

# Nanofluid Flow and Heat Transfer in Channel Entrance Region

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**Abstract** The present work uses the continuum description of nanofluid flow to study the flow, heat and mass transfer in the entrance and developing region of channels or tubes, where the viscous and heat conduction layers are thin and the heat transfer is much more intense than fully developed flow. Instead of supplementing the formulation with thermodynamic properties based on mixture calculations, use is made of recent molecular dynamical computations of such properties, specifically, the density and heat capacity of gold-water nanofluids. The more general formulation results, within the Rayleigh-Stokes (plug flow) approximation and perturbation for small volume fraction, show that the nanofluid density-heat capacity has an enormous effect in the inertia mechanism in causing the nanofluid temperature profile to steepen. The nanofluid thermal conductivity though has an explicit enhancement of the surface heat transfer rate has the almost hidden effect of stretching the nanofluid temperature profile thus giving the opposite effect of enhancement.

**Keywords:** Nanofluid Flow, Heat Transfer, Channel Leading Edge

## 1. Introduction

The purpose of the present contribution is to incorporate the recent results molecular dynamics computations of nanofluid thermodynamic properties, Puliti, et al. [1], in the continuum formulation of the dynamics and thermodynamics of nanofluid flow. The formulation is applied to the entrance region of channels and tubes where the transport layers are thin and where unusual high heat transfer rates have been observed (Wen & Ding 2004; Jung, et al., 2009) even for small volume fractions of the nanoparticles. The physical mechanisms of nanofluid contribution to heat transfer are elucidated from the analytical solutions obtained from the approximate continuum description in this paper.

## 2. Description of the formulation

Buongiorno (2006) gave significant impetus to the continuum description of nanofluid flow by generalizing the multiphase flow description (Bird, et al. 1960). This is applied by Pfautsch (2008) to the numerical solution

of nanofluid boundary layer flow over a flat plate. Recently the author (Liu 2012) studied the leading edge flow in the channel or tube entrance region approximately. The nonlinear effects of momentum, heat and mass transport owing to fluid advection are replaced by the advection effect of a uniform entrance velocity (the plug flow approximation).

The viscous, heat conduction and mass transfer layers are simplified into the Rayleigh-Stokes form of coupled “heat equations”. The problem is further simplified by noticing that the experimentally observed heat transfer enhancement occurs even at very low nanoparticle volume concentrations of a few percent, which allows a perturbation procedure about a pure fluid, in ascending powers of the volume fraction. In this case, the nanofluid problem becomes first order in the volume fraction, while “forced” by the solutions of the zeroth order which are the Rayleigh-Stokes solutions in terms error functions for the momentum, heat and mass transfer problems. Analytical solutions for the first order problems result.

Relative to a clean fluid, it is known that

the nanofluid thermal conductivity would enhance the surface heat transfer rate as coefficient of the temperature gradient at the wall. On the other hand, the enhanced thermal conductivity also has the implicit effect of stretching the modified temperature profile through more efficient thermal conduction and is, at the same time, detrimental to enhanced heat transfer. The nanofluid thermodynamic property in terms of the density-heat capacity product has the competing effect on the structure of the temperature profile: the product is an inertial effect and contributes to the steepening of the temperature profile.

The continuum descriptions in the literature have used the mixture theory to obtain the nanofluid density and heat capacity, with all the shortcomings discussed by Puliti, et al. (2012). Since it is recognized (Liu 2012) that the density-heat capacity product give rise to the mechanism of steepening of the modified temperature profile, the recent work on molecular dynamics computation of the nanofluid properties could provide a most welcomed alternative to that obtained through mixtures.

In this paper, we explore the possibilities of incorporating the molecular dynamics results in terms of analytical solutions obtainable (Liu 2012) for the leading edge plug flow approximation and the perturbation scheme in terms of small nanofluid volume fraction.

