda)v^*(t)), \varepsilon(y+v^*(t)) \\ = \;& (x-(1+\lambda)v^*(t))^{\gamma-1}\left\{-e^{r\delta t}(1+\lambda)\mathbf{E}_{\varepsilon}\frac{\partial V_{t+\delta t}}{\partial x_{t+\delta t}}(e^{r\delta t}, \varepsilon\zeta_b(t))\right. \\ + \;& \left.\mathbf{E}_{\varepsilon}\frac{\partial V_{t+\delta t}}{\partial y_{t+\delta t}}(e^{r\delta t}, \varepsilon\zeta_b(t))\right\} \\ = \;& (x-(1+\lambda)v^*(t))^{\gamma-1}\frac{\partial^{(r)}\phi}{\partial v(t)}(t, 0, 1, \zeta_b(t)), \end{aligned}$$

which equals zero, by Lemma 4, and the proof is complete.

Proof of Theorem 1

With the identification $v^*(t) = \delta L(t) > 0$ and $v^{\dagger}(t) = -\delta M(t) < 0$ we that Lemmas 3 and 5 are equivalent to Theorem 1.

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