### 3. Thermodynamic properties of Au-H<sub>2</sub>O nanofluid according to molecular dynamics computations (Puliti, et al. 2012)

The thermodynamic properties of pure gold and of water are computed as well as gold-water nanofluid at nanoparticle volume fractions of 1%, 10% and 15%. The heat capacity result (Puliti, et al. 2012) is obtained in terms of J/mol K and would thus need to be converted to units of J/kg K for use in the continuum conservation equations. To this end, the effective nanofluid molecular weight would be obtained according to procedures in mixtures and that an effective ideal volume fraction is obtained from the mixture relation

for the density to be consistent with that computed from molecular dynamics. This means that the effective volume fraction thus obtained would differ slightly from that used in the molecular dynamics computation.

The result obtained by Puliti, et al. (2012) is for a nanofluid consisting of gold nanoparticles and water and is shown in the Table 1 that follows. The dimensionless density-heat capacity product is normalized by that for the pure fluid (which is water in this case, for which  $\rho_f c_f = 1.42$  kJ/gK at  $\infty$ ). Subscript  $f$  denotes that of the pure fluid,  $\rho$  is the nanofluid density,  $c$  is the nanofluid heat capacity at constant pressure and  $\phi$  is the volume fraction of the nanoparticles, subscript  $\infty$  denotes the free stream, entrance region values, asterisks denote quantities normalized by that of the pure fluid:

$\phi_\infty$	0.017	0.109	0.168	1.0
$\rho^* c^*$	1.62	7.29	16.36	69.42

Table 1. Density-heat capacity product vs. volume fraction, gold-water nanofluid (adapted from Puliti, et al. 2012)

In anticipation of the perturbation scheme for  $\phi \ll 1$ , nanofluid properties are expanded in ascending powers of  $\phi_\infty$ :

$$\rho^* c^* = 1 + \phi_\infty (d\rho^* c^* / d\phi)_0 \Phi^{(1)} + \mathcal{O}(\phi_\infty^2), \quad (3.1)$$

where  $\Phi^{(1)} = \phi / \phi_\infty$  is the normalized volume fraction in the nanofluid diffusion layer. An approximate estimate from Table 1 gives the value of the slope  $(d\rho^* c^* / d\phi)_0 \cong 36.5$  in regions of small  $\phi_\infty$ . In view of the relatively large values of the initial slope in this specific case of gold-water nanofluid, the allowable free stream volume fraction would have to be limited to about 0.1% or less for the perturbation scheme to be valid. Experiments in metal oxide nanofluids are normally carried out in nanoparticle volume fraction ranges of the order of 0.1%.

From ideal mixture considerations, the density-heat capacity product is an “exact” relation to first order of the volume fraction:

$$\rho^* c^*]_{mixture} = 1 + \phi_\infty \Phi^{(1)} (\rho_p^* c_p^* - 1), \quad (3.2)$$

where  $\rho_p^* c_p^*$  is the dimensionless particle phase density-heat capacity, normalized by that of the fluid phase  $\rho_f c_f$ . Molecular dynamics computations (Puliti, et al. 2012) of the nanoparticle density and heat capacity, after normalization, give  $\rho_p^* c_p^* - 1 = 68.42$ . In the analyses where mixture theory is used (Liu 2012) to obtain the density-heat capacity product, wherever the factor  $\rho_p^* c_p^* - 1$  occurs, it would be replaced by  $(d\rho^* c^*/d\phi)_0$  for an extension to the molecular dynamics computation (Puliti, et al. 2012) of the density-heat capacity product.

### 3.1 Other thermophysical properties

Other thermophysical properties of the composite nanofluid are expressed tentatively, pending possible further molecular dynamics contributions. On the basis of correlations, the nanofluid viscosity is expressed as

$$\mu^* = \mu/\mu_f = 1 + a_1 \phi_\infty \Phi^{(1)} + a_2 \phi_\infty^2 \Phi^{(2)} + \dots$$

The nanofluid thermal conductivity is similarly expressed as

$$k^* = k/k_f = 1 + b_1 \phi_\infty \Phi^{(1)} + b_2 \phi_\infty^2 \Phi^{(2)} + \dots$$

where  $a_1, \dots, b_1, \dots$  are nanoparticle material dependent correlation constants. For Brownian diffusion, the Stokes-Einstein relation is

$$D_B = \sigma_B T / 3\pi\mu d_p,$$

where  $\sigma_B$  is the Boltzmann constant,  $d_p$  is the (average) nanoparticle diameter.

### 4. The Rayleigh-Stokes (plug flow) approximation for the entrance region

The nonlinear continuum description of the dynamics and thermodynamics of nanofluid flow is given in Buongiorno (2006), that for boundary layer flows in Pfautsch (2008). Here, we follow the discussion in Liu (2012) and state the boundary layer form of the conservation equations after subjecting them to the Rayleigh-Stokes approximation, where the nonlinear advective effects are replaced by

that of the known entrance velocity  $U$ ,

$$u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \equiv U \frac{\partial}{\partial x}$$

where  $u, v$  are the streamwise and normal velocity components in the  $x, y$  directions, respectively;  $y$  is measured from the wall surface and  $x$  measured from the leading edge of the entrance region. In this case, the momentum equation becomes

$$\rho U \frac{\partial u}{\partial x} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right), \quad (4.1)$$

where  $\rho$ , already defined, is the density and  $\mu$  the viscosity of the nanofluid. In the leading edge of the entrance region, approximated by parallel plates, the upper and lower wall transport layers are not yet intersecting so that the streamwise pressure gradient can still be set to zero. The energy transport equation in terms of the static enthalpy for low Mach numbers is

$$\rho c U \frac{\partial T}{\partial x} = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) - j_{p,y} \left( c_p \frac{\partial T}{\partial y} \right), \quad (4.2)$$

Where  $c_p \partial T / \partial y = \partial h_p / \partial y$ ,  $h_p$  and  $c_p$  are the enthalpy and heat capacity of the nanoparticle phase. The nanoparticle phase is assumed to be in thermal equilibrium with the base fluid at the local temperature  $T$  of the nanofluid. The diffusion flux  $j_{p,y}$  consists of Brownian diffusion and thermal diffusion (Buongiorno 2006). The latter is estimated to be a weak effect compared to the former mechanism so that at the outset, the diffusion flux is written simply as

$$j_{p,y} = -\rho_p D_B \frac{\partial \phi}{\partial y}.$$

The continuity equation for the nanoparticle phase in the nanofluid leads to the diffusion equation.

$$\rho_p U \frac{\partial \phi}{\partial x} = -\frac{\partial j_{p,y}}{\partial y}. \quad (4.3)$$

The boundary conditions are

$$\begin{aligned} y=0: u=0, T=T_w, j_{p,y}=0 \\ y=\infty: u=U, T=T_\infty, \phi=\phi_\infty \end{aligned} \quad (4.4)$$

where the subscript  $w$  denotes condition at the wall and  $\infty$  that in the free stream far away from the wall. The zero flux wall-boundary condition  $j_{p,y=0}=0$  is suggested by Buongiorno (2006) as the natural one to impose for non-porous, inert solid walls.

The global continuity equation, under the Rayleigh-Stokes approximation, is

$$U \frac{\partial \ln \rho}{\partial x} + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \quad (4.5)$$

Although the individual phases are “incompressible”, the composite nanofluid does not necessarily behave like an incompressible fluid unless the volume fraction is identically zero as indicated by the mixture calculation

$$\rho_{mix} = \phi \rho_p + (1 - \phi) \rho_f, \quad (4.6)$$

or indicated by the only available molecular dynamics computation (Puliti, et al. 2012) which gives the density ratio in the form

$$\rho_{MD} / \rho_f = 1 + (\partial \rho / \partial \phi)_0 \phi + \mathcal{O}(\phi^2), \quad (4.7)$$

In the present simplified leading edge formulation, the global continuity equation serves as a means to calculate the normal velocity  $v$ , if desired, from the  $u$  velocity and the nanofluid density (which can be cast in terms of the volume fraction  $\phi$ ).

The characteristic flow quantities of the entrance region are used to render the simplified conservation equations and nanofluid properties dimensionless

$$\rho^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\text{Re}_D} \frac{\partial}{\partial y^*} \left( \mu^* \frac{\partial u^*}{\partial y^*} \right), \quad (4.8)$$

$$\begin{aligned} \rho^* c^* \frac{\partial \theta}{\partial y^*} &= \frac{1}{\text{Pr}_f \text{Re}_D} \frac{\partial}{\partial y^*} \left( k^* \frac{\partial \theta}{\partial y^*} \right) \\ &+ \frac{\phi_\infty}{\text{Pr}_f \text{Re}_D \text{Le}} D_B^* \frac{\partial \Phi}{\partial y^*} \frac{\partial \theta}{\partial y^*}, \end{aligned} \quad (4.9)$$

$$\frac{\partial \Phi}{\partial x^*} = \frac{1}{\text{Sc}_f \text{Re}_D} \frac{\partial}{\partial y^*} \left( D_B^* \frac{\partial \Phi}{\partial y^*} \right), \quad (4.10)$$

and the global continuity equation becomes,

$$v^* - v^*(0) = - \int_0^{y^*} \frac{\partial}{\partial x^*} (u^* + \ln \rho^*) dy^* \quad (4.11)$$

once integrated, becomes an independent calculation of the normal velocity.

The boundary conditions in dimensionless form are

$$\begin{aligned} y^*=0: u^*=0, \theta=0, \partial \Phi / \partial y^*=0, \\ y^*=\infty: u^*=1, \theta=1, \Phi=1. \end{aligned} \quad (4.12)$$

The dimensionless quantities above that have not been defined are:

$$u^*, v^* = u/U, v/U, \theta = (T - T_w) / (T_\infty - T_w),$$

$$c^* = c/c_f, D_B^* = D_B / D_{B,ref},$$

$$x^*, y^* = x/D, y/D, \text{Re}_D = UD/v_f, \text{Pr}_f = \nu_f / \alpha_f,$$

$$\text{Le} = k_f / \rho_p c_p D_{B,ref}, \text{Sc}_f = \nu_f / D_{B,ref},$$

The reference diffusion coefficient is  $D_{B,ref} = \sigma_B T_\infty / 3\pi \mu_\infty d_p$ ,  $D$  is the channel width,  $\text{Re}_D$ ,  $\text{Pr}_f$ ,  $\text{Le}$ ,  $\text{Sc}_f$  are the Reynolds, Prandtl, Lewis and Schmidt numbers, respectively. The Lewis number definition differs from that in Buongiorno (2006) in that the parameter  $\phi_\infty$  has been taken out of the denominator here as it will serve as an explicit perturbation parameter that follows.

## 5. The perturbation problem for $\phi_\infty \ll 1$

The thermodynamic properties discussed are already cast into the form of ascending powers of  $\phi_\infty$ . The simplified conservation equations, however, are still nonlinear through the dependence on the volume fraction. In general, experiments on nanofluid heat transfer (e.g., Jung, et al. 2009, Wen & Ding 2004) are performed with  $\phi_\infty = \mathcal{O}(10^{-2})$ . In this case, it seems opportune to apply a perturbation procedure to the conservation equations and the thermodynamic properties for small  $\phi_\infty$ .

Thus for  $\phi_\infty \ll 1$

$$\begin{aligned} u &= u^{(0)} + \phi_\infty u^{(1)} + \mathcal{O}(\phi_\infty^2), \\ \theta &= \theta^{(0)} + \phi_\infty \theta^{(1)} + \mathcal{O}(\phi_\infty^2), \\ \phi &= \phi_\infty \Phi^{(1)} + \mathcal{O}(\phi_\infty^2). \end{aligned} \quad (5.1)$$

The zeroth order solution is that of a clean

fluid. Similarly, the thermodynamic properties are expressed as

$$\rho^* = 1 + \phi_\infty (d\rho^* / d\phi)_0 \Phi^{(1)} + \vartheta(\phi_\infty^2) \quad (5.2)$$

$$\rho^* c^* = 1 + \phi_\infty (d\rho^* c^* / d\phi)_0 \Phi^{(1)} + \vartheta(\phi_\infty^2). \quad (5.3)$$

where the derivatives  $(\rho^*)_0'$ ,  $(\rho^* c^*)_0'$  are obtained from the molecular dynamics computations of Puliti, et al. (2012). Other properties, which are not obtained from molecular dynamics computations, would be obtained from the literature:

$$\begin{aligned} \mu^* &= 1 + \phi_\infty a_1 \Phi^{(1)} + \vartheta(\phi_\infty^2), \\ k^* &= 1 + \phi_\infty b_1 \Phi^{(1)} + \vartheta(\phi_\infty^2), \end{aligned} \quad (5.4)$$

$$D_B^* = 1 + \phi_\infty D_B^{(1)} + \vartheta(\phi_\infty^2),$$

the dimensionless perturbation diffusion coefficient  $D_B^{(1)}$  does not enter into problems of interest in the approximations to follow.

### 5.1 The zeroeth order problem

The zeroeth order effects, though not of primary interest, are nevertheless inputs to the  $\vartheta(\phi_\infty)$  effects which is sought here. It suffices to summarize the zeroeth order solutions, which are for  $u^{(0)}$  and  $\theta^{(0)}$ , satisfying the zeroeth order boundary conditions from (4.12),

$$\begin{aligned} u^{(0)}(\eta_u) &= \text{erf}(\eta_u), \\ \theta^{(0)}(\eta_T) &= \text{erf}(\eta_T), \end{aligned} \quad (5.5)$$

where  $\text{erf}(\eta_u)$  and  $\text{erf}(\eta_T)$  are the error functions in terms of the respective velocity and temperature similarity variables

$$\begin{aligned} \eta_u &= y^* / 2\sqrt{x^* / \text{Re}_D}, \\ \eta_T &= y^* / 2\sqrt{x^* / \text{Re}_D \text{Pr}_f}. \end{aligned} \quad (5.6)$$

The respective momentum and thermal diffusion layers are thus estimated to grow like  $\delta_u \approx \sqrt{x^* / \text{Re}_D}$ ,  $\delta_T \approx \sqrt{x^* / \text{Re}_D \text{Pr}_f}$ . For water,  $\text{Pr}_f = 7.01, 0.98$  at  $T^0C = 20, 180$ , respectively; it is thus expected that  $\delta_u / \delta_T \geq \vartheta(1)$ .

The first order effects are obtained by substitution of (5.1)-(5.4) into (4.8)-4.10) and equating effects to  $\vartheta(\phi_\infty)$ . The perturbation expansions are carried out in the dimensionless physical coordinates and, as was performed for

the zeroeth order problem, are subsequently transformed to the respective similarity variables,  $\eta_u, \eta_T$  and the volume fraction (nanoparticle concentration) diffusion variable

$$\eta_\phi = y^* / 2\sqrt{x^* / \text{Re}_D \text{Sc}_f},$$

for the coupled problems of momentum, heat and mass transfer (4.8)-(4.11).

### 5.2 The first order problems

The first order momentum equation becomes

$$\begin{aligned} \frac{d^2 u^{(1)}}{d\eta_u^2} + 2\eta_u \frac{du^{(1)}}{d\eta_u} = & \\ - \left[ \left( d\rho^* / d\phi \right)_0 \Phi^{(1)}(\eta_\phi) 2\eta_u \frac{du^{(0)}(\eta_u)}{d\eta_u} \right. & \\ \left. + a_1 \frac{d}{d\eta_u} \left( \Phi^{(1)}(\eta_\phi) \frac{du^{(0)}(\eta_u)}{d\eta_u} \right) \right] & \end{aligned} \quad (5.8)$$

The left side differential operator is identical to that for  $u^{(0)}(\eta_u)$  which satisfied the boundary conditions of (4.12). Eq. (5.8) is an inhomogeneous heat equation in similarity variables and satisfies homogeneous boundary conditions  $u^{(1)}(0) = 0, u^{(1)}(\infty) = 0$ .

The energy conservation equation to first order in similarity form is

$$\begin{aligned} \frac{d\theta^{(1)}}{d\eta_T^2} + 2\eta_T \frac{d\theta^{(1)}}{d\eta_T} = & \\ - \left[ \left( d\rho^* c^* / d\phi \right)_0 \Phi^{(1)}(\eta_\phi) 2\eta_T \frac{d\theta^{(0)}(\eta_T)}{d\eta_T} \right. & \\ \left. + b_1 \frac{d}{d\eta_T} \left( \Phi^{(1)}(\eta_\phi) \frac{d\theta^{(0)}(\eta_T)}{d\eta_T} \right) \right] & \quad (5.9) \\ + \frac{1}{Le} \left[ \frac{d\Phi^{(1)}(\eta_\phi)}{d\eta_T} \frac{d\theta^{(0)}(\eta_T)}{d\eta_T} \right], & \end{aligned}$$

Eq. (5.9), which is indeed an inhomogeneous heat equation, which satisfies homogeneous boundary conditions  $\theta^{(1)}(0) = \theta^{(1)}(\infty) = 0$ . The definition of similarity variables in the different diffusive regions will be used to evaluate, for instance,  $d\eta_T / d\eta_\phi = \sqrt{Le_f}$ , where  $Le_f = \text{Sc}_f / \text{Pr}_f$ , (which differs from the definition of  $Le$  defined earlier by a factor of  $\rho^* c^*$ ).

It is recognized earlier that thermal diffusion is weak compared to Brownian diffusion in arriving at (4.10), thus in similarity form, the diffusion equation is

$$\frac{d^2\Phi^{(1)}}{d\eta_\phi^2} + 2\eta_\phi \frac{d\Phi^{(1)}}{d\eta_\phi} = 0, \quad (5.10)$$

The boundary conditions are zero flux at the wall, which, in the absence of thermal diffusion, is  $(d\Phi^{(1)}/d\eta_\phi)_{\eta_\phi=0} = 0$ ,

and  $\Phi^{(1)}(\infty) = 1$ . With the zero-flux wall boundary condition, a solution of (5.10) is simply uniform volume fraction throughout the concentration diffusion layer  $\Phi^{(1)}(\eta_\phi) = 1$ .

In addition, Brownian diffusion coefficients are estimated to be much small than the kinematic viscosity and thermal diffusivity of water so that

$$\delta_u/\delta_\phi \approx \sqrt{Sc_f} \gg 1, \quad \delta_T/\delta_\phi \approx \sqrt{Le_f} \gg 1.$$

These observations will bring enormous simplification to the uncoupling of the first order momentum and temperature problems.

Under these simplifications, the first order momentum problem becomes

$$\frac{d^2u^{(1)}}{d\eta_u^2} + 2\eta_u \frac{du^{(1)}}{d\eta_u} = - \left[ (d\rho^*/d\phi)_0 2\eta_u \frac{du^{(0)}}{d\eta_u} + a_1 \frac{d^2u^{(0)}}{d\eta_u^2} \right], \quad (5.11)$$

and the energy equation becomes

$$\frac{d^2\theta^{(1)}}{d\eta_T^2} + 2\eta_T \frac{d\theta^{(1)}}{d\eta_T} = - \left[ (d\rho^*c^*/d\phi)_0 2\eta_T \frac{d\theta^{(0)}}{d\eta_T} + b_1 \frac{d^2\theta^{(0)}}{d\eta_T^2} \right]. \quad (5.12)$$

We shall concentrate on the thermal problem (5.12), while the momentum problem (5.11) is now of peripheral interest to the heat transfer problem and can always be similarly solved.

## 6. The temperature distribution

The right side of (5.12), upon differentiation of the error function profile from (5.5), (5.12) become explicit functions of  $\eta_T$  and the coefficients  $(d\rho^*c^*/d\phi)_0$ ,  $b_1$ :

$$\frac{d^2\theta^{(1)}}{d\eta_T^2} + 2\eta_T \frac{d\theta^{(1)}}{d\eta_T} = - \left[ (d\rho^*c^*/d\phi)_0 - b_1 \right] \frac{4\eta_T}{\sqrt{\pi}} \exp(-\eta_T^2), \quad (6.1)$$

Subjected to the homogeneous boundary conditions  $\theta^{(1)}(0) = \theta^{(1)}(\infty) = 0$ . Eq. (6.1) is a first order inhomogeneous differential equation for  $d\theta^{(1)}/d\eta_T$  which has the solution (Murphy 1960, p. 14)

$$d\theta^{(1)}/d\eta_T = \left[ - \left( 2/\sqrt{\pi} \right) \left( (d\rho^*c^*/d\phi)_0 - b_1 \right) \eta_T^2 + C_1 \right] \exp(-\eta_T^2),$$

where  $C_1$  is an integration constant. Upon a straight forward integration, the first order temperature profile becomes

$$\theta^{(1)}(\eta_T) = C_1 \left( \sqrt{\pi}/2 \right) \text{erf}(\eta_T) - \left( 1/\sqrt{\pi} \right) \left[ (d\rho^*c^*/d\phi)_0 - b_1 \right] \times \left[ \left( \sqrt{\pi}/2 \right) \text{erf}(\eta_T) - \eta_T \exp(-\eta_T^2) \right] + C_2.$$

The condition at the wall gives  $C_2 = 0$ . The condition far from the wall gives

$$C_1 = \left( 1/\sqrt{\pi} \right) \left[ (d\rho^*c^*/d\phi)_0 - b_1 \right], \quad (6.2)$$

which is the important first order temperature gradient at the wall

$$\frac{d\theta^{(1)}(0)}{d\eta_T} = C_1 \left( (d\rho^*c^*/d\phi)_0 - b_1 \right). \quad (6.3)$$

The first order temperature profile thus becomes

$$\theta^{(1)}(\eta_T) = \left( 1/\sqrt{\pi} \right) \left[ (d\rho^*c^*/d\phi)_0 - b_1 \right] \eta_T \exp(-\eta_T^2). \quad (6.4)$$

The overall temperature profile to first order is

$$\theta = \text{erf}(\eta_T) + \phi_\infty \theta^{(1)}(\eta_T). \quad (6.5)$$

## 7. Heat transfer at the wall

For the zero concentration diffusion flux condition at the wall, the surface heat transfer rate is accomplished by conduction alone,

$$q_0 = - \left( \frac{\partial T}{\partial y} \right)_0. \quad (7.1)$$

In terms of the similarity results, (7.1)

becomes

$$q_0 = k_f (1 + \phi_\infty b_1) \frac{T_\infty - T_w}{2D} \sqrt{\frac{\text{Re}_D \text{Pr}_f}{x/D}} \times \left[ \frac{2}{\sqrt{\pi}} + \phi_\infty \frac{1}{\sqrt{\pi}} \left[ (d\rho^*c^*/d\phi)_0 - b_1 \right] \right], \quad (7.2)$$

where we have used  $(d\theta^{(0)}/d\eta_T)_0 = 2/\sqrt{\pi}$  and (6.2) and (6.3). The pure fluid heat transfer rate is obtainable from (7.2) by setting  $\phi_\infty = 0$ ,

$$q_{0,f} = k_f \frac{T_\infty - T_w}{2D} \sqrt{\frac{\text{Re}_D \text{Pr}_f}{x/D}} \frac{2}{\sqrt{\pi}}. \quad (7.3)$$

The normalized nanofluid surface heat transfer rate becomes

$$\frac{q_0}{q_{0,f}} = (1 + \phi_\infty b_1) \left[ 1 + \phi_\infty \frac{(d\rho^*c^*/d\phi)_0 - b_1}{2} \right], \quad (7.3)$$

where the first bracket  $(1 + \phi_\infty b_1)$  is the heat transfer enhancement owing to the enhanced thermal conductivity; the second bracket reflects the nanofluid effect on the temperature profile at the wall, or temperature profile in general (6.4). The inertial effect of  $(d\rho^*c^*/d\phi)_0 > 0$  steepens the temperature profile (6.4) and the temperature gradient at the wall (6.2), (6.3), hence it contributes heat transfer enhancement. The negative  $-b_1$  indicates that the enhanced nanofluid thermal conductivity actually has the opposite effect: it stretches the temperature profile and renders it less steep. With these clear indications of nanofluid effects, we could now combine the explicit thermal conductivity effect and the effects of inertia and conductivity on the temperature profile for small  $\phi_\infty$ :

$$q_0^* = 1 + (dq_0^*/d\phi)\phi_\infty, \quad (7.4)$$

where

$$(dq_0^*/d\phi) = \left[ (d\rho^*c^*/d\phi)_0 + b_1 \right] / 2$$

$$q_0^* = q_0/q_{0,f}$$

## 8. Numerical applications of the results

Adapted from Puliti, et al. (2012), from

Table 1:  $(d\rho^*c^*/d\phi)_0 \cong 36.5$  for gold-water nanofluid. The constant  $b_1$  for gold nanofluid is more difficult to estimate from the literature. From Buongiorno, et al. (2009), Table VI, Set 2, Sample 1 is for gold nanoparticles (10nm) in water plus stabilizer at a very low volume fraction of 0.001%, the measured ratio is  $k/k_f \approx 1.004 - 1.010$  for sample average including standard error of the mean, thus giving a very large range for  $b_1 \approx 700 - 1000$ . The calculated conductivity ratio, using the assumptions described in Buongiorno, et al. (2009), Appendix B in the model of Nan, et al. (1997) gave, unfortunately, an upper and lower bound of unity for this ratio so that  $b_1 = 0$ . Thus the  $b_1$  deduced is too wide a range to be acceptable. On the other hand, Maxwell's (1881) model gives, for very large nanoparticle conductivity ratio and very small volume ratio, the coefficient  $b_1 = 3$ .

Pending on further experimental and theoretical results on the conductivity of gold-water nanofluids, we use Maxwell (1881) conductivity and the density-heat capacity from Puliti, et al. (2012) to estimate

$$q_0^* \Big|_{\text{gold-water}} - 1 \approx 20\phi_\infty. \quad (8.1)$$

In this case, for a 1% volume fraction, the heat transfer enhancement would be about 20%.

Though the numerical estimates appear to be somewhat optimistic, the fluid dynamics and thermodynamics rationale have gone into such estimate.

## 9. Concluding remarks

Further molecular dynamics computations of thermodynamic properties and transport properties, including applications to other nanofluids, would be of significant interest in uncovering the role of nanofluids in heat transfer. Simultaneously, improving the continuum description to include the nonlinear effects of fluid advection while still retaining the perturbation scheme for small volume fraction, would go a long way not only in the "prediction" of nanofluid heat transfer but also to understand the underlying role of inertia and

of transport by diffusive actions.

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